In this note we present Baker's proof of the Lindemann-Weierstrass Theorem [1]. Let $\overline{\mathbf{Q}}$ denote the algebraic closure of $\mathbf{Q}$ inside $\mathbf{C}$.

Theorem 1. *Let $S \subset \overline{\mathbf{Q}}$ be a finite set. Then the numbers $e^\alpha \in \mathbf{C}$ are linearly independent over $\overline{\mathbf{Q}}$. In other words, in any linear relation of the form*

$$\sum_{\alpha\in S} \lambda_\alpha e^\alpha = 0, \qquad (\text{with } \lambda_\alpha \in \overline{\mathbf{Q}}),$$

the coefficients necessarily satisfy $\lambda_\alpha = 0$ for all $\alpha \in S$.

Before proving Theorem 1, we give two applications. The first one says that 'transcendental looking' numbers like e, π, $\log 2$, $\sin(1)$, $\cos(\sqrt{2})$, $e^{\sqrt[3]{2}}$, ... etc. are indeed all transcendental.

Corollary 2. *For any non-zero $\alpha \in \overline{\mathbf{Q}}$, the numbers e^α, $\sin\alpha$ and $\cos\alpha$ are transcendental. The number $\log\alpha$ is either zero or transcendental for any choice of a branch of the logarithm.*

Proof. Suppose that $\alpha \in \overline{\mathbf{Q}}$ is not zero. Let $S = \{0, \alpha\}$. If $e^\alpha \in \overline{\mathbf{Q}}$, the relation

$$e^\alpha \cdot e^0 + (-1)\cdot e^\alpha = 0$$

contradicts Theorem 1. Therefore e^α is transcendental. Applying this argument to $i\alpha$, we see that $e^{i\alpha}$ and hence $\sin\alpha$ and $\cos\alpha$ are transcendental. Finally, if $\log\alpha$ is a non-zero element of $\overline{\mathbf{Q}}$, then we let $S = \{0, \log\alpha\}$. Since $\alpha \in \overline{\mathbf{Q}}$, the relation

$$\alpha \cdot e^0 + (-1)\cdot e^{\log\alpha} = 0$$

contradicts Theorem 1. Therefore $\log\alpha$ must be transcendental when it is not zero.

The second application is often called the 'Lindemann-Weierstrass Theorem.

Corollary 3. *Let $\alpha_1, \ldots, \alpha_n \in \overline{\mathbf{Q}}$ and suppose that*

$$F(e^{\alpha_1}, \ldots, e^{\alpha_n}) = 0, \qquad \text{for a non-zero polynomial } F \in \overline{\mathbf{Q}}[X_1, \ldots, X_n],$$

then $\sum_{j=1}^n \mu_j \alpha_j = 0$ for certain $\mu_j \in \mathbf{Q}$ not all of which are zero.

Proof. Let

$$S = \{m_1\alpha_1 + \ldots + m_n\alpha_n : m_1, \ldots, m_n \in \mathbf{Z}_{\geq 0}\}.$$

Suppose that $\alpha_1, \ldots, \alpha_n$ are *independent over* $\mathbf{Q}$. Then the numbers $\alpha = m_1\alpha_1 + \ldots + m_n\alpha_n$ are all distinct. Therefore, the polynomial $F \in \overline{\mathbf{Q}}[X_1, \ldots, X_n]$ can be written as

$$F(e^{\alpha_1}, \ldots, e^{\alpha_n}) = \sum_{\alpha\in S} \lambda_\alpha e^\alpha,$$

where for $\alpha = m_1\alpha_1 + \ldots + m_n\alpha_n$, the number λ_α is precisely the $X_1^{m_1}\cdots X_n^{m_n}$-coefficient of F. Since $F(e^{\alpha_1}, \ldots, e^{\alpha_n}) = 0$, Theorem 1 implies that $F = 0$. Contradiction. We conclude that $\alpha_1, \ldots, \alpha_n$ are dependent over $\mathbf{Q}$ as required.

The first prove Theorem 1 in a special case. Recall that two numbers in $\overline{\mathbf{Q}}$ are called *conjugate* if their minimum polynomials over $\mathbf{Q}$ are the same. A number in $\overline{\mathbf{Q}}$ is called *integral* if it is a zero of a monic polynomial in $\mathbf{Z}[X]$. This is equivalent to saying that its minimum polynomial is in $\mathbf{Z}[X]$.

The following theorem deals with a particular symmetric case of Theorem 1.

Theorem 4. *Let S be a finite subset of $\overline{\mathbf{Q}}$ containing all conjugates of each of its elements. For every $\alpha \in S$, let $\lambda_\alpha \in \mathbf{Q}$ and suppose that $\lambda_\alpha = \lambda_\beta$ whenever α and β are conjugate. Then for any relation of the form*

$$\sum_{\alpha \in S} \lambda_\alpha e^\alpha \;=\; 0,$$

the coefficients λ_α are 0 for all $\alpha \in S$.

Proof. By removing all $\alpha \in S$ for which $\lambda_\alpha = 0$, we may assume that all coefficients λ_α are different from zero. We now derive a contradiction from the assumption $S \neq \emptyset$ and the relation $\sum_{\alpha \in S} \lambda_\alpha e^\alpha = 0$.

Let $\Phi \in \mathbf{Z}[X]$ be a polynomial having each alement of S as a simple zero. Let $c \in \mathbf{Z}_{>0}$ be the leading coefficient of Φ and let d be its degree. So, $c\beta$ is an algebraic integer for every $\beta \in S$. Let n be a positive integer. Fix $\alpha \in S$ and let

$$f_n(X) \;=\; \frac{\Phi(X)^n}{X - \alpha}.$$

Then $f_n(X)$ is in $\overline{\mathbf{Q}}[X]$. Let $F_n(t) = f_n(t) + f_n^{(1)}(t) + f_n^{(2)}(t) + \ldots$. Here $f_n^{(k)}$ denotes the k-th derivative of f_n. Both polynomials f_n and F_n have degree $nd - 1$. Since $-e^{-x}F_n(x)$ is a primitive function of $e^{-x}f_n(x)$, we have 'Hermite's formula'

$$\int_0^1 e^{z(1-t)} f_n(zt) z dt \;=\; e^z \int_0^z e^{-t} f_n(t) dt \;=\; e^z F_n(0) - F_n(z), \qquad \text{for all } z \in \mathbf{C}.$$

We apply this to $z = \beta \in S$. By hypothesis we have $\sum_{\beta \in S} \lambda_\beta e^\beta = 0$. It follows that

$$\Sigma_n = \sum_{\beta \in S} \lambda_\beta F_n(\beta)$$

is equal to

$$-\sum_{\beta \in S} \lambda_\beta \int_0^1 e^{\beta(1-t)} f_n(\beta t) \beta dt.$$

Since $f_n(X)$ is a polynomial of degree $nd - 1$, the integrals are $O(M^{nd})$ and hence

$$\Sigma_n \;=\; O(M^{nd}).$$

Here M is a large positive real number that only depends on S and the polynomial Φ.

If $\beta \in S$ and $\beta \neq \alpha$, we compute $f_n^{(k)}(\beta)$ by writing $f_n(X) = (X - \beta)^n h(X)$ for a certain polyomial $h(X) \in \overline{\mathbf{Q}}[X]$ and applying Leibniz's formula. We find that

$$f_n^{(k)}(\beta) = \begin{cases} 0, & \text{for } 0 \leq k < n; \\ n! h^{(k-n)}(\beta), & \text{for } k \geq n. \end{cases}$$

Since $\deg h < \deg f_n = dn-1$, it follows easily that $F_n(\beta)$ is equal to $n!/c^{dn-1}$ times an algebraic integer. Similarly, when $\beta = \alpha$, we have $f_n(X) = (X-\alpha)^{n-1}g(X)^n$ where $g(X) = \Phi(X)/(X-\alpha)$. We have

$$f_n^{(k)}(\alpha) = \begin{cases} 0, & \text{for } 0 \le k < n-1; \\ (n-1)!\Phi^{(1)}(\alpha)^n, & \text{for } k = n-1; \\ (n-1)!\binom{k}{n-1}G_n^{(k-n+1)}(\alpha), & \text{for } k \ge n. \end{cases}$$

Here $G_n(X)$ denotes the n-th power of $g(X)$. In particular, the first derivative of $G_n(X)$ is equal to $ng(X)^{n-1}g^{(1)}(X)$. It follows that for $k \ne n-1$, the number $f_n^{(k)}(\alpha)$ is equal to $n!/c^{dn-1}$ times an algebraic integer. Therefore we have

$$F_n(\alpha) \;=\; (n-1)!\Phi^{(1)}(\alpha)^n + \frac{\xi n!}{c^{dn-1}}$$

for some algebraic integer ξ. Since α is a *simple* zero of Φ, there are infinitely many n for which the term $\lambda_\alpha\Phi^{(1)}(\alpha)^n$ is not divisible by n. For this choice of n we now know that

$$\sum_{\beta\in S}\lambda_\beta F_n(\beta) \;\ne\; 0.$$

In other words, $\Sigma_n \ne 0$. In addition, we know that $c^{nd-1}\Sigma_n$ is an algebraic integer divisible by $(n-1)!$.

Recall that the number Σ depends on our choice of α. To finish the proof, we take the product over α of the corresponding numbers Σ_n. Since for every $\alpha \in S$, all conjugates are also in S, the product is of the form $m/c^{d(dn-1)}$, where m is an ordinary non-zero integer divisible by $(n-1)!^d$. It follows that

$$(n-1)!^d \;=\; O((c^dM)^{dn}), \quad \text{for infinitely many } n.$$

This contradicts Stirling's formula for very large n! This completes the proof of the theorem.

Proof of Theorem 1. Let $S \ne \emptyset$ and suppose that $\lambda_\alpha \ne 0$ for every $\alpha \in S$. We derive a contradiction from the assumption $\sum_{\alpha\in S}\lambda_\alpha e^\alpha = 0$.

Step 1. Let F be the extension of $\mathbf{Q}$ generated by the coefficients λ_α and their conjugates. Let $G = \mathrm{Aut}(F)$ and put $N = \#G$. We introduce variables X_α for each $\alpha \in S$ and consider the polynomial

$$\prod_{\sigma\in G}\sum_{\alpha\in S}\sigma(\lambda_\alpha)X_\alpha.$$

By symmetry it has coefficients in $\mathbf{Q}$. Next we substitute $X_\alpha = e^\alpha$. Since $\sum_{\alpha\in S}\lambda_\alpha e^\alpha = 0$ we obtain 0. On the other hand, we expand the product. Each monomial becomes equal to e^β where β is a sum of N not necessarily distinct elements of S. We collect monomials that give rise to the same value of β. It may happen that certain monomials with the same β cancel out. When we omit those, we obtain

$$\sum_{\beta \in S'} \mu_\beta e^\beta \;=\; 0,$$

with non-zero coefficients $\mu_\beta \in \mathbf{Q}$. Note that not all terms on the left cancel out. For instance, let α be the lexicographically first element of $S \subset \mathbf{C}$. This means that it is an element of S with maximal real part and among the elements with maximal real part it has the largest imaginary part. Then $\beta = N\alpha$ is not equal to any other sum of N elements of S and therefore the corresponding term certainly does not cancel out. So the set S' is not empty. The conclusion is that we obtain the same kind of sum as we started with, but this time the coefficients λ_α are non-zero rational numbers.

We denote this new sum again by $\sum_{\alpha \in S} \lambda_\alpha e^\alpha$ and we make the second step.

Step 2. As before, let F be the extension of $\mathbf{Q}$ generated by the coefficients λ_α and their conjugates. Let $G = \mathrm{Aut}(F)$ and put $N = \#G$. We consider the product

$$\prod_{\sigma \in G} \sum_{\alpha \in S} \lambda_\alpha e^{\sigma(\alpha)}.$$

Since $\sum_{\alpha \in S} \lambda_\alpha e^\alpha = 0$, it is zero. On the other hand, we expand the product and we obtain a sum with rational coefficients of numbers of the form e^β where β is a sum of conjugates of n not necessarily distinct elements of S. We collect the terms with the same β. Again it may happen that certain terms with the same β cancel out. When we omit those, we obtain

$$\sum_{\beta \in S''} \mu_\beta e^\beta \;=\; 0,$$

with all coefficients $\mu_\beta \neq 0$. By symmetry, for every $\beta \in S''$ also all conjugates of β are in S'' and we have $\mu_\beta = \mu_{\beta'}$ whenever β and β' are conjugate. To see that $S'' \neq \emptyset$, we pick in each conjugacy class of elements in S the lexicographically first element. Then the sum β of those elements is in S'' because β is not equal to any other sum of conjugates of elements of S. The coefficient of the corresponding exponential is the product of the λ_α and therefore does not vanish.

An application of Theorem 4 now leads to a contradiction. This proves the theorem.

Bibliography.

[1] Baker, Alan: *Transcendental number theory*, (1975), Cambridge University Press, New York.