In this note we show that the ring $A = \mathbf{R}[X,Y]/(X^2+Y^2+1)$ is *not* Euclidean, but is a principal ideal domain.

Elements of the $\mathbf{R}$-algebra A can be written in a unique way as $a+bY$ with $a,b \in \mathbf{R}[X]$. When $f = a+bY \in A$, we put $\overline{f} = a - bY$. The map $f \to \overline{f}$ is an $\mathbf{R}[X]$-algebra automorphism of A. The map $A \longrightarrow \mathbf{R}[X]$ given by $f \mapsto f\overline{f}$ is multiplicative. For $f = a+bX$, we have $f\overline{f} = a^2 + (X^2+1)b^2$. This can be used to give easy proofs of the facts that A is a domain and that $A^* = \mathbf{R}^*$.

The $\mathbf{R}$-algebra $\mathbf{R}[X,Y]/(X^2+Y^2+1)$ cannot be a Euclidean domain. Indeed, if $N : A - \{0\} \longrightarrow \mathbf{N}$ were a Euclidean norm, then let $a \in A - A^*$ with $N(a)$ *minimal.* Then the natural map

$$\mathbf{R} = A^* \cup \{0\} \longrightarrow A/(a).$$

is a surjective, and hence bijective, $\mathbf{R}$-algebra homomorphism. However, since the conic $X^2+Y^2+1=0$ does not have any real points, there does not exist any $\mathbf{R}$-algebra morphism $A \longrightarrow \mathbf{R}$. Therefore A is not Euclidean.

Lemma. *Let f be a non-zero element of A. Then $A/(f)$ is a finite dimensional real vector space and* $\dim A/(f) = \deg \overline{f}f$.

Proof. Since A is free of rank 2 over $\mathbf{R}[X]$, the lemma is true for $f \in \mathbf{R}[X]$. For any $f \in A$ we have $\dim A/(f) = \dim A/(\overline{f})$. Since the natural map

$$A/(f) \xrightarrow{\overline{f}} A/(f\overline{f})$$

is injective with cokernel $A/(\overline{f})$, we have

$$2\dim A/(f) = \dim A/(f) + \dim A/(\overline{f}) = \dim A/(f\overline{f}) = 2\deg \overline{f}f,$$

as required.

To see that A is a principal ideal domain, we first observe that A-ideals of even codimension are principal. Indeed, if an ideal I has codimension $2d$ for some natural number d, then the $2d+1$ elements $1, X, Y, X^2, XY, \ldots, X^d, X^{d-1}Y$ are $\mathbf{R}$-linearly dependent in A/I. This implies that I contains a non-zero element f for which $f\overline{f}$ has degree $\leq 2d$. By the lemma the ideals I and (f) have the same codimension, so that $I = (f)$, as required.

We finish the proof by showing that *all* non-zero A-ideals have even codimension. Since non-zero ideals have finite codimension, one does not need Zorn's Lemma to see that any ideal is contained in a maximal ideal. It suffices to show that all maximal ideals of A have even codimension. This is clear if you know that $\mathbf{C}$ is an algebraic closure of $\mathbf{R}$, but this argument can easily be avoided as follows.

Let $\mathfrak{m}$ be a maximal ideal of A and let $\phi \in \mathbf{R}[X]$ denote a generator of the ideal $\mathfrak{m} \cap \mathbf{R}[X]$. Then we have the following composition of injective $\mathbf{R}$-algebra homomorphisms

$$\mathbf{R} \hookrightarrow \mathbf{R}[X]/(\phi) \hookrightarrow A/\mathfrak{m}.$$

If ϕ has even degree, we are done. If it has odd degree, then the Mean Value Theorem implies that it has a zero $\lambda \in \mathbf{R}$. Then $X - \lambda$ divides zero in the subring $\mathbf{R}[X]/(\phi)$ of the field $A/\mathfrak{m}$. It follows that $\phi = c(X-\lambda)$ for some $c \in \mathbf{R}^*$, so that the map $\mathbf{R} \hookrightarrow \mathbf{R}[X]/(\phi)$ is an isomorphism. Since Y is a zero of a quadratic polynomial over $\mathbf{R}[X]$, the dimension of $A/\mathfrak{m}$ is at most 2. Since there are no $\mathbf{R}$-algebra homomorphisms $A \longrightarrow \mathbf{R}$, the dimension must be 2, which is even.