In this note we prove a weak form of the Prime Number Theorem due to P. Chebyshev. It is sufficient for properly estimating the running times of several algorithms. The proof is based on arithmetical properties of the binomial coefficients $\binom{2n}{n}$. For any real number $x > 0$ let $\pi(x)$ denote the number of primes smaller than x.

Lemma. *Let $n \in \mathbf{Z}_{>0}$. Let p be a prime and let $e_p = \mathrm{ord}_p\binom{2n}{n}$. Then $p^{e_p} \le 2n$. In particular, for $p > \sqrt{2n}$ we have $e_p \le 1$;*

Proof. For every $m \in \mathbf{Z}_{>0}$ we have $\mathrm{ord}_p m! = \sum_{i\ge1}[\frac{m}{p^i}]$. It follows that

$$e_p \;=\; \mathrm{ord}_p\binom{2n}{n} \;=\; \sum_{i\ge1}\left([\frac{2n}{p^i}] - 2[\frac{n}{p^i}]\right).$$

Since for every $t \in \mathbf{R}$ the integer $[2t] - 2[t]$ is either 0 or 1, the summands are at most 1. Therefore e_p is not larger than the number of non-zero summands, which is at most $\ln 2n/\ln p$. It follows that p^{e_p} is at most $2n$ as required.

Proposition 1. *There exists $c > 0$ for which $\pi(x) \,<\, c\frac{x}{\ln x}$ for all $x > 0$.*

Proof. The binomial coefficient $\binom{2n}{n}$ is divisible by all primes p between n and $2n$. Since we have $\binom{2n}{n} < 4^n$ for all $n \ge 1$, this leads to the inequality

$$\prod_{n<p\le 2n} p \;<\; 4^n = 2^{2n}, \qquad \text{for all } n \ge 1.$$

Let $x \in \mathbf{R}_{>0}$ and 2^k be the smallest power of 2 for which $x \le 2^k$. We have $2^k < 2x$. Taking logarithms and summing up the inequalities for $n = 1, 2, 2^2, \ldots, 2^{k-1}$ we find

$$\sum_{p<x}\ln p \;\le\; \sum_{p\le 2^k}\ln p \;\le\; \ln 2 + 2\ln 2 + \ldots + 2^k\ln 2 \;<\; 4x\ln 2.$$

On the other hand we have trivially

$$\sum_{p<x}\ln p \;>\; \sum_{\sqrt{x}<p<x}\ln p \;>\; (\pi(x) - \sqrt{x})\ln\sqrt{x}.$$

Combining the two inequalities gives

$$\pi(x) \;<\; 8\ln 2\frac{x}{\ln x} + \sqrt{x}.$$

This implies the proposition.

Proposition 2. *There exists $c > 0$ for which $\pi(x) > c\frac{x}{\ln x}$ for all $x > 1$.*

Proof. Let $n \in \mathbf{Z}_{>0}$. By Lemma 2 we have

$$\binom{2n}{n} = \prod_{p \leq 2n} p^{e_p} \leq 2n^{\sqrt{2n}} \prod_{\sqrt{2n} < p \leq 2n} p \leq 2n^{\sqrt{2n}} \prod_{p \leq 2n} p.$$

Since $\binom{2n}{n}$ is the largest of the binomial coefficients $\binom{2n}{k}$ with $k = 0, 1, \ldots, 2n$, we have $\binom{2n}{n} > \frac{1}{2n+1}4^n$. Combining the inequalities we get

$$\sum_{p \leq 2n} \ln p \geq n \ln 4 - \ln(2n+1) - \sqrt{2n} \ln(2n).$$

Since $\sum_{p \leq x} \ln p \leq \pi(x) \ln(x)$ for every $x \geq 1$, we obtain the inequality

$$\pi(x) \geq \ln 2 \frac{x}{\ln x} - \frac{\ln(x+1)}{\ln(x)} - \sqrt{x}$$

and the proposition follows.

Corollary. *(Chebyshev 1850) There exists constants $c_1, c_2 \in \mathbf{R}_{>0}$ for which*

$$c_1 \frac{x}{\ln x} < \pi(x) < c_2 \frac{x}{\ln x}, \qquad \textit{for all } x > 0.$$