In this note we present Baker's proof of the Lindemann-Weierstrass Theorem. Let $\overline{\mathbf{Q}}$ denote the algebraic closure of $\mathbf{Q}$ inside $\mathbf{C}$.

Theorem 1. *Let $S \subset \overline{\mathbf{Q}}$ be a finite set. If*

$$\sum_{\alpha \in S} \lambda_\alpha e^\alpha \;=\; 0, \qquad \textit{for certain } \lambda_\alpha \in \overline{\mathbf{Q}},$$

then $\lambda_\alpha = 0$ for all $\alpha \in S$.

Before proving Theorem 1, we give two applications. The first one says that 'transcendental looking' numbers like e, π, $\log 2$, $\sin(1)$, $\cos(\sqrt{2})$, $e^{\sqrt[3]{2}}$, ... etc. are indeed all transcendental.

Corollary 2. *For any non-zero $\alpha \in \overline{\mathbf{Q}}$, the numbers e^α, $\sin\alpha$ and $\cos\alpha$ are transcendental. If $\alpha \neq 1$, then $\log\alpha$ is transcendental for any choice of a branch of the logarithm.*

Proof. Suppose that $\alpha \in \overline{\mathbf{Q}}$ is not zero. Let $S = \{0, \alpha\}$. If $e^\alpha \in \overline{\mathbf{Q}}$, the relation

$$e^\alpha \cdot e^0 + (-1) \cdot e^\alpha \;=\; 0$$

contradicts Theorem 1. Therefore e^α is transcendental. Applying this argument to $i\alpha$, we see that $e^{i\alpha}$ and hence $\sin\alpha$ and $\cos\alpha$ are transcendental. Finally, if $\log\alpha$ is a non-zero element of $\overline{\mathbf{Q}}$, the argument shows that α is transcendental. Since this is not true, $\log\alpha$ must be transcendental when $\alpha \neq 1$, as required.

The second application is often called the 'Lindemann-Weierstrass Theorem.

Corollary 3. *Let $\alpha_1, \ldots, \alpha_n \in \overline{\mathbf{Q}}$ and suppose that*

$$F(e^{\alpha_1}, \ldots, e^{\alpha_n}) \;=\; 0, \qquad \textit{for a non-zero polynomial } F \in \overline{\mathbf{Q}}[X_1, \ldots, X_n],$$

then $\sum_{j=1}^n \mu_j \alpha_j = 0$ for certain $\mu_j \in \mathbf{Q}$ not all of which are zero.

Proof. Let

$$S \;=\; \{m_1\alpha_1 + \ldots + m_n\alpha_n : m_1, \ldots, m_n \in \mathbf{Z}_{\geq 0}\}.$$

Suppose that $\alpha_1, \ldots, \alpha_n$ are *independent over* $\mathbf{Q}$. Then the numbers $\alpha = m_1\alpha_1 + \ldots + m_n\alpha_n$ are all distinct. Therefore, for the polynomial $F \in \overline{\mathbf{Q}}[X_1, \ldots, X_n]$ we have

$$F(e^{\alpha_1}, \ldots, e^{\alpha_n}) \;=\; \sum_{\alpha \in S} \lambda_\alpha e^\alpha,$$

where for $\alpha = m_1\alpha_1 + \ldots + m_n\alpha_n$, the number λ_α is precisely the $X_1^{m_1} \cdots X_n^{m_n}$-coefficient of F. Since $F(e^{\alpha_1}, \ldots, e^{\alpha_n}) = 0$, Theorem 1 implies therefore that $F = 0$. Contradiction. We conclude that $\alpha_1, \ldots, \alpha_n$ are dependent over $\mathbf{Q}$ as required.

The main step in the proof of Theorem 1 is the following result. We prove it and then deduce Theorem 1 from it.

Theorem 4. *Let $S \subset \overline{\mathbf{Q}}$ be a finite set that is stable under conjugation. For every $\alpha \in S$ let $n_\alpha \in \mathbf{Q}$ and suppose that $n_\alpha = n_\beta$ whenever α and β are conjugate. Then,*

$$\text{if} \quad \sum_{\alpha\in S} n_\alpha e^\alpha \;=\; 0, \quad \text{then} \quad n_\alpha \;=\; 0 \quad \text{for all } \alpha \in S.$$

Proof. By removing all $\alpha \in S$ for which $n_\alpha = 0$, we may assume that all coefficients n_α are different from zero. We now derive a contradiction from the assumption $S \neq \emptyset$ and the relation $\sum_{\alpha\in S} n_\alpha e^\alpha = 0$.

Let $\Phi \in \mathbf{Z}[X]$ be a polynomial of which every $\alpha \in S$ is a simple zero. Let $c \in \mathbf{Z}_{>0}$ be the leading coefficient of Φ and let d be its degree. Fix $\alpha \in S$ and let

$$f(X) \;=\; \frac{\Phi(X)^n}{X-\alpha}.$$

Let $F(t) = f(t) + f^{(1)}(t) + f^{(2)}(t) + \ldots$. Since $-e^{-x}F(x)$ is a primitive function of $e^{-x}f(x)$, we have 'Hermite's formula'

$$e^z F(0) - F(z) \;=\; e^z \int_0^z e^{-t} f(t)dt \;==\; \int_0^1 e^{z(1-t)} f(zt)zdt, \qquad \text{for all } z \in \mathbf{C}.$$

Since $\sum_{\beta\in S} n_\beta e^\beta = 0$, taking the sum over $z = \beta \in S$ we find

$$\sum_{\beta\in S} n_\beta F(\beta) \;=\; -\sum_{\beta\in S} n_\beta \int_0^1 e^{\beta(1-t)} f(\beta t)\beta dt.$$

The integral is $O(M^n)$ for some $M \in \mathbf{R}_{>0}$ depending on S and the polynomial Φ.

If $\beta \in S$ and $\beta \neq \alpha$, the number $F(\beta)$ is equal to $n!/c^{dn-1}$ times an algebraic integer. When $\beta = \alpha$, the number $F(\alpha)$ is equal to $(n-1)!\Phi^{(1)}(\alpha)^n + \sum_{k\geq n} f^{(k)}(\alpha)$ and the sum $\sum_{k\geq n} f^{(k)}(\alpha)$ is equal to $n!/c^{dn-1}$ times an algebraic integer.

There are infinitely many n for which the expression $n_\alpha \Phi^(1)(\alpha)^n$ is not divisible by n. For this choice of n we know that $\sum_{\beta\in S} n_\beta F(\beta)$ is a non-zero algebraic integer divisible by $(n-1)!$.

We can also find infinitely many n for which $n_\alpha \Phi^{(1)}(\alpha)^n$ is not divisible by n *for each* $\alpha \in S$. Since S is stable under conjugation, taking the product over α of $\sum_{\beta\in S} n_\beta F(\beta)$, we obtain a non-zero integer divisible by $(n-1)!^d$. This leads to the contradiction

$$(n-1)!^d \;=\; O(M^{dn}), \quad \text{for infinitely many } n.$$

This completes the proof of the theorem.

Proof of Theorem 1. Let $S \neq \emptyset$ and suppose that $\lambda_\alpha \neq 0$ for every $\alpha \in S$. We derive a contradiction from the assumption $\sum_{\alpha\in S} \lambda_\alpha e^\alpha = 0$. The coefficients λ_α are contained in some finite Galois extension F of $\mathbf{Q}$. Let $G = \mathrm{Gal}(F/\mathbf{Q})$ have order n. We introduce variables X_α and consider the polynomial

$$\prod_{\sigma \in G} \sum_{\alpha \in S} \sigma(\lambda_\alpha) X_\alpha.$$

By symmetry it has coefficients in $\mathbf{Q}$. Next we substitute $X_\alpha = e^\alpha$. Since $\sum_{\alpha \in S} \lambda_\alpha e^\alpha = 0$ we obtain 0. On the other hand, we expand the product. Each monomial becomes equal to e^β where β is a sum of n not necessarily distinct elements of S. We collect monomials that give rise to the same value of β. It may happen that certain monomials with the same β cancel out. When we omit those, we obtain

$$\sum_{\beta \in S'} \mu_\beta e^\beta \ = \ 0,$$

with non-zero coefficients $\mu_\beta \in \mathbf{Q}$. Note that not all terms on the left cancel out. For instance, let α be the lexicographically first element of $S \subset \mathbf{C}$. This means that it is an element of S with maximal real part and among the elements with maximal real part it has the largest imaginary part. Then $\beta = n\alpha$ is not equal to any other sum of n elements of S and therefore the corresponding term certainly does not cancel out. So the set S' is not empty.

In conclusion, we obtain the same kind of sum as we started with, but this time the coefficients λ_α are non-zero rational numbers. We denote this new sum again by $\sum_{\alpha \in S} \lambda_\alpha e^\alpha$. We make the second step.

The set S is contained in some finite Galois extension K of $\mathbf{Q}$. Let $H = \mathrm{Gal}(K/\mathbf{Q})$ have order m. We consider the product

$$\prod_{\sigma \in H} \sum_{\alpha \in S} \lambda_\alpha e^{\sigma(\alpha)}.$$

Since $\sum_{\alpha \in S} \lambda_\alpha e^\alpha = 0$, it is zero. On the other hand, we expand the product and we obtain a sum with rational coefficients of numbers of the form e^β where β is a sum of conjugates of m not necessarily distinct elements of S. We collect the terms with the same β. Again it may happen that certain terms with the same β cancel out. When we omit those, we obtain

$$\sum_{\beta \in S''} \mu_\beta e^\beta \ = \ 0,$$

with all coefficients $\mu_\beta \neq 0$. By symmetry, the set S'' is stable under the Galois group H and we have $\mu_\beta = \mu_{\beta'}$ whenever β and β' are conjugate. To see that $S'' \neq \emptyset$, we denote for each $\sigma \in G$ by $\alpha_\sigma \in \mathbf{C}$ the lexicographically first element in $\{\sigma(\alpha) : \alpha \in S\}$. Then $\beta = \sum_{\sigma \in H} \alpha_\sigma$ is in S'' because β is not equal to any other sum of conjugates of elements of S. The coefficient of the corresponding exponential is $\prod_{\sigma \in H} \lambda_{\alpha_\sigma} \neq 0$.

An application of Theorem 1 now leads to a contradiction. This proves the theorem.