1. Split k-algebras.

Let k be a field. A finite k-algebra is called *split* if it isomorphic to k^m for some $m \geq 0$.

Lemma 1.1. *Let A be a split k-algebra. Then the set $\mathrm{Hom}_k(A, k)$ of k-algebra homomorphisms or k-points $A \longrightarrow k$ has precisely $m = \dim_k A$ elements. In addition, if $a \in A$ satisfies $\varphi(a) = 0$ for every $\varphi \in \mathrm{Hom}_k(A, k)$, we have $a = 0$.*

Proof. We may assume that $A = k^m$ for some $m \geq 0$. The standard basis vectors e_i of k^m form a set of orthogonal idempotents. Therefore any $\varphi \in \mathrm{Hom}_k(A, k)$ maps precisely one idempotent e_i to 1 and all the others to 0. It follows that the k-algebra homomorphisms $k^m \longrightarrow k$ are precisely the m projections. The second statement is therefore also clear.

This proves the lemma.

Let $\underline{C}$ be the category of split k-algebras and let $\underline{Sets}$ be the category of *finite* sets. Let $F : \underline{C} \longrightarrow \underline{Sets}$ be the contravariant functor that maps an object A of $\underline{C}$ to the set $\mathrm{Hom}_k(A, k)$ of its k-points. It maps a k-algebra morphism $s : A \longrightarrow B$ to the map $\mathrm{Hom}_k(B, k)$ to $\mathrm{Hom}_k(A, k)$ given by $f \mapsto f \cdot s$.

In the other direction, let $G : \underline{Sets} \longrightarrow \underline{C}$ be the contravariant functor that sends a finite set X to the algebra $\mathrm{Map}(X, k)$. It sends a map $g : X \longrightarrow Y$ in $\underline{Sets}$ to the k-algebra homomorphism $\mathrm{Map}(Y, k) \longrightarrow \mathrm{Map}(X, k)$ given by $\varphi \mapsto \varphi \cdot g$.

For any split k-algebra A we have $\#F(A) = \dim_k(A)$ and for every finite set X we have $\#X = \dim_k G(X)$.

Theorem 1.2. *The functors F and G defined above induce an anti-equivalence of categories*

$$\underline{C} \quad \cong \quad \underline{Sets}.$$

Proof. In order to show that $GF \cong \mathrm{id}_{\underline{C}}$ we observe that the evaluation map

$$A \;\longrightarrow\; GF(A) \;=\; \mathrm{Map}(\mathrm{Hom}_k(A, k), k)$$

given by $a \mapsto g_a$ where $g_a(\varphi) = \varphi(a)$ is an isomorphism. Indeed, by Lemma 1.1 the map is injective. Since both k-algebras have the same dimension, they are isomorphic.

Similarly, we have $FG \cong \mathrm{id}_{\underline{Sets}}$. This follows from the fact that the evaluation map

$$X \;\longrightarrow\; FG(X) \;=\; \mathrm{Hom}_k(\mathrm{Map}_k(X, k), k)$$

is an isomorphism. This in turn follows from the fact that it is injective and the equality $\dim_k \mathrm{Map}_k(X, k) = \#X$. Lemma 1.1 implies that both sides have the same cardinality.

The initial objects of the category $\underline{C}$ are isomorphic to k. In this equivalence of categories they correspond to the final objects of $\underline{Sets}$ which are the one point sets. Similarly, the initial obeject $\emptyset$ of $\underline{Sets}$ corresponds to the final object of $\underline{C}$. The latter is the zero algebra.

Finite products and sums of the category $\underline{C}$ correspond to finite sums and products respectively of the category $\underline{Sets}$. This means that for two finite split k-algebras A and B we have $F(A \times B) = F(A) \cup F(B)$ and $F(A \otimes_k B) = F(A) \times F(B)$. Similarly for two finite sets X and Y we have $G(X \cup Y) = G(X) \times G(Y)$ and $G(X \times Y) = G(X) \otimes_k G(Y)$. Here $X \cup Y$ denotes the disjoint union of X and Y. In a similar way, fibred sums and products correspond to one another.

Quotients of split k-algebras are again split. Indeed, let A be quotient of $\mathrm{Map}(X, k)$. Then the kernel of the morphism $k^m \to A$ is a product of ideals I_x of k. Then $A \cong \mathrm{Map}(Y, k)$, where Y is the set of $x \in X$ for which $I_x = 0$. Subalgebras of split algebras are also split. Indeed, let A be a subalgebra of $\mathrm{Map}(X, k)$. We write $x \sim y$ if $f(x) = f(y)$ for all $f \in A$. This is an equivalence relation. Let X' be the set of equivalence classes. Then we have natural inclusions $A \subset \mathrm{Map}(X', k) \subset \mathrm{Map}(X, k)$. For every $x \in X$ let e_x denote the map $X \to k$ for which $f(y) = 1$ or 0 depending on whether $y \sim x$ or not. The maps e_x only depend on the equivalencee classes of x. They form a k-basis for $\mathrm{Map}(X', k)$. In order to prove that A is equal to $\mathrm{Map}(X', k)$, it suffices to prove that $e_x \in A$ for all $x \in X$.

Let $x \in X$. For each $y \not\sim x$ there exists a map $f_y \in A$ for which $f_y(x) \neq f_y(y)$. Then we have (Lagrange interpolation)

$$e_x = \prod_{y \not\sim x} \frac{f_y - f_y(y)}{f_y(x) - f_y(y)}.$$

Here the product runs over y up to equivalence.

2. Finite Galois extensions.

Let k be a field and let $k \subset l$ be a finite extension of k. Put $\pi = \mathrm{Hom}_k(l, l) = \mathrm{Aut}_k(l)$. We have

$$\pi \quad = \quad \mathrm{Hom}_k(l, l) \quad = \quad \mathrm{Hom}_l(l \otimes_k l, l).$$

If $l \otimes_k l$ is a split l-algebra, Lemma 1.1 implies that the rightmost set has cardinality $\dim_l l \otimes_k l = [l : k]$.

Lemma 2.1. *Suppose that $l \otimes_k l$ is a split l-algebra. Then for every finite π-set X the natural map*

$$\mathrm{Map}_\pi(X, l) \otimes_k l \longrightarrow \mathrm{Map}(X, l)$$

is injective.

Proof. Since $\mathrm{Map}_\pi(-, l)$ and $\mathrm{Map}(-, l)$ map disjoint unions to direct products, we may assume that X is *transitive.* This means that X is isomorphic to the set of cosets π/H for some subgroup H of π.

Consider the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Map}_\pi(\pi/H, l) \otimes_k l & \longrightarrow & \mathrm{Map}(\pi/H, l) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Map}_\pi(\pi, l) \otimes_k l & \longrightarrow & \mathrm{Map}(\pi, l) \\
\big\downarrow \cong & & \big\downarrow \cong \\
l \otimes_k l & \xrightarrow{\cong} & l^{(\pi)}
\end{array}$$

Here the top horizontal maps are given by $f \otimes \lambda \mapsto [\sigma \mapsto \lambda f(\sigma)]$. The top vertical arrows are both injective. The bottom vertical maps are given by $f \otimes \lambda \mapsto \lambda f(1)$ and $f \mapsto (f(\sigma))_\sigma$ respectively. Both are isomorphisms. Finally the bottom row sends $x \otimes y$ to the vector $(y\sigma(x))_\sigma$ in $l^{(\pi)}$.

To prove that the bottom row is an isomorphism, we oberve that both $l \otimes_k l$ and $l^{(\pi)}$ are split l-algebras and apply the functor of Theorem 1.2: it suffices to show that the map

$$\mathrm{Hom}_l(l^{(\pi)}, l) \longrightarrow \mathrm{Hom}_l(l \otimes_k l, l),$$

that maps the projection on the σ-coordinate to the homomorphism $l \otimes_k l \to l$ given by $x \otimes y \mapsto y\sigma(x)$ is an isomorphism. This follows immediately from the fact that the map from $\pi = \mathrm{Hom}_k(l, l)$ to $\mathrm{Hom}_l(l^{(\pi)}, l)$ given by mapping $\sigma \in \pi$ to the k-algebra morphism given by $x \otimes y \mapsto y\sigma(x)$, is an isomorphism.

We conclude that the top row is *injective*, as required.

In this section, let $\underline{C}$ be the category of k-algebras A that are *split over* l, i.e. for which $A \otimes_k l$ is a split l-algebra. Let $\underline{Sets}_\pi$ be the category of *finite* π-sets. Let

$$F : \underline{C} \longrightarrow \underline{Sets}_\pi$$

be the contravariant functor that maps an object A of $\underline{C}$ to the set $\mathrm{Hom}_k(A, l)$ of its l-points. In the other direction, let

$$G : \underline{Sets}_\pi \longrightarrow \underline{C}$$

be the contravariant functor that maps a finite π-set X to the k-algebra $\mathrm{Map}_\pi(X, l)$.

The group π acts on the finite set $F(A) = \mathrm{Hom}_k(A, l)$ by composition. By Lemma 2.1 the algebra $\mathrm{Map}_\pi(X, l) \otimes_k l$ is isomorphic to a subalgebra of the split l-algebra $\mathrm{Map}(X, l)$. Therefore it is itself split. It follows that the F and G are well defined contravariant functors.

Theorem 2.2. *The functors F and G defined above induce an anti-equivalence of categories*

$$\underline{C} \quad \cong \quad \underline{Sets}_\pi.$$

Proof. *Step 1.* We show that $GF \cong \mathrm{id}_{\underline{C}}$. We consider the evaluation map

$$A \longrightarrow GF(A) \;=\; \mathrm{Map}_\pi(\mathrm{Hom}_k(A, l), l)$$

given by $a \mapsto g_a$ where $g_a(\varphi) = \varphi(a)$. We tensor this isomorphism with l over k and obtain the following diagram

$$\begin{tikzcd}
A \otimes_k l \arrow[r] \arrow[dr] & \mathrm{Map}_\pi(\mathrm{Hom}_k(A, l), l) \otimes_k l \arrow[d] \\
 & \mathrm{Map}(\mathrm{Hom}_k(A, l), l) \arrow[d, equal] \\
 & \mathrm{Map}(\mathrm{Hom}_l(A \otimes_k l, l), l)
\end{tikzcd}$$

The diagonal map is the evaluation of section 1. Since $A \otimes_k l$ is a split l-algebra, Theorem 1.2 implies that it is an isomorphism. By Lemma 2.1 the vertical arrow is injective. It follows that the horizontal map is an isomorphism, as required.

Step 2. As a byproduct we see that the vertical map in the diagram above is an isomorphism. This is important for the rest of the proof. We state this result explicitly for future reference:

Corollary. Let $X = \mathrm{Hom}_k(A, l)$ for some k-algebra in the category $\underline{C}$. Then the natural map

$$\mathrm{Map}_\pi(X, l) \otimes_k l \quad \longrightarrow \quad \mathrm{Map}(X, l)$$

is an isomorphism of k-algebras. In particular, this applies to $\pi = \mathrm{Hom}_k(A, l)$ with $A = l$ and hence to π-sets that are disjoint unions of copies of π.

Step 3. We show that the natural map

$$\mathrm{Map}_\pi(X, l) \otimes_k l \quad \longrightarrow \quad \mathrm{Map}(X, l)$$

is an isomorphism for *every* finite π-set X.

It suffices to show this for transitive set X. Therefore, let $X = \pi/H$ for some subgroup H of π. The trick is to relate X to π-sets that are unions of copies of π and for which we know the result already by Step 2. This can be done as follows. Consider the fiber product

$$\pi \times_{\pi/H} \pi \quad = \quad \{(\sigma, \tau) \in \pi \times \pi : \sigma H = \tau H\}$$

The map

$$\pi \times_{\pi/H} \pi \quad \longrightarrow \quad \pi \times (H)$$

given by $(\sigma, \tau) \mapsto (\sigma, \sigma^{-1}\tau)$ is an isomorphism in the category Sets_π of π-sets. Here (H) denotes the set H with *trivial* π-action. We see that the π-set $\pi \times_{\pi/H} \pi$ is a union of copies of π. Consider the following exact sequence *of k-vector spaces*

$$0 \longrightarrow \mathrm{Map}_\pi(\pi/H, l) \longrightarrow \mathrm{Map}_\pi(\pi, l) \xrightarrow{r} \mathrm{Map}_\pi(\pi \times_{\pi/H} \pi, l).$$

Here the map r is the map that maps f to the map that sends (σ, τ) to the difference $f(\sigma) - f(\tau)$. We tensor this sequence with l over k and consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Map}_\pi(\pi/H, l) \otimes_k l & \longrightarrow & \mathrm{Map}_\pi(\pi, l) \otimes_k l & \xrightarrow{r \otimes 1} & \mathrm{Map}_\pi(\pi \times_{\pi/H} \pi, l) \otimes_k l \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Map}(\pi/H, l) & \longrightarrow & \mathrm{Map}(\pi, l) & \xrightarrow{r} & \mathrm{Map}(\pi \times_{\pi/H} \pi, l)
\end{array}$$

By the corollary of the proof of Step 1, we see that the two rightmost vertical arrows are bijections. Therefore the same is true for the leftmost vertical map. This completes Step 3.

Step 4. To see that $FG \cong \mathrm{id}_{\underline{Sets}}$, we consider the evaluation map

$$X \ \longrightarrow \ FG(X) \ = \ \mathrm{Hom}_k(\mathrm{Map}_\pi(X, l), l)$$

It fits in the following commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Hom}_k(\mathrm{Map}_\pi(X,l),l) \\ & & \| \\ & \searrow & \mathrm{Hom}_l(\mathrm{Map}_\pi(X,l)\otimes_k l,l) \\ & & \downarrow \cong \\ & & \mathrm{Hom}_l(\mathrm{Map}(X,l),l) \end{array}$$

The bottom vertical map is a bijection by Step 3. Since the diagonal map is an isomorphism by Theorem 1.2, we conclude that the top horizontal map is an isomorphism, as required.

Remark. Let l be a field that is split over itself and let $\pi = \mathrm{Aut}_k(l)$. Under the functors F and G, objects of $\underline{C}$ that are *fields*, correspond to *transitive* π-sets.

Indeed, any transitive π-set X is of the form π/H for some subgroup $H \subset \pi$ that is unique up to conjugation. The map from $G(X) = \mathrm{Map}_\pi(X,l)$ to l given by $\phi \mapsto \phi(1)$ is an injective k-algebra homomorphism that identifies $G(X)$ with the subfield of H-invariants of l.

Conversely, let A be a k-algebra that is a field and is split over l. Then $F(A)$ is a transitive π-set, because if $F(A)$ is the disjoint union of two non-empty π-sets Y and Z, then Theorem 2.2 implies that $A \cong G(Y) \times G(Z)$ and this is imposisble if A is a field. Writing $F(A) = \pi/H$ for some subgroup H of π and applying G to the epimorphism $\pi \longrightarrow \pi/H$ we see that A is actually isomorphic to a subfield of l that contains k.

3. The classical theory.

Let K be a field. An element of a finite extension of K is called *separable* over K if its minimum polynomial over K has no multiple zeroes. A finite extension $K \subset L$ is called separable if every $a \in L$ is separable. A finite extension $K \subset L$ is called *normal* if the minimum polynomial of any element $a \in L$ is a product of linear factors in the ring $L[X]$. A finite extension $K \subset L$ is called *Galois* if it is both normal and separable. The *Galois group* of L over K is the group of automorphisms of L that fix K. It is denoted by $\mathrm{Gal}(L/K)$.

Theorem 3.1. *Let $K \subset L$ be a finite Galois extension with Galois group G. Then the maps f and g*

$$\{\text{fields } K \subset E \subset L\} \quad \underset{g}{\overset{f}{\rightleftarrows}} \quad \{\text{subgroups } H \subset G\}$$

given by

$$f(E) = \mathrm{Aut}_E(L) \qquad \text{and} \qquad g(H) = L^H,$$

are inclusion reversing bijections that are inverse to one another.

Proof. Since L is Galois over K, it is the splitting field of a separable polynomial $h \in K[X]$. This means that the K-algebra $A = K[X]/(h)$ has the property that $A \otimes_K L$ is a split

L-algebra. For each zero $\alpha \in L$ of h let $\phi_\alpha : A \longrightarrow L$ denote the K-algebra morphism given by $X \mapsto \alpha$. The morphism

$$\bigotimes_{h(\alpha)=0} A \longrightarrow L$$

induces by the various morphisms ϕ_α, is a surjective morphism of K-algebras. Since any tensor products of copies of A is split over L, so is the quotient L. Therefore Theorem 2.2 applies with $\pi = G = \mathrm{Gal}(L/K)$.

Let E be a field with $K \subset E \subset L$. Then the functor F maps the embedding $E \subset L$ of K-algebras to an epimorphism $\pi = F(L) \longrightarrow F(E)$. This shows that the π-set $F(E)$ is isomorphic to π/H where H is the subgroup $\mathrm{Aut}_E(L)$ of π. It follows that $F(E) = \mathrm{Aut}_E(L) = H$.

Conversely, let H be a subgroup of π. Then the functor G maps the surjection $\pi \longrightarrow \pi/H$ to the K-algebra monomorphism $L^H \subset L$. In particular we have $K \subset L^H$.

It is easy to see that F and G reverse inclusions. Therefore the equivalence of categories of Theorem 2.2 implies the theorem.

Remark 3.2. In the notation of Theorem 3.1, the field L is Galois over each subfield E with Galois group $\mathrm{Aut}_E(L)$. On the other hand, the field L^H is Galois over K if and only if H is normal in G. In this case the natural map $\pi/H \longrightarrow \mathrm{Aut}_K(E)$ is an isomorphism of groups.

Proof. If the K-algebra L splits over K, it certainly also splits over each subfield E. Therefore L is Galois over E if it is Galois over K. This takes care of the first statement. A subgroup $H \subset \pi$ is normal if and only if $\tau H \tau^{-1} = H$ for all $\tau \in \pi$. This shows that $\tau(L^H) = l^H$ for all $\tau \in \pi$. In other words, L^H is Galois over K.

3. Finite étale algebras.

Let k be a field. Recall that a finite k-algebra A is called split if its isomorphic to k^d for some d. The algebra A is called *potentially split* if and only if it splits over an algebraic closure $\overline{k}$ of k. It is easy to see that for any potentially split k-algebra A there actually exists a *finite* extension $k \subset l$ for which $A \otimes_k l$ is split over l.

For any finite k-algebra A, we define the *trace* $\mathrm{Tr}(x)$ of $x \in A$ as the trace of the multiplication by x map. The *discriminant* $\Delta(A/k)$ of A is the determinant of the matrix $(\mathrm{Tr}(e_i e_j))_{1 \le i,j \le d}$. Here $e_1, \ldots, e_d \in A$ denote a k-basis for A. Changing the basis, multiplies $\Delta(A/k)$ by a non-zero square in k. Therefore the discriminant is only well defined up to multiplication by non-zero squares of k.

For any two finite k-algebra A, B we have $\Delta((A \times B)/k) = \Delta(A/k) \cdot \Delta(B/k)$. For any extension field $k \subset l$, the discriminant $\Delta(A \otimes_k l/l)$ is, up to squares in l^*, equal to $\Delta(A/k)$. Since the discriminant of the k-algebra is not zero, the same is true for any split k-algebra and even for any potentially split k-algebra.

The pairing $A \times A \longrightarrow k$ given by $(a, b) \mapsto \mathrm{Tr}(ab)$ is non-degenerate if and only if the discriminant $\Delta(A/k)$ is not zero.

Definition 3.1. Let A be a k-algebra. The *module of Kähler differentials* $\Omega_{A/k}$ of A is the A-module generated by symbols dx for $x \in A$ modulo the A-submodule generated by elements that reflect the usual rules of differential calculus:

(a) $d\lambda = 0$ for all $\lambda \in k \subset A$;
(b) $d(x+y) - dx - dy$ for all $x, y \in A$;
(c) $d(xy) - xdy - ydx$ for all $x, y \in A$.

For any two finite k-algebra A, B we have $\Omega_{(A\times B)/k} = \Omega_{A/k} \times \Omega_{B/k}$. For any field extension $k \subset l$ we have $\Omega_{(A\otimes_k l)/l} = \Omega_{A/k} \otimes_k l$, Since $\Omega_{A/k} = 0$ for $A = k$, the same is true for any split k-algebra and any potentially split k-algebra. For every A-module M we have $\mathrm{Der}(A, M) = \mathrm{Hom}(\Omega_{A/k}, M)$. Here $\mathrm{Der}(A, M)$ denotes the A-module of derivations $A \longrightarrow M$.

Definition 3.2. A finite k-algebra A is called *étale* if it satisfies one of the four equivalent conditions of the following proposition.

Proposition 3.2. *Let A be a finite k-algebra. Then the following are equivalent.*
(a) The discriminant $\Delta(A/k)$ is not zero.
(b) The module of Kähler differentials $\Omega_{A/k}$ is zero.
(c) A is potentially split.
(d) A is potentially reduced

By what has been said above, we see that the product of two étale algebras is again étale and that a k-algebra A is étale if and only if $A \otimes_k l$ is an étale l-algebra for some, or equivalently, every field l containing k. In particular, since k itself is étale, split k-algebras are also étale.

Proof. We may assume that k is algebraically closed. If A is a product of two k-algebras B and C then conditions (a), (b), (c) and (d) hold if and only if they hold for both B and C.

We proceed by induction on the k-dimension of A. If A has dimension 1, then $A = k$ and all four conditions hold true. If $\dim_k A > 1$, it cannot be a field. If A has a proper ideal I with $I = I^2$, Lemma 3.4 below implies $A \cong A/I \times A/\mathrm{Ann}\, I$. Since $0 < \dim A/I < \dim A$, we are done by induction. If there is no such ideal, choose a maximal ideal $\mathfrak{m} \subset A$ and consider the sequence of ideals $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \ldots$. Since this sequence stabilizes, we have $\mathfrak{m}^k = \mathfrak{m}^{k+1}$ for some $k \geq 1$. It follows that $\mathfrak{m}^k = \mathfrak{m}^{2k}$ and hence by assupmtion $\mathfrak{m}^k = 0$ This implies that A contains non-zero nilpotents so that (a), (c) and (d) do not hold. To see that (b) does not hold either, we define the derivation $d : A \longrightarrow \mathfrak{m}/\mathfrak{m}^2$ by $d(a) = a - \overline{a}$, where $\overline{a} \in k \subset A$ denote is the unique element in $k \subset A$ congruent to a modulo $\mathfrak{m}$. Since d is surjective, it is not zero and hence $\Omega_{A/k}$ is not zero, as required.

Lemma 3.4. *Let A be a finite k-algebra and let $I \subset A$ be an ideal satisfying $I = I^2$. Let $J = \mathrm{Ann}\, I$. Then the natural map $A \longrightarrow A/I \times A/J$ is a k-algebra isomorphism.*

Proof. Let S be the multiplicative subset $1 + I$ of A and let A_S denote the corresponding localization of A. Then the A_S-ideal generated by I is zero. Indeed, let $e_1, \ldots, e_m$ be a minimal set of generators. Since $I = I^2$ we have $e_1 = \sum_{i=1}^m \lambda_i e_i$ for certain $\lambda_i \in I$. It follows that

$$(1 - \lambda_1)e_1 = \sum_{i=2}^m \lambda_i e_i,$$

contradicting the minimality of the set $e_1, \ldots, e_m$.

The A-ideal I is finitely generated and for each of its generators there exists an element in S that annihilates it. The product b of these elements annihilates the ideal I. Since $b \equiv 1 \pmod{I}$, we see that the ideals I and $\mathrm{Ann}(I)$ are coprime. The lemma now follows from the Chinese Remainder Theorem.

4. Infinite Galois theory.

Let k be a field and let $\overline{k}$ be a separable closure of k. Let π denote the group $\mathrm{Aut}_k(\overline{k})$ of k-automorphisms of $\overline{k}$. Equipped with the Krull topology, it is a compact Hausdorff topological group.

In this section, let $\underline{C}$ be the category of finite étale k-algebras A and let $\underline{Sets}_\pi$ be the category of *finite* sets equipped with *continuous* action by π. Here the finite sets have the discrete topology. The continuity of the action of π means that each point has an open stabilizer.

Let $F : \underline{C} \longrightarrow \underline{Sets}_\pi$ be the contravariant functor that maps an object A of $\underline{C}$ to the set $\mathrm{Hom}_k(A, \overline{k})$ of its $\overline{k}$-points. In the other direction, let Let $G : \underline{Sets}_\pi \longrightarrow \underline{C}$ be the contravariant functor that maps a finite π-set X to the k-algebra $\mathrm{Map}_\pi(X, \overline{k})$. Since A is étale, the $\overline{k}$-algebra $A \otimes_k \overline{k}$ is split and the set

$$F(A) \;=\; \mathrm{Hom}_k(A, \overline{k}) \;=\; \mathrm{Hom}_{\overline{k}}(A \otimes_k \overline{k}, \overline{k})$$

is finite of cardinality $\dim_k A$. The group π acts on $F(A)$ by composition. To see that the action is continuous, let $f \in \mathrm{Hom}_k(A, \overline{k})$. Choose a k-basis $e_1, \ldots, e_d$ for A and put $Z = \{f(e_1), \ldots, f(e_d)\}$. Then Z is a finite subset of $\overline{k}$ and the open subgroup $U_Z = \{\sigma \in \pi : \sigma \text{ fixes } Z\}$ of π acts trivially on Z and hence fixes f. We see that the stabilizer of f is an open subgroup of π. Therefore the π-action is continuous.

It follows that the functor F is well defined. We have no direct proof of the fact that the functor G is well defined. This will be established in the course of the proof of the following theorem.

Theorem 4.1. *The functors F and G induce an anti-equivalence of categories*

$$\underline{C} \quad \cong \quad \underline{Sets}_\pi.$$

Proof. We show that $GF \cong \mathrm{id}_{\underline{C}}$. We consider the evaluation map

$$A \;\longrightarrow\; GF(A) \;=\; \mathrm{Map}_\pi(\mathrm{Hom}_k(A, \overline{k}), \overline{k})$$

given by $a \mapsto g_a$ where $g_a(\varphi) = \varphi(a)$. As in Sections 1 and 2, it is immediate that this map is injective. Since A étale, it is a product of fields, each of which is a finite étale extension of k. Since the functor F maps products in the category $\underline{C}$ to disjoint unions in $\underline{Sets}_\pi$ and the functor G does the converse, it suffices to consider the case where $A = l$ is a finite field extension of k. In this case the π-set $\mathrm{Hom}_k(l, \overline{k})$ is transitive. This follows from the fact that every field homomorphism $l \hookrightarrow \overline{k}$ extends to a field automorphism $\overline{k} \hookrightarrow \overline{k}$ (?). Therefore $\mathrm{Hom}_k(l, \overline{k})$ is isomorphic to π/H for some open subgroup H of π.