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MEAN FIELD GAMES WITH STATE CONSTRAINTS

Author: ROSSANA CAPUANI

Supervisors: PIERMARCO CANNARSA
PIERRE CARDALIAGUET

Coordinator: ANDREA BRAIDES

UNIVERSITA' degli STUDI di ROMA
TOR VERGATA

A Giuseppe, Anna e Gabriella

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Abstract

The aim of this Thesis is to study deterministic mean field games for agents who operate in a bounded domain. In this case, the existence and uniqueness of Nash equilibria cannot be deduced as for unrestricted state space because, for a large set of initial conditions, the uniqueness of the solution to the associated minimization problem is no longer guaranteed. We attack the problem by interpreting equilibria as measures in a space of arcs. In such a relaxed environment the existence of constrained MFG equilibrium follows by set-valued fixed point arguments. Then, we give a uniqueness result for such equilibria under a classical monotonicity assumption.

At this point, it is natural to define a mild solution of the constrained MFG problem as a pair $(u, m) \in C([0,T] \times \overline{\Omega}) \times C([0,T]; \mathcal{P}(\overline{\Omega}))$, where m is given by $m(t) = e_t \sharp \eta$ for some constrained MFG equilibrium η and

$$u(t,x) = \inf_{\substack{\gamma \in \Gamma \\ \gamma(t) = x}} \left\{ \int_t^T \Big[L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m(s)) \Big] \, ds + G(\gamma(T), m(T)) \right\}.$$

Under suitable assumptions on the data, we have analyzed the regularity and sensitivity of the mild solutions. Finally, using the regularity of mild solutions and the structure of our problem, we show that (u, m) satisfies the MFG system in suitable point-wise sense.

INTRODUCTION

Mean field games (MFG) theory has been introduced simultaneously by Lasry and Lions ([51], [52], [53]) and by Huang, Malhamé and Caines ([47], [48]) in order to study large population differential games. The main idea of such a theory is to borrow from statistical physics the general principle of a mean-field approach to describe equilibria in a system of many interacting particles. Interestingly, the concept was previously developed in economic literature—in discrete time and often in a stationary regime—under the terminology of "heterogenous agent models" [4, 16, 49, 50].

In game theory, for a system with a finite number of players, the natural notion of equilibrium is the one introduced by John Nash [54]. So, the notion of mean-field equilibrium suggested by Lasry-Lions is justified as being the limit, as $N \to \infty$, of the Nash equilibria for N-player games, under the assumption that players are symmetric and rational.

Since the seminal works of Lasry and Lions and of Caines, Huang and Malhamé, the subject has known a very fast growth. We refer to the notes [1, 26], the survey papers [17, 44] and the books [9, 31, 45] for a general presentation.

In deterministic settings, the equilibrium found in the mean field limit turns out to be a solution of the forward-backward system of PDEs

$$(MFG)\begin{cases} -\partial_t u + H(x, Du) = F(x, m) \quad \text{in } [0, T] \times \Omega, \\ \partial_t m - div(m D_p H(x, Du)) = 0 \quad \text{in } [0, T] \times \Omega, \\ m(0) = m_0 \quad u(x, T) = G(x, m(T)) \end{cases} \tag{I.1}$$

which couples a Hamilton-Jacobi-Bellman equation (for the value function u of the generic player) with a continuity equation (for the density m of players). Here $\Omega \subset \mathbb{R}^n$ represents the domain in the state space in which agents are supposed to operate.

The well-posedness of system (I.1) was developed for special geometries of the domain Ω, namely when Ω equals the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, or the whole space $\mathbb{R}^n$ (see, e.g., [26], [28], [52], [53]).

The goal of the present Thesis is to study the well-posedness of the MFG problem subject to state constraints, that is, when players are confined into a compact domain $\overline{\Omega} \subseteq \mathbb{R}^n$. MFG with state constraint appear very naturally in the applications. For instance, the heterogeneous agent models always involve state constraints. In fact these constraints play even a central role in the analysis since they mainly explain the heterogeneity in the economy: see for instance Huggett's model of income and wealth distribution as discussed in [2, 3]. It is also very natural to introduce state constraints in pedestrians MFG models, although this has just been discussed so far in an informal way. Here again the constraints are important to explain the behavior of the crowd and it is largely believed that they should help to regulate the traffic: see for instance [29, 36] on related issues. Let us also note that other type of constraints can be considered for the MFG systems: for instance the density constraints were first discussed in [59] and then analyzed in [57]. Although an important issue in terms of application, MFG with state constraints have attracted little attention up to now. To the best of our knowledge, the only reference is the analysis of Huggett's model in [3].

In the absence of state constraints, the solution of (I.1) on $[0, T] \times \mathbb{T}^n$ is obtained by a fixed point argument which uses in an essential way the fact that viscosity solutions of the Hamilton-Jacobi equation

$$-\partial_t u + H(x, Du) = F(x, m) \quad \text{in } [0, T] \times \mathbb{T}^n$$

are smooth on a sufficiently large set to allow the continuity equation

$$\partial_t m - div(mD_pH(x, Du)) = 0 \quad \text{in } [0, T] \times \mathbb{T}^n$$

to be solvable. Specifically, it is known that u is of class $C^{1,1}_{loc}$ outside a closed singular set of zero Lebesgue measure and that m remains absolutely continuous with a bounded density. In this way, the coefficient $D_pH(x, Du)$ in the continuity equation turns out to be locally Lipschitz continuous on a "sufficiently large" open set, ensuring uniqueness for the continuity equation. The presence of the state constraints destroys this structure in two different ways. First, the measure can stop being absolutely continuous after a while and may develop a singular part. This phenomenon is known in Huggett's model, where the singular part of the measure appears at the boundary of the constraints ([3]). In Huggett's model, the singular part is trapped at the boundary, but this to only a particular case and one expects in general the singular part to propagate into the interior of the constraint. This makes the interpretation of the continuity equation difficult. Second, the "almost smooth" structure of the value function described above is lost in the presence of state constraints. Indeed, the value function cannot be semiconcave in general, see [27, Example 1.1]. Because of these two phenomena, proving the existence of solutions to (I.1) requires a complete change of paradigm.

In this direction, the first step is a better understanding of optimal control problems with state constraints. There is a huge literature on the subject; we refer to the survey paper [40] for a general presentation and recent results. Two approaches can be considered. The first one consists in deriving optimality conditions for the optimal trajectories of the constrained problem (maximum principle): it yields regularity properties of the optimal trajectories. The second approach focuses on the value function and its characterization in terms of Hamilton-Jacobi equation, which will allow us ultimately to derive the MFG system for problems with state constraints. We now discuss both approaches successively.

The maximum principle under state constraints was first established by Dubovitskii and Milyutin [37] (see also the monograph [61] for different forms of maximum principle under state constraints). It may happen that the maximum principle is degenerate and does not yield much information (abnormal maximum principle). As explained in [10, 14, 38, 39] in various contexts, the so-called "inward pointing condition" generally ensures the normality of the maximum principle under state constraints. In our setting (calculus of variation problem, without constraints on the velocity), this will never be an issue. The maximum principle under state constraints generally involves an adjoint state which is the sum of a $W^{1,1}$ map and a map with bounded variations. This later mapping may be very irregular and have infinitely many jumps [58], which implies the discontinuity of the optimal control. However, under suitable assumptions (requiring in particular some regularity of the data and the fact that the dynamics is affine with respect to the control), it has been shown that the optimal control and the adjoint state are continuous, and even Lipschitz continuous: see the seminal work of Hager [46] (in the convex setting) and the subsequent contributions of Malanowski [56] and Galbraith and Vinter [43] (in much more general frameworks). Generalization to less smooth frameworks can also be found in [38, 11].

In a seminal paper, Soner [60] characterized the value function of optimal control problems with state constraints as the unique solution of a suitable Hamilton-Jacobi equations (constrained viscosity solutions). When the domain is backward invariant, it is also proved in [42] that the value function can also be characterized in terms of viscosity solutions involving with only equalities. As explained [19] for problems in the whole space, there is a strong relation between the value function and the maximum principle: the key idea is that the adjoint variable in the maximum principle satisfies some relations involving the superdifferential of the value function. This "sensitivity analysis" is also known for problems with state constraints: [15, 13, 12, 32, 41].

PRESENTATION OF THE MAIN RESULTS

First of all, we need some preliminary results that we use during this Thesis. To this aim, in CHAPTER 1 we introduce the notation and recall some know result on the Measure Theory. Moreover, in SECTION 1.3, we introduce the concept of superdifferential and subdifferential on a closed domain and we give some of their properties. We also focus our attention on the definition of directional derivatives in a boundary point (SUBSECTION 1.3.1).

In CHAPTER 2, we give useful results for the constrained minimization problem that we will use in the following chapters. More precisely, let Ω be a bounded open subset of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary. Let Γ be the metric subspace of $AC(0,T;\mathbb{R}^n)$ defined by

$$\Gamma = \Big\{ \gamma \in AC(0,T;\mathbb{R}^n) : \ \gamma(t) \in \overline{\Omega}, \ \ \forall t \in [0,T] \Big\}.$$

For any $x \in \overline{\Omega}$, we set

$$\Gamma[x] = \{\gamma \in \Gamma : \gamma(0) = x\}.$$

In SECTION 2.1, we show that any constrained trajectory $\gamma \in \Gamma[x]$ can be approximated in suitable way (PROPOSITION 2.1.0.2). Let $U \subset \mathbb{R}^n$ be an open set such that $\overline{\Omega} \subset U$. Given $x \in \overline{\Omega}$, we consider the constrained minimization problem

$$\inf_{\gamma\in\Gamma[x]} J[\gamma], \quad \text{where} \quad J[\gamma]=\left\{\int_0^T f(t,\gamma(t),\dot\gamma(t))\,dt+g(\gamma(T))\right\}, \tag{I.2}$$

where $f:[0,T]\times U\times\mathbb{R}^n\to\mathbb{R}$ and $g:U\to\mathbb{R}$. Let $\mathcal{X}[x]$ be the set of solutions of (2.2.0.1). We denote by $H:[0,T]\times U\times\mathbb{R}^n\to\mathbb{R}$ the Hamiltonian

$$H(t,x,p)=\sup_{v\in\mathbb{R}^n}\left\{-\langle p,v\rangle-f(t,x,v)\right\}, \qquad \forall\,(t,x,p)\in[0,T]\times U\times\mathbb{R}^n.$$

Under suitable assumptions on Ω, f and g (SECTION 2.2), we obtain the following necessary conditions for our problem.

THEOREM 0.0.0.1. For any $x\in\overline{\Omega}$ and any $\gamma^\star\in\mathcal{X}[x]$ the following holds true.

(i) $\gamma^\star$ is of class $C^{1,1}([0,T];\overline{\Omega})$.

(ii) There exist:

(a) a Lipschitz continuous arc $p:[0,T]\to\mathbb{R}^n$,

(b) a bounded measurable function $\Lambda:[0,T]\to[0,\infty)$,

(c) a constant $\nu\in\mathbb{R}$ such that

$$0\le\nu\le\max\left\{1,2\mu\sup_{x\in U}\left|D_pH(T,x,Dg(x))\right|\right\},$$

which satisfy the adjoint system

$$\begin{cases}\dot\gamma^\star(t)=-D_pH(t,\gamma^\star(t),p(t)) & \text{for all } t\in[0,T],\\ \dot p(t)=D_xH(t,\gamma^\star(t),p(t))-\Lambda(t)Db_\Omega(\gamma^\star(t)) & \text{for a.e. } t\in[0,T],\end{cases} \tag{I.3}$$

and the transversality condition

$$p(T)=Dg(\gamma^\star(T))+\nu Db_\Omega(\gamma^\star(T))\mathbf{1}_{\partial\Omega}(\gamma^\star(T)).$$

Moreover,

(iii) the following estimate holds

$$||\dot\gamma^\star||_\infty\le L^\star, \quad \forall\gamma^\star\in\mathcal{X}[x], \tag{I.4}$$

where $L^\star=L^\star(\mu,M',M,\kappa,T,||Dg||_\infty,||g||_\infty)$;

(iv) for all $t\in[0,T]$, $\Lambda(t)$ is given by

$$\Lambda(t)=\frac{1}{\theta(t)}\Big[-\Big\langle D^2b_\Omega(\gamma^\star(t))D_pH\big(t,\gamma^\star(t),p(t)\big),D_pH\big(t,\gamma^\star(t),p(t)\big)\Big\rangle$$

$$
\begin{aligned}
&- \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pt}H\big(t,\gamma^\star(t),p(t)\big)\Big\rangle - \\
&\Big\langle Db_\Omega(\gamma^\star(t)), D^2_{px}H\big(t,\gamma^\star(t),p(t)\big)D_pH\big(t,\gamma^\star(t),p(t)\big)\Big\rangle \\
&+ \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H\big(t,\gamma^\star(t),p(t)\big)D_xH\big(t,\gamma^\star(t),p(t)\big)\Big\rangle\Big]\mathbf{1}_{\partial\Omega}(\gamma^\star(t)),
\end{aligned}
$$

where $\theta(t) := \langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H(t,\gamma^\star(t),p(t))Db_\Omega(\gamma^\star(t))\rangle$.

Note that THEOREM 0.0.0.1 also holds if we consider trajectories starting at $x \in \overline{\Omega}$ at time $t \in [0,T]$. It suffices to replace $J[\gamma]$ with $J_t[\gamma]$ (with obvious notation) and then, accordingly, the set $\mathcal{X}_t[x]$. Let us recall that the existence of relatively smooth optimal trajectories is not new and goes back to Hager [46]. Beside a new technique of proof, the improvement with respect to the previous literature on the subject (in particular [43]) is the explicit representation of the derivative of the adjoint p. This representation will allow us later to obtain explicit Lipschitz bounds for $\dot\gamma$, which is necessary to show the existence of a Lipschitz continuous mild solution to the MFG problem with state constraints.
In SECTION 2.5, under suitable assumptions on f and using THEOREM 0.0.0.1 we deduce the sensitivity relations for the value function associated to (I.2).

THEOREM 0.0.0.2. For any $\varepsilon \geq 0$ there exists a constant $c_\varepsilon \geq 0$ such that for any $(t,x) \in [0,T-\varepsilon]\times\overline{\Omega}$ and for any $\gamma \in \mathcal{X}_t[x]$, denoting by $p \in \mathrm{Lip}(t,T,\mathbb{R}^n)$ a dual arc associated with γ, one has that

$$u(t+\sigma,x+h) - u(t,x) \leq \sigma H(t,x,p(t)) + \langle p(t),h\rangle + c_\varepsilon(|h|+|\sigma|)^{\frac{3}{2}} \quad \forall (t,x) \in [0,T-\epsilon]\times\overline{\Omega},$$

for all $h \in \mathbb{R}^n$ small enough such that $x+h \in \overline{\Omega}$, and for all $\sigma \in \mathbb{R}$ such that $0 \leq t+\sigma \leq T-\varepsilon$.

A direct consequence of THEOREM 0.0.0.2 is the local semiconcavity of the value function associated to (I.2) in $(0,T)\times\overline{\Omega}$ (COROLLARY 2.5.0.3). Let us point out that this semiconcavity is unexpected, since it was known so far that the value function is *not* semiconcave with linear modulus in general [27].
In CHAPTER 3, we are ready to start to analyze MFG problem with state constraints. Following the Lagrangian formulation of the unconstrained MFG problem proposed in [8], we define a "relaxed" notion of equilibria and solutions. Such a formulation consists of replacing probability measures on $\overline{\Omega}$ with measures on arcs in $\overline{\Omega}$. More precisely, on the metric space Γ with the uniform metric, for any $t \in [0,T]$ we consider the evaluation map $e_t : \Gamma \to \overline{\Omega}$ defined by

$$e_t(\gamma) = \gamma(t) \qquad (\gamma \in \Gamma).$$

Given any probability measure m_0 on $\overline{\Omega}$, we denote by $\mathcal{P}_{m_0}(\Gamma)$ the set of all Borel probability measures η on Γ such that $e_0\sharp\eta = m_0$ and we consider, for any $\eta \in \mathcal{P}_{m_0}(\Gamma)$, the functional

$$J_\eta[\gamma] = \int_0^T \Big[L(\gamma(t),\dot\gamma(t)) + F(\gamma(t),e_t\sharp\eta)\Big]\,dt + G(\gamma(T),e_T\sharp\eta) \qquad (\gamma \in \Gamma). \tag{I.5}$$

Then, we call a measure $\eta \in \mathcal{P}_{m_0}(\Gamma)$ a *constrained MFG equilibrium* for m_0 if η is supported on the set of all curves $\overline{\gamma} \in \Gamma$ such that

$$J_\eta[\overline{\gamma}] \leq J_\eta[\gamma] \;\; \forall \gamma \in \Gamma,\; \gamma(0) = \overline{\gamma}(0).$$

In THEOREM 3.1.2.4, we prove the existence of constrained MFG equilibria for m_0 by applying the Kakutani fixed point theorem [55]. At this point, in SECTION 3.2, we give the definition of *mild solutions of the constrained MFG problem* in $\overline{\Omega}$ as a pair $(u, m) \in C([0,T] \times \overline{\Omega}) \times C([0,T]; \mathcal{P}(\overline{\Omega}))$, where m is given by $m(t) = e_t \sharp \eta$ for some constrained MFG equilibrium η for m_0 and

$$u(t,x) = \inf_{\substack{\gamma \in \Gamma \\ \gamma(t)=x}} \left\{ \int_t^T \Big[L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m(s)) \Big] \, ds + G(\gamma(T), m(T)) \right\}.$$

In this way, the existence of mild solutions of the constrained MFG problem in $\overline{\Omega}$ (COROLLARY 3.2.0.2) becomes an easy corollary of the existence of equilibria for m_0 (THEOREM 3.1.2.4), whereas the uniqueness issue for such a problem remains a more challenging question. As observed by Lasry and Lions, in absence of state constraints uniqueness can be addressed by imposing suitable monotonicity assumptions on the data. We show that the same general strategy can be adopted even for constrained problems (THEOREM 3.2.0.4). However, we have to interpret the Lasry-Lions method differently because, as recalled above, solutions are highly nonsmooth in our case.

In CHAPTER 4, we apply our necessary conditions given in CHAPTER 2 to deduce the existence of more regular equilibria than those constructed in CHAPTER 3, assuming the data F and G to be Lipschitz continuous. More precisely, applying THEOREM 0.0.0.1, we show that the minimizers of J_η belong to $C^{1,1}([0,T], \overline{\Omega})$ and

$$||\dot{\gamma}||_\infty \leq L_0, \quad \forall \gamma \in \Gamma^\eta[x], \tag{I.6}$$

where $L_0 = L_0(\mu, M', M, \kappa, T, ||DG||_\infty, ||g||_\infty)$ depends only on the data. At this point, denoted by Γ_{L_0} the set of all γ of Γ such that (I.6) holds, we define by $\mathcal{P}^{\text{Lip}}_{m_0}(\Gamma_{L0})$ the set of $\eta \in \mathcal{P}_{m_0}(\Gamma)$ such that $m(t) = e_t \sharp \eta$ is Lipschitz continuous. Arguing as in SUBSECTION 3.1.2 we show that there exists at least one constrained MFG equilibrium $\eta \in \mathcal{P}^{\text{Lip}}_{m_0}(\Gamma_{L0})$ (THEOREM 4.2.0.5). The main consequence of THEOREM 4.2.0.5 is the existence of Lipschitz mild solutions under the assumptions in SECTION 4.1.

THEOREM 0.0.0.3. There exists at least one mild solution (u, m) of the constrained MFG problem in $\overline{\Omega}$. Moreover,

(i) u is Lipschitz continuous in $(0,T) \times \overline{\Omega}$;

(ii) $m \in \text{Lip}(0,T; \mathcal{P}(\overline{\Omega}))$ and $\text{Lip}(m) = L_0$, where L_0 is the constant in (I.6).

Owing to state constraints, it is not possible to have local semiconcavity with linear modulus in $(0,T) \times \overline{\Omega}$. Remarkably, applying THEOREM 0.0.0.2, we deduce a local semiconcavity with modulus $\omega(r) = Cr^{\frac{1}{2}}$ for u in $(0,T) \times \overline{\Omega}$.

In CHAPTER 5, we prove that mild solutions satisfy MFG system in suitable point-wise sense. Let (u, m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$. By the regularity of u, proved in CHAPTER 4, we deduce that u is a constrained viscosity solutions of

$$-\partial_t u(t,x) + H(x, Du(t,x)) = F(x, m(t)) \quad \text{in } (0,T) \times \overline{\Omega}, \tag{I.7}$$

that is

$$
\begin{aligned}
&-p_1 + H(x,p_2) \le F(x,m(t)) \quad \forall\,(t,x) \in (0,T)\times\Omega,\ \forall\,(p_1,p_2) \in D^+u(t,x)\\
&-p_1 + H(x,p_2) \ge F(x,m(t)) \quad \forall\,(t,x) \in (0,T)\times\overline{\Omega},\ \forall\,(p_1,p_2) \in D^-u(t,x).
\end{aligned}
$$

Thanks to the structure of mild solutions we can prove more. For simplicity, we set

$$
Q_m = \{(t,x) \in (0,T)\times\Omega : x \in supp(m(t))\}, \quad \partial Q_m = \{(t,x) \in (0,T)\times\partial\Omega : x \in supp(m(t))\}.
$$

where $supp(m(t))$ is the support of the measure $m(t)$. It is easly to prove that for any $(t,x) \in Q_m$ u is differentiable at (t,x) (PROPOSITION 5.1.0.6). While, for any $(t,x) \in \partial Q_m$ we give the following full description of $D^+u(t,x)$.

PROPOSITION 0.0.0.4. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$ and let $(t,x) \in \partial Q_m$. The following holds true.

(a) The partial derivative of u with respect to t, denoted by $\partial_t u(t,x)$, does exist and

$$
D^+u(t,x) = \{\partial_t u(t,x)\} \times D_x^+u(t,x).
$$

(b) All $p \in D_x^+u(t,x)$ have the same tangential component, which will be denoted by $p^\tau(t,x)$, that is,

$$
\left\{p^\tau \in \mathbb{R}^n : p \in D_x^+u(t,x)\right\} = \left\{p^\tau(t,x)\right\}. \tag{I.8}
$$

(c) For all $\theta \in \mathbb{R}^n$ such that $|\theta| = 1$ and $\langle \theta, \nu(x)\rangle = 0$ one has that

$$
\partial_\theta^+ u(t,x) = \langle p^\tau(t,x), \theta\rangle. \tag{I.9}
$$

Moreover,

$$
\partial_{-\nu}^+ u(t,x) = -\lambda_+(t,x) := -\max\{\lambda_p(t,x) : p \in D_x^+u(t,x)\}, \tag{I.10}
$$

where

$$
\lambda_p(t,x) = \max\{\lambda \in \mathbb{R} : p^\tau + \lambda\nu(x) \in D_x^+u(t,x)\}, \ \ \forall p \in D_x^+u(t,x). \tag{I.11}
$$

(d) $D_x^+u(t,x) = \{p \in \mathbb{R}^n : p = p^\tau(t,x) + \lambda\nu(x),\ \lambda \in (-\infty, \lambda_+(t,x)]\}$.

By the regularity of u and PROPOSITION 0.0.0.4 we deduce the following result.

THEOREM 0.0.0.5. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$. The following holds true.

(i) Let $(t,x) \in (0,T)\times\Omega$. Then, one has that

$$
\limsup_{\substack{(s,y) \in (0,T)\times\Omega \\ (s,y)\to(t,x)}} D^+u(s,y) \subset D^+u(t,x). \tag{I.12}
$$

In particular, for all $(t,x) \in Q_m$ we have that

$$\limsup_{\substack{(s,y)\in(0,T)\times\Omega \\ (s,y)\to(t,x)}} D^+u(s,y) = \Big\{\Big(\partial_t u(t,x), D_x u(t,x)\Big)\Big\}. \tag{I.13}$$

(ii) Let $(t,x) \in \partial Q_m$. Then,

$$\limsup_{\substack{(s,y)\in Q_m \\ (s,y)\to(t,x)}} D^+u(s,y) = \Big\{\Big(\partial_t u(t,x),\ p^\tau(t,x) + \lambda_+(t,x)\nu(x)\Big)\Big\}, \tag{I.14}$$

where $p^\tau(t,x)$ and $\lambda_+(t,x)$ are given in (I.8) and (I.10), respectively.

Therefore, any mild solutions (u,m) of constrained MFG problem in $\overline{\Omega}$ solve a mean field games system. More precisely, the following holds true.

(a) u is a constrained viscosity solution of

$$\begin{cases} -\partial_t u + H(x,Du) = F(x,m(t)) & \text{in } (0,T)\times\overline{\Omega} \\ u(x,T) = G(x,m(T)) & \text{in } \overline{\Omega}. \end{cases}$$

Moreover, u is solution in point-wise sense of

$$-\partial_t u + H(x,Du) = F(x,m(t)) \quad \text{in } Q_m,$$

and

$$-\partial_t u + H^\tau(x,p^\tau(t,x)) = F(x,m(t)) \quad \text{in } \partial Q_m$$

where

$$H^\tau(x,p) = \sup_{\substack{v\in\mathbb{R}^n \\ \langle v,\nu(x)\rangle = 0}} \{-\langle p,v\rangle - L(x,v)\}.$$

(b) There exists $V : [0,T]\times\overline{\Omega} \to \mathbb{R}^n$ Borel measurable vector field such that m is a solution in the sense of distributions of the continuity equation

$$\begin{cases} \partial_t m + div(V\, m) = 0, & \text{in } (0,T)\times\overline{\Omega} \\ m(0,x) = m_0(x) & \text{in } \overline{\Omega}. \end{cases}$$

Moreover, V is continuous on Q_m and

$$V(t,x) = \begin{cases} -D_pH(x,Du(t,x)) & \text{if } (t,x)\in Q_m, \\ -D_pH(x,p^\tau(t,x)+\lambda_+(t,x)\nu(x)) & \text{if } (t,x)\in\partial Q_m. \end{cases} \tag{I.15}$$

CHAPTER 3 are contained in an work joint with Piermarco Cannarsa [20], while CHAPTER 2, CHAPTER 4 and CHAPTER 5 are contained in two works joint with Piermarco Cannarsa and Pierre Cardaliaguet [21] and [22].

Chapter 1

Preliminaries

1.1 Notation

We denote by $|\cdot|$, $\langle\cdot\rangle$, respectively, the Euclidean norm and scalar product in $\mathbb{R}^n$. We denote by B_R the ball of radius $R > 0$ and center 0. Let $A \in \mathbb{R}^{n\times n}$ be a matrix. We denote by $||\cdot||$ the norm of A defined as follows

$$||A|| = \max_{||x||=1} ||Ax|| \quad (x \in \mathbb{R}^n).$$

For any subset $S \subset \mathbb{R}^n$, $\overline{S}$ stands for its closure, ∂S for its boundary and $S^c = \mathbb{R}^n \setminus S$ for the complement of S. We denote by $\mathbf{1}_S : \mathbb{R}^n \to \{0, 1\}$ the characteristic function of S, i.e.,

$$\mathbf{1}_S(x) = \begin{cases} 1 & x \in S, \\ 0 & x \in S^c. \end{cases}$$

We write $\mathrm{Lip}(0, T; \mathbb{R}^n)$ for the space of all Lischitz continuous $\mathbb{R}^n$-valued functions on $[0, T]$. We write $AC(0, T; \mathbb{R}^n)$ for the space of all absolutely continuous $\mathbb{R}^n$-valued functions on $[0, T]$, equipped with the uniform norm $||\gamma||_\infty = \sup_{[0,T]} |\gamma(t)|$. We observe that $AC(0, T; \mathbb{R}^n)$ is not a Banach space.
For any measurable function $f : [0, T] \to \mathbb{R}^n$, we set

$$||f||_2 = \left(\int_0^T |f|^2 \, dt \right)^{\frac{1}{2}}.$$

Let U be an open subset of $\mathbb{R}^n$. $C(U)$ is the space of all continuous functions on U and $C_b(U)$ is the space of all bounded continuous functions on U. $C^k(U)$ is the space of all functions $\phi : U \to \mathbb{R}$ that are k-times continuously differentiable. Let $\phi \in C^1(U)$. The gradient vector of ϕ is denoted by $D\phi = (D_{x_1}\phi, \cdots, D_{x_n}\phi)$, where $D_{x_i}\phi = \frac{\partial \phi}{\partial x_i}$. Let $\phi \in C^k(U)$ and let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ be a multiindex. We define $D^\alpha \phi = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n} \phi$. $C_b^k(U)$ is the space of all function $\phi \in C^k(U)$ and such that

$$\|\phi\|_{k,\infty} := \sup_{\substack{x \in U \\ |\alpha| \le k}} |D^\alpha \phi(x)| < \infty$$

Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. $C^{1,1}(\overline{\Omega})$ is the space of all the functions C^1 in a neighborhood U of Ω and with locally Lipschitz continuous first order derivates.
The distance function from $\overline{\Omega}$ is the function $d_\Omega : \mathbb{R}^n \to [0, +\infty[$ defined by

$$d_\Omega(x) := \inf_{y \in \overline{\Omega}} |x - y| \quad (x \in \mathbb{R}^n).$$

We define the oriented boundary distance from $\partial\Omega$ by

$$b_\Omega(x) = d_\Omega(x) - d_{\Omega^c}(x) \quad (x \in \mathbb{R}^n). \tag{1.1.0.1}$$

We recall that, since the boundary of Ω is of class $\mathcal{C}^2$, there exists $\rho_0 > 0$ such that

$$b_\Omega(\cdot) \in C_b^2 \text{ on } \partial\Omega + B_{\rho_0} = \Big\{ y \in B(x, \rho_0) : x \in \partial\Omega \Big\}. \tag{1.1.0.2}$$

Throughout the thesis, we suppose that ρ_0 is fixed so that (1.1.0.2) holds.

1.2 RESULTS FROM MEASURE THEORY

In this section we introduce, without proof, some basic tools needed in the thesis (see, e.g., [6]).
Let X be a separable metric space. $C_b(X)$ is the space of all bounded continuous functions on X. We denote by $\mathcal{B}(X)$ the family of the Borel subset of X and by $\mathcal{P}(X)$ the family of all Borel probability measures on X. The support of $\eta \in \mathcal{P}(X)$, $supp(\eta)$, is the closed set defined by

$$supp(\eta) := \Big\{ x \in X : \eta(V) > 0 \text{ for each neighborhood V of } x \Big\}. \tag{1.2.0.1}$$

We say that a sequence $(\eta_i) \subset \mathcal{P}(X)$ is narrowly convergent to $\eta \in \mathcal{P}(X)$ if

$$\lim_{n \to \infty} \int_X f(x)\, d\eta_i(x) = \int_X f(x)\, d\eta(x) \quad \forall f \in C_b^0(X),$$

where $C_b^0(X)$ is the set of all bounded continuous functions on X.
We recall an interesting link between narrow convergence of probability measures and Kuratowski convergence of their supports.

PROPOSITION 1.2.0.1. If $(\eta_i) \subset \mathcal{P}(X)$ is a sequence narrowly converging to $\eta \in \mathcal{P}(X)$ then $supp(\eta) \subset K - \liminf_{n\to\infty} supp(\eta_i)$, i.e.

$$\forall x \in supp(\eta)\, \exists\, x_i \in supp(\eta_i) : \lim_{n\to\infty} x_i = x.$$

The following theorem is a useful characterization of relatively compact sets with respect to narrow topology.

THEOREM 1.2.0.2. (Prokhorov's Theorem) If a set $\mathcal{K} \subset \mathcal{P}(X)$ is tight, i.e.

$$\forall \epsilon > 0\, \exists\, K_\epsilon \; compact \; in \; X \; such \; that \; \widehat{\eta}(K_\epsilon) \geq 1 - \epsilon \;\; \forall \widehat{\eta} \in \mathcal{K},$$

then $\mathcal{K}$ is relatively compact in $\mathcal{P}(X)$ with respect to narrow topology. Conversely, if X is a separable complete metric space then every relatively compact subset of $\mathcal{P}(X)$ is tight.

Let X be a separable metric space. We recall that X is a Radon space if every Borel probability measure $\eta \in \mathcal{P}(X)$ satisfies

$$\forall B \in \mathcal{B}(X), \forall \epsilon > 0,\; \exists K_\epsilon \text{ compact with } K_\epsilon \Subset B \text{ such that } \eta(B \setminus K_\epsilon) \leq \epsilon.$$

Let us denote by d the distance on X and, for $p \in [1, +\infty)$, by $\mathcal{P}_p(X)$ the set of probability measures m on X such that

$$\int_X d^p(x_0, x)\, dm(x) < +\infty, \quad \forall x_0 \in X.$$

The *Monge-Kantorowich distance* on $\mathcal{P}_p(X)$ is given by

$$d_p(m, m') = \inf_{\eta \in \Pi(m,m')} \left[\int_{X^2} d(x,y)^p \, d\eta(x,y) \right]^{1/p}, \tag{1.2.0.2}$$

where $\Pi(m, m')$ is the set of Borel probability measures on $X \times X$ such that $\eta(A \times \mathbb{R}^n) = m(A)$ and $\eta(\mathbb{R}^n \times A) = m'(A)$ for any Borel set $A \subset X$. In the particular case when $p = 1$, the distance d_p takes the name of Kantorovich-Rubinstein distance and the following formula holds

$$d_1(m, m') = sup\Big\{ \int_X f(x)\, dm(x) - \int_X f(x)\, dm'(x) \mid f : X \to \mathbb{R} \text{ is 1-Lipschitz} \Big\}, \tag{1.2.0.3}$$

for all $m, m' \in \mathcal{P}_1(X)$.
Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. We write $\mathrm{Lip}(0, T; \mathcal{P}(\overline{\Omega}))$ for the space of all maps $m : [0, T] \to \mathcal{P}(\overline{\Omega})$ that are Lipschitz continuous with respect to d_1, i.e.,

$$d_1(m(t), m(s)) \leq C|t - s|, \quad \forall t, s \in [0, T], \tag{1.2.0.4}$$

for some constant $C \geq 0$. We denote by $\mathrm{Lip}(m)$ the smallest constant that verifies (1.2.0.4).
In the next result, we recall the relationship between the weak-$*$ convergence of measures and convergence with respect to d_p.

PROPOSITION 1.2.0.3. If a sequence of measures $\{\eta_i\}_{n\geq 1} \subset \mathcal{P}_p(X)$ converges to η for d_p, then $\{\eta_i\}_{n\geq 1}$ weakly converges to η. "Conversely", if η_i is concentrated on a fixed compact subset of X for all $n \geq 1$ and $\{\eta_i\}_{n\geq 1}$ weakly converges to η, then the $\{\eta_i\}_{n\geq 1}$ converges to η in d_p.

Given separable metric spaces X_1 and X_2 and a Borel map $f : X_1 \to X_2$, we recall that the push-forward of a measure $\eta \in \mathcal{P}(X_1)$ through f is defined by

$$f\sharp\eta(B) := \eta(f^{-1}(B)) \quad \forall B \in \mathcal{B}(X_2). \tag{1.2.0.5}$$

The push-forward is characterized by the fact that

$$\int_{X_1} g(f(x))\, d\eta(x) = \int_{X_2} g(y)\, df\sharp\eta(y) \tag{1.2.0.6}$$

for every Borel function $g : X_2 \to \mathbb{R}$.

We conclude this preliminary session by recalling the disintegration theorem.

THEOREM 1.2.0.4. Let X, Y be Radon separable metric spaces, $\mu \in \mathcal{P}(X)$, let $\pi : X \to Y$ be a Borel map and let $\eta = \pi\sharp\mu \in \mathcal{P}(Y)$. Then there exists an η-a.e. uniquely determined Borel measurable family[1] of probabilities $\{\mu_y\}_{y\in Y} \subset \mathcal{P}(X)$ such that

$$\mu_y(X \setminus \pi^{-1}(y)) = 0 \;\text{ for } \eta\text{-a.e. } y \in Y \tag{1.2.0.7}$$

and

$$\int_X f(x)\, d\mu(x) = \int_Y \Big(\int_{\pi^{-1}(y)} f(x)\, d\mu_y(x) \Big)\, d\eta(y) \tag{1.2.0.8}$$

for every Borel map $f : X \to [0, +\infty]$.

1.3 SEMICONCAVE FUNCTION AND GENERALIZED GRADIENTS

DEFINITION 1.3.0.1. We say that $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a modulus if it is a nondecreasing upper semicontinuous function such that $\lim_{r\to 0^+} \omega(r) = 0$.

DEFINITION 1.3.0.2. Let Ω be a bounded subset of $\mathbb{R}^n$ with C^2 boundary. Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a modulus. We say that a function $u : \overline{\Omega} \to \mathbb{R}$ is semiconcave with modulus ω if

$$\lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda)y) \leq \lambda(1-\lambda)|x-y|\omega(|x-y|) \tag{1.3.0.1}$$

for any pair $x, y \in \overline{\Omega}$, such that the segment $[x, y]$ is contained in $\overline{\Omega}$ and for any $\lambda \in [0, 1]$. We call ω a modulus of semiconcavity for u in $\overline{\Omega}$.

[1] We say that $\{\mu_y\}_{y\in Y}$ is a Borel family (of probability measures) if $y \in Y \longmapsto \eta_y(B) \in \mathbb{R}$ is Borel for any Borel set $B \subset X$.

A function u is called semiconvex in $\overline{\Omega}$ if $-u$ is semiconcave.
When the right-side of (1.3.0.1) is replaced by a term of form $C|x-y|^2$ we say that u is semiconcave with linear modulus.
For any $x \in \overline{\Omega}$, the sets

$$D^-u(x) = \Big\{p \in \mathbb{R}^n : \liminf_{\substack{y \to x \\ y \in \overline{\Omega}}} \frac{u(y)-u(x)-\langle p, y-x\rangle}{|y-x|} \geq 0\Big\} \tag{1.3.0.2}$$

$$D^+u(x) = \Big\{p \in \mathbb{R}^n : \limsup_{\substack{y \to x \\ y \in \overline{\Omega}}} \frac{u(y)-u(x)-\langle p, y-x\rangle}{|y-x|} \leq 0\Big\} \tag{1.3.0.3}$$

are called, respectively, the (Fréchet) subdifferential and superdifferential of u at x.
We note that if $x \in \Omega$ then, $D^+u(x)$, $D^-u(x)$ are both nonempty if and only if u is differentiable in x. In this case we have that

$$D^+u(x) = D^-u(x) = \{Du(x)\}.$$

PROPOSITION 1.3.0.3. Let Ω be a bounded subset of $\mathbb{R}^n$ with C^2 boundary. Let u be a real-valued function defined on $\overline{\Omega}$. Let $x \in \partial\Omega$ and let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. If $p \in D^+u(x)$, $\lambda \leq 0$ then, $p + \lambda\nu(x)$ belongs to $D^+u(x)$ for all $\lambda \leq 0$.

Proof. Let $x \in \partial\Omega$ and let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. Let $p \in D^+u(x)$. Let us take $\lambda \leq 0$ and $y \in \overline{\Omega}$. Since $p \in D^+u(x)$ and $\lambda \leq 0$, one has that

$$u(y)-u(x)-\langle p+\lambda\nu(x), y-x\rangle = u(y)-u(x)-\langle p, y-x\rangle - \lambda\langle \nu(x), y-x\rangle \leq o(|y-x|).$$

Hence, $p + \lambda\nu(x)$ belongs to $D^+u(x)$. □

$D^+u(x)$, $D^-u(x)$ can be described in terms of test functions as shown in the next lemma.

PROPOSITION 1.3.0.4. Let $u \in C(\overline{\Omega})$, $p \in \mathbb{R}^n$, and $x \in \overline{\Omega}$. Then the following properties are equivalent:

(a) $p \in D^+u(x)$ (resp. $p \in D^-u(x)$);

(b) $p = D\phi(x)$ for some function $\phi \in C^1(\mathbb{R}^n)$ touching u from above (resp. below);

(c) $p = D\phi(x)$ for some function $\phi \in C^1(\mathbb{R}^n)$ such that $f - \phi$ attains a local maximum (resp. minimum) at x.

In the proof of PROPOSITION 1.3.0.4 it is possible to follow the same method of [24, Proposition 3.1.7]. Before giving the proof, let us prove a technical lemma.

LEMMA 1.3.0.5. Let $\omega : (0,+\infty) \to [0,+\infty)$ be an upper semicontinuous function such that $\lim_{r\to 0} \omega(r) = 0$. Then there exists a continuous nondecreasing function $\omega_1 : [0,+\infty) \to [0,+\infty)$ such that

(i) $\omega_1(r) \to 0$ as $r \to 0$,

(ii) $\omega(r) \le \omega_1(r)$ for any $r \ge 0$,

(iii) the function $\xi(r) := r\omega_1(r)$ is in $C^1([0,+\infty))$ and satisfies $\dot{\xi}(0) = 0$.

Proof. Let us first set

$$\overline{\omega}(r) = \max_{\rho\in(0,r]} \omega(\rho).$$

Then $\overline{\omega}$ is nondecreasing, not smaller than ω and tends to 0 as $r \to 0$. Next we define for $r > 0$

$$\omega_0(r) = \frac{1}{r}\int_r^{2r} \overline{\omega}(\rho)\,d\rho, \qquad \omega_1(r) = \frac{1}{r}\int_r^{2r} \omega_0(\rho)\,d\rho,$$

and so we set $\omega_1(0) = 0$. We first observe that, since $\overline{\omega}$ is nondecreasing, the same holds for ω_0 and ω_1. Then we have that $\omega(r) \le \omega(r_0) \le \overline{\omega}(2r)$, and so $\omega_0(r) \to 0$ as $r \to 0$. Arguing in the same way with ω_1 we deduce that properties (i) and (ii) hold. To prove (iii), let us set $\xi(r) = r\omega_1(r)$. Then $\xi \in C^1((0+\infty))$ with derivate $\dot{\xi}(r) = 2\omega_0(2r) - \omega_0(r)$. Thus $\dot{\xi}(r) \to 0$ as $r \to 0$ and so ξ in C^1 in the closed half-line $[0,+\infty)$. □

Proof of PROPOSITION 1.3.0.4. The implications $(b) \implies (c)$ and $(c) \implies (a)$ are obvious; so it is enough to prove that (a) implies (b). Given $p \in D^+u(x)$, let us define, for $r > 0$,

$$\omega(r) = \max_{\substack{y\,\in\,\overline{\Omega} \\ y\,:\,|y-x|\,\le\, r}} \left[\frac{u(y)-u(x)-\langle p, y-x\rangle}{|y-x|}\right]_+, \tag{1.3.0.4}$$

where $[\cdot]_+$ denotes the positive part. The function ω is continuous and tends to 0 as $r \to 0$, by the definition of D^+u. Let ω_1 be the function given by the previous lemma. Then, setting

$$\phi(y) = u(x) + \langle p, y-x\rangle + |y-x|\omega_1(|y-x|),$$

we have that $\phi \in C^1(\mathbb{R}^n)$ and touches u from above at x. □

PROPOSITION 1.3.0.6. Let $u : \overline{\Omega} \to \mathbb{R}$ be a semiconcave with modulus ω and let $x \in \overline{\Omega}$. Then, a vector $p \in \mathbb{R}^n$ belongs to $D^+u(x)$ if and only if

$$u(y) - u(x) - \langle p, y-x\rangle \le |y-x|\omega(|y-x|) \tag{1.3.0.5}$$

for any point $y \in \overline{\Omega}$ such that $[y,x] \subset \overline{\Omega}$.

Proof. If $p \in \mathbb{R}^n$ satisfies (1.3.0.5), then, by the definition of superdifferential, $p \in D^+u(x)$.
In order to prove the converse, let $p \in D^+u(x)$. Then, dividing the semiconcavity inequality (1.3.0.1) by $(1-\lambda)|y-x|$, we have that

$$\frac{u(y)-u(x)}{|y-x|} \le \frac{u(x+(1-\lambda)(y-x))-u(x)}{(1-\lambda)|y-x|} + \lambda\omega(|x-y|), \;\; \forall\lambda\in[0,1].$$

Hence, taking the limit as $\lambda \to 1^-$, we obtain that

$$\frac{u(y)-u(x)}{|y-x|} \leq \frac{\langle p, y-x\rangle}{|y-x|} + \omega(|y-x|),$$

since $p \in D^+u(x)$. This completes the proof. □

A direct consequence of PROPOSITION 1.3.0.6 is the following result.

PROPOSITION 1.3.0.7. Let $u : \overline{\Omega} \to \mathbb{R}$ be a semiconcave function with modulus ω and let $x \in \overline{\Omega}$. Let $\{x_k\} \subset \overline{\Omega}$ be a sequence converging to x and let $p_k \in D^+u(x_k)$. If p_k converges to a vector $p \in \mathbb{R}^n$, then $p \in D^+u(x)$.

Proof. Let $x \in \overline{\Omega}$ and let $\{x_k\} \subset \overline{\Omega}$ be such that $x_k \to x$ as $k \to +\infty$. Let $p \in \mathbb{R}^n$ and let $p_k \in D^+u(x_k)$ be such that $p_k \to p$ as $k \to +\infty$. By PROPOSITION 1.3.0.6 we have that

$$u(y) - u(x_k) - \langle p_k, y - x_k\rangle \leq |y - x_k|\omega(|y - x_k|)$$

for all $y \in \overline{\Omega}$ such that $[y, x_k] \subset \overline{\Omega}$. Taking the limit as $k \to +\infty$, and using the upper semicontinuity of ω one has that p satisfies (1.3.0.5). This completes the proof. □

REMARK 1.3.0.8. If the function u depends on $(t, x) \in (0, T) \times \overline{\Omega}$, for some $T > 0$, it is natural to consider the generalized partial differentials with respect to x as follows

$$D_x^+u(t,x) := \left\{\eta \in \mathbb{R}^n : \limsup_{h\to 0} \frac{u(t, x+h) - u(t,x) - \langle \eta, h\rangle}{h} \leq 0\right\}.$$

1.3.1 DIRECTIONAL DERIVATES

Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let us first recall the definition of contingent cone.

DEFINITION 1.3.1.1. Let $x \in \overline{\Omega}$ be given. The contingent cone (or Bouligand's tangent cone) to Ω at x is the set

$$T_{\overline{\Omega}}(x) = \left\{\lim_{i\to\infty} \frac{x_i - x}{t_i} : x_i \in \Omega, x_i \to x,\ t_i \in \mathbb{R}_+,\ t_i \downarrow 0\right\}.$$

REMARK 1.3.1.2. Since Ω is a bounded open subset of $\mathbb{R}^n$ with C^2 boundary, then

$$\begin{aligned} &\text{if}\quad x \in \Omega \ \ \Rightarrow\ \ T_{\overline{\Omega}}(x) = \mathbb{R}^n,\\ &\text{if}\quad x \in \partial\Omega \Rightarrow\ \ T_{\overline{\Omega}}(x) = \left\{\theta \in \mathbb{R}^n : \langle \theta, \nu(x)\rangle \leq 0\right\}, \end{aligned}$$

where $\nu(x)$ is the outward unit normal vector to $\partial\Omega$ in x.

DEFINITION 1.3.1.3. Let $x \in \overline{\Omega}$ and $\theta \in T_{\overline{\Omega}}(x)$. The upper and lower Dini derivates of u at x in direction θ are defined as

$$\partial^{\uparrow} u(x;\theta) = \limsup_{\substack{h \to 0^+ \\ \theta' \to \theta \\ x + h\theta' \in \overline{\Omega}}} \frac{u(x+h\theta') - u(x)}{h} \tag{1.3.1.1}$$

and

$$\partial^{\downarrow} u(x;\theta) = \liminf_{\substack{h \to 0^+ \\ \theta' \to \theta \\ x + h\theta' \in \overline{\Omega}}} \frac{u(x+h\theta') - u(x)}{h}, \tag{1.3.1.2}$$

respectively.
The one-sided derivative of u at x in direction θ is defined as

$$\partial^{+}_{\theta} u(x) = \lim_{\substack{h \to 0^+ \\ \theta' \to \theta \\ x + h\theta' \in \overline{\Omega}}} \frac{u(x+h\theta') - u(x)}{h} \tag{1.3.1.3}$$

Let $x \in \partial\Omega$ and let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. In the next result, we show that any semiconcave function admits one-sided derivative in all θ such that $\langle \theta, \nu(x) \rangle \leq 0$.

LEMMA 1.3.1.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. Let $u : \overline{\Omega} \to \mathbb{R}$ be Lipschitz continuous and semiconcave with modulus ω in $\overline{\Omega}$. Let $x \in \partial\Omega$ and let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. Then, for any $\theta \in \mathbb{R}^n$ such that $\langle \theta, \nu(x) \rangle \leq 0$ one has that

$$\partial^{\uparrow} u(x;\theta) = \min_{p \in D^+ u(x)} \langle p, \theta \rangle = \partial^{\downarrow} u(x;\theta). \tag{1.3.1.4}$$

Proof. The idea of the proof is based on [25, Theorem 4.5]. Let $x \in \partial\Omega$ and let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. Let $\theta \in \mathbb{R}^n$ be such that $\langle \theta, \nu(x) \rangle \leq 0$. Let us set

$$M(\theta, x) = \min_{p \in D^+ u(x)} \langle p, \theta \rangle.$$

It suffices to prove that

$$\limsup_{\substack{h \to 0^+ \\ \theta' \to \theta \\ x + h\theta' \in \overline{\Omega}}} \frac{u(x+h\theta') - u(x)}{h} \leq M(\theta, x) \leq \liminf_{\substack{h \to 0^+ \\ \theta' \to \theta \\ x + h\theta' \in \overline{\Omega}}} \frac{u(x+h\theta') - u(x)}{h}. \tag{1.3.1.5}$$

The first inequality in (1.3.1.5) is straightforward. Indeed,

$$\limsup_{\substack{h \to 0^+ \\ \theta' \to \theta \\ x + h\theta' \in \overline{\Omega}}} \frac{u(x+h\theta') - u(x) - \langle hp, \theta' \rangle}{h} \leq \limsup_{\zeta \to 0^+} \frac{u(x+\zeta) - u(x) - \langle p, \zeta \rangle}{|\zeta|} \leq 0$$

for any $p \in D^+u(x)$. So,

$$\limsup_{\substack{h \to 0^+ \\ \theta' \to \theta \\ x+h\theta' \in \overline{\Omega}}} \frac{u(x+h\theta)-u(x)}{h} \leq \langle p, \theta \rangle, \quad \forall\, p \in D^+u(x).$$

In order to prove the last inequality in (1.3.1.5), pick up a sequence $h_k \to 0$ and $\theta_k \to \theta$ such that

$$\lim_{k\to\infty} \frac{u(x+h_k\theta_k)-u(x)}{h_k} = \liminf_{\substack{h \to 0^+ \\ \theta' \to \theta \\ x+h\theta' \in \overline{\Omega}}} \frac{u(x+h\theta')-u(x)}{h} \tag{1.3.1.6}$$

and define

$$\mathcal{Q}(x,\theta_k) = \Big\{ x' \in \Omega : \langle x'-x, \theta_k \rangle > 0, |\langle x'-x, \theta_k\rangle\theta_k - (x'-x)| \leq |x'-x|^2 \Big\}.$$

We observe that the interior of $\mathcal{Q}(x,\theta_k)$ is nonempty. Since u is Lipschitz there exists a sequence x_k such that

(i) $x_k \in \mathcal{Q}(x,\theta_k)$, $x_k \to x$ as $k \to \infty$;

(ii) u is differentiable at x_k and there exists $\overline{p} \in D^+u(x)$ such that $\nabla u(x_k) \to \overline{p}$ as $k \to \infty$;

(iii) $|s_k - h_k| \leq h_k^2$, where $s_k = \langle x_k - x, \theta_k \rangle$.

By the Lipschitz continuity of u, we note that (iii) yields

$$\begin{aligned}
&\Big| \frac{u(x+h_k\theta_k)-u(x)}{h_k} - \frac{u(x+s_k\theta_k)-u(x)}{s_k} \Big| \leq \frac{|u(x+h_k\theta_k)-u(x+s_k\theta_k)|}{h_k} \\
&+ \left| \frac{1}{h_k} - \frac{1}{s_k} \right| [|u(x+s_k\theta_k)-u(x)|] \leq 2\text{Lip}(u)h_k.
\end{aligned}$$

So, by (1.3.1.6) we have that

$$\lim_{n\to\infty} \frac{u(x+s_k\theta_k)-u(x)}{s_k} = \liminf_{\substack{h \to 0^+ \\ \theta' \to \theta \\ x+h\theta' \in \overline{\Omega}}} \frac{u(x+h\theta')-u(x)}{h}. \tag{1.3.1.7}$$

Moreover,

$$\begin{aligned}
u(x+s_k\theta_k)-u(x) &= [u(x+s_k\theta_k)-u(x_k)] + [u(x_k)-u(x)-\langle \nabla u(x_k), x_k - x\rangle] \\
&+ \langle \nabla u(x_k), x_k - x - s_k\theta_k \rangle + \langle s_k \nabla u(x_k), \theta_k \rangle.
\end{aligned}$$

Since u is locally Lipschitz function, and $x_k \in \mathcal{Q}(x,\theta_k)$ one has that

$$\big|u(x+s_k\theta_k)-u(x_k)\big| + \big|\langle \nabla u(x_k), x_k - x - s_k\theta_k\rangle\big| \leq 2\text{Lip}(u)|x_k - x - s_k\theta_k|$$

$$\leq 2\text{Lip}(u)|x_k - x|^2.$$

Since u is semiconcave we deduce that

$$u(x_k) - u(x) - \langle \nabla u(x_k), x_k - x \rangle \geq -C|x_k - x|\omega(|x_k - x|),$$

for some constant $C > 0$. Therefore

$$\frac{u(x + s_k\theta_k) - u(x)}{s_k} \geq \langle \nabla u(x_k), \theta_k \rangle - \frac{2\text{Lip}(u)|x_k - x|^2 + C|x_k - x|\omega(|x_k - x|)}{s_k}.$$

By the definition of $\mathcal{Q}(x, \theta_k)$ one has that $s_k|\theta_k| \geq |x_k - x| - |x_k - x|^2$, so that, as $x_k \to x$, $|x_k - x| \leq 2s_k$ for k large enough. Recalling (ii), (1.3.1.7) and $\theta_k \to \theta$, we conclude that

$$\liminf_{\substack{h \to 0^+ \\ \theta' \to \theta \\ x + h\theta' \in \overline{\Omega}}} \frac{u(x + h\theta') - u(x)}{h} \geq \langle \overline{p}, \theta \rangle \geq M(\theta, x). \tag{1.3.1.8}$$

This completes the proof. □

REMARK 1.3.1.5. We observe that LEMMA 1.3.1.4 also holds when $x \in \Omega$. In this case, (1.3.1.4) is a direct consequence of [25, Theorem 4.5].

Fix $x \in \partial\Omega$ and let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. All $p \in D_x^+ u(x)$ can be written as

$$p = p^\tau + p^\nu$$

where p^ν is the normal component of p, i.e.,

$$p^\nu = \langle p, \nu(x) \rangle \nu(x),$$

and p^τ is the tangential component of p which satisfies

$$\langle p^\tau, \nu(x) \rangle = 0.$$

PROPOSITION 1.3.1.6. Let $x \in \partial\Omega$ and let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. Let $u : \overline{\Omega} \to \mathbb{R}$ be Lipschitz continuous and semiconcave with modulus ω in $\overline{\Omega}$. Then,

$$\partial^+_{-\nu} u(x) = -\lambda_+(x) := \max\{\lambda_p(x) : p \in D^+ u(x)\}, \tag{1.3.1.9}$$

where

$$\lambda_p(x) := \max\{\lambda \in \mathbb{R} : p^\tau + \lambda\nu(x)\}, \quad \forall p \in D^+ u(x). \tag{1.3.1.10}$$

Proof. Let $x \in \partial\Omega$ and let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. Let $\{x_k\} \subset \Omega$ be a sequence such that $x_k \to x$ as $k \to +\infty$. Let θ_k be a sequence of unit vectors $\theta_k := \frac{x_k - x}{|x_k - x|}$ such that

$$\lim_{k \to +\infty} \theta_k = -\nu(x), \quad x + |x_k - x|\theta_k \in \Omega.$$

Let $\overline{\lambda} \in \mathbb{R}$ be such that $p = p^\tau + \overline{\lambda}\nu(x) \in D^+u(x)$. By PROPOSITION 1.3.0.6 one has that

$$u(x + |x_k - x|\theta_k) - u(x) - \langle p^\tau + \overline{\lambda}\nu(x), |x_k - x|\theta_k\rangle \leq |x_k - x|\omega(|x_k - x|).$$

Hence,

$$-\overline{\lambda}\langle \nu(x), |x_k - x|\theta_k\rangle \leq u(x) - u(x + |x_k - x|\theta_k) + \langle p^\tau, |x_k - x|\theta_k\rangle + |x_k - x|\omega(|x_k - x|).$$

Dividing by $|x_k - x|$, we obtain that

$$-\overline{\lambda}\langle \nu(x), \theta_k\rangle \leq \frac{u(x) - u(x + |x_k - x|\theta_k)}{|x_k - x|} + \langle p^\tau, \theta_k\rangle + \omega(|x_k - x|).$$

Taking the limit as $k \to +\infty$, and by LEMMA 1.3.1.4 we have that

$$\overline{\lambda} \leq \lim_{k\to+\infty} \frac{u(x) - u(x + |x_k - x|\theta_k)}{|x_k - x|} = -\partial^+_{-\nu}u(x).$$

Moreover, by PROPOSITION 1.3.0.3 we have that $p^\tau + (\overline{\lambda} - \lambda)\nu(x) \in D^+u(x)$ for all $\lambda \in (-\infty, \overline{\lambda}]$. Set

$$\lambda_p(x) = \max\{\lambda \in \mathbb{R} : p^\tau + \lambda\nu(x) \in D^+u(x)\}, \;\; \forall p \in D^+u(x)$$

and by LEMMA 1.3.1.4 we conclude that

$$\begin{aligned} &-\partial^+_{-\nu}u(x) = -\min_{p\in D^+u(x)}\{-\langle p, \nu(x)\rangle\} = \max_{p\in D^+u(x)}\{\langle p, \nu(x)\rangle\} \\ &= \max\{\lambda_p(x) : p \in D^+u(x)\} =: \lambda_+(x). \end{aligned}$$

This completes the proof. □

1.3.2 LIMITING SUBGRADIENTS OF d_Ω

Take a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ and a point $x \in \mathbb{R}^n$. A vector $p \in \mathbb{R}^n$ is said to be a proximal subgradient of f at x if there exists $\epsilon > 0$ and $C \geq 0$ such that

$$p \cdot (y - x) \leq f(y) - f(x) + C|y - x|^2 \;\text{ for all } y \text{ that satisfy } |y - x| \leq \epsilon.$$

The set of all proximal subgradients of f at x is called the proximal subdifferential of f at x and is denoted by $\partial^p f(x)$. A vector $p \in \mathbb{R}^n$ is said to be a limiting subgradient of f at x if there exist sequences $x_i \in \mathbb{R}^n$, $p_i \in \partial^p f(x_i)$ such that $x_i \to x$ and $p_i \to p$ $(i \to \infty)$.
The set of all limiting subgradients of f at x is called the limiting subdifferential and is denoted by $\partial f(x)$. In particular, for the distance function we have the following result.

LEMMA 1.3.2.1. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Then, for every $x \in \mathbb{R}^n$ it holds

$$\partial^p d_\Omega(x) = \partial d_\Omega(x) = \begin{cases} Db_\Omega(x) & 0 < b_\Omega(x) < \rho_0, \\ Db_\Omega(x)[0,1] & x \in \partial\Omega, \\ 0 & x \in \Omega, \end{cases}$$

where ρ_0 is such that (1.1.0.2) holds.

Proof. The only case which needs to be analyzed is when $x \in \partial\Omega$. We recall that $p \in \partial^p d_\Omega(x)$ if and only if there exists $\epsilon > 0$ such that

$$d_\Omega(y) - d_\Omega(x) - \langle p, y - x\rangle \geq C|y - x|^2, \quad \text{for any } y \text{ such that } |y - x| \leq \epsilon, \tag{1.3.2.1}$$

for some constant $C \geq 0$. Let us show that $\partial^p d_\Omega(x) = Db_\Omega(x)[0,1]$. By the regularity of b_Ω, one has that

$$d_\Omega(y) - d_\Omega(x) - \langle Db_\Omega(x), y - x\rangle \geq b_\Omega(y) - b_\Omega(x) - \langle Db_\Omega(x), y - x\rangle \geq C|y - x|^2.$$

This shows that $Db_\Omega(x) \in \partial^p d_\Omega(x)$. Moreover, since

$$d_\Omega(y) - d_\Omega(x) - \langle \lambda Db_\Omega(x), y - x\rangle \geq \lambda\left(d_\Omega(y) - d_\Omega(x) - \langle Db_\Omega(x), y - x\rangle\right) \quad \forall \lambda \in [0,1],$$

we further obtain the inclusion

$$Db_\Omega(x)[0,1] \subset \partial d_\Omega(x).$$

Next, in order to show the reverse inclusion, let $p \in \partial^p d_\Omega(x) \setminus \{0\}$ and let $y \in \Omega^c$. Then, we can rewrite (1.3.2.1) as

$$b_\Omega(y) - b_\Omega(x) - \langle p, y - x\rangle \geq C|y - x|^2, \quad |y - x| \leq \epsilon. \tag{1.3.2.2}$$

Since $y \in \Omega^c$, by the regularity of b_Ω one has that

$$b_\Omega(y) - b_\Omega(x) \leq \langle Db_\Omega(x), y - x\rangle + C|y - x|^2 \tag{1.3.2.3}$$

for some constant $C \in \mathbb{R}$. By (1.3.2.2) and (1.3.2.3) one has that

$$\left\langle Db_\Omega(x) - p, \frac{y - x}{|y - x|}\right\rangle \geq C|y - x|.$$

Hence, passing to the limit for $y \to x$, we have that

$$\langle Db_\Omega(x) - p, v\rangle \geq 0, \quad \forall v \in T_{\Omega^c}(x),$$

where $T_{\Omega^c}(x)$ is the contingent cone to Ω^c at x. Therefore, by the regularity of $\partial\Omega$,

$$Db_\Omega(x) - p = \lambda v(x),$$

where $\lambda \geq 0$ and $v(x)$ is the exterior unit normal vector to $\partial\Omega$ in x. Since $v(x) = Db_\Omega(x)$, we have that

$$p = (1-\lambda)Db_\Omega(x).$$

Now, we prove that $\lambda \leq 1$. Suppose that $y \in \Omega$, then, by (1.3.2.1) one has that

$$0 = d_\Omega(y) \geq (1-\lambda)\langle Db_\Omega(x), y-x\rangle + C|y-x|^2.$$

Hence,

$$(1-\lambda)\left\langle Db_\Omega(x), \frac{y-x}{|y-x|}\right\rangle \leq -C|y-x|.$$

Passing to the limit for $y \to x$, we obtain

$$(1-\lambda)\langle Db_\Omega(x), w\rangle \leq 0, \qquad \forall w \in T_{\overline{\Omega}}(x),$$

where $T_{\overline{\Omega}}(x)$ is the contingent cone to Ω at x. We now claim that $\lambda \leq 1$. If $\lambda > 1$, then $\langle Db_\Omega(x), w\rangle \geq 0$ for all $w \in T_{\overline{\Omega}}(x)$ but this is impossible since $Db_\Omega(x)$ is the exterior unit normal vector to $\partial\Omega$ in x. Using the regularity of b_Ω, simple limit-taking procedures permit us to prove that $\partial d_\Omega(x) = Db_\Omega(x)[0,1]$ when $x \in \partial\Omega$. This completes the proof.

□

CHAPTER 2

NECESSARY CONDITIONS AND SENSITIVITY RELATIONS

In this Chapter, we collect some results that we will use throughout this Thesis. More precisely, in SECTION 2.3, we introduce the constrained minimization problem. Under suitable assumptions (SECTION 2.2) we give the formulation of necessary conditions for this problem. Finally, in SECTION 2.5 we deduce a local semiconcavity estimate for the value function associated with the constrained minimization problem.

2.1 APPROXIMATION OF CONSTRAINED TRAJECTORIES

Let Ω be a bounded open subset of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary. Let Γ be the metric subspace of $AC(0,T;\mathbb{R}^n)$ defined by

$$\Gamma = \Big\{\gamma \in AC(0,T;\mathbb{R}^n) : \gamma(t) \in \overline{\Omega}, \quad \forall t \in [0,T]\Big\}.$$

For any $x \in \overline{\Omega}$, we set

$$\Gamma[x] = \{\gamma \in \Gamma : \gamma(0) = x\}. \tag{2.1.0.1}$$

LEMMA 2.1.0.1. Let $\rho_0 > 0$ be such that (1.1.0.2) holds. Let $\gamma \in AC(0,T;\mathbb{R}^n)$ and suppose that $d_\Omega(\gamma(t)) < \rho_0$ for all $t \in [0,T]$. Then $d_\Omega \circ \gamma \in AC(0,T)$ and

$$\frac{d}{dt}(d_\Omega \circ \gamma)(t) = \langle Db_\Omega(\gamma(t)), \dot\gamma(t)\rangle \mathbf{1}_{\Omega^c}(\gamma(t)) \text{ a.e. } t \in [0,T]. \tag{2.1.0.2}$$

Moreover,

$$N_\gamma := \{t \in [0,T] : \gamma(t) \in \partial\Omega,\ \exists\, \dot\gamma(t),\ \langle Db_\Omega(\gamma(t)), \dot\gamma(t)\rangle \neq 0\} \tag{2.1.0.3}$$

is a discrete set.

Proof. First we prove that N_γ is a discrete set. Let $t \in N_\gamma$, then there exists $\epsilon > 0$ such that $\gamma(s) \notin \partial\Omega$ for any $s \in (]t-\epsilon, t+\epsilon[\,\setminus\{t\}) \cap [0,T]$. Therefore, N_γ is composed of isolated points and so it is a discrete set.

Let us now set $\phi(t) = (d_\Omega \circ \gamma)(t)$ for all $t \in [0,T]$. We note that $\phi \in AC(0,T)$ because it is the composition of $\gamma \in AC(0,T;\mathbb{R}^n)$ with the Lipschitz continuous function $d_\Omega(\cdot)$. Denote by D the set of $t \in [0,T]$ such that there exists the first order derivative of γ in t, i.e.,

$$D = \{t \in [0,T] :\ \exists\, \dot\gamma(t)\,\}\,.$$

We observe that D has full Lebesgue measure and we decompose D in the following way:

$$D = \underbrace{\{t \in D : \gamma(t) \notin \partial\Omega\}}_{D_0} \cup \underbrace{\{t \in D : \gamma(t) \in \partial\Omega\}}_{D_1}.$$

By [35, Theorem 4, pg 129], for all $t \in D_0$ the first order derivative of ϕ is equal to

$$\dot\phi(t) = \begin{cases} 0 & \gamma(t) \in \Omega \\ \langle Db_\Omega(\gamma(t)), \dot\gamma(t)\rangle & \gamma(t) \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Now, consider $t \in D_1 \setminus N_\gamma$. Since $\gamma(t) \in \partial\Omega$, one has that

$$\frac{\phi(t+h) - \phi(t)}{h} = \frac{d_\Omega(\gamma(t+h))}{h},$$

for all $h > 0$. Since $\gamma(t+h) = \gamma(t) + h\dot\gamma(t) + o(h)$ and d_Ω is Lipschitz continuous, we obtain

$$0 \leq \frac{d_\Omega(\gamma(t+h))}{h} \leq \frac{o(h)}{h} + \frac{d_\Omega(\gamma(t) + h\dot\gamma(t))}{h}.$$

Hence, one has that

$$0 \leq \liminf_{h\to 0} \frac{d_\Omega(\gamma(t+h))}{h} \leq \limsup_{h\to 0} \frac{d_\Omega(\gamma(t+h))}{h} \leq \limsup_{h\to 0} \frac{d_\Omega(\gamma(t) + h\dot\gamma(t))}{h}. \tag{2.1.0.4}$$

Moreover, by the regularity of b_Ω, we obtain

$$d_\Omega(\gamma(t) + h\dot\gamma(t)) \leq |b_\Omega(\gamma(t) + h\dot\gamma(t))| \leq |h|\,|\langle Db_\Omega(\gamma(t)), \dot\gamma(t)\rangle| + o(h). \tag{2.1.0.5}$$

Thus, since $t \in D \setminus N_\gamma$, we conclude that

$$\limsup_{h\to 0} \frac{d_\Omega(\gamma(t) + h\dot\gamma(t))}{h} \leq |\langle Db_\Omega(\gamma(t)), \dot\gamma(t)\rangle| = 0. \tag{2.1.0.6}$$

So $\dot\phi(t) = 0$ and the proof is complete. □

PROPOSITION 2.1.0.2. Let $x_i \in \overline{\Omega}$ be such that $x_i \to x$ and let $\gamma \in \Gamma[x]$. Then there exists $\gamma_i \in \Gamma[x_i]$ such that:

(i) $\gamma_i \to \gamma$ uniformly on $[0, T]$;

(ii) $\dot\gamma_i \to \dot\gamma$ a.e. on $[0, T]$;

(iii) $|\dot\gamma_i(t)| \leq C|\dot\gamma(t)|$ for any $i \geq 1$, a.e. $t \in [0, T]$, and some constant $C \geq 0$.

Proof. Let $\widehat\gamma_i$ be the trajectory defined by

$$\widehat\gamma_i(t) = \gamma(t) + x_i - x. \tag{2.1.0.7}$$

We observe that $d_\Omega(\widehat\gamma_i(t)) \leq \rho_0$ for all $t \in [0, T]$ and all sufficiently large i, say $i \geq i_0$. Indeed,

$$d_\Omega(\widehat\gamma_i(t)) \leq |\widehat\gamma_i(t) - \gamma(t)| = |x_i - x|.$$

Since $x_i \to x$, we have that $d_\Omega(\widehat\gamma_i(t)) \leq \rho_0$ for all $t \in [0, T]$ and $i \geq i_0$. We denote by γ_i the projection of $\widehat\gamma_i$ on $\overline{\Omega}$, i.e.,

$$\gamma_i(t) = \widehat\gamma_i(t) - d_\Omega(\widehat\gamma_i(t))Db_\Omega(\widehat\gamma_i(t)) \quad \forall t \in [0, T]. \tag{2.1.0.8}$$

We note that $\gamma_i \in \Gamma[x_i]$. Moreover, γ_i converges uniformly to γ on $[0, T]$. Indeed,

$$|\gamma_i(t) - \gamma(t)| = |x_i - x - d_\Omega(\widehat\gamma_i(t))Db_\Omega(\widehat\gamma_i(t))| \leq 2|x_i - x|, \quad \forall\, t \in [0, T].$$

By LEMMA 2.1.0.1, $d_\Omega(\widehat\gamma_i(\cdot)) \in AC(0, T)$ and $\frac{d}{dt}\left(d_\Omega(\widehat\gamma_i(t))\right) = \left\langle Db_\Omega(\widehat\gamma_i(t)), \dot{\widehat\gamma}_i(t)\right\rangle \mathbf{1}_{\Omega^c}(\widehat\gamma_i(t))$ a.e. $t \in [0, T]$. Using the regularity of b_Ω, we obtain

$$\dot\gamma_i(t) = \dot\gamma(t) - \left\langle Db_\Omega(\widehat\gamma_i(t)), \dot\gamma(t)\right\rangle Db_\Omega(\widehat\gamma_i(t))\mathbf{1}_{\Omega^c}(\widehat\gamma_i(t)) - d_\Omega(\widehat\gamma_i(t))D^2b_\Omega(\widehat\gamma_i(t))\dot\gamma(t), \ \text{a.e. } t \in [0, T].$$

Therefore, there exists a constant $C \geq 0$ such that $|\dot\gamma_i(t)| \leq C|\dot\gamma(t)|$ for any $i \geq i_0$, a.e. $t \in [0, T]$. Finally, we have to show that $\dot\gamma_i \to \dot\gamma$ almost everywhere on $[0, T]$. Since $\widehat\gamma_i \to \gamma$ and $\gamma \in \Gamma[x]$, one has that

$$d_\Omega(\widehat\gamma_i(t))D^2b_\Omega(\widehat\gamma_i(t))\dot\gamma(t) \xrightarrow{i\to\infty} 0, \quad \forall t \in [0, T].$$

So, we have to prove that

$$-\left\langle Db_\Omega(\widehat\gamma_i(t)), \dot\gamma(t)\right\rangle Db_\Omega(\widehat\gamma_i(t))\mathbf{1}_{\Omega^c}(\widehat\gamma_i(t)) \xrightarrow{i\to\infty} 0, \ \text{a.e. } t \in [0, T]. \tag{2.1.0.9}$$

We note that

$$\Big|\big\langle Db_\Omega(\widehat{\gamma}_i(t)),\dot{\gamma}(t)\big\rangle Db_\Omega(\widehat{\gamma}_i(t))\mathbf{1}_{\Omega^c}(\widehat{\gamma}_i(t))\Big| \le \Big|\big\langle Db_\Omega(\widehat{\gamma}_i(t)),\dot{\gamma}(t)\big\rangle\Big|, \quad \text{a.e. } t\in[0,T]. \tag{2.1.0.10}$$

Fix $t\in[0,T]$ such that (2.1.0.10) holds. If $\gamma(t)\in\Omega$ then $\widehat{\gamma}_i(t)\in\Omega$ for i large enough and (2.1.0.9) holds. On the other hand, if $\gamma(t)\in\partial\Omega$, then passing to the limit in (2.1.0.10), we have that

$$\limsup_{i\to\infty}\Big|\big\langle Db_\Omega(\widehat{\gamma}_i(t)),\dot{\gamma}(t)\big\rangle Db_\Omega(\widehat{\gamma}_i(t))\mathbf{1}_{\Omega^c}(\widehat{\gamma}_i(t))\Big| \le \limsup_{i\to\infty}\Big|\big\langle Db_\Omega(\widehat{\gamma}_i(t)),\dot{\gamma}(t)\big\rangle\Big|.$$

Since $\gamma_i\to\gamma$ uniformly on $[0,T]$, one has that

$$\limsup_{i\to\infty}\Big|\big\langle Db_\Omega(\widehat{\gamma}_i(t)),\dot{\gamma}(t)\big\rangle\Big| = \Big|\big\langle Db_\Omega(\gamma(t)),\dot{\gamma}(t)\big\rangle\Big|. \tag{2.1.0.11}$$

By LEMMA 2.1.0.1, we have that $\langle Db_\Omega(\gamma(t)),\dot{\gamma}(t)\rangle=0$ for $t\in[0,T]\setminus N_\gamma$, where N_γ is the discret set defined in (2.1.0.3). So (2.1.0.9) holds for a.e. $t\in[0,T]$. Thus, $\dot{\gamma}_i$ converges almost everywhere to $\dot{\gamma}$ on $[0,T]$. This completes the proof. □

2.2 ASSUMPTIONS AND SOME REMARKS

For reader's convenience we collect in this section all assumptions that we will use throughout this Chapter.

Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with C^2 boundary. Let $U\subset\mathbb{R}^n$ be an open set such that $\overline{\Omega}\subset U$. Given $x\in\overline{\Omega}$, we consider the constrained minimization problem

$$\inf_{\gamma\in\Gamma[x]} J[\gamma], \quad \text{where} \quad J[\gamma]=\Big\{\int_0^T f(t,\gamma(t),\dot{\gamma}(t))\,dt+g(\gamma(T))\Big\}. \tag{2.2.0.1}$$

We denote by $\mathcal{X}[x]$ the set of solutions of (2.2.0.1), that is

$$\mathcal{X}[x]=\Big\{\gamma^\star\in\Gamma[x] : J[\gamma^\star]=\inf_{\Gamma[x]}J[\gamma]\Big\}.$$

We assume that $f:[0,T]\times U\times\mathbb{R}^n\to\mathbb{R}$ and $g:U\to\mathbb{R}$ satisfy the following conditions.

(g1) $g\in C^1_b(U)$.

(f0) $f\in C\big([0,T]\times U\times\mathbb{R}^n\big)$ and for all $t\in[0,T]$ the function $(x,v)\longmapsto f(t,x,v)$ is differentiable. Moreover, D_xf, D_vf are continuous on $[0,T]\times U\times\mathbb{R}^n$ and there exists a constant $M\ge 0$ such that

$$|f(t,x,0)|+|D_xf(t,x,0)|+|D_vf(t,x,0)|\le M, \quad \forall\,(t,x)\in[0,T]\times U. \tag{2.2.0.2}$$

(f1) For all $t \in [0,T]$ the map $(x,v) \longmapsto D_v f(t,x,v)$ is continuously differentiable and there exists a constant $\mu \geq 1$ such that

$$\frac{I}{\mu} \leq D^2_{vv} f(t,x,v) \leq I\mu, \tag{2.2.0.3}$$

$$||D^2_{vx} f(t,x,v)|| \leq \mu(1+|v|), \tag{2.2.0.4}$$

for all $(t,x,v) \in [0,T] \times U \times \mathbb{R}^n$, where I denotes the identity matrix.

(f2) For all $(x,v) \in U \times \mathbb{R}^n$ the function $t \longmapsto f(t,x,v)$ and the map $t \longmapsto D_v f(t,x,v)$ are Lipschitz continuous. Moreover, there exists a constant $\kappa \geq 0$ such that

$$|f(t,x,v) - f(s,x,v)| \leq \kappa(1+|v|^2)|t-s|, \tag{2.2.0.5}$$

$$|D_v f(t,x,v) - D_v f(s,x,v)| \leq \kappa(1+|v|)|t-s|, \tag{2.2.0.6}$$

for all t, $s \in [0,T]$, $x \in U$, $v \in \mathbb{R}^n$.

(f3) For all $s \in [0,T]$, for all $x \in U$ and for all v, $w \in B_R$, there exists a constant $C(R) \geq 0$ such that

$$|D_x f(s,x,v) - D_x f(s,x,w)| \leq C(R)|v-w|. \tag{2.2.0.7}$$

(f4) For any $R > 0$ the map $x \longmapsto f(t,x,v)$ is semiconcave with linear modulus ω_R, i.e., for any $(t,v) \in [0,T] \times B_R$ one has that

$$\lambda f(t,y,v) + (1-\lambda) f(t,x,v) - f(t, \lambda y + (1-\lambda)x, v) \leq \lambda(1-\lambda)|x-y|\omega_R(|x-y|),$$

for any pair x, $y \in U$ such that the segment $[x,y]$ is contained in U and for any $\lambda \in [0,1]$.

REMARK 2.2.0.1. By classical results in the calculus of variation (see, e.g., [34, Theorem 11.1i]), there exists at least one minimizer of (2.2.0.1) in Γ for any fixed point $x \in \overline{\Omega}$.

In the next lemma we show that (f0)-(f2) imply the useful growth conditions for f and for its derivatives.

LEMMA 2.2.0.2. Suppose that (f0)-(f2) hold. Then, there exists a positive constant $C(\mu, M)$ depends only on μ and M such that

$$|D_v f(t,x,v)| \leq C(\mu,M)(1+|v|), \tag{2.2.0.8}$$

$$|D_x f(t,x,v)| \leq C(\mu,M)(1+|v|^2), \tag{2.2.0.9}$$

$$\frac{1}{4\mu}|v|^2 - C(\mu,M) \leq f(t,x,v) \leq 4\mu|v|^2 + C(\mu,M), \tag{2.2.0.10}$$

for all $(t,x,v) \in [0,T] \times U \times \mathbb{R}^n$.

Proof. By (2.2.0.2), and by (2.2.0.3) one has that

$$|D_v f(t,x,v)| \le |D_v f(t,x,v) - D_v f(t,x,0)| + |D_v f(t,x,0)|$$
$$\le \int_0^1 |D^2_{vv} f(t,x,\tau v)||v|\, d\tau + |D_v f(t,x,0)| \le \mu|v| + M \le C(\mu,M)(1+|v|)$$

and so (2.2.0.8) holds. Furthermore, by (2.2.0.2), and by (2.2.0.4) we have that

$$|D_x f(t,x,v)| \le |D_x f(t,x,v) - D_x f(t,x,0)| + |D_x f(t,x,0)| \le \int_0^1 |D^2_{xv} f(t,x,\tau v)||v|\, d\tau + M$$
$$\le \mu(1+|v|)|v| + M \le C(\mu,M)(1+|v|^2).$$

Therefore, (2.2.0.9) holds. Moreover, fixed $v \in \mathbb{R}^n$ there exists a point ξ of the segment with endpoints 0, v such that

$$f(t,x,v) = f(t,x,0) + \langle D_v f(t,x,0), v\rangle + \frac{1}{2}\langle D^2_{vv} f(t,x,\xi)v, v\rangle.$$

By (2.2.0.2), (2.2.0.3), and by (2.2.0.8) we have that

$$-C(\mu,M) + \frac{1}{4\mu}|v|^2 \le -M - C(\mu,M)|v| + \frac{1}{2\mu}|v|^2 \le f(t,x,v) \le M + C(\mu,M)|v| + \frac{\mu}{2}|v|^2$$
$$\le C(\mu,M) + 4\mu|v|^2,$$

and so (2.2.0.10) holds. This completes the proof. □

In the next result we show a special property of the minimizers of (2.2.0.1).

LEMMA 2.2.0.3. For any $x \in \overline{\Omega}$ and for any $\gamma^\star \in \mathcal{X}[x]$ we have that

$$\int_0^T \frac{1}{4\mu}|\dot\gamma^\star(t)|^2\, dt \le K,$$

where

$$K := T\Big(C(\mu,M) + M\Big) + 2\max_U |g(x)|. \tag{2.2.0.11}$$

Proof. Let $x \in \overline{\Omega}$ and let $\gamma^\star \in \mathcal{X}[x]$. By comparing the cost of $\gamma^\star$ with the cost of the constant trajectory $\gamma^\star(t) \equiv x$, one has that

$$\int_0^T f(t,\gamma^\star(t),\dot\gamma^\star(t))\, dt + g(\gamma^\star(T)) \le \int_0^T f(t,x,0)\, dt + g(x) \tag{2.2.0.12}$$
$$\le T \max_{[0,T]\times U} |f(t,x,0)| + \max_U |g(x)|.$$

Using (2.2.0.2) and (2.2.0.10) in (3.1.2.10), one has that

$$\int_0^T \frac{1}{4\mu}|\dot\gamma^\star(t)|^2\, dt \le K,$$

where

$$K := T\Big(C(\mu,M) + M\Big) + 2\max_U |g(x)|.$$

□

We denote by $H : [0,T] \times U \times \mathbb{R}^n \to \mathbb{R}$ the Hamiltonian

$$H(t,x,p) = \sup_{v\in\mathbb{R}^n} \Big\{ -\langle p, v\rangle - f(t,x,v)\Big\}, \qquad \forall\,(t,x,p) \in [0,T] \times U \times \mathbb{R}^n.$$

LEMMA 2.2.0.4. Suppose that (f0)-(f4) hold. Then, H satisfies the following conditions.

(H0) $H \in C\big([0,T] \times U \times \mathbb{R}^n\big)$ and for all $t \in [0,T]$ the function $(x,p) \longmapsto H(t,x,p)$ is differentiable. Moreover, D_xH, D_pH are continuous on $[0,T] \times U \times \mathbb{R}^n$ and there exists a constant $M' \geq 0$ such that

$$|H(t,x,0)| + |D_xH(t,x,0)| + |D_pH(t,x,0)| \leq M', \quad \forall\,(t,x) \in [0,T] \times U. \tag{2.2.0.13}$$

(H1) For all $t \in [0,T]$ the map $(x,p) \longmapsto D_pH(t,x,p)$ is continuously differentiable and

$$\frac{I}{\mu} \leq D^2_{pp}H(t,x,p) \leq I\mu, \tag{2.2.0.14}$$

$$||D^2_{px}H(t,x,p)|| \leq C(\mu, M')(1+|p|), \tag{2.2.0.15}$$

for all $(t,x,p) \in [0,T] \times U \times \mathbb{R}^n$, where μ is the constant given in (f1) and $C(\mu, M')$ depends only on μ and M'.

(H2) For all $(x,p) \in U \times \mathbb{R}^n$ the function $t \longmapsto H(t,x,p)$ and the map $t \longmapsto D_pH(t,x,p)$ are Lipschitz continuous. Moreover

$$|H(t,x,p) - H(s,x,p)| \leq \kappa C(\mu, M')(1+|p|^2)|t-s|, \tag{2.2.0.16}$$

$$|D_pH(t,x,p) - D_pH(s,x,p)| \leq \kappa C(\mu, M')(1+|p|)|t-s|, \tag{2.2.0.17}$$

for all t, $s \in [0,T]$, $x \in U$, $p \in \mathbb{R}^n$, where κ is the constant given in (f2) and $C(\mu, M')$ depends only on μ and M'.

(H3) For all $s \in [0,T]$, for all $x \in U$ and for all p, $q \in B_R$, there exists a constant $C(R) \geq 0$ such that

$$|D_xH(s,x,p) - D_xH(s,x,q)| \leq C(R)|p-q|. \tag{2.2.0.18}$$

(H4) For any $R > 0$ the map $x \longmapsto H(t,x,p)$ is semiconvex with linear modulus ω_R, i.e., for any $(t,p) \in [0,T] \times B_R$ one has that

$$\lambda H(t,y,p) + (1-\lambda)H(t,x,p) - H(t,\lambda y + (1-\lambda)x, p) \geq \lambda(1-\lambda)|x-y|\omega_R(|x-y|),$$

for any pair x, $y \in U$ such that the segment $[x,y]$ is contained in U and for any $\lambda \in [0,1]$.

Moreover,

$$|D_pH(t,x,p)| \leq C(\mu, M')(1+|p|), \tag{2.2.0.19}$$
$$|D_xH(t,x,p)| \leq C(\mu, M')(1+|p|^2), \tag{2.2.0.20}$$
$$\frac{1}{4\mu}|p|^2 - C(\mu, M') \leq H(t,x,p) \leq 4\mu|p|^2 + C(\mu, M'), \tag{2.2.0.21}$$

for all $(t,x,p) \in [0,T] \times U \times \mathbb{R}^n$ and $C(\mu, M')$ depends only on μ and M'.

Proof. Assumption (H0) and (H4) are a direct consequence of (f0) and (f4), respectively. Moreover, by (f1), and by the definition of H one has that for all $t \in [0,T]$ the map $(x,p) \longmapsto D_pH(t,x,p)$ is continuously differentiable. Let $v_p \in \mathbb{R}^n$ be such that $p = -D_vf(t,x,v_p)$ or, equivalently, $v_p = -D_pH(t,x,p)$. As $D^2_{pp}H(t,x,p) = [D^2_{vv}f(t,x,v_p)]^{-1}$, (2.2.0.14) follows. Arguing as in LEMMA 2.2.0.2 we deduce that

$$|D_pH(t,x,p)| \leq C(\mu, M')(1+|p|), \tag{2.2.0.22}$$

for all $(t,x,p) \in [0,T] \times U \times \mathbb{R}^n$. Since $D^2_{px}H(t,x,p) = -D^2_{vx}f(t,x,v_p)D^2_{pp}H(t,x,p)$, and by (2.2.0.22) one has that

$$||D^2_{px}H(t,x,p)|| = ||D^2_{vx}f(t,x,v_p)D^2_{pp}H(t,x,p)|| \leq \mu||D^2_{vx}f(t,x,v_p)|| \leq \mu^2(1+|v_p|) \tag{2.2.0.23}$$
$$\leq C(\mu, M')(1+|p|). \tag{2.2.0.24}$$

Arguing as in LEMMA 2.2.0.2, and using (2.2.0.22) and (2.2.0.23), we deduce that

$$|D_xH(t,x,p)| \leq C(\mu, M')(1+|p|^2),$$
$$\frac{1}{4\mu}|p|^2 - C(\mu, M') \leq H(t,x,p) \leq 4\mu|p|^2 + C(\mu, M'),$$

for all $(t,x,p) \in [0,T] \times U \times \mathbb{R}^n$. By assumption (f2) one has that for all $(x,p) \in U \times \mathbb{R}^n$ the function $t \longmapsto H(t,x,p)$ and the map $t \longmapsto D_pH(t,x,p)$ are Lipschitz continuous. Now, we prove that (2.2.0.16) holds. Let $v_p \in \mathbb{R}^n$ be such that $H(t,x,p) = -\langle p, v_p\rangle - f(t,x,v_p)$. Hence, by (f2) one has that

$$H(t,x,p) - H(s,x,p) \leq -\langle p, v_p\rangle - f(t,x,v_p) + \langle p, v_p\rangle + f(s,x,v_p)$$
$$\leq \kappa(1+|v_p|^2)|t-s| \leq \kappa C(\mu, M')(1+|p|^2)|t-s|.$$

Now, we show that (2.2.0.17) holds. For semplicity, we set

$$v = -D_pH(t,x,-D_vf(t,x,v)) = -D_pH(s,x,-D_vf(s,x,v)),$$

and for all $t, s \in [0,T]$

$$p_t = -D_vf(t,x,v), \quad p_s = -D_vf(s,x,v).$$

Hence, for all $t, s \in [0,T]$ we have that

$$0{=}D_pH(t,x,p_t) - D_pH(s,x,p_s){=}D_pH(t,x,p_t) + D_pH(t,x,p_s) - D_pH(t,x,p_s) - D_pH(s,x,p_s).$$

By (2.2.0.6) one has that

$$\begin{aligned} &|D_pH(t,x,p_s) - D_pH(s,x,p_s)| = |D_pH(t,x,p_t) - D_pH(t,x,p_s)| \leq C|p_t - p_s| \\ &= C|D_vf(t,x,v) - D_vf(s,x,v)| \leq C\kappa|t-s|(1+|v|) = C\kappa|t-s|(1+|D_pH(s,x,p_s)|) \\ &\leq \kappa C(\mu, M')|t-s|(1+|p_s|). \end{aligned} \tag{2.2.0.25}$$

Since (2.2.0.25) holds for all p_s, then (2.2.0.17) holds. Finally, let v_p, $v_q \in \mathbb{R}^n$ be such that $p = -D_vf(t,x,v_p)$ and $q = -D_vf(t,x,v_q)$, respectively. Recalling that $D_xH(t,x,p) = -D_xf(t,x,v_p)$ and $D_xH(t,x,q) = -D_xf(t,x,v_q)$, and by (f3) one has that

$$\begin{aligned} &|D_xH(t,x,p) - D_xH(t,x,q)| = |D_xf(t,x,v_q) - D_xf(t,x,v_p)| \leq C(R)|v_q - v_p| \\ &= C(R)|D_pH(t,x,p) - D_pH(t,x,q)| \leq C(R)|p-q|, \end{aligned}$$

for all $t \in [0,T]$ and for all $x \in U$. This completes the proof. □

2.3 NECESSARY CONDITIONS AND SMOOTHNESS OF MINIMIZERS

Throughout this section we assume that $f : [0,T] \times U \times \mathbb{R}^n \to \mathbb{R}$ and $g : U \to \mathbb{R}$ satisfy (f0)-(f2) and (g1), respectively. Under the assumptions on Ω, f and g our necessary conditions can be stated as follows.

THEOREM 2.3.0.1. For any $x \in \overline{\Omega}$ and any $\gamma^\star \in \mathcal{X}[x]$ the following holds true.

(i) $\gamma^\star$ is of class $C^{1,1}([0,T]; \overline{\Omega})$.

(ii) There exist:

   (a) a Lipschitz continuous arc $p : [0,T] \to \mathbb{R}^n$,

   (b) a bounded measurable function $\Lambda : [0,T] \to [0,\infty)$,

   (c) a constant $\nu \in \mathbb{R}$ such that

$$0 \leq \nu \leq \max\left\{1, 2\mu \sup_{x\in U} \left|D_pH(T,x,Dg(x))\right|\right\},$$

which satisfy the adjoint system

$$\begin{cases} \dot\gamma^\star(t) = -D_pH(t,\gamma^\star(t),p(t)) & \text{for all } t\in[0,T], \\ \dot p(t) = D_xH(t,\gamma^\star(t),p(t)) - \Lambda(t)Db_\Omega(\gamma^\star(t)) & \text{for a.e. } t\in[0,T], \end{cases} \tag{2.3.0.1}$$

and the transversality condition

$$p(T) = Dg(\gamma^\star(T)) + \nu Db_\Omega(\gamma^\star(T))\mathbf{1}_{\partial\Omega}(\gamma^\star(T)). \tag{2.3.0.2}$$

Moreover,

(iii) the following estimate holds

$$||\dot\gamma^\star||_\infty \le L^\star, \quad \forall \gamma^\star \in \mathcal{X}[x], \tag{2.3.0.3}$$

where $L^\star = L^\star(\mu, M', M, \kappa, T, ||Dg||_\infty, ||g||_\infty)$;

(iv) for all $t\in[0,T]$, $\Lambda(t)$ is given by

$$\begin{aligned}\Lambda(t) =& \frac{1}{\theta(t)}\Big[-\Big\langle D^2b_\Omega(\gamma^\star(t))D_pH\big(t,\gamma^\star(t),p(t)\big), D_pH\big(t,\gamma^\star(t),p(t)\big)\Big\rangle \\ & -\Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pt}H\big(t,\gamma^\star(t),p(t)\big)\Big\rangle - \\ & \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{px}H\big(t,\gamma^\star(t),p(t)\big)D_pH\big(t,\gamma^\star(t),p(t)\big)\Big\rangle \\ & + \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H\big(t,\gamma^\star(t),p(t)\big)D_xH\big(t,\gamma^\star(t),p(t)\big)\Big\rangle\Big]\mathbf{1}_{\partial\Omega}(\gamma^\star(t)),\end{aligned}$$

where $\theta(t) := \langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H(t,\gamma^\star(t),p(t))Db_\Omega(\gamma^\star(t))\rangle$.

Observe that (2.2.0.14) ensures that $\theta(t) > 0$ for all $t\in[0,T]$ such that $\gamma^\star(t)\in\partial\Omega + B_{\rho_0}$.

2.3.1 PROOF OF THEOREM 2.3.0.1 FOR $U = \mathbb{R}^n$

In this section we prove THEOREM 2.3.0.1 in the special case of $U=\mathbb{R}^n$. The proof for a general open set U will be given in the next section.

The idea of proof is based on [23, Theorem 2.1] where the Maximum Principle under state constraints is proved for a Mayer problem. The reasoning requires several intermediate steps.

Fixed $x\in\overline\Omega$, the key point is to approximate the constrained problem by penalized problems as follows

$$\inf_{\substack{\gamma\in AC(0,T;\mathbb{R}^n)\\ \gamma(0)=x}} \left\{\int_0^T \Big[f(t,\gamma(t),\dot\gamma(t)) + \frac{1}{\epsilon}\, d_\Omega(\gamma(t))\Big]\,dt + \frac{1}{\delta}\, d_\Omega(\gamma(T)) + g(\gamma(T))\right\}. \tag{2.3.1.1}$$

Then, we will show that, for $\epsilon>0$ and $\delta\in(0,1]$ small enough, the solutions of the penalized problem remain in $\overline\Omega$.

Observe that the Hamiltonian associated with the penalized problem is given by

$$H_\epsilon(t,x,p) = \sup_{v\in\mathbb{R}^n}\Big\{-\langle p,v\rangle - f(t,x,v) - \frac{1}{\epsilon}\,d_\Omega(x)\Big\} = H(t,x,p) - \frac{1}{\epsilon}\,d_\Omega(x), \tag{2.3.1.2}$$

for all $(t, x, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$.
By classical results in the calculus of variation (see, e.g., [34, Section 11.2]), there exists at least one mimimizer of (2.3.1.1) in $AC(0, T; \mathbb{R}^n)$ for any fixed initial point $x \in \overline{\Omega}$. We denote by $\mathcal{X}_{\epsilon,\delta}[x]$ the set of solutions of (2.3.1.1).

REMARK 2.3.1.1. Arguing as in LEMMA 2.2.0.3 we have that, for any $x \in \overline{\Omega}$, all $\gamma \in \mathcal{X}_{\epsilon,\delta}[x]$ satisfy

$$\int_0^T \left[\frac{1}{4\mu}|\dot{\gamma}(t)|^2 + \frac{1}{\epsilon}\, d_\Omega(\gamma(t))\right] dt \leq K, \tag{2.3.1.3}$$

where K is the constant given in (2.2.0.11).

The first step of the proof consists in showing that the solutions of the penalized problem remain in a neighborhood of $\overline{\Omega}$.

LEMMA 2.3.1.2. Let ρ_0 be such that (1.1.0.2) holds. For any $\rho \in (0, \rho_0]$, there exists $\epsilon(\rho) > 0$ such that for all $\epsilon \in (0, \epsilon(\rho)]$ and all $\delta \in (0, 1]$ we have that

$$\forall\, x \in \overline{\Omega},\ \gamma \in \mathcal{X}_{\epsilon,\delta}[x] \quad \Longrightarrow \quad \sup_{t\in[0,T]} d_\Omega(\gamma(t)) \leq \rho. \tag{2.3.1.4}$$

Proof. We argue by contradiction. Assume that, for some $\rho > 0$, there exist sequences $\{\epsilon_k\}$, $\{\delta_k\}$, $\{t_k\}$, $\{x_k\}$ and $\{\gamma_k\}$ such that

$$\epsilon_k \downarrow 0,\ \delta_k > 0,\ t_k \in [0, T],\ x_k \in \overline{\Omega},\ \gamma_k \in \mathcal{X}_{\epsilon_k,\delta_k}[x_k] \text{ and } d_\Omega(\gamma_k(t_k)) > \rho, \quad \text{for all } k \geq 1.$$

By REMARK 2.3.1.1, one has that for all $k \geq 1$

$$\int_0^T \left[\frac{1}{4\mu}|\dot{\gamma}_k(t)|^2 + \frac{1}{\epsilon_k}\, d_\Omega(\gamma_k(t))\right] dt \leq K,$$

where K is the constant given in (2.2.0.11). The above inequality implies that γ_k is $1/2-$Hölder continuous with Hölder constant $(4\mu K)^{1/2}$. Then, by the Lipschitz continuity of d_Ω and the regularity of γ_k, we have that

$$d_\Omega(\gamma_k(t_k)) - d_\Omega(\gamma_k(s)) \leq (4\mu K)^{1/2}|t_k - s|^{1/2}, \quad s \in [0, T].$$

Since $d_\Omega(\gamma_k(t_k)) > \rho$, one has that

$$d_\Omega(\gamma_k(s)) > \rho - (4\mu K)^{1/2}|t_k - s|^{1/2}.$$

Hence, $d_\Omega(\gamma_k(s)) \geq \rho/2$ for all $s \in J := [t_k - \frac{\rho^2}{16\mu K}, t_k + \frac{\rho^2}{16\mu K}] \cap [0, T]$ and all $k \geq 1$. So,

$$K \geq \frac{1}{\epsilon_k}\int_0^T d_\Omega(\gamma_k(t))\, dt \geq \frac{1}{\epsilon_k}\int_J d_\Omega(\gamma_k(t))\, dt \geq \frac{1}{\epsilon_k}\frac{\rho^3}{32\mu K}.$$

But the above inequality contradicts the fact that $\epsilon_k \downarrow 0$. So, (2.3.1.4) holds true. □

In the next lemma, we show the necessary conditions for the minimizers of the penalized problem.

LEMMA 2.3.1.3. Let $\rho \in (0, \rho_0]$ and let $\epsilon \in (0, \epsilon(\rho)]$, where $\epsilon(\rho)$ is given by LEMMA 2.3.1.2. Fix $\delta \in (0, 1]$, let $x_0 \in \overline{\Omega}$, and let $\gamma \in \mathcal{X}_{\epsilon,\delta}[x_0]$. Then,

(i) γ is of class $C^{1,1}([0, T]; \mathbb{R}^n)$;

(ii) there exists an arc $p \in \mathrm{Lip}(0, T; \mathbb{R}^n)$, a measurable map $\lambda : [0, T] \to [0, 1]$, and a constant $\beta \in [0, 1]$ such that

$$\begin{cases} \dot{\gamma}(t) = -D_p H(t, \gamma(t), p(t)), & \text{for all } t \in [0, T], \\ \dot{p}(t) = D_x H(t, \gamma(t), p(t)) - \frac{\lambda(t)}{\epsilon}\, Db_\Omega(\gamma(t)), & \text{for a.e. } t \in [0, T], \\ p(T) = Dg(\gamma(T)) + \frac{\beta}{\delta}\, Db_\Omega(\gamma(T)), \end{cases} \tag{2.3.1.5}$$

where

$$\lambda(t) \in \begin{cases} \{0\} & \text{if } \gamma(t) \in \Omega, \\ \{1\} & \text{if } 0 < d_\Omega(\gamma(t)) < \rho, \\ [0, 1] & \text{if } \gamma(t) \in \partial\Omega, \end{cases} \tag{2.3.1.6}$$

and

$$\beta \in \begin{cases} \{0\} & \text{if } \gamma(T) \in \Omega, \\ \{1\} & \text{if } 0 < d_\Omega(\gamma(T)) < \rho, \\ [0, 1] & \text{if } \gamma(T) \in \partial\Omega. \end{cases} \tag{2.3.1.7}$$

Moreover,

(iii) the function

$$r(t) := H(t, \gamma(t), p(t)) - \frac{1}{\epsilon}\, d_\Omega(\gamma(t)), \quad \forall t \in [0, T]$$

belongs to $AC(0, T; \mathbb{R})$ and satisfies

$$\int_0^T |\dot{r}(t)|\, dt \leq \kappa(T + 4\mu K),$$

where K is the constant given in (2.2.0.11) and μ, κ are the constants in (2.2.0.5) and (2.2.0.10), respectively;

(iv) the following estimate holds

$$|p(t)|^2 \leq 4\mu \left[\frac{1}{\epsilon} d_\Omega(\gamma(t)) + \frac{C_1}{\delta^2} \right], \qquad \forall t \in [0, T], \tag{2.3.1.8}$$

where $C_1 = 8\mu + 8\mu ||Dg||_\infty^2 + 2C(\mu, M') + \kappa(T + 4\mu K)$.

Proof. In order to use the Maximum Principle in the version of [61, Theorem 8.7.1], we rewrite (2.3.1.1) as a Mayer problem in a higher dimensional state space. Define $X(t) \in \mathbb{R}^n \times \mathbb{R}$ as

$$X(t) = \begin{pmatrix} \gamma(t) \\ z(t) \end{pmatrix},$$

where $z(t) = \int_0^t \left[f(s, \gamma(s), \dot\gamma(s)) + \frac{1}{\epsilon}\, d_\Omega(\gamma(s)) \right] ds$. Then the state equation becomes

$$\begin{cases} \dot X(t) = \begin{pmatrix} \dot\gamma(t) \\ \dot z(t) \end{pmatrix} = \mathcal{F}_\epsilon(t, X(t), u(t)), \\ \\ X(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}. \end{cases}$$

where

$$\mathcal{F}_\epsilon(t, X, u) = \begin{pmatrix} u \\ \mathcal{L}_\epsilon(t, x, u) \end{pmatrix}$$

and $\mathcal{L}_\epsilon(t,x,u) = f(t,x,u) + \frac{1}{\epsilon}\, d_\Omega(x)$ for $X = (x, z)$ and $(t, x, z, u) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. Thus, (2.3.1.1) can be written as

$$\min \left\{ \Phi(X^u(T)) : u \in L^1 \right\}, \tag{2.3.1.9}$$

where $\Phi(X) = g(x) + \frac{1}{\delta}\, d_\Omega(x) + z$ for any $X = (x, z) \in \mathbb{R}^n \times \mathbb{R}$. The associated unmaximized Hamiltonian is given by

$$\mathcal{H}_\epsilon(t, X, P, u) = -\langle P, \mathcal{F}_\epsilon(t, X, u) \rangle, \qquad \forall (t, X, P, u) \in [0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^n.$$

We observe that, as $\gamma(\cdot)$ is minimizer for (2.3.1.1), X is minimizer for (2.3.1.9). Hence, the hypotheses of [61, Theorem 8.7.1] are satisfied. It follows that there exist $P(\cdot) = (p(\cdot), b(\cdot)) \in AC(0,T;\mathbb{R}^{n+1})$, $r(\cdot) \in AC(0,T;\mathbb{R})$, and $\lambda_0 \geq 0$ such that

(i) $(P, \lambda_0) \not\equiv (0, 0)$,

(ii) $\big(\dot r(t), \dot P(t)\big) \in co\, \partial_{t,X} \mathcal{H}_\epsilon\big(t, X(t), P(t), \dot\gamma(t)\big)$, a.e $t \in [0,T]$,

(iii) $P(T) \in \lambda_0 \partial \Phi(X^u(T))$,

(iv) $\mathcal{H}_\epsilon\big(t, X(t), P(t), \dot\gamma(t)\big) = \max_{u \in \mathbb{R}^n} \mathcal{H}_\epsilon\big(t, X(t), P(t), u\big)$, a.e. $t \in [0,T]$,

(v) $\mathcal{H}_\epsilon\big(t, X(t), P(t), \dot\gamma(t)\big) = r(t)$, a.e. $t \in [0,T]$, where $\partial_{t,X}\mathcal{H}_\epsilon$ and $\partial\Phi$ denote the limiting subdifferential of $\mathcal{H}_\epsilon$ and Φ with respect to (t, X) and X respectively, while co stands for the closed convex hull. Using the definition of $\mathcal{H}_\epsilon$ we have that

$$(p, b, \lambda_0) \not\equiv (0, 0, 0), \tag{2.3.1.10}$$

$$(\dot{r}(t), \dot{p}(t)) \in -b(t)\; co\, \partial_{t,x}\mathcal{L}_\epsilon(t, \gamma(t), \dot{\gamma}(t)), \tag{2.3.1.11}$$

$$\dot{b}(t) = 0, \tag{2.3.1.12}$$

$$p(T) \in \lambda_0\, \partial(g + \frac{1}{\delta}\, d_\Omega)(\gamma(T)), \tag{2.3.1.13}$$

$$b(T) = \lambda_0, \tag{2.3.1.14}$$

$$r(t) = H_\epsilon(t, \gamma(t), p(t)), \tag{2.3.1.15}$$

where $\partial_{t,x}\mathcal{L}_\epsilon$ and $\partial(g + \frac{1}{\delta}\, d_\Omega)$ stands for the limiting subdifferential of $\mathcal{L}_\epsilon(\cdot, \cdot, u)$ and $g(\cdot) + \frac{1}{\delta} d_\Omega(\cdot)$. We claim that $\lambda_0 > 0$. Indeed, suppose that $\lambda_0 = 0$. Then $b \equiv 0$ by (2.3.1.12) and (2.3.1.14). Moreover, $p(T) = 0$ by (2.3.1.13). It follows from (2.3.1.11) that $p \equiv 0$, which is in contradiction with (2.3.1.10). So, $\lambda_0 > 0$ and we may rescale p and b so that $b(t) = \lambda_0 = 1$ for any $t \in [0, T]$.
Note that the Weierstrass Condition (iv) becomes

$$-\langle p(t), \dot{\gamma}(t)\rangle - f(t, \gamma(t), \dot{\gamma}(t)) = \sup_{u\in\mathbb{R}^n} \Big\{ -\langle p(t), u\rangle - f(t, \gamma(t), u)\Big\}. \tag{2.3.1.16}$$

Therefore

$$\dot{\gamma}(t) = -D_pH(t, \gamma(t), p(t)), \qquad \text{a.e. } t \in [0, T]. \tag{2.3.1.17}$$

By LEMMA 1.3.2.1, by the definition of ρ, and by (2.2.0.5) we have that

$$\partial_{t,x}\mathcal{L}_\epsilon(t, x, u) \subset \begin{cases} [-\kappa(1 + |u|^2), \kappa(1 + |u|^2)] \times D_xf(t, x, u) & \text{if } x \in \Omega,\\ [-\kappa(1 + |u|^2), \kappa(1 + |u|^2)] \times \big(D_xf(t, x, u) + \frac{1}{\epsilon}\, Db_\Omega(x)\big) & \text{if } 0 < b_\Omega(x) < \rho,\\ [-\kappa(1 + |u|^2), \kappa(1 + |u|^2)] \times \big(D_xf(t, x, u) + \frac{1}{\epsilon}[0, 1]\, Db_\Omega(x)\big) & \text{if } x \in \partial\Omega. \end{cases}$$

Thus (2.3.1.11) implies that there exists $\lambda(t) \in [0, 1]$ as in (2.3.1.6) such that

$$|\dot{r}(t)| \leq \kappa(1 + |\dot{\gamma}(t)|^2), \;\; \forall t \in [0, T], \tag{2.3.1.18}$$

$$\dot{p}(t) = -D_xf(t, \gamma(t), \dot{\gamma}(t)) - \frac{\lambda(t)}{\epsilon}\, Db_\Omega(\gamma(t)), \;\; \text{a.e. } t \in [0, T]. \tag{2.3.1.19}$$

Hence, by (2.3.1.18), and by REMARK 2.3.1.1 we conclude that

$$\int_0^T |\dot{r}(t)|\, dt \leq \kappa \int_0^T (1 + |\dot{\gamma}(t)|^2)\, dt \leq \kappa(T + 4\mu K). \tag{2.3.1.20}$$

Moreover, by LEMMA 1.3.2.1, and by assumption on g, one has that

$$\partial\Big(g + \frac{1}{\delta}\, d_\Omega\Big)(x) \subset \begin{cases} Dg(x) & \text{if } x \in \Omega,\\ Dg(x) + \frac{1}{\delta}\, Db_\Omega(x) & \text{if } 0 < b_\Omega(x) < \rho,\\ Dg(x) + \frac{1}{\delta}[0, 1]\, Db_\Omega(x) & \text{if } x \in \partial\Omega. \end{cases}$$

So, by (2.3.1.13), there exists $\beta \in [0,1]$ as in (2.3.1.7) such that

$$p(T) = Dg(x) + \frac{\beta}{\delta}\, Db_\Omega(x). \tag{2.3.1.21}$$

Finally, by well-known properties of the Legendre transform one has that

$$D_x H(t,x,p) = -D_x f\big(t,x,-D_p H(t,x,p)\big).$$

So, recalling (2.3.1.17), (2.3.1.19) can be rewritten as

$$\dot p(t) = D_x H(t,\gamma(t),p(t)) - \frac{\lambda(t)}{\epsilon}\, Db_\Omega(\gamma(t)), \ \text{a.e. } t \in [0,T].$$

We have to prove estimate (2.3.1.8). Recalling (2.3.1.2) and (2.2.0.21), we have that

$$H_\epsilon(t,\gamma(t),p(t)) = H(t,\gamma(t),p(t)) - \frac{1}{\epsilon}\, d_\Omega(\gamma(t)) \geq \frac{1}{4\mu}|p(t)|^2 - C(\mu,M') - \frac{1}{\epsilon}\, d_\Omega(\gamma(t)).$$

So, using (2.3.1.20) one has that

$$|H_\epsilon(T,\gamma(T),p(T)) - H_\epsilon(t,\gamma(t),p(t))| = |r(T)-r(t)| \leq \int_t^T |\dot r(s)|\, ds \leq \kappa(T+4\mu K).$$

Moreover, (2.3.1.21) implies that $|p(T)| \leq \frac{1}{\delta} + ||Dg||_\infty$. Therefore, using again (2.2.0.21), we obtain

$$\begin{aligned}
&\frac{1}{4\mu}|p(t)|^2 - C(\mu,M') - \frac{1}{\epsilon}\, d_\Omega(\gamma(t)) \leq H_\epsilon(t,\gamma(t),p(t)) \leq H_\epsilon(T,\gamma(T),p(T)) + \kappa(T+4\mu K)\\
&\leq 4\mu|p(T)|^2 + C(\mu,M') + \kappa(T+4\mu K) \leq 8\mu\left[\frac{1}{\delta^2} + ||Dg||_\infty^2\right] + C(\mu,M') + \kappa(T+4\mu K).
\end{aligned}$$

Hence,

$$|p(t)|^2 \leq 4\mu\left[\frac{1}{\epsilon} d_\Omega(\gamma(t)) + \frac{C_1}{\delta^2}\right],$$

where $C_1 = 8\mu + 8\mu||Dg||_\infty^2 + 2C(\mu,M') + \kappa(T+4\mu K)$. This completes the proof of (2.3.1.8). Finally, by the regularity of H, we have that $p \in \mathrm{Lip}(0,T;\mathbb{R}^n)$. So, $\gamma \in C^{1,1}([0,T];\mathbb{R}^n)$. Observing that the right-hand side of the equality $\dot\gamma(t) = D_p H(t,\gamma(t),p(t))$ is continuous we conclude that this equality holds for all t in $[0,T]$. □

LEMMA 2.3.1.4. Let $\rho \in (0,\rho_0]$ and let $\epsilon \in (0,\epsilon(\rho)]$, where $\epsilon(\rho)$ is given by LEMMA 2.3.1.2. Fix $\delta \in (0,1]$, let $x \in \overline{\Omega}$, and let $\gamma \in \mathcal{X}_{\epsilon,\delta}[x]$. If $\gamma(\bar t) \notin \partial\Omega$ for some $\bar t \in [0,T]$, then there exists $\tau > 0$ such that $\gamma \in C^2\left((\bar t-\tau,\bar t+\tau)\cap[0,T];\mathbb{R}^n\right)$.

Proof. Let $\gamma \in \mathcal{X}_{\epsilon,\delta}[x]$ and let $\bar t \in [0,T]$ be such that $\gamma(\bar t) \in \Omega \cup (\mathbb{R}^n \setminus \overline{\Omega})$. If $\gamma(\bar t) \in \mathbb{R}^n \setminus \overline{\Omega}$, then there exists $\tau > 0$ such that $\gamma(t) \in \mathbb{R}^n \setminus \overline{\Omega}$ for all $t \in I := (\bar t-\tau,\bar t+\tau)\cap[0,T]$. By LEMMA 2.3.1.3, we have that there exists $p \in \mathrm{Lip}(0,T;\mathbb{R}^n)$ such that

$$\dot\gamma(t) = -D_p H(t,\gamma(t),p(t)),$$

$$\dot{p}(t) = D_x H(t, \gamma(t), p(t)) - \frac{1}{\epsilon} Db_\Omega(\gamma(t)),$$

for $t \in I$. Since $p(t)$ is Lipschitz continuous for $t \in I$, and $\dot{\gamma}(t) = -D_p H(t, \gamma(t), p(t))$, then γ belongs to $C^1(I; \mathbb{R}^n)$. Moreover, by the regularity of H, b_Ω, p, and γ one has that $\dot{p}(t)$ is continuous for $t \in I$. Then $p \in C^1(I; \mathbb{R}^n)$. Hence, $\dot{\gamma} \in C^1(I; \mathbb{R}^n)$. So, $\gamma \in C^2(I; \mathbb{R}^n)$. Finally, if $\gamma(\bar{t}) \in \Omega$, the conclusion follows by a similar argument. □

In the next two lemmas, we show that, for $\epsilon > 0$ and $\delta \in (0, 1]$ small enough, any solution γ of problem (2.3.1.1) belongs to $\overline{\Omega}$ for all $t \in [0, T]$. For this we first establish that, if $\delta \in (0, 1]$ is small enough and $\gamma(T) \notin \overline{\Omega}$, then the function $t \longmapsto b_\Omega(\gamma(t))$ has nonpositive slope at $t = T$. Then we prove that the entire trajectory γ remains in $\overline{\Omega}$ provided ϵ is small enough. Hereafter, we set

$$\epsilon_0 = \epsilon(\rho_0), \quad \text{where } \rho_0 \text{ is such that (1.1.0.2) holds and } \epsilon(\cdot) \text{ is given by LEMMA 2.3.1.2.}$$

LEMMA 2.3.1.5. Let

$$\delta = \frac{1}{2\mu N} \wedge 1, \tag{2.3.1.22}$$

where

$$N = \sup_{x \in \mathbb{R}^n} |D_p H(T, x, Dg(x))|.$$

Fix any $\delta_1 \in (0, \delta]$ and let $x \in \overline{\Omega}$. Let $\epsilon \in (0, \epsilon_0]$. If $\gamma \in \mathcal{X}_{\delta_1, \epsilon}[x]$ is such that $\gamma(T) \notin \overline{\Omega}$, then

$$\langle \dot{\gamma}(T), Db_\Omega(\gamma(T)) \rangle \leq 0.$$

Proof. As $\gamma(T) \notin \overline{\Omega}$, by LEMMA 2.3.1.3 we have that $p(T) = Dg(\gamma(T)) + \frac{1}{\delta} Db_\Omega(\gamma(T))$. Hence,

$$\Big\langle D_p H\big(T, \gamma(T), p(T)\big), Db_\Omega(\gamma(T)) \Big\rangle = \Big\langle D_p H\big(T, \gamma(T), Dg(\gamma(T))\big), Db_\Omega(\gamma(T)) \Big\rangle$$
$$+ \Big\langle D_p H\big(T, \gamma(T), Dg(\gamma(T)) + \frac{1}{\delta} Db_\Omega(\gamma(T))\big) - D_p H\big(T, \gamma(T), Dg(\gamma(T))\big), Db_\Omega(\gamma(T)) \Big\rangle.$$

Recalling that $D^2_{pp} H(t, x, p) \geq \frac{I}{\mu}$, one has that

$$\Big\langle D_p H\big(T, \gamma(T), Dg(\gamma(T)) + \frac{1}{\delta} Db_\Omega(\gamma(T))\big) - D_p H\big(T, \gamma(T), Dg(\gamma(T))\big), \frac{1}{\delta} Db_\Omega(\gamma(T)) \Big\rangle$$
$$\geq \frac{1}{2\mu} \frac{1}{\delta^2} |Db_\Omega(\gamma(T))|^2 = \frac{1}{2\delta^2 \mu}.$$

So,

$$\Big\langle D_p H\big(T, \gamma(T), p(T)\big), Db_\Omega(\gamma(T)) \Big\rangle \geq \frac{1}{2\delta\mu} - |D_p H\big(T, \gamma(T), Dg(\gamma(T))\big)|.$$

Therefore, we obtain

$$\langle \dot{\gamma}(T), Db_\Omega(\gamma(T)) \rangle = -\Big\langle D_p H\big(T, \gamma(T), p(T)\big), Db_\Omega(\gamma(T)) \Big\rangle$$
$$\leq -\frac{1}{2\delta\mu} + |D_p H(T, \gamma(T), Dg(\gamma(T)))|.$$

Thus, choosing δ as in (2.3.1.22) gives the result. □

LEMMA 2.3.1.6. Fix δ as in (2.3.1.22). Then there exists $\epsilon_1 \in (0, \epsilon_0]$, such that for any $\epsilon \in (0, \epsilon_1]$

$$\forall x \in \overline{\Omega},\ \gamma \in \mathcal{X}_{\epsilon,\delta}[x] \implies \gamma(t) \in \overline{\Omega} \quad \forall t \in [0,T].$$

Proof. We argue by contradiction. Assume that there exist sequences $\{\epsilon_k\}$, $\{t_k\}$, $\{x_k\}$, $\{\gamma_k\}$ such that

$$\epsilon_k \downarrow 0,\ t_k \in [0,T],\ x_k \in \overline{\Omega},\ \gamma_k \in \mathcal{X}_{\epsilon_k,\delta}[x_k] \text{ and } \gamma_k(t_k) \notin \overline{\Omega}, \quad \text{for all } k \geq 1. \tag{2.3.1.23}$$

Then, for each $k \geq 1$ one could find an interval with end-points $0 \leq a_k < b_k \leq T$ such that

$$\begin{cases} d_\Omega(\gamma_k(a_k)) = 0, \\ d_\Omega(\gamma_k(t)) > 0 \quad t \in (a_k, b_k), \\ d_\Omega(\gamma_k(b_k)) = 0 \ \text{ or else } \ b_k = T. \end{cases}$$

Let $\bar{t}_k \in (a_k, b_k]$ be such that

$$d_\Omega(\gamma_k(\bar{t}_k)) = \max_{t \in [a_k,b_k]} d_\Omega(\gamma_k(t)).$$

We note that, by LEMMA 2.3.1.4, γ_k is of class C^2 in a neighborhood of $\bar{t}_k$.

Step 1

We claim that

$$\frac{d^2}{dt^2} d_\Omega(\gamma_k(t))\Big|_{t=\bar{t}_k} \leq 0. \tag{2.3.1.24}$$

Indeed, (2.3.1.24) is trivial if $\bar{t}_k \in (a_k, b_k)$. Suppose $\bar{t}_k = b_k$. Since $\bar{t}_k$ is a maximum point of the map $t \longmapsto d_\Omega(\gamma_k(t))$ and $\gamma_k(\bar{t}_k) \notin \overline{\Omega}$, we have that $d_\Omega(\gamma_k(\bar{t}_k)) \neq 0$. So, $b_k = T = \bar{t}_k$ and we get

$$\frac{d}{dt} d_\Omega(\gamma_k(t))\Big|_{t=\bar{t}_k} \geq 0.$$

Moreover, LEMMA 2.3.1.5 yields

$$\frac{d}{dt} d_\Omega(\gamma_k(t))\Big|_{t=\bar{t}_k} \leq 0.$$

So,

$$\frac{d}{dt} d_\Omega(\gamma_k(t))\Big|_{t=\bar{t}_k} = 0,$$

and we have that (2.3.1.24) holds true at $\bar{t}_k = T$.

Step 2

Now, we prove that

$$\frac{1}{\mu\epsilon_k} \leq C(\mu, M', \kappa)\left[1 + 4\mu\frac{C_1}{\delta^2} + \frac{4\mu}{\epsilon_k}\, d_\Omega(\gamma_k(\bar{t}_k))\right], \quad \forall k \geq 1, \tag{2.3.1.25}$$

where $C_1 = 8\mu + 8\mu||Dg||^2_\infty + 2C(\mu, M') + \kappa(T + 4\mu K)$ and the constant $C(\mu, M', \kappa)$ depends only on μ, M' and κ. Indeed, since γ is of class C^2 in a neighborhood of $\bar{t}_k$ one has that

$$\ddot{\gamma}(\bar{t}_k) = -\, D^2_{pt}H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)) - \langle D^2_{px}H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)), \dot{\gamma}(\bar{t}_k)\rangle \tag{2.3.1.26}$$

$$- \left\langle D^2_{pp}H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)), \dot{p}(\bar{t}_k) \right\rangle .$$

Developing the second order derivative of $d_\Omega \circ \gamma$, by (2.3.1.26) and the expression of the derivatives of γ and p in LEMMA 2.3.1.3 one has that

$$\begin{aligned} 0 \ &\geq \ \left\langle D^2 d_\Omega(\gamma(\bar{t}_k))\dot{\gamma}(\bar{t}_k), \dot{\gamma}(\bar{t}_k) \right\rangle + \left\langle Dd_\Omega(\gamma(\bar{t}_k)), \ddot{\gamma}(\bar{t}_k) \right\rangle \\ &= \ \left\langle D^2 d_\Omega(\gamma(\bar{t}_k)) D_pH(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)), D_pH(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)) \right\rangle \\ &\quad - \left\langle Dd_\Omega(\gamma(\bar{t}_k)), D^2_{pt}H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)) \right\rangle \\ &\quad + \left\langle Dd_\Omega(\gamma(\bar{t}_k)), D^2_{px}H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)) D_pH(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)) \right\rangle \\ &\quad - \left\langle Dd_\Omega(\gamma(\bar{t}_k)), D^2_{pp}H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)) D_xH(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)) \right\rangle \\ &\quad + \frac{1}{\epsilon} \left\langle Dd_\Omega(\gamma(\bar{t}_k)), D^2_{pp}H(\bar{t}_k, \gamma(\bar{t}_k), p(\bar{t}_k)) Dd_\Omega(\gamma(\bar{t}_k)) \right\rangle . \end{aligned}$$

We now use the growth properties of H in (2.2.0.15), and (2.2.0.17)-(2.2.0.20), the lower bound for $D^2_{pp}H$ in (2.2.0.14), and the regularity of the boundary of Ω to obtain:

$$\frac{1}{\mu\epsilon_k} \leq C(\mu, M')(1 + |p(\bar{t}_k)|)^2 + \kappa C(\mu, M')(1 + |p(\bar{t}_k)|) \leq C(\mu, M', \kappa)(1 + |p(\bar{t}_k)|^2),$$

where the constant $C(\mu, M', \kappa)$ depends only on μ, M' and κ. By our estimate for p in (2.3.1.8) we get:

$$\frac{1}{\mu\epsilon_k} \leq C(\mu, M', \kappa) \left[1 + 4\mu \frac{C_1}{\delta^2} + \frac{4\mu}{\epsilon_k} d_\Omega(\gamma(\bar{t}_k)) \right], \ \ \forall\, k \geq 1,$$

where $C_1 = 8\mu + 8\mu ||Dg||^2_\infty + 2C(\mu, M') + \kappa(T + 4\mu K)$.

Conclusion

Let $\rho = \min\left\{ \rho_0, \frac{1}{32C(\mu,M',\kappa)\mu^2} \right\}$. Owing to LEMMA 2.3.1.2, for all $\epsilon \in (0, \epsilon(\rho)]$ we have that

$$\sup_{t\in[0,T]} d_\Omega(\gamma(t)) \leq \rho, \quad \forall \gamma \in \mathcal{X}_{\epsilon,\delta}[x].$$

Hence, using (2.3.1.25), we deduce that

$$\frac{1}{2\mu\epsilon_k} \leq 4C(\mu, M', \kappa) \left[1 + 4\mu \frac{C_1}{\delta^2} \right].$$

Since the above inequality fails for k large enough, we conclude that (2.3.1.23) cannot hold true. So, $\gamma(t)$ belongs to $\overline{\Omega}$ for all $t \in [0, T]$. □

An obvious consequence of LEMMA 2.3.1.6 is the following:

COROLLARY 2.3.1.7. Fix δ as in (2.3.1.22) and take $\epsilon = \epsilon_1$, where ϵ_1 is defined as in LEMMA 2.3.1.6. Then an arc $\gamma(\cdot)$ is a solution of problem (2.3.1.1) if and only if it is also a solution of (2.2.0.1).

We are now ready to complete the proof of THEOREM 2.3.0.1.

Proof of THEOREM 2.3.0.1. Let $x \in \overline{\Omega}$ and $\gamma^\star \in \mathcal{X}[x]$. By COROLLARY 2.3.1.7 we have that $\gamma^\star$ is a solution of problem (2.3.1.1) with δ as in (2.3.1.22) and $\epsilon = \epsilon_1$ as in LEMMA 2.3.1.6. Let $p(\cdot)$ be the associated adjoint map such that $(\gamma^\star(\cdot), p(\cdot))$ satisfies (2.3.1.5). Moreover, let $\lambda(\cdot)$ and β be defined as in LEMMA 2.3.1.3. Define $\nu = \frac{\beta}{\delta}$. Then we have $0 \leq \nu \leq \frac{1}{\delta}$ and, by (2.3.1.5),

$$p(T) = Dg(\gamma^\star(T)) + \nu\, Db_\Omega(\gamma^\star(T)). \tag{2.3.1.27}$$

By LEMMA 2.3.1.3 $\gamma^\star \in C^{1,1}([0,T];\overline{\Omega})$ and

$$\dot\gamma^\star(t) = -D_pH(t,\gamma^\star(t),p(t)), \quad \forall\, t \in [0,T]. \tag{2.3.1.28}$$

Moreover, $p(\cdot) \in \mathrm{Lip}(0,T;\mathbb{R}^n)$, and by (2.3.1.8) one has that

$$|p(t)| \leq 2\frac{\sqrt{\mu C_1}}{\delta}, \quad \forall t \in [0,T],$$

where $C_1 = 8\mu + 8\mu||Dg||_\infty^2 + 2C(\mu,M') + \kappa(T+4\mu K)$. Hence, p is bounded. By (2.3.1.28), and by (2.2.0.19) one has that

$$||\dot\gamma^\star||_\infty = \sup_{t\in[0,T]} |D_pH(t,\gamma^\star(t),p(t))| \leq C(\mu,M')\Big(\sup_{t\in[0,T]}|p(t)|+1\Big) \leq C(\mu,M')\Big(2\frac{\sqrt{\mu C_1}}{\delta}+1\Big)\Big)=L^\star,$$

where $L^\star = L^\star(\mu, M', M, \kappa, T, ||Dg||_\infty, ||g||_\infty)$. Thus, (2.3.0.3) holds.
Finally, we want to find an explicit expression for $\lambda(t)$. For this, we set

$$D = \Big\{t \in [0,T] : \gamma^\star(t) \in \partial\Omega\Big\} \text{ and } D_{\rho_0} = \Big\{t \in [0,T] : |b_\Omega(\gamma^\star(t))| < \rho_0\Big\},$$

where ρ_0 is as in assumption (1.1.0.2). Note that $\psi(t) := b_\Omega \circ \gamma^\star$ is of class $C^{1,1}$ on the open set D_{ρ_0}, with

$$\dot\psi(t) = \Big\langle Db_\Omega(\gamma^\star(t)), \dot\gamma^\star(t)\Big\rangle = \Big\langle Db_\Omega(\gamma^\star(t)), -D_pH(t,\gamma^\star(t),p(t))\Big\rangle.$$

Since $p \in \mathrm{Lip}(0,T;\mathbb{R}^n)$, $\dot\psi$ is absolutely continuous on D_{ρ_0} with

$$\begin{aligned}
\ddot\psi(t) &= -\Big\langle D^2b_\Omega(\gamma^\star(t))\dot\gamma^\star(t), D_pH\big(t,\gamma^\star(t),p(t)\big)\Big\rangle - \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pt}H\big(t,\gamma^\star(t),p(t)\big)\Big\rangle \\
&- \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{px}H\big(t,\gamma^\star(t),p(t)\big)\dot\gamma^\star(t)\Big\rangle - \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H\big(t,\gamma^\star(t),p(t)\big)\dot p(t)\Big\rangle \\
&= \Big\langle D^2b_\Omega(\gamma^\star(t))D_pH\big(t,\gamma^\star(t),p(t)\big), D_pH\big(t,\gamma^\star(t),p(t)\big)\Big\rangle \\
&- \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pt}H\big(t,\gamma^\star(t),p(t)\big)\Big\rangle \\
&+ \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{px}H\big(t,\gamma^\star(t),p(t)\big)D_pH\big(t,\gamma^\star(t),p(t)\big)\Big\rangle \\
&- \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H\big(t,\gamma^\star(t),p(t)\big)D_xH\big(t,\gamma^\star(t),p(t)\big)\rangle \\
&+ \frac{\lambda(t)}{\epsilon}\Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H\big(t,\gamma^\star(t),p(t)\big)Db_\Omega(\gamma^\star(t))\Big\rangle.
\end{aligned}$$

Let $N_{\gamma^\star} = \{t \in D \cap (0,T) | \, \dot{\psi}(t) \neq 0\}$. Let $t \in N_{\gamma^\star}$, then there exists $\sigma > 0$ such that $\gamma^\star(s) \notin \partial\Omega$ for any $s \in ((t-\sigma, t+\sigma) \setminus \{t\}) \cap (0,T)$. Therefore, $N_{\gamma^\star}$ is composed of isolated points and so it is a discrete set. Hence, $\dot{\psi}(t) = 0$ a.e. $t \in D \cap (0,T)$. So, $\ddot{\psi}(t) = 0$ a.e. in D, because $\dot{\psi}$ is absolutely continuous. Moreover, since $D^2_{pp}H(t,x,p) > 0$ and $|Db_\Omega(\gamma^\star(t))| = 1$, we have that

$$\theta(t) := \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pp}H\big(t,\gamma^\star(t),p(t)\big) Db_\Omega(\gamma^\star(t)) \Big\rangle > 0, \qquad \text{a.e. } t \in D_{\rho_0}.$$

So, for a.e. $t \in D$, $\lambda(t)$ is given by

$$\begin{aligned}\frac{\lambda(t)}{\epsilon} =& \frac{1}{\theta(t)} \Big[-\Big\langle D^2 b_\Omega(\gamma^\star(t)) D_p H\big(t,\gamma^\star(t),p(t)\big), D_p H\big(t,\gamma^\star(t),p(t)\big) \Big\rangle \\ & - \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pt} H\big(t,\gamma^\star(t),p(t)\big) \Big\rangle \\ & - \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{px} H\big(t,\gamma^\star(t),p(t)\big) D_p H\big(t,\gamma^\star(t),p(t)\big) \Big\rangle \\ & + \Big\langle Db_\Omega(\gamma^\star(t)), D^2_{pp} H\big(t,\gamma^\star(t),p(t)\big) D_x H\big(t,\gamma^\star(t),p(t)\big) \Big\rangle \Big].\end{aligned}$$

Since $\lambda(t) = 0$ for all $t \in [0,T] \setminus D$ by (2.3.1.6), taking $\Lambda(t) = \frac{\lambda(t)}{\epsilon}$, we obtain the conclusion.

□

2.3.2 PROOF OF THEOREM 2.3.0.1 FOR GENERAL U

We now want to remove the extra assumption $U = \mathbb{R}^n$. For this purpose, it suffices to show that the data f and g—a priori defined just on U—can be extended to $\mathbb{R}^n$ preserving the conditions in (f0)-(f2) and (g1). So, we proceed to construct such an extension by taking a cut-off function $\xi \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} \xi(x) = 0 & \text{if } x \in (-\infty, \frac{1}{3}], \\ 0 < \xi(x) < 1 & \text{if } x \in (\frac{1}{3}, \frac{2}{3}), \\ \xi = 1 & \text{if } x \in [\frac{2}{3}, +\infty). \end{cases} \tag{2.3.2.1}$$

LEMMA 2.3.2.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. Let U be a open subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset U$ and

$$0 < dist(\overline{\Omega}, \mathbb{R}^n \setminus U) =: \sigma_0.$$

Suppose that $f : [0,T] \times U \times \mathbb{R}^n \to \mathbb{R}$ and $g : U \to \mathbb{R}$ satisfy (f0)-(f2) and (g1), respectively. Set $\sigma = \sigma_0 \wedge \rho_0$. Then, the function f admits the extension

$$\widetilde{f}(t,x,v) = \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\frac{|v|^2}{2} + \left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right) f(t,x,v), \quad \forall\, (t,x,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n,$$

that verifies the conditions (f0)-(f2) with $U = \mathbb{R}^n$. Moreover, the function g admits the extension

$$\widetilde{g}(x) = \left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right) g(x), \quad \forall x \in \mathbb{R}^n,$$

that satisfies the condition (g1) with $U = \mathbb{R}^n$.

Proof. By construction we note that $\widetilde{f} \in C([0,T] \times \mathbb{R}^n \times \mathbb{R}^n)$. Moreover, for all $t \in [0,T]$ the function $(x,v) \longmapsto \widetilde{f}(t,x,v)$ is differentiable and the map $(x,v) \longmapsto D_v\widetilde{f}(t,x,v)$ is continuously differentiable by construction. Furthermore, $D_x\widetilde{f}$, $D_v\widetilde{f}$ are continuous on $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ and $\widetilde{f}$ satisfies (2.2.0.2).

$$D_v\widetilde{f}(t,x,v) = \xi\left(\frac{b_\Omega(x)}{\sigma}\right) v + \left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right) D_v f(t,x,v).$$

In order to prove (2.2.0.3) for $\widetilde{f}$, we observe that

$$D_v\widetilde{f}(t,x,v) = \xi\left(\frac{b_\Omega(x)}{\sigma}\right) v + \left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right) D_v f(t,x,v),$$

and

$$D^2_{vv}\widetilde{f}(t,x,v) = \xi\left(\frac{b_\Omega(x)}{\sigma}\right) I + \left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right) D^2_{vv} f(t,x,v).$$

Hence, by the definition of ξ and (2.2.0.3) we obtain that

$$\left(1 \wedge \frac{1}{\mu}\right) I \leq D^2_{vv}\widetilde{f}(t,x,v) \leq (1 \vee \mu) I, \quad \forall\, (t,x,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n.$$

Since $\mu \geq 1$, we have that $\widetilde{f}$ verifies the estimate in (2.2.0.3).
Moreover, since

$$\begin{aligned} D_x(D_v\widetilde{f}(t,x,v)) &= \dot{\xi}\left(\frac{b_\Omega(x)}{\sigma}\right) v \otimes \frac{Db_\Omega(x)}{\sigma} + \left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right) D^2_{vx} f(t,x,v) \\ &- \dot{\xi}\left(\frac{b_\Omega(x)}{\sigma}\right) D_v f(t,x,v) \otimes \frac{Db_\Omega(x)}{\sigma}, \end{aligned}$$

and by (2.2.0.4) we obtain that

$$||D^2_{vx}\widetilde{f}(t,x,v)|| \leq C(\mu, M)(1 + |v|) \quad \forall (t,x,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n.$$

For all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$ the function $t \longmapsto \widetilde{f}(t,x,v)$ and the map $t \longmapsto D_v\widetilde{f}(t,x,v)$ are Lipschitz continuous by construction. Moreover, by (2.2.0.5) and the definition of ξ one has that

$$\left|\widetilde{f}(t,x,v) - \widetilde{f}(s,x,v)\right| = \left|\left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right)[f(t,x,v) - f(s,x,v)]\right| \leq \kappa(1 + |v|^2)|t - s|$$

for all t, $s \in [0,T]$, $x \in \mathbb{R}^n$, $v \in \mathbb{R}^n$. Now, we have to prove that (2.2.0.6) holds for $\widetilde{f}$. Indeed, using (2.2.0.6) we deduce that

$$\begin{aligned} &|D_v\widetilde{f}(t,x,v)) - D_v\widetilde{f}(s,x,v))| \leq \left|\left(1 - \xi\left(\frac{b_\Omega(x)}{\sigma}\right)\right)[D_v f(t,x,v) - D_v f(s,x,v))]\right| \\ &\leq \kappa(1 + |v|)|t - s|, \end{aligned}$$

for all t, $s \in [0,T]$, $x \in \mathbb{R}^n$, $v \in \mathbb{R}^n$. Therefore, $\widetilde{f}$ verifies the assumptions (f0)-(f2).
Finally, by the regularity of b_Ω, ξ, and g we have that $\widetilde{g}$ is of class $C^1_b(\mathbb{R}^n)$. This completes the proof. □

2.4 REGULARITY FOR CONSTRAINED MINIMIZATION PROBLEMS

Suppose that $f : [0,T] \times U \times \mathbb{R}^n \to \mathbb{R}$ and $g : U \to \mathbb{R}$ satisfy the assumptions (f0)-(f4) and (g1), respectively. For any $(t,x) \in [0,T] \times \overline{\Omega}$, we set

$$\Gamma_t[x] = \{\gamma \in \Gamma : \gamma(t) = x\}.$$

Given $(t,x) \in [0,T] \times \overline{\Omega}$, we consider the constrained minimization problem

$$\inf_{\gamma \in \Gamma_t[x]} J_t[\gamma], \quad \text{where} \quad J_t[\gamma] = \left\{ \int_t^T f(s,\gamma(s),\dot\gamma(s))\,ds + g(\gamma(T)) \right\}. \tag{2.4.0.1}$$

We denote by $\mathcal{X}_t[x]$ the set of solutions of (2.2.0.1), that is

$$\mathcal{X}_t[x] = \left\{ \gamma \in \Gamma_t[x] : J_t[\gamma] = \inf_{\Gamma_t[x]} J_t[\gamma] \right\}.$$

REMARK 2.4.0.1. Let $x \in \overline{\Omega}$ and let $\gamma \in \mathcal{X}_t[x]$. By THEOREM 2.3.0.1 one has that

(i) γ is of class $C^{1,1}([t,T];\overline{\Omega})$;

(ii) there exist:

  (a) $p \in \mathrm{Lip}(t,T;\mathbb{R}^n)$,

  (b) a bounded measurable function $\Lambda : [t,T] \to [0,\infty)$,

  (c) a constant $\nu \in \mathbb{R}$ such that

$$0 \le \nu \le \max\left\{1, 2\mu \sup_{x \in U} \left| D_p H(T,x,Dg(x)) \right| \right\},$$

  which satisfy the adjoint system

$$\begin{cases} \dot p(s) = -D_x f(s,\gamma(s),p(s)) - \Lambda(s) Db_\Omega(\gamma(s)) & \text{for a.e. } s \in [t,T], \\ p(T) = Dg(\gamma(T)) + \nu Db_\Omega(\gamma(T)) \mathbf{1}_{\partial\Omega}(\gamma(T)) \end{cases} \tag{2.4.0.2}$$

  and

$$-\langle p(t)\dot\gamma(t)\rangle - f(t,\gamma(t),p(t)) = \sup_{v \in \mathbb{R}^n} \{-\langle p(t),v\rangle - f(t,x,v)\}. \tag{2.4.0.3}$$

Moreover, γ satisfies the following estimate

$$||\dot\gamma||_\infty \le L^\star$$

where $L^\star = L^\star(\mu, M', M, \kappa, T, ||Dg||_\infty, ||g||_\infty)$.
Following the terminology of control theory, given $\gamma \in \mathcal{X}_t[x]$ any arc p satisfing (2.4.0.2) and (2.4.0.3) is called a *dual arc* associated with γ.

DEFINITION 2.4.0.2. Define $u : [0,T] \times \overline{\Omega} \to \mathbb{R}$ as the value function of the minimization problem (2.2.0.1), i.e.,

$$u(t,x) = \inf_{\gamma \in \Gamma_t[x]} \int_t^T f(s, \gamma(s), \dot{\gamma}(s))\, ds + g(\gamma(T)). \tag{2.4.0.4}$$

PROPOSITION 2.4.0.3. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Suppose that f and g satisfy (f0)-(f2) and (g1), respectively. Then, u is Lipschitz continuous in $(0,T) \times \overline{\Omega}$.

Proof. First, we shall prove that u is Lipschitz continuous in space, uniformly in time. Let $x_0 \in \Omega$ and choose $0 < r < 1$ such that $B_r(x_0) \subset B_{2r}(x_0) \subset B_{4r}(x_0) \subset \Omega$. To prove that u is Lipschitz continuous in $B_r(x_0)$, we take $x \neq y$ in $B_r(x_0)$ and t $\in (0,T)$. Let γ be an optimal trajectory for u at (t,x) and let $\overline{\gamma}$ be the trajectory defined by

$$\begin{cases} \overline{\gamma}(t) = y \\ \dot{\overline{\gamma}}(s) = \dot{\gamma}(s) + \frac{x-y}{\tau} & \text{if } s \in [t, t+\tau] \text{ a.e.} \\ \dot{\overline{\gamma}}(s) = \dot{\gamma}(s) & \text{otherwise,} \end{cases}$$

where $\tau = \frac{|x-y|}{2L^\star} < T - t$. We claim that

(a) $\overline{\gamma}(t+\tau) = \gamma(t+\tau)$;

(b) $\overline{\gamma}(s) = \gamma(s)$ for any $s \in [t+\tau, T]$;

(c) $|\overline{\gamma}(s) - \gamma(s)| \leq |y - x|$ for any $s \in [t, t+\tau]$;

(d) $\overline{\gamma}(s) \in \overline{\Omega}$ for any $s \in [t, T]$.

Indeed, by the definition of $\overline{\gamma}$ we have that

$$\overline{\gamma}(t+\tau) - \overline{\gamma}(t) = \overline{\gamma}(t+\tau) - y = \int_t^{t+\tau} \left(\dot{\gamma}(s) + \frac{x-y}{\tau}\right) ds = \gamma(t+\tau) - y,$$

and this gives (a).
Moreover, by (a), and by the definition of $\overline{\gamma}$ one has that $\overline{\gamma}(s) = \gamma(s)$ for any $s \in [t+\tau, T]$. Hence, $\overline{\gamma}$ verifies (b).
By the definition of $\overline{\gamma}$, for any $s \in [t, t+\tau]$ we obtain that

$$\left|\overline{\gamma}(s) - \gamma(s)\right| \leq \left|y - x + \int_t^s (\dot{\overline{\gamma}}(\sigma) - \dot{\gamma}(\sigma))\, d\sigma\right| = \left|y - x + \int_t^s \frac{x-y}{\tau}\, d\sigma\right| \leq |y - x|$$

and so (c) holds.
Since γ is an optimal trajectory for u and by $\overline{\gamma}(s) = \gamma(s)$ for all $s \in [t+\tau, T]$, we only have to prove that $\overline{\gamma}(s)$ belongs to $\overline{\Omega}$ for all $s \in [t, t+\tau]$. Let $s \in [t, t+\tau]$, by THEOREM 2.3.0.1 one has that

$$|\overline{\gamma}(s) - x_0| \leq |\overline{\gamma}(s) - y| + |y - x_0| \leq \left|\int_t^s \dot{\overline{\gamma}}(\sigma)\, d\sigma\right| + r \leq \int_t^s \left|\dot{\gamma}(\sigma) + \frac{x-y}{\tau}\right| d\sigma + r$$

$$\leq \int_t^s \left[|\dot{\gamma}(\sigma)| + \frac{|x-y|}{\tau}\right] d\sigma + r \leq L^\star(s-t) + \frac{|x-y|}{\tau}(s-t) + r \leq L^\star\tau + |x-y| + r.$$

Since

$$\tau = \frac{|x-y|}{2L^\star} \leq \frac{1}{2L^\star}$$

one has that

$$|\overline{\gamma}(s) - x_0| \leq \frac{|x-y|}{2} + |x-y| + r \leq 4r.$$

Therefore, $\overline{\gamma}(s) \in B_{4r}(x_0) \subset \overline{\Omega}$ for all $s \in [t, t+\tau]$.

Using the dynamic programming principle, by (a) one has that

$$u(t,y) \leq \int_t^{t+\tau} f(s, \overline{\gamma}(s), \dot{\overline{\gamma}}(s))\, ds + u(t+\tau, \gamma(t+\tau)). \tag{2.4.0.5}$$

Since γ is an optimal trajectory for u at (t,x), we obtain that

$$u(t,y) \leq u(t,x) + \int_t^{t+\tau} \left[f(s, \overline{\gamma}(s), \dot{\overline{\gamma}}(s)) - f(s, \gamma(s), \dot{\gamma}(s))\right] ds.$$

By (2.2.0.8), (2.2.0.9), and the definition of $\overline{\gamma}$, for $s \in [t, t+\tau]$ we have that

$$\begin{aligned}
&|f(s, \overline{\gamma}(s), \dot{\overline{\gamma}}(s)) - f(s, \gamma(s), \dot{\gamma}(s))| \\
&\leq |f(s, \overline{\gamma}(s), \dot{\overline{\gamma}}(s)) - f(s, \overline{\gamma}(s), \dot{\gamma}(s))| + |f(s, \overline{\gamma}(s), \dot{\gamma}(s)) - f(s, \gamma(s), \dot{\gamma}(s))| \\
&\leq \int_0^1 |\langle D_v f(s, \overline{\gamma}(s), \lambda\dot{\overline{\gamma}}(s) + (1-\lambda)\dot{\gamma}(s)), \dot{\overline{\gamma}}(s) - \dot{\gamma}(s)\rangle|\, d\lambda \\
&+ \int_0^1 |D_x f(s, \lambda\overline{\gamma}(s) + (1-\lambda)\gamma(s), \dot{\gamma}(s)), \overline{\gamma}(s) - \gamma(s)\rangle|\, d\lambda \\
&\leq C(\mu, M)|\dot{\overline{\gamma}}(s) - \dot{\gamma}(s)| \int_0^1 (1 + |\lambda\dot{\overline{\gamma}}(s) + (1-\lambda)\dot{\gamma}(s)|)\, d\lambda \\
&+ C(\mu, M)|\overline{\gamma}(s) - \gamma(s)| \int_0^1 (1 + |\dot{\gamma}(s)|^2)\, d\lambda.
\end{aligned}$$

By THEOREM 2.3.0.1 one has that

$$\int_0^1 (1 + |\lambda\dot{\overline{\gamma}}(s) + (1-\lambda)\dot{\gamma}(s)|)\, d\lambda \leq 1 + 4L^\star, \tag{2.4.0.6}$$

$$\int_0^1 (1 + |\dot{\gamma}(s)|^2)\, d\lambda \leq 1 + (L^\star)^2. \tag{2.4.0.7}$$

Using (2.4.0.6), (2.4.0.7), and (c), by the definition of $\overline{\gamma}$ one has that

$$\begin{aligned}
|f(s, \overline{\gamma}(s), \dot{\overline{\gamma}}(s)) - f(s, \gamma(s), \dot{\gamma}(s))| \leq C(\mu, M)(1 + 4L^\star)\frac{|x-y|}{\tau} \\
+ C(\mu, M)(1 + (L^\star)^2)|x-y|,
\end{aligned} \tag{2.4.0.8}$$

for a.e. $s \in [t, t+\tau]$.
By (2.4.0.8), and the choice of τ we deduce that

$$\begin{aligned}
u(t,y) &\leq u(t,x) + C(\mu,M)(1+4L^\star)\int_t^{t+\tau}\frac{|x-y|}{\tau}\,ds + C(\mu,M)(1+(L^\star)^2)\int_t^{t+\tau}|x-y|\,ds\\
&\leq u(t,x) + C(\mu,M)(1+4L^\star)\big|x-y\big| + \tau C(\mu,M)(1+(L^\star)^2)\big|x-y\big|\\
&\leq u(t,x) + C_{L^\star}|x-y|
\end{aligned}$$

where $C_{L^\star} = C(\mu,M)(1+4L^\star) + \frac{1}{2L^\star}C(\mu,M)(1+(L^\star)^2)$. Thus, u is locally Lipschitz continuous in space and one has that $||Du||_\infty \leq \vartheta$, where ϑ is a constant not depending on Ω. Owing to the smoothness of Ω, u is globally Lipschitz continuous in space.
Let $x \in \overline{\Omega}$. Let t_1, $t_2 \in (0,T)$ and, without loss of generality, suppose that $t_2 \geq t_1$. Let γ be an optimal trajectory for u at (t_1, x). Then,

$$|u(t_2,x) - u(t_1,x)| \leq |u(t_2,x) - u(t_2,\gamma(t_2))| + |u(t_2,\gamma(t_2)) - u(t_1,x)|. \tag{2.4.0.9}$$

The first term on the right-side of (2.4.0.9) can be estimated using the Lipschitz continuity in space of u and THEOREM 2.3.0.1. Indeed, we get

$$|u(t_2,x) - u(t_2,\gamma(t_2))| \leq C_{L^\star}|x-\gamma(t_2)| \leq C_{L^\star}\int_{t_1}^{t_2}|\dot\gamma(s)|\,ds \leq L^\star C_{L^\star}(t_2-t_1). \tag{2.4.0.10}$$

We only have to estimate the second term on the right-side of (2.4.0.9). By dynamic programming principle, (2.2.0.10), and the assumptions on F we deduce that

$$\begin{aligned}
|u(t_2,\gamma(t_2)) - u(t_1,x)| &\leq \left|\int_{t_1}^{t_2} f(s,\gamma(s),\dot\gamma(s))\,ds\right| \leq \int_{t_1}^{t_2}|f(s,\gamma(s),\dot\gamma(s))|\,ds \qquad (2.4.0.11)\\
&\leq \int_{t_1}^{t_2}\Big[C(\mu,M) + 4\mu|\dot\gamma(s)|^2\Big]\,ds \leq \Big[C(\mu,M) + 4\mu L^\star\Big](t_2-t_1)
\end{aligned}$$

Using (2.4.0.10) and (2.4.0.11) in (2.4.0.9), we obtain that u is Lipschitz continuous in time. This completes the proof. □

In the next result we show that u is locally semiconcave with linear modulus in Ω.

PROPOSITION 2.4.0.4. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Suppose that f satisfies (f0)-(f2) and (f4). Then u is locally semiconcave with linear modulus in Ω.

Proof. For a given $R > 0$, let us set $\Omega_R = \Big\{x \in \Omega : dist(x, \partial\Omega) > R\Big\}$. Fix $(t,x) \in (0,T) \times \Omega_R$ and let γ be an optimal trajectory for u at (t,x). Let $h \in \mathbb{R}^n$ be such that $|h| \leq \frac{R}{4}$ and $x \pm h \in \Omega_R$. We want to prove that u satisfies

$$u(t,x+h) + u(t,x-h) - 2u(t,x) \leq c|h|^2, \tag{2.4.0.12}$$

where c denotes the semiconcavity constant of u. Set $\sigma_x = t + \frac{R}{4(L^\star+1)}$, where $L^\star$ is given in (2.3.0.3). We define the following trajectories

$$\begin{cases} \gamma_\pm^h(s) = \gamma(s) \pm \Big(1 - \frac{s-t}{\sigma_x - t}\Big)h, & s \in (t, \sigma_x], \\ \gamma_\pm^h(t) = x \pm h. \end{cases}$$

First we observe that the trajectory $\gamma_+^h(s)$ belongs to $\overline{\Omega}$ for all $s \in [t, \sigma_x]$. Indeed, by THEOREM 2.3.0.1 we have that

$$|\gamma_+^h(s) - \gamma_+^h(t)| \le \int_t^s |\dot\gamma_+^h(r)|\,dr = \int_t^s \Big|\dot\gamma(r) - \frac{h}{\sigma_x - t}\Big|\,dr \le \int_t^s L^\star + \frac{|h|}{\sigma_x - t}\,dr$$
$$= \Big(L^\star + \frac{|h|}{\sigma_x - t}\Big)(s-t) \le L^\star(\sigma_x - t) + |h| = \frac{RL^\star}{4(L^\star+1)} + |h| \le \frac{R}{4} + \frac{R}{4} = \frac{R}{2}.$$

Thus $\gamma_+^h(s)$ remains in $\overline{\Omega}$ for all $s \in [t, \sigma_x]$. Arguing as above we also have that $\gamma_-^h(s)$ belongs to $\overline{\Omega}$ for all $s \in [t, \sigma_x]$.

Recalling that γ is optimal trajectory for u at (t,x), and using the dynamical programming principle, one has that

$$u(t, x+h) + u(t, x-h) - 2u(x,t) \le \int_t^{\sigma_x} f(s, \gamma_+^h(s), \dot\gamma_+^h(s))\,ds + u(\sigma_x, \gamma_+^h(\sigma_x))$$
$$+ \int_t^{\sigma_x} f(s, \gamma_-^h(s), \dot\gamma_-^h(s))\,ds + u(\sigma_x, \gamma_-^h(\sigma_x)) - 2\int_t^{\sigma_x} f(s, \gamma(s), \dot\gamma(s))\,ds - 2u(\sigma_x, \gamma(\sigma_x)).$$

Since $\gamma_+^h(\sigma_x) = \gamma_-^h(\sigma_x) = \gamma(\sigma_x)$, we obtain that

$$u(t, x+h) + u(t, x-h) - 2u(x,t) \le$$
$$\int_t^{\sigma_x} \Big[f(s, \gamma_+^h(s), \dot\gamma_+^h(s)) + f(s, \gamma_-^h(s), \dot\gamma_-^h(s)) - 2f(s, \gamma(s), \dot\gamma(s))\Big]\,ds.$$

Moreover, by the definition of $\gamma_\pm^h$ one has that

$$|\gamma_+^h(s) - \gamma_-^h(s)|^2 \le 4|h|^2 \quad \forall\, s \in [t, \sigma_x]. \tag{2.4.0.13}$$

By our assumptions on f, and by (2.4.0.13), for all $s \in [t, \sigma_x]$ we deduce that

$$f(s, \gamma_+^h(s), \dot\gamma_+^h(s)) + f(s, \gamma_-^h(s), \dot\gamma_-^h(s)) - 2f(s, \gamma(s), \dot\gamma(s)) \le \Big[f(s, \gamma_+^h(s), \dot\gamma_+^h(s)) - f(s, \gamma_+^h(s), \dot\gamma(s))\Big]$$
$$+ \Big[f(s, \gamma_-^h(s), \dot\gamma_-^h(s)) - f(s, \gamma_-^h(s), \dot\gamma(s))\Big] + c|\gamma_+^h(s) - \gamma_-^h(s)|^2 \tag{2.4.0.14}$$
$$\le \Big[f(s, \gamma_+^h(s), \dot\gamma_+^h(s)) - f(s, \gamma_+^h(s), \dot\gamma(s))\Big] + \Big[f(s, \gamma_-^h(s), \dot\gamma_-^h(s)) - f(s, \gamma_-^h(s), \dot\gamma(s))\Big] + 4c|h|^2.$$

Moreover, by the convexity of f with respect to the third variable we obtain that

$$f(s, \gamma_+^h(s), \dot\gamma_+^h(s)) - f(s, \gamma_+^h(s), \dot\gamma(s)) \le \langle D_v f(s, \gamma_+^h(s), \dot\gamma_+^h(s)), \dot\gamma_+^h(s) - \dot\gamma(s)\rangle; \tag{2.4.0.15}$$

$$f(s,\gamma_-^h(s),\dot\gamma_-^h(s))-f(s,\gamma_-^h(s),\dot\gamma(s))\le\langle D_vf(s,\gamma_-^h(s),\dot\gamma_-^h(s)),\dot\gamma_-^h(s)-\dot\gamma(s)\rangle, \qquad (2.4.0.16)$$

for all $s\in[t,\sigma_x]$. Using (2.4.0.13), (2.4.0.15) and (2.4.0.16) in (2.4.0.14), and the definition of $\gamma_\pm^h$ we get

$$\begin{aligned}&f(s,\gamma_+^h(s),\dot\gamma_+^h(s))+f(s,\gamma_-^h(s),\dot\gamma_-^h(s))-2f(s,\gamma(s),\dot\gamma(s))\\&\le\langle D_vf(s,\gamma_-^h(s),\dot\gamma_-^h(s))-D_vf(s,\gamma_+^h(s),\dot\gamma_+^h(s)),\frac{h}{\sigma_x-t}\rangle+c|h|^2,\end{aligned}$$

for all $s\in[t,\sigma_x]$. By our assumptions on f, for all $s\in[t,\sigma_x]$ one has that

$$\begin{aligned}&\langle D_vf(s,\gamma_-^h(s),\dot\gamma_-^h(s))-D_vf(s,\gamma_+^h(s),\dot\gamma_+^h(s)),\frac{h}{\sigma_x-t}\rangle\\&\le\frac{4(L^\star+1)}{R}|D_vf(s,\gamma_-^h(s),\dot\gamma_-^h(s))-D_vf(s,\gamma_+^h(s),\dot\gamma_+^h(s))||h|\\&\le C\frac{4(L^\star+1)}{R}\Big[|\gamma_-^h(s)-\gamma_+^h|+|\dot\gamma_-^h(s)-\dot\gamma_+^h(s)|\Big]|h|\\&=C\frac{4(L^\star+1)}{R}\Big[2|h|\Big(1-\frac{s-t}{\sigma_x-t}\Big)+\frac{2|h|}{\sigma_x-t}\Big]|h|\\&\le 8C\frac{L^\star+1}{R}\Big[1+\frac{4(L^\star+1)}{R}\Big]|h|^2.\end{aligned}$$

Therefore, we deduce that

$$f(s,\gamma_+^h(s),\dot\gamma_+^h(s))+f(s,\gamma_-^h(s),\dot\gamma_-^h(s))-2f(s,\gamma(s),\dot\gamma(s))\le C|h|^2,\ \ \forall\, s\in[t,\sigma_x] \qquad (2.4.0.17)$$

where the constant C depends only on $L^\star$ and R. Hence, we conclude that

$$u(t,x+h)+u(t,x-h)-u(t,x)\le C_{L^\star,R}|h|^2,$$

where $C_{L^\star,R}$ depends only on $L^\star$ and R. Thus, u is locally semiconcave with linear modulus in Ω. □

2.5 SENSITIVITY RELATIONS AND SEMICONCAVITY ESTIMATE

In our setting, the sensitivity relations can be stated as follows.

THEOREM 2.5.0.1. For any $\varepsilon\ge 0$ there exists a constant $c_\varepsilon\ge 0$ such that for any $(t,x)\in[0,T-\varepsilon]\times\overline{\Omega}$ and for any $\gamma\in\mathcal{X}_t[x]$, denoting by $p\in \mathrm{Lip}(t,T,\mathbb{R}^n)$ a dual arc associated with γ, one has that

$$u(t+\sigma,x+h)-u(t,x)\le\sigma H(t,x,p(t))+\langle p(t),h\rangle+c_\varepsilon(|h|+|\sigma|)^{\frac32}\quad\forall(t,x)\in[0,T-\epsilon]\times\overline{\Omega},$$

for all $h\in\mathbb{R}^n$ small enough such that $x+h\in\overline{\Omega}$, and for all $\sigma\in\mathbb{R}$ such that $0\le t+\sigma\le T-\varepsilon$.

COROLLARY 2.5.0.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. Let $(t,x) \in [0,T) \times \overline{\Omega}$. Let $\gamma \in \mathcal{X}_t[x]$ and let $p \in \mathrm{Lip}(t,T;\mathbb{R}^n)$ be a dual arc associated with γ. Then,

$$\Big(p(s), H(s,\gamma(s),p(s))\Big) \in D^+u(s,\gamma(s)) \quad \forall\, s \in [t,T]. \tag{2.5.0.1}$$

A direct consequence of THEOREM 2.5.0.1 is that u is a semiconcave function.

COROLLARY 2.5.0.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. The value function (2.4.0.4) is locally semiconcave with modulus $\omega(r) = Cr^{\frac{1}{2}}$ in $(0,T) \times \overline{\Omega}$.

Proof. Let $\varepsilon \geq 0$ and let $(t,x) \in [0,T-\varepsilon] \times \partial\Omega$. Let $\gamma \in \mathcal{X}_t[x]$ and let $p \in \mathrm{Lip}(t,T;\mathbb{R}^n)$ be a dual arc assosiated with γ. Let $h \in \mathbb{R}^n$ be such that $x+h,\ x-h \in \overline{\Omega}$. Let $\sigma > 0$ be such that $0 \leq t-\sigma \leq t \leq t+\sigma \leq T-\varepsilon$. By THEOREM 2.5.0.1, there exists a constant $c_\varepsilon \geq 0$ such that

$$\begin{aligned} &\frac{1}{2}u(t+\sigma,x+h) + \frac{1}{2}u(t-\sigma,x-h) - u(t,x) \leq \frac{1}{2}\Big[u(t,x) + \langle p(t),h\rangle + \sigma H(t,x,p(t))\Big] \\ &+ \frac{1}{2}\Big[u(t,x) - \langle p(t),h\rangle - \sigma H(t,x,p(t))\Big] + c_\varepsilon(|h|+\sigma)^{\frac{3}{2}} - u(t,x) \\ &= c_\varepsilon(|h|+\sigma)^{\frac{3}{2}}. \end{aligned} \tag{2.5.0.2}$$

Inequality (2.5.0.2) yields (1.3.0.1) for $\lambda = \frac{1}{2}$. By [24, Theorem 2.1.10] this is enough to conclude that u is semiconcave, because u is continuous on $(0,T) \times \overline{\Omega}$. □

2.5.1 PROOF OF THEOREM 2.5.0.1

It is convenient to divide the proof of THEOREM 2.5.0.1 in several lemmas. First, we show that u is semiconcave with modulus $\omega(r) = Cr^{\frac{1}{2}}$ in $\overline{\Omega}$.

LEMMA 2.5.1.1. For any $\varepsilon > 0$ there exists a constant $c_\varepsilon \geq 0$ such that for any $(t,x) \in [0,T-\varepsilon] \times \overline{\Omega}$ and for any $\gamma \in \mathcal{X}_t[x]$, denoting by $p \in \mathrm{Lip}(t,T;\mathbb{R}^n)$ a dual arc associated with γ, one has that

$$u(t,x+h) - u(t,x) - \langle p(t),h\rangle \leq c_\varepsilon |h|^{\frac{3}{2}}, \tag{2.5.1.1}$$

for all $h \in \mathbb{R}^n$ small enough such that $x+h \in \overline{\Omega}$.

Proof. If $(t,x) \in [0,T) \times \Omega$, it is known that (2.5.1.1) holds. Let $\varepsilon \geq 0$ and let $(t,x) \in [0,T-\varepsilon] \times \partial\Omega$. Let $\gamma \in \mathcal{X}_t[x]$ and let $p \in \mathrm{Lip}(t,T;\mathbb{R}^n)$ be a dual arc assosiated with γ. Let $h \in \mathbb{R}^n$ be small enough and such that $x+h \in \overline{\Omega}$. Now, we want to prove the following estimate

$$u(t,x+h) - u(t,x) - \langle p(t),h\rangle \leq c_\varepsilon |h|^{\frac{3}{2}}, \tag{2.5.1.2}$$

where $c_\varepsilon \geq 0$ is a uniform constant that depends only on ε.

Let $r \geq 0$ be such that $0 \leq t+r \leq T-\varepsilon$. Suppose that $\frac{h}{r}$ is fixed, we denote by γ_h the trajectory defined by

$$\gamma_h(s) = \begin{cases} \gamma(s) + \left(1 + \frac{t-s}{r}\right)_+ h & s \in [t, t+r], \\ \gamma(s) & s \in [t+r, T]. \end{cases}$$

We observe that $d_\Omega(\gamma_h(s)) \leq \rho_0$ for all $s \in [t, t+r]$. Indeed,

$$d_\Omega(\gamma_h(s)) \leq |\gamma_h(s) - \gamma(s)| \leq \left|\left(1 + \frac{t-s}{r}\right)_+ h\right|.$$

Thus, we have that $d_\Omega(\gamma_h(s)) \leq \rho_0$ for all $s \in [t, t+r]$ and for h small enough.
Denote by $\widehat{\gamma}_h$ the projection of γ_h on $\overline{\Omega}$, i.e.,

$$\widehat{\gamma}_h(s) = \gamma_h(s) - d_\Omega(\gamma_h(s)) Db_\Omega(\gamma_h(s)) \ \ \forall\, s \in [t, t+r].$$

By construction $\widehat{\gamma}_h \in AC(0, T; \mathbb{R}^n)$ and for $s = t$ one has that $\widehat{\gamma}_h(t) = x + h$. Moreover,

$$|\widehat{\gamma}_h(s) - \gamma(s)| \leq 4|h|, \quad \forall s \in [t, t+r]. \tag{2.5.1.3}$$

Indeed,

$$\begin{aligned} \left|\widehat{\gamma}_h(s) - \gamma(s)\right| = \left|\gamma_h(s) - d_\Omega(\gamma_h(s)) Db_\Omega(\gamma_h(s)) - \gamma(s)\right| &\leq 2|h| + d_\Omega(\gamma_h(s)) \\ \leq 2|h| + |\gamma_h(s) - \gamma(s)| \leq 4|h|, \end{aligned}$$

for all $s \in [t, t+r]$. Furthemore, recalling LEMMA 2.1.0.1, we have that

$$\begin{aligned} \dot{\widehat{\gamma}}_h(s) =& \dot{\gamma}(s) - \frac{h}{r} - \langle Db_\Omega(\gamma_h(s)), \dot{\gamma}(s) - \frac{h}{r} \rangle Db_\Omega(\gamma_h(s)) \mathbf{1}_{\Omega^c}(\gamma_h(s)) \\ & - d_\Omega(\gamma_h(s)) D^2 b_\Omega(\gamma_h(s)) \Big(\dot{\gamma}(s) - \frac{h}{r}\Big), \end{aligned} \tag{2.5.1.4}$$

for a.e. $s \in [t, t+r]$.
Since γ is an optimal trajectory for u at (t, x), by the dynamic programming principle, and by the definition of $\widehat{\gamma}_h$ we have that

$$\begin{aligned} u(t, x+h) - u(t, x) - \langle p(t), h \rangle &\leq \int_t^{t+r} f(s, \widehat{\gamma}_h(s), \dot{\widehat{\gamma}}_h(s))\, ds + u(t+r, \overbrace{\widehat{\gamma}_h(t+r)}^{=\gamma(t+r)}) \\ - \int_t^{t+r} & f(s, \gamma(s), \dot{\gamma}(s))\, ds - u(t+r, \gamma(t+r)) - \langle p(t), h \rangle \\ = \int_t^{t+r} & \Big[f(s, \widehat{\gamma}_h(s), \dot{\widehat{\gamma}}_h(s)) - f(s, \gamma(s), \dot{\gamma}(s))\Big]\, ds - \langle p(t), h \rangle. \end{aligned} \tag{2.5.1.5}$$

Integrating by parts, $\langle p(t), h \rangle$ can be rewritten as

$$-\langle p(t), h \rangle = -\big\langle p(t+r), \overbrace{\widehat{\gamma}_h(t+r) - \gamma(t+r)}^{=0} \big\rangle + \int_t^{t+r} \frac{d}{ds}\Big[\langle p(s), \widehat{\gamma}_h(s) - \gamma(s) \rangle\Big]\, ds$$

$$= \int_t^{t+r} \langle \dot{p}(s), \widehat{\gamma}_h(s) - \gamma(s) \rangle \, ds + \int_t^{t+r} \langle p(s), \dot{\widehat{\gamma}}_h(s) - \dot{\gamma}(s) \rangle \, ds.$$

Recalling that p satisfies (2.4.0.2) and (2.4.0.3), we deduce that

$$-\langle p(t), h \rangle = -\int_t^{t+r} \Big[\big\langle D_x f(s, \gamma(s), \dot{\gamma}(s)), \widehat{\gamma}_h(s) - \gamma(s) \big\rangle + \Lambda(s) \big\langle Db_\Omega(\gamma(s)), \widehat{\gamma}_h(s) - \gamma(s) \big\rangle \Big] \, ds$$
$$- \int_t^{t+r} \big\langle D_v f(s, \gamma(s), \dot{\gamma}(s)), \dot{\widehat{\gamma}}_h(s) - \dot{\gamma}(s) \big\rangle \, ds \tag{2.5.1.6}$$

Therefore, using (2.5.1.6), (2.5.1.5) can be rewritten as

$$u(t, x+h) - u(t, x) - \langle p(t), h \rangle \leq$$
$$\int_t^{t+r} \Big[f(s, \widehat{\gamma}_h(s), \dot{\widehat{\gamma}}_h(s)) - f(s, \gamma(s), \dot{\widehat{\gamma}}_h(s)) - \big\langle D_x f(s, \gamma(s), \dot{\widehat{\gamma}}_h(s)), \widehat{\gamma}_h(s) - \gamma(s) \big\rangle \Big] \, ds$$
$$+ \int_t^{t+r} \Big[f(s, \gamma(s), \dot{\widehat{\gamma}}_h(s)) - f(s, \gamma(s), \dot{\gamma}(s)) - \big\langle D_v f(s, \gamma(s), \dot{\gamma}(s)), \dot{\widehat{\gamma}}_h(s) - \dot{\gamma}(s) \big\rangle \Big] \, ds$$
$$+ \int_t^{t+r} \big\langle D_x f(s, \gamma(s), \dot{\widehat{\gamma}}_h(s)) - D_x f(s, \gamma(s), \dot{\gamma}(s)), \widehat{\gamma}_h(s) - \gamma(s) \big\rangle \, ds \tag{2.5.1.7}$$
$$- \int_t^{t+r} \Lambda(s) \big\langle Db_\Omega(\gamma(s)), \widehat{\gamma}_h(s) - \gamma(s) \big\rangle \, ds.$$

Using the assumptions (f1), (f3) and (f4) in (2.5.1.7) we have that

$$u(t, x+h) - u(t, x) - \langle p(t), h \rangle \leq c \int_t^{t+r} \left| \widehat{\gamma}_h(s) - \gamma(s) \right|^2 ds + c \int_t^{t+r} \left| \dot{\widehat{\gamma}}_h(s) - \dot{\gamma}(s) \right|^2 ds$$
$$+ C(R) \int_t^{t+r} \left| \dot{\widehat{\gamma}}_h(s) - \dot{\gamma}(s) \right| \left| \widehat{\gamma}_h(s) - \gamma(s) \right| ds - \int_t^{t+r} \Lambda(s) \big\langle Db_\Omega(\gamma(s)), \widehat{\gamma}_h(s) - \gamma(s) \big\rangle \, ds,$$

for some constant $c \geq 0$. By (2.5.1.3) we observe that

$$\int_t^{t+r} \left| \widehat{\gamma}_h(s) - \gamma(s) \right|^2 ds \leq 4r|h|^2. \tag{2.5.1.8}$$

Moreover, recalling (2.5.1.4) one has that

$$\int_t^{t+r} \left| \dot{\widehat{\gamma}}_h(s) - \dot{\gamma}(s) \right|^2 ds \leq \frac{|h|^2}{r} + \int_t^{t+r} \Big\langle Db_\Omega(\gamma_h(s)), \dot{\gamma}(s) - \frac{h}{r} \Big\rangle^2 \mathbf{1}_{\Omega^c}(\gamma_h(s)) \, ds$$
$$+ \int_t^{t+r} 2 \Big\langle Db_\Omega(\gamma_h(s)), \frac{h}{r} \Big\rangle \Big\langle Db_\Omega(\gamma_h(s)), \dot{\gamma}(s) - \frac{h}{r} \Big\rangle \mathbf{1}_{\Omega^c}(\gamma_h(s)) \, ds$$
$$+ \int_t^{t+r} \Big[\Big| d_\Omega(\gamma_h(s)) D^2 b_\Omega(\gamma_h(s)) \Big(\dot{\gamma}(s) - \frac{h}{r} \Big) \Big|^2 + 2 d_\Omega(\gamma_h(s)) \Big\langle D^2 b_\Omega(\gamma_h(s)) \Big(\dot{\gamma}(s) - \frac{h}{r} \Big), \frac{h}{r} \Big\rangle \Big] \, ds$$
$$+ 2 \int_t^{t+r} d_\Omega(\gamma_h(s)) \Big\langle D^2 b_\Omega(\gamma_h(s)) \Big(\dot{\gamma}(s) - \frac{h}{r} \Big), Db_\Omega(\gamma_h(s)) \Big\rangle \Big\langle Db_\Omega(\gamma_h(s)), \dot{\gamma}(s) - \frac{h}{r} \Big\rangle \mathbf{1}_{\Omega^c}(\gamma_h(s)) \, ds.$$

By LEMMA 2.1.0.1 we obtain that

$$\int_t^{t+r} \Big[\Big\langle Db_\Omega(\gamma_h(s)), \dot{\gamma}(s) - \frac{h}{r} \Big\rangle^2 \mathbf{1}_{\Omega^c}(\gamma_h(s)) \, ds$$

$$
\begin{aligned}
&+ 2\int_t^{t+r} \Big\langle Db_\Omega(\gamma_h(s)), \frac{h}{r}\Big\rangle \Big\langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) - \frac{h}{r}\Big\rangle \mathbf{1}_{\Omega^c}(\gamma_h(s))\Big]\, ds \\
&= \int_t^{t+r} \Big\langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) - \frac{h}{r}\Big\rangle \Big\langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) + \frac{h}{r}\Big\rangle \mathbf{1}_{\Omega^c}(\gamma_h(s))\, ds \\
&= \int_t^{t+r} \frac{d}{ds}\Big[d_\Omega(\gamma_h(s))\Big] \langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) + \frac{h}{r}\rangle \mathbf{1}_{\Omega^c}(\gamma_h(s))\, ds.
\end{aligned}
$$

Recalling that $\gamma_h(t)$, $\gamma_h(t+r) \in \overline{\Omega}$, we observe that

$$
\Big\{ s \in\ [t, t+r] : \gamma_h(s) \in \overline{\Omega}^c \Big\} = \Big\{ s \in\ (t, t+r) : \gamma_h(s) \in \overline{\Omega}^c \Big\} = \bigcup_{i\in\mathbb{N}} (s_i, t_i),
$$

where $(s_i, t_i) \cap (s_j, t_j) = \emptyset$ for all $i \neq j$. Hence,

$$
\begin{aligned}
&\int_t^{t+r} \frac{d}{ds}\Big[d_\Omega(\gamma_h(s))\Big] \Big\langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) + \frac{h}{r}\Big\rangle \mathbf{1}_{\Omega^c}(\gamma_h(s))\, ds \\
&= \sum_{i\in\mathbb{N}} \int_{s_i}^{t_i} \frac{d}{ds}\Big[d_\Omega(\gamma_h(s))\Big] \Big\langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) + \frac{h}{r}\Big\rangle\, ds.
\end{aligned}
$$

Integrating by parts, we get

$$
\begin{aligned}
&\sum_{i\in\mathbb{N}} \int_{s_i}^{t_i} \frac{d}{ds}\Big[d_\Omega(\gamma_h(s))\Big] \Big\langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) + \frac{h}{r}\Big\rangle\, ds = \sum_{i\in\mathbb{N}} \Big[d_\Omega(\gamma_h(s)) \Big\langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) + \frac{h}{r}\Big\rangle\Big]\Big|_{s_i}^{t_i} \\
&- \sum_{i\in\mathbb{N}} \int_{s_i}^{t_i} d_\Omega(\gamma_h(s)) \frac{d}{ds}\Big[\Big\langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) + \frac{h}{r}\Big\rangle\Big]\, ds.
\end{aligned}
$$

Owing to $d_\Omega(\gamma_h(s_i)) = d_\Omega(\gamma_h(t_i)) = 0$ for $i \in \mathbb{N}$, $d_\Omega(\gamma_h(t+r)) = d_\Omega(\gamma(t+r)) = 0$ and $d_\Omega(\gamma_h(t)) = d_\Omega(\gamma(t)) = 0$, one has that

$$
\sum_{i\in\mathbb{N}} \Big[d_\Omega(\gamma_h(s)) \Big\langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) + \frac{h}{r}\Big\rangle\Big]\Big|_{s_i}^{t_i} = 0. \tag{2.5.1.9}
$$

Since $\frac{h}{r}$ is fixed then, recalling that $\gamma \in C^{1,1}([0,T]; \overline{\Omega})$, one has that

$$
\frac{d}{ds}\Big[\Big\langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) + \frac{h}{r}\Big\rangle\Big] \leq C,
$$

where the constant C is not dependent on h. Hence, we deduce that

$$
\left|\sum_{i\in\mathbb{N}} \int_{s_i}^{t_i} d_\Omega(\gamma_h(s)) \frac{d}{ds}\Big[\Big\langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) + \frac{h}{r}\Big\rangle\Big]\, ds\right| \leq C|h|r,
$$

and so

$$
\int_t^{t+r} \frac{d}{ds}\Big[d_\Omega(\gamma_h(s))\Big] \Big\langle Db_\Omega(\gamma_h(s)), \dot\gamma(s) + \frac{h}{r}\Big\rangle \mathbf{1}_{\Omega^c}(\gamma_h(s))\, ds \leq C|h|r. \tag{2.5.1.10}
$$

Moreover, we have that

$$\int_t^{t+r} \Big|d_\Omega(\gamma_h(s))D^2b_\Omega(\gamma_h(s))\Big(\dot\gamma(s)-\frac{h}{r}\Big)\Big|^2 ds \le C\int_t^{t+r}\Big|d_\Omega(\gamma_h(s))\Big|^2\Big|\dot\gamma(s)-\frac{h}{r}\Big|^2 ds$$
$$\le C\left[r|h|^2+\frac{|h|^4}{r}+|h|^3\right],$$

and

$$\int_t^{t+r} d_\Omega(\gamma_h(s))\Big\langle D^2b_\Omega(\gamma_h(s))\Big(\dot\gamma(s)-\frac{h}{r}\Big),\frac{h}{r}\Big\rangle ds \le C\left(|h|^2+\frac{|h|^3}{r}\right),$$

for some constant $C\ge 0$ independent on h and r. Since $\langle D^2b_\Omega(x), Db_\Omega(x)\rangle = 0\ \forall x\in\mathbb{R}^n$ one has that

$$\int_t^{t+r} d_\Omega(\gamma_h(s))\Big\langle D^2b_\Omega(\gamma_h(s))\Big(\dot\gamma(s)-\frac{h}{r}\Big), Db_\Omega(\gamma_h(s))\Big\rangle\Big\langle Db_\Omega(\gamma_h(s)),\dot\gamma(s)-\frac{h}{r}\Big\rangle \mathbf{1}_{\Omega^c}(\gamma_h(s))\, ds = 0.$$

Hence,

$$\int_t^{t+r}\Big|\dot{\widehat\gamma}_h(s)-\dot\gamma(s)\Big|^2 ds \le c\left[\frac{|h|^2}{r}+r|h|^2+|h|^2+|h|r+\frac{|h|^4}{r}+|h|^3+\frac{|h|^3}{r}\right]. \tag{2.5.1.11}$$

Moreover, using Young's inequality, (2.5.1.11) and (2.5.1.8), we deduce that

$$\int_t^{t+r}|\dot{\widehat\gamma}_h(s)-\dot\gamma(s))||\widehat\gamma_h(s)-\gamma(s)|\,ds \le \frac12\int_t^{t+r}|\dot{\widehat\gamma}_h(s)-\dot\gamma(s))|^2 ds+\frac12\int_t^{t+r}|\widehat\gamma_h(s)-\gamma(s)|^2 ds$$
$$\le \frac12 c\Big(\frac{|h|^2}{r}+r|h|^2+|h|^2+|h|r+\frac{|h|^4}{r}+|h|^3+\frac{|h|^3}{r}\Big), \tag{2.5.1.12}$$

where c is a constant independent on h and r. Moreover, since

$$\int_t^{t+r}\Lambda(s)\langle Db_\Omega(\gamma(s)),\widehat\gamma_h(s)-\gamma(s)\rangle\,ds\le r|h|,$$

and using (2.5.1.11) and (2.5.1.12) we have that

$$u(t,x+h)-u(t,x)-\langle p(t),h\rangle \le c\Big(\frac{|h|^2}{r}+r|h|^2+|h|^2+r|h|+\frac{|h|^4}{r}+|h|^3+\frac{|h|^3}{r}\Big). \tag{2.5.1.13}$$

Thus, choosing $r=|h|^{\frac12}$ in (2.5.1.13), we conclude that (2.5.1.2) holds. This completes the proof.

□

LEMMA 2.5.1.2. For any $\varepsilon\ge 0$ there exists a constant $c_\varepsilon \ge 0$ such that for all $(t,x)\in[0,T-\varepsilon]\times\overline{\Omega}$ and for all $\gamma\in\mathcal{X}_t[x]$, denoting by $p\in \mathrm{Lip}(t,T;\mathbb{R}^n)$ a dual arc associated with γ, one has that

$$u(t+\sigma,x+h)-u(t,x)\le\langle p(t),h\rangle+\sigma H(t,x,p(t))+c_\varepsilon(|h|+|\sigma|)^{\frac32},$$

for any $h\in\mathbb{R}^n$ small enough such that $x+h\in\overline{\Omega}$, and for any $\sigma>0$ such that $0\le t+\sigma\le T-\varepsilon$.

Proof. Let $\varepsilon \geq 0$ and let $(t,x) \in [0, T-\varepsilon] \times \overline{\Omega}$. Let $\sigma > 0$ be such that $0 \leq t \leq t+\sigma \leq T-\varepsilon$ and let $h \in \mathbb{R}^n$ be small enough and such that $x+h \in \overline{\Omega}$. Let $\gamma \in \mathcal{X}_t[x]$ and let $p \in \mathrm{Lip}(t,T;\mathbb{R}^n)$ be a dual arc associated with γ. By dynamical programming principle one has that

$$u(t+\sigma, x+h) - u(t,x) = u(t+\sigma, x+h) - u(t+\sigma, \gamma(t+\sigma)) - \int_t^{t+\sigma} f(s,\gamma(s),\dot\gamma(s))\,ds.$$

By LEMMA 2.5.1.1 there exists a constant $c_\varepsilon \geq 0$ such that

$$u(t+\sigma, x+h) - u(t+\sigma, \gamma(t+\sigma)) \leq \langle p(t+\sigma), x+h-\gamma(t+\sigma)\rangle + c_\varepsilon\big(\big|x+h-\gamma(t+\sigma)\big|\big)^{\frac{3}{2}}. \quad (2.5.1.14)$$

By REMARK 2.4.0.1, we have that

$$\big|x+h-\gamma(t+\sigma)\big| \leq |h| + \big|x-\gamma(t+\sigma)\big| = |h| + \left|\int_t^{t+\sigma} \dot\gamma(s)\,ds\right| \leq |h| + L^\star|\sigma|. \quad (2.5.1.15)$$

Since $\gamma \in C^{1,1}([0,T];\overline{\Omega})$, $p \in \mathrm{Lip}(t,T;\mathbb{R}^n)$, we deduce that

$$\begin{aligned}
&\big\langle p(t+\sigma), x+h-\gamma(t+\sigma)\big\rangle = \langle p(t+\sigma), h\rangle + \langle p(t+\sigma), \gamma(t) - \gamma(t+\sigma)\rangle \\
&= \langle p(t+\sigma) - p(t), h\rangle + \langle p(t), h\rangle + \int_{t+\sigma}^{t} \langle p(t+\sigma), \dot\gamma(s)\rangle\,ds \\
&\leq \mathrm{Lip}(p)|\sigma||h| + \langle p(t), h\rangle - \int_t^{t+\sigma} \langle p(s), \dot\gamma(s)\rangle\,ds + \mathrm{Lip}(p)|\sigma|^2.
\end{aligned} \quad (2.5.1.16)$$

Using (2.5.1.15) and (2.5.1.16) in (2.5.1.14) one has that

$$\begin{aligned}
u(t+\sigma, x+h) - u(t,x) \leq \langle p(t), h\rangle - \int_t^{t+\sigma} \big[f(s,\gamma(s),\dot\gamma(s)) + \langle p(s), \dot\gamma(s)\rangle\big]\,ds + \mathrm{Lip}(p)|\sigma||h| \\
+ \mathrm{Lip}(p)|\sigma|^2 + c_\varepsilon(|h| + |\sigma|)^{\frac{3}{2}}.
\end{aligned} \quad (2.5.1.17)$$

By the definition of H we have that

$$-\int_t^{t+\sigma} \big[f(s,\gamma(s),\dot\gamma(s)) + \langle p(s), \dot\gamma(s)\rangle\big]\,ds = \int_t^{t+\sigma} H(s,\gamma(s),\dot\gamma(s))\,ds.$$

Since $\gamma \in C^{1,1}([0,T];\overline{\Omega})$, we get

$$\begin{aligned}
H(s,\gamma(s),\dot\gamma(s)) = H(t,\gamma(t),\dot\gamma(t)) + |s-t| + |\gamma(s)-\gamma(t)| + |\dot\gamma(s) - \dot\gamma(t)| \\
\leq |\sigma| + L^\star|\sigma| + C|\sigma|,
\end{aligned} \quad (2.5.1.18)$$

where C is a positive constant independent on h and σ.
Using SUBSECTION 2.5.1 in (2.5.1.17) we conclude that

$$u(t+\sigma, x+h) - u(t,x) \leq \langle p(t), h\rangle + \sigma H(t,x,p(t)) + c_\varepsilon(|h| + |\sigma|)^{\frac{3}{2}}.$$

This completes the proof. □

Lemma 2.5.1.3. For any $\varepsilon \geq 0$ there exists a constant $c_\varepsilon \geq 0$ such that for any $(t,x) \in [0, T-\varepsilon] \times \overline{\Omega}$ and for any $\gamma \in \mathcal{X}_t[x]$, denoting by $p \in \mathrm{Lip}(t,T;\mathbb{R}^n)$ a dual arc associated with γ, one has that

$$u(t-\sigma, x+h) - u(t,x) \leq \langle p(t), h\rangle - \sigma H(t,x,p(t)) + c_\varepsilon(|h|+|\sigma|)^{\frac{3}{2}}, \tag{2.5.1.19}$$

for any $h \in \mathbb{R}^n$ small enough such that $x+h \in \overline{\Omega}$, and for any $\sigma > 0$ such that $0 \leq t-\sigma \leq T-\varepsilon$.

Proof. Let $\varepsilon \geq 0$ and let $(t,x) \in [0,T-\varepsilon] \times \overline{\Omega}$. Let $\sigma > 0$ be such that $0 \leq t-\sigma \leq T-\sigma \leq T-\varepsilon$ and let $h \in \mathbb{R}^n$ be small enough and such that $x+h \in \overline{\Omega}$. Let $\gamma \in \mathcal{X}_t[x]$ and let $p \in \mathrm{Lip}(t,T;\mathbb{R}^n)$ be a dual arc associated with γ. We define the trajectory γ_σ as

$$\gamma_\sigma(s) = \begin{cases} \gamma(\sigma+s)+h & s \in [t-\sigma, T-\sigma], \\ \gamma(T)+h & s \in [T-\sigma, T]. \end{cases}$$

We denote by $\widehat{\gamma}_\sigma$ the projecton of γ_σ on $\overline{\Omega}$, i.e.,

$$\widehat{\gamma}_\sigma(s) = \gamma_\sigma(s) - d_\Omega(\gamma_\sigma(s))Db_\Omega(\gamma_\sigma(s)), \quad \forall s \in [t-\sigma, T]. \tag{2.5.1.20}$$

For all $s \in [t-\sigma, T]$ we have that

$$|\widehat{\gamma}_\sigma(s) - \gamma(s)| \leq C(\sigma + |h|), \tag{2.5.1.21}$$

where $C \geq 0$ independent on σ and h. Indeed, by the definition of $\widehat{\gamma}_\sigma$ one has that

$$\begin{aligned} |\widehat{\gamma}_\sigma(s) - \gamma(s)| &= \big|\gamma(s+\sigma) + h - d_\Omega(\gamma_\sigma(s))Db_\Omega(\gamma_\sigma(s)) - \gamma(s)\big| \leq \int_s^{s+\sigma} |\dot{\gamma}(s)|\,ds \\ &+ |\gamma_\sigma(s) - \gamma(s)| + |h| \leq C(\sigma+|h|), \end{aligned}$$

for all $s \in [t-\sigma, T]$.

Furthermore, recalling Lemma 2.1.0.1, we have that

$$\begin{aligned} \dot{\widehat{\gamma}}_\sigma(s) =& \dot{\gamma}_\sigma(s) - \big\langle Db_\Omega(\gamma_\sigma(s)), \dot{\gamma}_\sigma(s)\big\rangle Db_\Omega(\gamma_\sigma(s))\mathbf{1}_{\Omega^c}(\gamma_\sigma(s)) \\ &- d_\Omega(\gamma_\sigma(s))D^2 b_\Omega(\gamma_\sigma(s))\dot{\gamma}_\sigma(s), \end{aligned} \tag{2.5.1.22}$$

for all $s \in [t-\sigma, T]$. By the dynamical programming principle we obtain that

$$u(t-\sigma, x+h) - u(t,x) \leq \int_{t-\sigma}^t f(s, \widehat{\gamma}_\sigma(s), \dot{\widehat{\gamma}}_\sigma(s))\,ds + \big[u(t, \widehat{\gamma}_\sigma(t)) - u(t,x)\big]. \tag{2.5.1.23}$$

Now, we estimate the first term on the right-side of (2.5.1.23) in the following way

$$\begin{aligned} \int_{t-\sigma}^t f(s, \widehat{\gamma}_\sigma(s), \dot{\widehat{\gamma}}_\sigma(s))\,ds &= \int_{t-\sigma}^t \big[f(s, \widehat{\gamma}_\sigma(s), \dot{\widehat{\gamma}}_\sigma(s)) - f(s, \widehat{\gamma}_\sigma(s), \dot{\gamma}_\sigma(s))\big]\,ds \\ + \int_{t-\sigma}^t \big[f(s, \widehat{\gamma}_\sigma(s), \dot{\gamma}_\sigma(s)) &- f(s, \gamma(s+\sigma), \dot{\gamma}_\sigma(s))\big]\,ds + \int_{t-\sigma}^t f(s, \gamma(s+\sigma), \dot{\gamma}_\sigma(s))\,ds. \end{aligned}$$

Using the change of variables $r = s + \sigma$, and by REMARK 2.4.0.1 we get

$$\int_{t-\sigma}^{t} f(s,\gamma(s+\sigma),\dot{\gamma}_\sigma(s))\,ds = \int_{t}^{t+\sigma} f(r-\sigma,\gamma(r),\dot{\gamma}(r))\,dr \le \sigma f(t,x,\dot{\gamma}(t)) + C\sigma^2. \tag{2.5.1.24}$$

Moreover, by the regularity of f one has that

$$\int_{t-\sigma}^{t} \left[f(s,\widehat{\gamma}_\sigma(s),\dot{\widehat{\gamma}}_\sigma(s)) - f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s))\right] ds \le \int_{t-\sigma}^{t} \langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), \dot{\widehat{\gamma}}_\sigma(s) - \dot{\gamma}_\sigma(s)\rangle\, ds$$
$$+ C\int_{t-\sigma}^{t} |\dot{\widehat{\gamma}}_\sigma(s) - \dot{\gamma}_\sigma(s)|^2\, ds.$$

Recalling (2.5.1.22) we deduce that

$$\begin{aligned}
&\int_{t-\sigma}^{t} \langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), \dot{\widehat{\gamma}}_\sigma(s) - \dot{\gamma}_\sigma(s)\rangle\, ds \\
&= -\int_{t-\sigma}^{t} \langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), Db_\Omega(\gamma_\sigma(s))\rangle \langle Db_\Omega(\gamma_\sigma(s)), \dot{\gamma}_\sigma(s)\rangle \mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds \\
&- \int_{t-\sigma}^{t} d_\Omega(\gamma_\sigma(s)) \langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), D^2 b_\Omega(\gamma_\sigma(s))\dot{\gamma}_\sigma(s)\rangle\, ds.
\end{aligned} \tag{2.5.1.25}$$

Using the regularity of b_Ω and LEMMA 2.1.0.1, the first term on the right-side of (2.5.1.25) can be estimated in the following way

$$\begin{aligned}
&\left|\int_{t-\sigma}^{t} \langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), Db_\Omega(\gamma_\sigma(s))\rangle \langle Db_\Omega(\gamma_\sigma(s)), \dot{\gamma}_\sigma(s)\rangle \mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds\right| \\
&\le \left|\int_{t-\sigma}^{t} \langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), Db_\Omega(\gamma_\sigma(s))\rangle \langle Db_\Omega(\gamma_\sigma(s)) - Db_\Omega(\gamma(s+\sigma)), \dot{\gamma}_\sigma(s)\rangle \mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds\right| \\
&+ \left|\int_{t-\sigma}^{t} \langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), Db_\Omega(\gamma_\sigma(s))\rangle \langle Db_\Omega(\gamma(s+\sigma)), \dot{\gamma}_\sigma(s)\rangle \mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds\right| \\
&\le \int_{t-\sigma}^{t} |D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s))||Db_\Omega(\gamma_\sigma(s))||h||\dot{\gamma}(s+\sigma)|\mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds \\
&+ \left|\int_{t-\sigma}^{t} \langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), Db_\Omega(\gamma_\sigma(s))\rangle \mathbf{1}_{\Omega^c}(\gamma_\sigma(s)) \frac{d}{ds}\left[d_\Omega(\gamma(s+\sigma))\right] ds\right|.
\end{aligned}$$

By our assumptions on f, and by regularity of b_Ω one has that

$$\int_{t-\sigma}^{t} |D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s))||Db_\Omega(\gamma_\sigma(s))||h||\dot{\gamma}(s+\sigma)|\mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds \le C|h||\sigma|.$$

Integrating by parts, and since $\frac{d}{ds}\Big[d_\Omega(\gamma(s+\sigma))\Big] = 0$ we obtain that

$$\begin{aligned}
&\left|\int_{t-\sigma}^{t} \langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), Db_\Omega(\gamma_\sigma(s))\rangle \mathbf{1}_{\Omega^c}(\gamma_\sigma(s)) \frac{d}{ds}(d_\Omega(\gamma(s+\sigma)))\, ds\right| \\
&\le \left|\Big[\langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), Db_\Omega(\gamma_\sigma(s))\rangle \mathbf{1}_{\Omega^c}(\gamma_\sigma(s)) d_\Omega(\gamma(s+\sigma))\Big]\Big|_{t-\sigma}^{t}\right|
\end{aligned} \tag{2.5.1.26}$$

$$+\left|\int_{t-\sigma}^{t} d_\Omega(\gamma(s+\sigma))\frac{d}{ds}\Big[\langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), Db_\Omega(\gamma_\sigma(s))\rangle \mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\Big]\right| = 0.$$

Hence, we deduce that

$$\left|\int_{t-\sigma}^{t} \langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), Db_\Omega(\gamma_\sigma(s))\rangle\langle Db_\Omega(\gamma_\sigma(s)),\dot{\gamma}_\sigma(s)\rangle \mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds\right| \leq C|h|\sigma. \quad (2.5.1.27)$$

Moreover, the second term of (2.5.1.25) can be estimated as

$$\left|\int_{t-\sigma}^{t} d_\Omega(\gamma_\sigma(s))\langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), D^2 b_\Omega(\gamma_\sigma(s))\dot{\gamma}_\sigma(s)\rangle\, ds\right| \leq C|h|\sigma. \quad (2.5.1.28)$$

Using (2.5.1.27) and (2.5.1.28) in (2.5.1.25) one has that

$$\int_{t-\sigma}^{t} \big\langle D_v f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)), \dot{\widehat{\gamma}}_\sigma(s) - \dot{\gamma}_\sigma(s)\big\rangle\, ds \leq C|h|\sigma. \quad (2.5.1.29)$$

Furthermore, by (2.5.1.22) and by the regularity of b_Ω we obtain that

$$\begin{aligned}
&\int_{t-\sigma}^{t} \big|\dot{\widehat{\gamma}}_\sigma(s) - \dot{\gamma}_\sigma(s)\big|^2\, ds = \\
&\int_{t-\sigma}^{t} \Big| -\langle Db_\Omega(\gamma_\sigma(s)),\dot{\gamma}_\sigma(s)\rangle Db_\Omega(\gamma_\sigma(s))\mathbf{1}_{\Omega^c}(\gamma_\sigma(s)) - d_\Omega(\gamma_\sigma(s))D^2 b_\Omega(\gamma_\sigma(s))\dot{\gamma}_\sigma(s)\Big|^2\, ds \\
&= \int_{t-\sigma}^{t} \langle Db_\Omega(\gamma_\sigma(s)),\dot{\gamma}_\sigma(s)\rangle^2 \mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds + \int_{t-\sigma}^{t} d_\Omega^2(\gamma_\sigma(s))|D^2 b_\Omega(\gamma_\sigma(s))\dot{\gamma}_\sigma(s)|^2\, ds \\
&+ 2\int_{t-\sigma}^{t} d_\Omega(\gamma_\sigma(s))\langle Db_\Omega(\gamma_\sigma(s)),\dot{\gamma}_\sigma(s)\rangle\langle Db_\Omega(\gamma_\sigma(s)), D^2 b_\Omega(\gamma_\sigma(s))\dot{\gamma}_\sigma(s)\rangle \mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds \\
&\leq C(|h|+\sigma)^2 + \int_{t-\sigma}^{t} \langle Db_\Omega(\gamma_\sigma(s)),\dot{\gamma}_\sigma(s)\rangle^2 \mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds.
\end{aligned}$$

Arguing as in (2.5.1.26) we deduce that

$$\int_{t-\sigma}^{t} \langle Db_\Omega(\gamma_\sigma(s)),\dot{\gamma}_\sigma(s)\rangle^2 \mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds = 0,$$

and so

$$\int_{t-\sigma}^{t} \big|\dot{\widehat{\gamma}}_\sigma(s) - \dot{\gamma}_\sigma(s)\big|^2\, ds \leq C(|h|+\sigma)^2. \quad (2.5.1.30)$$

By (2.5.1.29) and (2.5.1.30) we conclude that

$$\int_{t-\sigma}^{t} \big[f(s,\widehat{\gamma}_\sigma(s),\dot{\widehat{\gamma}}_\sigma(s)) - f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s))\big]\, ds \leq C(|h|+\sigma)^2. \quad (2.5.1.31)$$

Moreover, by assumption (f4) we have that

$$\int_{t-\sigma}^{t} \big[f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)) - f(s,\gamma(s+\sigma),\dot{\gamma}_\sigma(s))\big]\, ds \leq C\int_{t-\sigma}^{t} |\widehat{\gamma}_\sigma(s) - \gamma(s+\sigma)|^2\, ds$$

$$+\int_{t-\sigma}^{t} \langle D_x f(s,\gamma(s+\sigma),\dot{\gamma}_\sigma(s)), \widehat{\gamma}_\sigma(s)-\gamma(s+\sigma)\rangle\, ds.$$

By the definition of $\widehat{\gamma}_\sigma$ we obtain that

$$\left|\int_{t-\sigma}^{t} \langle D_x f(s,\gamma(s+\sigma),\dot{\gamma}_\sigma(s)), h - d_\Omega(\gamma_\sigma(s))Db_\Omega(\gamma_\sigma(s))Db_\Omega(\gamma_\sigma(s))\rangle\, ds\right| \leq C|h|\sigma,$$

and by (2.5.1.21) one has that

$$\int_{t-\sigma}^{t} |\widehat{\gamma}_\sigma(s)-\gamma(s+\sigma)|^2\, ds \leq C(|h|+\sigma)^2.$$

Hence,

$$\int_{t-\sigma}^{t} \left[f(s,\widehat{\gamma}_\sigma(s),\dot{\gamma}_\sigma(s)) - f(s,\gamma(s+\sigma),\dot{\gamma}_\sigma(s))\right] ds \leq C(|h|+\sigma)^2. \tag{2.5.1.32}$$

Using (2.5.1.24), (2.5.1.31) and (2.5.1.32) we deduce that

$$\int_{t-\sigma}^{t} f(s,\widehat{\gamma}_\sigma(s),\dot{\widehat{\gamma}}_\sigma(s))\, ds \leq \sigma f(t,x,\dot{\gamma}(t)) + C(|h|+\sigma)^2. \tag{2.5.1.33}$$

We only have to estimate the second term in (2.5.1.23). By LEMMA 2.5.1.1 there exists a constant $c_\varepsilon \geq 0$ such that

$$u(t,\widehat{\gamma}_\sigma(t)) - u(t,x) \leq \langle p(t), \widehat{\gamma}_\sigma(t)-x\rangle + c_\varepsilon|\widehat{\gamma}_\sigma(t)-x|^{\frac{3}{2}}. \tag{2.5.1.34}$$

Moreover, by the definition of $\widehat{\gamma}_\sigma$ we have that

$$\widehat{\gamma}_\sigma(t) - x = \int_{t-\sigma}^{t} \dot{\widehat{\gamma}}_\sigma(s)\, ds + h. \tag{2.5.1.35}$$

Using (2.5.1.22), (2.5.1.35) and (2.5.1.21) in (2.5.1.34) we obtain that

$$\begin{aligned}
&u(t,\widehat{\gamma}_\sigma(t)) - u(t,x) \leq \langle p(t), \widehat{\gamma}_\sigma(s)-x\rangle + c_\varepsilon(\sigma+|h|)^{\frac{3}{2}} = \langle p(t), h\rangle \\
&+\int_{t-\sigma}^{t} \langle p(t), \dot{\gamma}(s+\sigma)\rangle\, ds - \int_{t-\sigma}^{t} \langle p(t), Db_\Omega(\gamma_\sigma(s))\rangle\langle Db_\Omega(\gamma_\sigma(s)), \dot{\gamma}_\sigma(s)\rangle \mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds \\
&-\int_{t-\sigma}^{t} d_\Omega(\gamma_\sigma(s))\langle p(t), D^2 b_\Omega(\gamma_\sigma(s))\dot{\gamma}_\sigma(s)\rangle\, ds + c_\varepsilon(|\sigma+|h|)^{\frac{3}{2}}.
\end{aligned}$$

Using the change of variables $r = s+\sigma$, and by REMARK 2.4.0.1 one has that

$$\int_{t-\sigma}^{t} \langle p(t), \dot{\gamma}(s+\sigma)\rangle\, ds = \int_{t}^{t+\sigma} \langle p(t), \dot{\gamma}(r)\rangle\, dr \leq \sigma\langle p(t), \dot{\gamma}(t)\rangle + c\sigma^2.$$

Arguing as in (2.5.1.26) we have that

$$\left|\int_{t-\sigma}^{t} \langle p(t), Db_\Omega(\gamma_\sigma(s))\rangle\langle Db_\Omega(\gamma_\sigma(s)), \dot{\gamma}_\sigma(s)\rangle \mathbf{1}_{\Omega^c}(\gamma_\sigma(s))\, ds\right| = 0.$$

Moreover,

$$\Big|\int_{t-\sigma}^{t} d_\Omega(\gamma_\sigma(s))\langle p(t), D^2 b_\Omega(\gamma_\sigma(s))\dot{\gamma}_\sigma(s)\rangle\, ds\Big| \leq C|\sigma||h|.$$

Hence, we deduce that

$$u(t,\widehat{\gamma}_\sigma(t)) - u(t,x) \leq \langle p(t), h\rangle + \sigma\langle p(t),\dot{\gamma}(t)\rangle + c_\varepsilon(|h|+\sigma)^{\frac{3}{2}}. \tag{2.5.1.36}$$

By the definition of H, using (2.5.1.33) and (2.5.1.36) in (2.5.1.23) we conclude that

$$u(t-\sigma, x+h) - u(t,x) \leq \langle p(t), h\rangle - \sigma H(t,x,p(t)) + c_\varepsilon(|h|+\sigma)^{\frac{3}{2}}. \tag{2.5.1.37}$$

This completes the proof.

□

We observe that THEOREM 2.5.0.1 is a direct consequence of LEMMA 2.5.1.2 and LEMMA 2.5.1.3.

Chapter 3

Existence and uniqueness for Mean Field Games with state constraints

In this Chapter, following the Lagrangian formulation of the unconstrained MFG problem proposed in [8], we define a notion of constrained MFG equilibria (Section 3.1) and mild solutions (Section 3.2), for which we give existence and uniqueness results.

3.1 Constrained MFG equilibria

3.1.1 Assumptions

Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let $\mathcal{P}(\overline{\Omega})$ be the set of all Borel probability measures on $\overline{\Omega}$ endowed with the Kantorovich-Rubinstein distance d_1 defined in (1.2.0.2). We suppose throughout this Chapter that $F, G : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ and $L : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ are given continuous functions. Moreover, we assume the following conditions.

(L1) $L \in C^1(\overline{\Omega} \times \mathbb{R}^n)$ and for all $(x, v) \in \overline{\Omega} \times \mathbb{R}^n$,

$$|D_x L(x, v)| \leq C(1 + |v|^2), \tag{3.1.1.1}$$

$$|D_v L(x,v)| \leq C(1+|v|), \tag{3.1.1.2}$$

for some constant $C > 0$.

(L2) There exist constants $c_1, c_0 > 0$ such that

$$L(x,v) \geq c_1|v|^2 - c_0, \quad \forall (x,v) \in \overline{\Omega} \times \mathbb{R}^n. \tag{3.1.1.3}$$

(L3) $v \longmapsto L(x,v)$ is convex for all $x \in \overline{\Omega}$.

REMARK 3.1.1.1. (i) As $\overline{\Omega} \times \mathcal{P}(\overline{\Omega})$ is a compact set, the continuity of F and G implies that they are bounded and uniformly continuous on $\overline{\Omega} \times \mathcal{P}(\overline{\Omega})$.
(ii) In (L1), L is assumed to be of class $C^1(\overline{\Omega} \times \mathbb{R}^n)$ just for simplicity. All the results of this chapter hold true if L is locally Lipschitz—hence, a.e. differentiable—in $\overline{\Omega} \times \mathbb{R}^n$ and satisfies the growth conditions (3.1.1.1) and (3.1.1.2) a.e. on $\overline{\Omega} \times \mathbb{R}^n$, see REMARK 3.1.2.7 below.

3.1.2 EXISTENCE OF CONSTRAINED MFG EQUILIBRIA

For any $t \in [0,T]$, we denote by $e_t : \Gamma \to \overline{\Omega}$ the evaluation map defined by

$$e_t(\gamma) = \gamma(t), \quad \forall \gamma \in \Gamma. \tag{3.1.2.1}$$

For any $\eta \in \mathcal{P}(\Gamma)$, we define

$$m^\eta(t) = e_t \sharp \eta, \quad \forall t \in [0,T]. \tag{3.1.2.2}$$

LEMMA 3.1.2.1. The following holds true.

(i) $m^\eta \in C([0,T]; \mathcal{P}(\overline{\Omega}))$ for any $\eta \in \mathcal{P}(\Gamma)$.

(ii) Let η_i, $\eta \in \mathcal{P}(\Gamma)$, $i \geq 1$, be such that η_i is narrowly convergent to η. Then $m^{\eta_i}(t)$ is narrowly convergent to $m^\eta(t)$ for all $t \in [0,T]$.

Proof. First, we prove point (i). By definition (3.1.2.2), it is obvious that $m^\eta(t)$ is a Borel probability measure on $\overline{\Omega}$ for any $t \in [0,T]$. Let $\{t_k\} \subset [0,T]$ be a sequence such that $t_k \to \bar{t}$. We want to show that

$$\lim_{t_k \to \bar{t}} \int_{\overline{\Omega}} f(x) m^\eta(t_k, dx) = \int_{\overline{\Omega}} f(x) m^\eta(\bar{t}, dx), \tag{3.1.2.3}$$

for any $f \in C(\overline{\Omega})$. Since $m^\eta(t_k) = e_{t_k}\sharp\eta$ and $e_{t_k}(\gamma) = \gamma(t_k)$, we have that

$$\lim_{t_k \to \bar{t}} \int_{\overline{\Omega}} f(x) m^\eta(t_k, dx) = \lim_{t_k \to \bar{t}} \int_\Gamma f(e_{t_k}(\gamma))\, d\eta(\gamma) = \lim_{t_k \to \bar{t}} \int_\Gamma f(\gamma(t_k))\, d\eta(\gamma).$$

Since $f \in C(\overline{\Omega})$ and $\gamma \in \Gamma$, then $f(\gamma(t_k)) \to f(\gamma(\bar{t}))$ and $|f(\gamma(t_k))| \leq ||f||_\infty$. Therefore, by Lebesgue's dominated convergence theorem, we have that

$$\lim_{t_k \to \bar{t}} \int_\Gamma f(\gamma(t_k))\, d\eta(\gamma) = \int_\Gamma f(\gamma(\bar{t}))\, d\eta(\gamma). \tag{3.1.2.4}$$

Thus, recalling the definition of m^η, we obtain (3.1.2.3). Moreover, by PROPOSITION 1.2.0.3, we conclude that $d_1(m^\eta(t_k), m^\eta(\bar{t})) \to 0$. This completes the proof of point (i).

In order to prove point (ii), we suppose that η_i is narrowly convergent to η. Then, for all $f \in C(\overline{\Omega})$ we have that

$$\lim_{i\to\infty} \int_{\overline{\Omega}} f(x) m^{\eta_i}(t,\, dx) = \lim_{i\to\infty} \int_\Gamma f(\gamma(t))\, d\eta_i(\gamma) = \int_\Gamma f(\gamma(t))\, d\eta(\gamma) = \int_{\overline{\Omega}} f(x) m^\eta(t,\, dx).$$

Hence, $m^{\eta_i}(t)$ is narrowly convergent to $m^\eta(t)$ for all $t \in [0,T]$. □

For any fixed $m_0 \in \mathcal{P}(\overline{\Omega})$, we denote by $\mathcal{P}_{m_0}(\Gamma)$ the set of all Borel probability measures η on Γ such that $e_0 \sharp \eta = m_0$. For all $\eta \in \mathcal{P}_{m_0}(\Gamma)$, we define

$$J_\eta[\gamma] = \int_0^T \Big[L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), m^\eta(t)) \Big]\, dt + G(\gamma(T), m^\eta(T)), \quad \gamma \in \Gamma. \tag{3.1.2.5}$$

REMARK 3.1.2.2. We note that $\mathcal{P}_{m_0}(\Gamma)$ is nonempty. Indeed, let $j : \overline{\Omega} \to \Gamma$ be the continuous map defined by

$$j(x)(t) = x \quad \forall t \in [0,T].$$

Then,

$$\eta := j \sharp m_0$$

is a Borel probability measure on Γ and $\eta \in \mathcal{P}_{m_0}(\Gamma)$.

For all $x \in \overline{\Omega}$ and $\eta \in \mathcal{P}_{m_0}(\Gamma)$, we define

$$\Gamma^\eta[x] = \left\{ \gamma \in \Gamma[x] : J_\eta[\gamma] = \min_{\Gamma[x]} J_\eta \right\}. \tag{3.1.2.6}$$

DEFINITION 3.1.2.3. Let $m_0 \in \mathcal{P}(\overline{\Omega})$. We say that $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a constrained MFG equilibrium for m_0 if

$$supp(\eta) \subseteq \bigcup_{x \in \overline{\Omega}} \Gamma^\eta[x]. \tag{3.1.2.7}$$

In other words, $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a constrained MFG equilibrium for m_0 if for η-a.e. $\overline{\gamma} \in \Gamma$ we have that

$$J_\eta[\overline{\gamma}] \leq J_\eta[\gamma], \qquad \forall \gamma \in \Gamma[\overline{\gamma}(0)].$$

The main result of this section is the existence of constrained MFG equilibria for m_0.

THEOREM 3.1.2.4. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary and let $m_0 \in \mathcal{P}(\overline{\Omega})$. Suppose that (L1)-(L3) hold true. Let $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ and $G : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ be continuous. Then, there exists at least one constrained MFG equilibrium for m_0.

THEOREM 3.1.2.4 will be proved in SUBSECTION 3.1.3. Now, we will show some properties of $\Gamma^\eta[x]$ that we will use in what follows.

LEMMA 3.1.2.5. For all $x \in \overline{\Omega}$ and $\eta \in \mathcal{P}_{m_0}(\Gamma)$ the following holds true.

(i) $\Gamma^\eta[x]$ is a nonempty set.

(ii) All $\gamma \in \Gamma^\eta[x]$ satisfy

$$||\dot{\gamma}||_2 \leq K, \tag{3.1.2.8}$$

where

$$K = \frac{1}{\sqrt{c_1}} \left[T \max_{\overline{\Omega}} L(x,0) + 2T \max_{\overline{\Omega}\times\mathcal{P}(\overline{\Omega})} |F| + 2 \max_{\overline{\Omega}\times\mathcal{P}(\overline{\Omega})} |G| + Tc_0 \right]^{\frac{1}{2}} \tag{3.1.2.9}$$

and c_0, c_1 are the constants in (3.1.1.3). Consequently, all minimizers $\gamma \in \Gamma^\eta[x]$ are $\frac{1}{2}$-Hölder continuous of constant K.

In addition, if $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a constrained MFG equilibrium for m_0, then $m^\eta(t) = e_t \sharp \eta$ is $\frac{1}{2}$-Hölder continuous of constant K.

Proof. By classical results in the calculus of variation (see, e.g., [24, Theorem 6.1.2]), there exists at least one mimimizer of $J_\eta[\cdot]$ on Γ for any fixed initial point $x \in \overline{\Omega}$. So $\Gamma^\eta[x]$ is a nonempty set.
Let $x \in \overline{\Omega}$ and let $\gamma \in \Gamma^\eta[x]$. By comparing the cost of γ with the cost of the constant trajectory $\gamma(0) \equiv x$, one has that

$$\begin{aligned}
&\int_0^T \Big[L(\gamma(t),\dot{\gamma}(t)) + F(\gamma(t), m^\eta(t)) \Big]\, dt + G(\gamma(T), m^\eta(T)) \\
&\leq \int_0^T \Big[L(x,0) + F(x, m^\eta(t)) \Big]\, dt + G(x, m^\eta(T)) \\
&\leq \Big[T \max_{\overline{\Omega}} L(x,0) + T \max_{\overline{\Omega}\times\mathcal{P}(\overline{\Omega})} |F| + \max_{\overline{\Omega}\times\mathcal{P}(\overline{\Omega})} |G| \Big].
\end{aligned} \tag{3.1.2.10}$$

Using (3.1.1.3) in ((3.1.2.10)), one has that

$$||\dot{\gamma}||_2 \leq \frac{1}{\sqrt{c_1}} \left[T \max_{\overline{\Omega}} L(x,0) + 2T \max_{\overline{\Omega}\times\mathcal{P}(\overline{\Omega})} |F| + 2 \max_{\overline{\Omega}\times\mathcal{P}(\overline{\Omega})} |G| + Tc_0 \right]^{\frac{1}{2}} = K, \tag{3.1.2.11}$$

where c_0, c_1 are the constants in (3.1.1.3). This completes the proof of point (ii) since the Hölder regularity of γ is a direct conseguence of the estimate (3.1.2.11).

Finally, we claim that, if η is a constrained MFG equilibrium for m_0, then the map $t \to m^\eta(t)$ is $\frac{1}{2}$-Hölder continuous with constant K. Indeed, for any $t_1, t_2 \in [0,T]$, we have that

$$d_1(m^\eta(t_2), m^\eta(t_1)) = \sup_\phi \int_{\overline{\Omega}} \phi(x)\,(m^\eta(t_2,dx) - m^\eta(t_1,dx)), \tag{3.1.2.12}$$

where the supremum is taken over the set of all 1-Lipschitz continuous maps $\phi : \overline{\Omega} \to \mathbb{R}$.
Since $m^\eta(t) = e_t \sharp \eta$ and the map ϕ is 1-Lipschitz continuous, one has that

$$\int_{\overline{\Omega}} \phi(x)\,(m^\eta(t_2,dx) - m^\eta(t_1,dx)) = \int_\Gamma \Big[\phi(e_{t_2}(\gamma)) - \phi(e_{t_1}(\gamma))\Big]\,d\eta(\gamma)$$
$$= \int_\Gamma \Big[\phi(\gamma(t_2)) - \phi(\gamma(t_1))\Big]\,d\eta(\gamma) \le \int_\Gamma |\gamma(t_2) - \gamma(t_1)|\,d\eta(\gamma).$$

Since η is a constrained MFG equilibrium for m_0, property (ii) yields

$$\int_\Gamma |\gamma(t_2) - \gamma(t_1)|\,d\eta(\gamma) \le K \int_\Gamma |t_2 - t_1|^{\frac{1}{2}}\,d\eta(\gamma) = K|t_2 - t_1|^{\frac{1}{2}}.$$

Hence, we conclude that

$$d_1(m^\eta(t_2), m^\eta(t_1)) \le K|t_2 - t_1|^{\frac{1}{2}}, \quad \forall\, t_1, t_2 \in [0,T]$$

and the map $t \longmapsto m^\eta(t)$ is 1/2-Hölder continuous. □

LEMMA 3.1.2.6. Let $\eta_i, \eta \in \mathcal{P}_{m_0}(\Gamma)$ be such that η_i narrowly converges to η. Let $x_i \in \overline{\Omega}$ be such that $x_i \to x$ and let $\gamma_i \in \Gamma^{\eta_i}[x_i]$ be such that $\gamma_i \to \overline{\gamma}$. Then $\overline{\gamma} \in \Gamma^\eta[x]$. Consequently, $\Gamma^\eta[\cdot]$ has closed graph.

Proof. We want to prove that

$$J_\eta[\overline{\gamma}] \le J_\eta[\gamma], \quad \forall \gamma \in \Gamma[x] \text{ such that } \int_0^T |\dot{\gamma}|^2\,dt < \infty. \tag{3.1.2.13}$$

We observe that the above request is not restrictive because, by assuption (L2), if $\int_0^T |\dot{\gamma}|^2\,dt = \infty$ then the above inequality is trivial.
Fix $\gamma \in \Gamma[x]$ with $\int_0^T |\dot{\gamma}|^2\,dt < \infty$, by PROPOSITION 2.1.0.2, we have that there exists $\widehat{\gamma}_i \in \Gamma[x_i]$ such that $\widehat{\gamma}_i \to \gamma$ uniformly on $[0,T]$, $\dot{\widehat{\gamma}}_i \to \dot{\gamma}$ a.e. on $[0,T]$ and $|\dot{\widehat{\gamma}}_i(t)| \le C|\dot{\gamma}(t)|$ for any $i \ge 1$, a.e. $t \in [0,T]$, and some constant $C \ge 0$. Since $\gamma_i \in \Gamma^{\eta_i}[x_i]$, one has that

$$J_{\eta_i}[\gamma_i] \le J_{\eta_i}[\widehat{\gamma}_i], \quad \forall i \ge 1. \tag{3.1.2.14}$$

So, in order to prove (3.1.2.13), we have to check that

(a) $J_\eta[\overline{\gamma}] \le \liminf_{i\to\infty} J_{\eta_i}[\gamma_i]$;

(b) $\lim_{i\to+\infty} J_{\eta_i}[\widehat{\gamma}_i] = J_\eta[\gamma]$.

First we show that (a) holds, that is,

$$\liminf_{i\to\infty}\left\{\int_0^T\Big[L\big(\gamma_i(t),\dot\gamma_i(t)\big)\big)+F\big(\gamma_i(t),m^{\eta_i}(t)\big)\Big]\,dt+G(\gamma_i(T),m^{\eta_i}(T))\right\}$$
$$\geq\int_0^T\big[L(\overline{\gamma}(t),\dot{\overline{\gamma}}(t))\big)+F\big(\overline{\gamma}(t),m^{\eta}(t)\big)\Big]\,dt+G(\overline{\gamma}(T),m^{\eta}(T)). \tag{3.1.2.15}$$

First of all, we recall that, by LEMMA 3.1.2.1, $m^{\eta_i}(t)$ narrowly converges to $m^\eta(t)$ for all $t\in[0,T]$. Owing to the convergence of γ_i to $\overline{\gamma}$, the narrow convergence of $m^{\eta_i}(t)$ to $m^\eta(t)$,our assumption on F and G, and recalling REMARK 3.1.1.1, we conclude that

$$\int_0^T F(\gamma_i(t),m^{\eta_i}(t))\,dt\xrightarrow{i\to\infty}\int_0^T F(\overline{\gamma}(t),m^{\eta}(t))\,dt,$$
$$G(\gamma_i(T),m^{\eta_i}(T))\xrightarrow{i\to\infty}G(\overline{\gamma}(T),m^{\eta}(T)).$$

Up to taking a subsequence of γ_i, we can assume that $\dot\gamma_i\rightharpoonup\dot{\overline{\gamma}}$ in $L^2(0,T;\mathbb{R}^n)$ without loss of generality. By assumption (L3), one has that

$$\int_0^T L(\gamma_i(t),\dot\gamma_i(t))=\int_0^T L(\overline{\gamma}(t),\dot\gamma_i(t))\,dt+\int_0^T\Big[L(\gamma_i(t),\dot\gamma_i(t))-L(\overline{\gamma}(t),\dot\gamma_i(t))\Big]\,dt$$
$$\geq\int_0^T\Big[L(\overline{\gamma}(t),\dot{\overline{\gamma}}(t))+\langle D_vL(\overline{\gamma}(t),\dot{\overline{\gamma}}(t)),\dot\gamma_i-\dot{\overline{\gamma}}\rangle\Big]\,dt+\int_0^T\Big[L(\gamma_i(t),\dot\gamma_i(t))-L(\overline{\gamma}(t),\dot\gamma_i(t))\Big]\,dt.$$

Since $\gamma_i\in\Gamma^{\eta_i}[x_i]$ and $\gamma_i\to\overline{\gamma}$, by (L1), we obtain

$$\int_0^T\Big[L(\gamma_i(t),\dot\gamma_i(t))-L(\overline{\gamma}(t),\dot\gamma_i(t))\Big]\,dt\xrightarrow{i\to\infty}0.$$

Moreover, since $\dot\gamma_i\rightharpoonup\dot{\overline{\gamma}}$ in $L^2(0,T;\mathbb{R}^n)$, one has that

$$\int_0^T\langle D_vL(\overline{\gamma}(t),\dot{\overline{\gamma}}(t)),\dot\gamma_i-\dot{\overline{\gamma}}\rangle\,dt\xrightarrow{i\to\infty}0.$$

Thus, (3.1.2.15) holds.
Finally, we prove (b), i.e.,

$$\lim_{i\to\infty}\left\{\int_0^T L(\widehat{\gamma}_i(t),\dot{\widehat{\gamma}}_i(t))+F(\widehat{\gamma}_i(t),m^{\eta_i}(t))\,dt+G(\widehat{\gamma}_i(T),m^{\eta_i}(T))\right\}$$
$$=\int_0^T L(\gamma(t),\dot\gamma(t))+F(\gamma(t),m^{\eta}(t))\,dt+G(\gamma(T),m^{\eta}(T)).$$

Owing to the convergence of $\widehat{\gamma}_i$ to γ, the narrow convergence of $m^{\eta_i}(t)$ to $m^\eta(t)$ for all $t\in[0,T]$, and our assumption on F and G, one has that

$$\int_0^T F(\widehat{\gamma}_i(t),m^{\eta_i}(t))\,dt\xrightarrow{i\to\infty}\int_0^T F(\gamma(t),m^{\eta}(t))\,dt,$$

$$G(\widehat{\gamma}_i(T), m^{\eta_i}(T)) \xrightarrow{i\to\infty} G(\gamma(T), m^{\eta}(T)).$$

Hence, we only need to prove that

$$\liminf_{i\to\infty} \int_0^T L(\widehat{\gamma}_i(t), \dot{\widehat{\gamma}}_i(t))\, dt = \int_0^T L(\gamma(t), \dot{\gamma}(t))\, dt. \tag{3.1.2.16}$$

Owing to (L1), one has that

$$\begin{aligned}
&\left| \int_0^T \left[L(\widehat{\gamma}_i(t), \dot{\widehat{\gamma}}_i(t)) - L(\gamma(t), \dot{\gamma}(t))\right] dt \right| \\
&\leq \int_0^T \left| L(\widehat{\gamma}_i(t), \dot{\widehat{\gamma}}_i(t)) - L(\gamma(t), \dot{\widehat{\gamma}}_i(t)) \right| dt + \int_0^T \left| L(\gamma(t), \dot{\widehat{\gamma}}_i(t)) - L(\gamma(t), \dot{\gamma}(t)) \right| dt \\
&\leq ||\widehat{\gamma}_i - \gamma||_\infty \int_0^T \left(1 + |\dot{\widehat{\gamma}}_i|^2\right) dt + \int_0^T \left| \int_0^1 \left\langle D_v L(\gamma(t), \lambda\dot{\widehat{\gamma}}_i + (1-\lambda)\dot{\gamma}), \dot{\widehat{\gamma}}_i(t) - \dot{\gamma}(t) \right\rangle d\lambda \right| dt \\
&\leq ||\widehat{\gamma}_i - \gamma||_\infty \int_0^T \left(1 + |\dot{\widehat{\gamma}}_i|^2\right) dt + C \int_0^T \int_0^1 \left(1 + |\dot{\widehat{\gamma}}_i| + |\dot{\gamma}|\right) |\dot{\widehat{\gamma}}_i(t) - \dot{\gamma}(t)|\, dt.
\end{aligned}$$

Since $\widehat{\gamma}_i \to \gamma$ uniformly on $[0,T]$ and $|\dot{\widehat{\gamma}}_i(t)| \leq C|\dot{\gamma}(t)|$ for any $i \geq 1$ and for any $t \in [0,T]$, we have that

$$||\widehat{\gamma}_i - \gamma||_\infty \int_0^T \left(1 + |\dot{\widehat{\gamma}}_i|^2\right) dt \xrightarrow{i\to\infty} 0.$$

In addition, since $\dot{\widehat{\gamma}}_i \to \dot{\gamma}$ a.e. on $[0,T]$, by Lebesgue's dominated convergence theorem we obtain

$$C \int_0^T \int_0^1 \left(1 + |\dot{\widehat{\gamma}}_i| + |\dot{\gamma}|\right) \left|\dot{\widehat{\gamma}}_i(t) - \dot{\gamma}(t)\right| dt \xrightarrow{i\to\infty} 0. \tag{3.1.2.17}$$

This gives (b) and the proof is complete. □

REMARK 3.1.2.7. The above proof can be adapted to treat the case of a locally Lipschitz Lagrangian L as was mentioned in REMARK 3.1.1.1. Indeed, it suffices to replace the gradient $D_v L(\overline{\gamma}(t), \dot{\overline{\gamma}}(t))$ with a measurable selection of the subdifferential $\partial_v L(\overline{\gamma}(t), \dot{\overline{\gamma}}(t))$.

3.1.3 PROOF OF THEOREM 3.1.2.4

In this section we prove THEOREM 3.1.2.4 using a fixed point argument. First of all, we recall that, by THEOREM 1.2.0.4, for any $\eta \in \mathcal{P}_{m_0}(\Gamma)$, there exists a unique Borel measurable family of probabilities $\{\eta_x\}_{x\in\overline{\Omega}}$ on Γ which disintegrates η in the sense that

$$\begin{cases} \eta(d\gamma) = \int_{\overline{\Omega}} \eta_x(d\gamma)\, dm_0(x), \\ supp(\eta_x) \subset \Gamma[x] \;\; m_0 - \text{a.e. } x \in \overline{\Omega}. \end{cases} \tag{3.1.3.1}$$

We introduce the set-valued map $E : \mathcal{P}_{m_0}(\Gamma) \rightrightarrows \mathcal{P}_{m_0}(\Gamma)$ by defining, for any $\eta \in \mathcal{P}_{m_0}(\Gamma)$,

$$E(\eta) = \left\{ \widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma) : supp(\widehat{\eta}_x) \subseteq \Gamma^{\eta}[x]\; m_0 - \text{a.e. } x \in \overline{\Omega} \right\}. \tag{3.1.3.2}$$

Then, it is immediate to realize that $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a constrained MFG equilibrium for m_0 if and only if $\eta \in E(\eta)$. We will therefore show that the set-valued map E has a fixed point. For this purpose, we will apply Kakutani's Theorem [55]. The following lemmas are intended to check that the assumptions of such a theorem are satisfied by E.

LEMMA 3.1.3.1. For any $\eta \in \mathcal{P}_{m_0}(\Gamma)$, $E(\eta)$ is a nonempty convex set.

Proof. First, we note that $E(\eta)$ is a nonempty set. Indeed, by (i) of LEMMA 3.1.2.5, LEMMA 3.1.2.6, and [7, Theorem 8.1.4] we have that $x \longmapsto \Gamma^\eta[x]$ is measurable. Moreover, by [7, Theorem 8.1.3], $x \longmapsto \Gamma^\eta[x]$ has a Borel measurable selection $x \longmapsto \gamma_x^\eta$. Thus, the measure $\widehat{\eta}$, defined by $\widehat{\eta}(\mathcal{B}) = \int_{\overline{\Omega}} \delta_{\{\gamma_x^\eta\}}(\mathcal{B})\, dm_0(x)$ for any $\mathcal{B} \in \mathcal{B}(\Gamma)$, belongs to $E(\eta)$.
Now we want to check that $E(\eta)$ is a convex set. Let $\{\eta_i\}_{i=1,2} \in E(\eta)$ and let $\lambda_1, \lambda_2 \geq 0$ be such that $\lambda_1 + \lambda_2 = 1$. Since η_i are Borel probability measures, $\widehat{\eta} := \lambda_1\eta_1 + \lambda_2\eta_2$ is a Borel probability measure as well. Moreover, for any Borel set $B \in \mathcal{B}(\overline{\Omega})$ we have that

$$e_0\sharp\widehat{\eta}(B) = \widehat{\eta}(e_0^{-1}(B)) = \sum_{i=1}^{2} \lambda_i\eta_i(e_0^{-1}(B)) = \sum_{i=1}^{2} \lambda_i e_0\sharp\eta_i(B) = \sum_{i=1}^{2} \lambda_i m_0(B) = m_0(B). \tag{3.1.3.3}$$

So, $\widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma)$. Since $\eta_1 \in E(\eta)$, there exists a unique Borel measurable family of probabilities $\{\eta_{1,x}\}_{x\in\overline{\Omega}}$ on Γ which disintegrates η_1 as in (3.1.3.1) and there exists $A_1 \subset \overline{\Omega}$, with $m_0(A_1) = 0$, such that

$$supp(\eta_{1,x}) \subset \Gamma^\eta[x], \quad x \in \overline{\Omega} \setminus A_1. \tag{3.1.3.4}$$

In the same way, $\eta_2(d\gamma) = \int_{\overline{\Omega}} \eta_{2,x}(\,d\gamma)\, dm_0(x)$ can be disintegrated and one has that

$$supp(\eta_{2,x}) \subset \Gamma^\eta[x] \quad x \in \overline{\Omega} \setminus A_2, \tag{3.1.3.5}$$

where A_2 is such that $m_0(A_2) = 0$. Hence, $\widehat{\eta}$ can be disintegrated in the following way:

$$\begin{cases} \widehat{\eta}(d\gamma) = \int_{\overline{\Omega}} \Big(\lambda_1\eta_{1,x} + \lambda_2\eta_{2,x}\Big)(d\gamma)dm_0(x), \\ supp(\lambda_1\eta_{1,x} + \lambda_2\eta_{2,x}) \subset \Gamma^\eta[x] \quad x \in \overline{\Omega} \setminus (A_1 \cup A_2), \end{cases} \tag{3.1.3.6}$$

where $m_0(A_1 \cup A_2) = 0$. This completes the proof. □

LEMMA 3.1.3.2. The map $E : \mathcal{P}_{m_0}(\Gamma) \rightrightarrows \mathcal{P}_{m_0}(\Gamma)$ has closed graph.

Proof. Let $\eta_i, \eta \in \mathcal{P}_{m_0}(\Gamma)$ be such that η_i is narrowly convergent to η. Let $\widehat{\eta}_i \in E(\eta_i)$ be such that $\widehat{\eta}_i$ is narrowly convergent to $\widehat{\eta}$. We have to prove that $\widehat{\eta} \in E(\eta)$. Since $\widehat{\eta}_i$ narrowly converges to $\widehat{\eta}$, we have that $\widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma)$ and there exists a unique Borel measurable family of probabilities $\{\widehat{\eta}_x\}_{x\in\overline{\Omega}}$ on Γ such that $\widehat{\eta}(d\gamma) = \int_{\overline{\Omega}} \widehat{\eta}_x(d\gamma)\, dm_0(x)$ and $supp(\widehat{\eta}_x) \subset \Gamma[x]$ for m_0-a.e. $x \in \overline{\Omega}$. Hence, $\widehat{\eta} \in E(\eta)$ if and only if $supp(\widehat{\eta}_x) \subseteq \Gamma^\eta[x]$ for m_0-a.e $x \in \overline{\Omega}$. Let $\Omega_0 \subset \overline{\Omega}$ be an m_0-null set such that

$$supp(\widehat{\eta}_x) \subset \Gamma[x] \quad \forall x \in \overline{\Omega} \setminus \Omega_0.$$

Let $x \in \overline{\Omega} \setminus \Omega_0$ and let $\widehat{\gamma} \in supp(\widehat{\eta}_x)$. Since $\widehat{\eta}_i$ narrowly converges to $\widehat{\eta}$, then, by PROPOSITION 1.2.0.1, one has that

$$\exists\, \widehat{\gamma}_i \in \; supp(\widehat{\eta}_i) \;\text{ such that } \lim_{i\to\infty} \widehat{\gamma}_i = \widehat{\gamma}.$$

Let $\widehat{\gamma}_i(0) = x_i$, with $x_i \in \overline{\Omega}$. Since $\widehat{\eta}_i \in E(\eta_i)$ and $\widehat{\gamma}_i \in supp(\widehat{\eta}_i)$, we have that $\widehat{\gamma}_i$ is an optimal trajectory for $J_{\eta_i}[\cdot]$, i.e.,

$$J_{\eta_i}[\widehat{\gamma}_i] \leq J_{\eta_i}[\gamma] \quad \forall \gamma \in \Gamma[x_i]. \tag{3.1.3.7}$$

As η_i narrowly converges to η, applying LEMMA 3.1.2.6, we conclude that $\widehat{\gamma} \in \Gamma^{\eta}[x]$. Since x is any point in $\overline{\Omega} \setminus \Omega_0$, we have shown that $\widehat{\eta} \in E(\eta)$. □

We denote by Γ_K the set of trajectories $\gamma \in \Gamma$ such that γ satisfies (3.1.2.8), i.e.,

$$\Gamma_K = \{\gamma \in \Gamma : ||\dot{\gamma}||_2 \leq K\} \tag{3.1.3.8}$$

where K is the constant given by (3.1.2.9). By the definition of $E(\eta)$ in (3.1.3.2) and LEMMA 3.1.2.5, we deduce that

$$E(\eta) \subseteq \mathcal{P}_{m_0}(\Gamma_K) \quad \forall \eta \in \mathcal{P}_{m_0}(\Gamma). \tag{3.1.3.9}$$

REMARK 3.1.3.3. In general Γ fails to be complete as a metric space. Then, by THEOREM 1.2.0.2, $\mathcal{P}_{m_0}(\Gamma)$ is not a compact set. On the other hand, if Γ is replaced by Γ_K then $\mathcal{P}_{m_0}(\Gamma_K)$ is a compact convex subset of $\mathcal{P}_{m_0}(\Gamma)$. Indeed, the convexity of $\mathcal{P}_{m_0}(\Gamma_K)$ follows by the same argument used in the proof of LEMMA 3.1.3.1. As for compactness, let $\{\eta_k\} \subset \mathcal{P}_{m_0}(\Gamma_K)$. Since Γ_K is a compact set, $\{\eta_k\}$ is tight. So, by PROPOSITION 1.2.0.1, one finds a subsequence, that we denote again by η_k, which narrowly converges to some probability measure $\eta \in \mathcal{P}_{m_0}(\Gamma_K)$.

We will restrict domain of interest to $\mathcal{P}_{m_0}(\Gamma_K)$, where Γ_K is given by (3.1.3.8). Hereafter, we denote by E the restriction $E_{|\mathcal{P}_{m_0}(\Gamma_K)}$.

Conclusion

By REMARK 3.1.3.3 and REMARK 3.1.2.2, $\mathcal{P}_{m_0}(\Gamma_K)$ is a nonempty compact convex set. Moreover, by LEMMA 3.1.3.1, $E(\eta)$ is nonempty convex set for any $\eta \in \mathcal{P}_{m_0}(\Gamma_K)$ and, by LEMMA 3.1.3.2, the set-valued map E has closed graph. Then, the assumptions of Kakutani's Theorem are satisfied and so there exists $\overline{\eta} \in \mathcal{P}_{m_0}(\Gamma_K)$ such that $\overline{\eta} \in E(\overline{\eta})$.

3.2 MILD SOLUTION OF THE CONSTRAINED MFG PROBLEM

In this section we define mild solutions of the constrained MFG problem in $\overline{\Omega}$. Moreover, under the assumptions of SUBSECTION 3.1.1, we prove the existence of such solutions. Then, we give a uniqueness result under a classical monotonicity assumption on F and G.

DEFINITION 3.2.0.1. We say that $(u, m) \in C([0,T] \times \overline{\Omega}) \times C([0,T]; \mathcal{P}(\overline{\Omega}))$ is a mild solution of the constrained MFG problem in $\overline{\Omega}$ if there exists a constrained MFG equilibrium $\eta \in \mathcal{P}_{m_0}(\Gamma)$ such that

(i) $m(t) = e_t \sharp \eta$ for all $t \in [0,T]$;

(ii) u is given by

$$u(t,x) = \inf_{\substack{\gamma \in \Gamma \\ \gamma(t)=x}} \left\{ \int_t^T [L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m(s))] \, ds + G(\gamma(T), m(T)) \right\}, \quad (3.2.0.1)$$

for $(t,x) \in [0,T] \times \overline{\Omega}$.

A direct consequence of THEOREM 3.1.2.4 is the following result.

COROLLARY 3.2.0.2. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary and let $m_0 \in \mathcal{P}(\overline{\Omega})$. Suppose that (L1)-(L3) hold true. Let $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ and $G : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ be continuous. There exists at least one mild solution (u, m) of the constrained MFG problem in $\overline{\Omega}$.

Before proving our uniqueness result, we recall the following definitions.

DEFINITION 3.2.0.3. We say that $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ is monotone if

$$\int_{\overline{\Omega}} (F(x,m_1) - F(x,m_2)) d(m_1 - m_2)(x) \geq 0, \quad (3.2.0.2)$$

for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$.
We say that F is strictly monotone if

$$\int_{\overline{\Omega}} (F(x,m_1) - F(x,m_2)) d(m_1 - m_2)(x) \ \geq \ 0, \quad (3.2.0.3)$$

for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$ and $\int_{\overline{\Omega}} (F(x,m_1) - F(x,m_2)) d(m_1 - m_2)(x) = 0$ if and only if $F(x,m_1) = F(x,m_2)$ for all $x \in \overline{\Omega}$.

Example 3.2.1. Assume that $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ is of the form

$$F(x,m) = \int_{\overline{\Omega}} f(y, (\phi \star m)(y)) \phi(x-y) \, dy, \quad (3.2.0.4)$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is a smooth even kernel with compact support and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a smooth function such that $z \to f(x,z)$ is strictly increasing for all $x \in \overline{\Omega}$. Then F satisfies condition (3.2.0.3). Indeed, for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$, we have that

$$\begin{aligned} &\int_{\overline{\Omega}} [F(x,m_1) - F(x,m_2)] \, d(m_1 - m_2)(x) \\ &= \int_{\overline{\Omega}} \int_{\overline{\Omega}} [f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))] \, \phi(x-y) \, dy \, d(m_1 - m_2)(x) \end{aligned}$$

$$= \int_{\overline{\Omega}} [f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))] \int_{\overline{\Omega}} \phi(x - y)\, d\,(m_1 - m_2)\,(x)\, dy$$
$$= \int_{\overline{\Omega}} [f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))]\, [(\phi \star m_1)(y) - (\phi \star m_2)(y)]\; dy.$$

Since $z \to f(x, z)$ is increasing, then one has that

$$\int_{\overline{\Omega}} [f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))]\, [(\phi \star m_1)(y) - (\phi \star m_2)(y)]\; dy \geq 0.$$

Moreover, if the term on the left-side above is equal to zero, then we obtain

$$[f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))]\, [(\phi \star m_1)(y) - (\phi \star m_2(y))] = 0 \quad \text{a.e. } y \in \overline{\Omega}.$$

As $z \to f(x, z)$ is strictly increasing, we deduce that $\phi \star m_1(y) = \phi \star m_2(y)$ for any $y \in \overline{\Omega}$ and so $F(\cdot, m_1) = F(\cdot, m_2)$.

THEOREM 3.2.0.4. Suppose that F and G satisfy (3.2.0.3). Let η_1, $\eta_2 \in \mathcal{P}_{m_0}(\Gamma)$ be constrained MFG equilibria and let J_{η_1} and J_{η_2} be the associated functionals. Then J_{η_1} is equal to J_{η_2}. Consequently, if (u_1, m_1), (u_2, m_2) are mild solutions of the constrained MFG problem in $\overline{\Omega}$, then $u_1 = u_2$.

Proof. Let η_1, $\eta_2 \in \mathcal{P}_{m_0}(\Gamma)$ be constrained MFG equilibria, such that $m_1(t) = e_t \sharp \eta_1$, $m_2(t) = e_t \sharp \eta_2$ and let u_1, u_2 be the value functions of J_{η_1} and J_{η_2}, respectively. Let $x_0 \in \overline{\Omega}$ and let γ be an optimal trajectory for u_1 at $(0, x_0)$. We get

$$u_1(0, x_0) = \int_0^T \Big[L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m_1(s))\Big]\, ds + G(\gamma(T), m_1(T)),$$
$$u_2(0, x_0) \leq \int_0^T \Big[L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m_2(s))\Big]\, ds + G(\gamma(T), m_2(T)).$$

Therefore,

$$G(\gamma(T), m_1(T)) - G(\gamma(T), m_2(T)) \leq u_1(0, x_0) - u_2(0, x_0)$$
$$- \int_0^T \Big[L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m_1(s))\Big]\, ds + \int_0^T \Big[L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m_2(s))\Big]\, ds$$
$$= u_1(0, x_0)) - u_2(0, x_0) + \int_0^T F(\gamma(s), m_2(s)) - F(\gamma(s), m_1(s))\, ds.$$

Recalling that $supp(\eta_1) \subset \Gamma^{\eta_1}[x_0]$, we integrate on Γ respect to $d\eta_1$ to obtain

$$\int_\Gamma \Big[G(\gamma(T), m_1(T)) - G(T, m_2(T))\Big] d\eta_1(\gamma) \leq$$
$$\leq \int_\Gamma \Big[u_1(0, \gamma(0)) - u_2(0, \gamma(0))\Big] d\eta_1(\gamma) + \int_\Gamma \int_0^T \Big[F(\gamma(s), m_2(s)) - F(\gamma(s), m_1(s))\Big]\, ds\, d\eta_1(\gamma).$$

Recalling the definition of e_t and using the change of variables $e_t(\gamma) = x$ in the above inequality, we get

$$\int_{\Gamma} \Big[G(\overbrace{\gamma(T)}^{e_T(\gamma)}, m_1(T)) - G(\overbrace{\gamma(T)}^{e_T(\gamma)}, m_2(T))\Big] d\eta_1(\gamma) = \int_{\overline{\Omega}} \Big[G(x, m_1(T)) - G(x, m_2(T))\Big]\, m_1(T, dx),$$

$$\int_{\Gamma} \Big[u_1(0, \overbrace{\gamma(0)}^{e_0(\gamma)}) - u_2(0, \overbrace{\gamma(0)}^{e_0(\gamma)})\Big] d\eta_1(\gamma) = \int_{\overline{\Omega}} \Big[u_1(0, x) - u_2(0, x)\Big]\, m_1(0, dx),$$

and

$$\begin{aligned} &\int_0^T \int_{\Gamma} \Big[F(\overbrace{\gamma(s)}^{e_s(\gamma)}, m_2(s)) - F(\overbrace{\gamma(s)}^{e_s(\gamma)}, m_1(s))\Big]\, d\eta_1(\gamma)\, ds \\ &= \int_0^T \int_{\overline{\Omega}} \Big[F(x, m_2(s)) - F(x, m_1(s))\Big]\, m_1(s, dx)\, ds. \end{aligned}$$

So, we deduce that

$$\begin{aligned} &\int_{\overline{\Omega}} \Big[G(x, m_1(T)) - G(x, m_2(T))\Big]\, m_1(T, dx) \\ &\leq \int_{\overline{\Omega}} \Big[u_1(0, x) - u_2(0, x)\Big]\, m_1(0, dx) + \int_0^T \int_{\overline{\Omega}} \Big[F(x, m_2(s)) - F(x, m_1(s))\, m_1(s, dx)\Big]\, ds. \end{aligned} \tag{3.2.0.5}$$

In a similar way, we obtain

$$\begin{aligned} &\int_{\overline{\Omega}} \Big[G(x, m_2(T)) - G(x, m_1(T))\Big]\, m_2(T, dx) \\ &\leq \int_{\overline{\Omega}} \Big[u_2(0, x) - u_1(0, x)\Big]\, m_2(0, dx) + \int_0^T \int_{\overline{\Omega}} \Big[F(x, m_1(s)) - F(x, m_2(s))\Big]\, m_2(s, dx)\, ds. \end{aligned} \tag{3.2.0.6}$$

Summing the inequalities (3.2.0.5) and (3.2.0.6), we deduce that

$$\begin{aligned} &\int_{\overline{\Omega}} [G(x, m_1(T)) - G(x, m_2(T))]\, (m_1(T, dx) - m_2(T, dx)) \\ &\leq \int_{\overline{\Omega}} [u_1(0, x) - u_2(0, x)]\, (m_1(0, dx) - m_2(0, dx)) \\ &+ \int_0^T \int_{\overline{\Omega}} \Big[F(x, m_2(s)) - F(x, m_1(s))\Big]\, (m_1(s, dx) - m_2(s, dx))\, ds \end{aligned}$$

Since $m_1(0, dx) = m_2(0, dx) = m_0(\, dx)$, by using the monotonicity assumption on G and F, we obtain that

$$\begin{aligned} 0 \geq \int_0^T \int_{\overline{\Omega}} \Big[F(x, m_2(s)) - F(x, m_1(s))\Big]\, (m_1(s, dx) - m_2(s, dx))\, ds \geq \\ \int_{\overline{\Omega}} \Big[G(x, m_1(T)) - G(x, m_2(T))\Big]\, (m_1(T, dx) - m_2(T, dx)) \geq 0. \end{aligned}$$

Therefore,

$$\int_{\overline{\Omega}} \Big[F(x, m_2(s)) - F(x, m_1(s))\Big]\, (m_1(s, dx) - m_2(s, dx)) = 0 \quad \text{a.e. } s \in [0, T],$$

and also
$$\int_{\overline{\Omega}} \Big[G(x,m_1(T)) - G(x,m_2(T))\Big] \left(m_1(T,dx) - m_2(T,dx)\right) = 0.$$
Thus, by the strict monotonicity of F and G, we conclude that $F(x,m_1) = F(x,m_2)$ for all $x \in \overline{\Omega}$ and $G(x,m_1) = G(x,m_2)$ for all $x \in \overline{\Omega}$. Consequently, we have that J_{η_1} is equal to J_{η_2}. □

COROLLARY 3.2.0.5. Suppose that G satisfies (3.2.0.2) and F satisfies the following condition
$$\int_{\overline{\Omega}} \Big[F(x,m_1) - F(x,m_2)\Big] d(m_1 - m_2)(x) \; > \; 0 \tag{3.2.0.7}$$
for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$ with $m_1 \neq m_2$. Then there exists a unique mild solution of the constrained MFG problem in $\overline{\Omega}$.

Proof. Let η_1, $\eta_2 \in \mathcal{P}_{m_0}(\Gamma)$ be constrained MFG equilibria, such that $m_1(t) = e_t\sharp\eta_1$, $m_2(t) = e_t\sharp\eta_2$ and let u_1, u_2 be the value functions of J_{η_1} and J_{η_2}, respectively. By THEOREM 3.2.0.4 we obtain that u_1 is equal to u_2. Moreover, by (3.2.0.7) we have that $m_1(t) = m_2(t)$ a.e. $t \in [0,T]$. This completes the proof. □

Example 3.2.2. Assume that $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ is of the form
$$F(x,m) = \int_{\overline{\Omega}} f(y,(\phi \star m)(y))\phi(x-y)\, dy \tag{3.2.0.8}$$
where $\phi : \mathbb{R}^n \to \mathbb{R}$ is a smooth, even kernel with compact support and its Fourier trasform is not equal to zero almost everywhere, and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a smooth map such that $z \to f(x,z)$ is strictly increasing for all $x \in \overline{\Omega}$. Then F satisfies the condition (3.2.0.7).
Indeed, for any m_1, $m_2 \in \mathcal{P}(\overline{\Omega})$, we have that
$$\begin{aligned}
&\int_{\overline{\Omega}} [F(x,m_1) - F(x,m_2)]\, d(m_1 - m_2)(x)\\
&= \int_{\overline{\Omega}}\int_{\overline{\Omega}} [f(y,(\phi\star m_1)(y)) - f(y,(\phi\star m_2)(y))]\phi(x-y)\, dy\, d(m_1-m_2)(x)\\
&= \int_{\overline{\Omega}} [f(y,(\phi\star m_1)(y)) - f(y,(\phi\star m_2)(y))] \int_{\overline{\Omega}} \phi(x-y)\, d(m_1-m_2)(x)\, dy\\
&= \int_{\overline{\Omega}} [f(y,(\phi\star m_1)(y)) - f(y,(\phi\star m_2)(y))][(\phi\star m_1)(y) - (\phi\star m_2(y))]\, dy.
\end{aligned}$$
Since $z \to f(x,z)$ is increasing, then one has that
$$\int_{\overline{\Omega}} [f(y,(\phi\star m_1)(y)) - f(y,(\phi\star m_2)(y))][(\phi\star m_1)(y) - (\phi\star m_2(y))]\, dy \geq 0.$$
Morover, if the term on the left-side above is equal to zero, then we obtain
$$[f(y,(\phi\star m_1)(y)) - f(y,(\phi\star m_2)(y))][(\phi\star m_1)(y) - (\phi\star m_2)(y)] = 0 \quad \text{a.e. } \; y \in \overline{\Omega}.$$

As $z \to f(x,z)$ is strictly increasing then $\phi \star m_1(y) = \phi \star m_2(y)$ for a.e. $y \in \overline{\Omega}$. Set $M = m_1 - m_2$ then one has that $\phi \star M = 0$ for a.e. $y \in \overline{\Omega}$. Since Fourier trasform $\mathcal{F}$ is an isomorphism in $L^2(\mathbb{R}^n)$ then $\mathcal{F}(\phi \star M) = 0$ a.e.. Recalling the properties of Fourier trasform, we have that $\mathcal{F}(\phi \star M) = \mathcal{F}(\phi)\mathcal{F}(M) = 0$ a.e. As $\mathcal{F}(\phi) \neq 0$ then $\mathcal{F}(M) = 0$ a.e. Moreover, since $\mathcal{F}$ is an injective function from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ then $M = 0$ a.e. and so $m_1 = m_2$ a.e.

CHAPTER 4

REGULARITY OF MILD SOLUTIONS

In this Chapter, we are interested to study the regularity of mild solutions. To this aim, in SECTION 4.2 we apply THEOREM 2.3.0.1 to deduce the existence of more regular equilibria than those constructed in CHAPTER 3. Under suitable assumptions (SECTION 4.1) we prove the existence of Lipschitz continuous mild solutions (THEOREM 4.3.0.1). Moreover, given a mild solution (u, m), we show that u is locally semiconcave with linear modulus in Ω. While, using THEOREM 2.5.0.1, we deduce that u is locally semiconcave with modulus $\omega(r) = Cr^{\frac{1}{2}}$ in $(0, T) \times \overline{\Omega}$.

4.1 ASSUMPTIONS

Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let U be an open subset of $\mathbb{R}^n$ and such that $\overline{\Omega} \subset U$. Assume that $F : U \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ and $G : U \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ satisfy the following hypotheses.

(D1) For all $x \in U$, the functions $m \longmapsto F(x, m)$ and $m \longmapsto G(x, m)$ are Lipschitz continuous, i.e., there exists $\kappa \geq 0$ such that

$$|F(x, m_1) - F(x, m_2)| + |G(x, m_1) - G(x, m_2)| \leq \kappa d_1(m_1, m_2), \tag{4.1.0.1}$$

for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$.

(D2) For all $m \in \mathcal{P}(\overline{\Omega})$, the functions $x \longmapsto G(x, m)$ and $x \longmapsto F(x, m)$ belong to $C^1_b(U)$. Moreover

$$|D_x F(x, m)| + |D_x G(x, m)| \leq \kappa, \quad \forall\, x \in U,\ \forall\, m \in \mathcal{P}(\overline{\Omega}).$$

(D3) For all $m \in \mathcal{P}(\overline{\Omega})$, the function $x \longmapsto F(x, m)$ is semiconcave with modulus linear.

Let $L : U \times \mathbb{R}^n \to \mathbb{R}$ be a function that satisfies the following assumptions.

(L0) $L \in C^1(U \times \mathbb{R}^n)$ and there exists a constant $M \geq 0$ such that

$$|L(x,0)| + |D_x L(x,0)| + |D_v L(x,0)| \leq M, \quad \forall\, x \in U. \tag{4.1.0.2}$$

(L1) $D_v L$ is differentiable on $U \times \mathbb{R}^n$ and there exists a constant $\mu \geq 1$ such that

$$\frac{I}{\mu} \leq D^2_{vv} L(x,v) \leq I\mu, \tag{4.1.0.3}$$

$$||D^2_{vx} L(x,v)|| \leq \mu(1 + |v|), \tag{4.1.0.4}$$

for all $(x,v) \in U \times \mathbb{R}^n$.

(L2) For all $x \in U$ and for all v, $w \in B_R$, there exists a constant $C(R) \geq 0$ such that

$$|D_x L(x,v) - D_x L(x,w)| \leq C(R)|v - w|. \tag{4.1.0.5}$$

(L3) For any $R > 0$ the map $x \longmapsto L(x,v)$ is semiconcave with linear modulus independent of v for $v \in B_R$.

REMARK 4.1.0.1. (i) F, G and L are assumed to be defined on $U \times \mathcal{P}(\overline{\Omega})$ and on $U \times \mathbb{R}^n$, respectively, just for simplicity. All the results of this Chapter hold true if we replace U by $\overline{\Omega}$. This fact can be easly checked by using well-known extension techniques (see, e.g. [5, Theorem 4.26]).
(ii) Arguing as LEMMA 2.2.0.4 we deduce that there exists a positive constant $C(\mu, M)$ that dependes only on M, μ such that

$$|D_x L(x,v)| \leq C(\mu, M)(1 + |v|^2), \tag{4.1.0.6}$$

$$|D_v L(x,v)| \leq C(\mu, M)(1 + |v|),, \tag{4.1.0.7}$$

$$\frac{|v|^2}{4\mu} - C(\mu, M) \leq L(x,v) \leq 4\mu|v|^2 + C(\mu, M), \tag{4.1.0.8}$$

for all $(x,v) \in U \times \mathbb{R}^n$.

Let $m \in \mathrm{Lip}(0,T;\mathcal{P}(\overline{\Omega}))$. If we set $f(t,x,v) = L(x,v) + F(x,m(t))$, then the associated Hamiltonian H takes the form

$$H(t,x,p) = H_L(x,p) - F(x,m(t)), \quad \forall\, (t,x,p) \in [0,T] \times U \times \mathbb{R}^n,$$

where

$$H_L(x,p) = \sup_{v \in \mathbb{R}^n} \Big\{ -\langle p, v\rangle - L(x,v) \Big\}, \quad \forall\, (x,p) \in U \times \mathbb{R}^n. \tag{4.1.0.9}$$

Hereafter, for simplicity, we denote by $H(x,p)$ the Hamiltonian $H_L(x,p)$ defined in (4.1.0.9).
Arguing as in LEMMA 2.2.0.4, the assumptions on L imply that H satisfies the following conditions.

(H0) $H \in C^1(U \times \mathbb{R}^n)$ and there exists a constant $M' \geq 0$ such that

$$|H(x,0)| + |D_x H(x,0)| + |D_p H(x,0)| \leq M', \quad \forall x \in U. \tag{4.1.0.10}$$

(H1) $D_p H$ is differentiable on $U \times \mathbb{R}^n$ and satisfies

$$\frac{I}{\mu} \leq D_{pp} H(x,p) \leq I\mu, \qquad \forall\,(x,p) \in U \times \mathbb{R}^n, \tag{4.1.0.11}$$

$$||D^2_{px} H(x,p)|| \leq C(\mu, M')(1+|p|), \quad \forall\,(x,p) \in U \times \mathbb{R}^n, \tag{4.1.0.12}$$

where $C(\mu, M')$ depends only on μ and M'.

(H2) For all $x \in U$ and for all $p, q \in B_R$, there exists a constant $C(R) \geq 0$ such that

$$|D_x H(x,p) - D_x H(x,q)| \leq C(R)|p-q|. \tag{4.1.0.13}$$

(H3) For any $R > 0$ the map $x \longmapsto H(x,p)$ is semiconvex with linear modulus independent of p for $p \in B_R$.

4.2 THE EXISTENCE RESULT

Let $m_0 \in \mathcal{P}(\overline{\Omega})$. Let Γ' be a nonempty subset of Γ. We denote by $\mathcal{P}_{m_0}(\Gamma')$ the set of all Borel probability measures η on Γ' such that $e_0\sharp\eta = m_0$. We now introduce special subfamilies of $\mathcal{P}_{m_0}(\Gamma)$ that play a key role in what follows.

DEFINITION 4.2.0.1. We define by $\mathcal{P}^{\text{Lip}}_{m_0}(\Gamma')$ the set of $\eta \in \mathcal{P}_{m_0}(\Gamma')$ such that $m^\eta(t) = e_t\sharp\eta$ is Lipschitz continuous, i.e.,

$$\mathcal{P}^{\text{Lip}}_{m_0}(\Gamma') = \{\eta \in \mathcal{P}_{m_0}(\Gamma') : m \in \text{Lip}(0,T;\mathcal{P}(\overline{\Omega}))\}.$$

REMARK 4.2.0.2. Arguing as in REMARK 3.1.2.2, we observe that $\mathcal{P}^{\text{Lip}}_{m_0}(\Gamma)$ is a nonempty set. Moreover, $\mathcal{P}^{\text{Lip}}_{m_0}(\Gamma)$ is a convex set. Indeed, let $\{\eta_i\}_{i=1,2} \subset \mathcal{P}^{\text{Lip}}_{m_0}(\Gamma)$ be such that $m_1(t) = e_t\sharp\eta_1$ and $m_2(t) = e_t\sharp\eta_2$ belong to $(0,T;\mathcal{P}(\overline{\Omega}))$. Let $\lambda_1, \lambda_2 \geq 0$ be such that $\lambda_1 + \lambda_2 = 1$. Since η_i are Borel probability measures, $\eta := \lambda\eta_1 + (1-\lambda)\eta_2$ is a Borel probability measure as well. Moreover, for any Borel set $B \in \mathcal{B}(\overline{\Omega})$ we have that

$$e_0\sharp\eta(B) = \eta(e_0^{-1}(B)) = \sum_{i=1}^{2} \lambda_i \eta_i(e_0^{-1}(B)) = \sum_{i=1}^{2} \lambda_i e_0\sharp\eta_i(B) = \sum_{i=1}^{2} \lambda_i m_0(B) = m_0(B).$$

So, $\eta \in \mathcal{P}_{m_0}(\Gamma)$. Since $m_1, m_2 \in \text{Lip}(0,T;\mathcal{P}(\overline{\Omega}))$, we have that $m(t) = \lambda_1 m_1(t) + \lambda_2 m_2(t)$ belongs to $\text{Lip}(0,T;\mathcal{P}(\overline{\Omega}))$.

In the next result, we apply THEOREM 2.3.0.1 to prove a useful property of minimizers of J_η.

PROPOSITION 4.2.0.3. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary and let $m_0 \in \mathcal{P}(\overline{\Omega})$. Suppose that (L0), (L1), (D1), and (D2) hold true. Let $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$ and fix $x \in \overline{\Omega}$. Then $\Gamma^\eta[x] \subset C^{1,1}([0,T];\mathbb{R}^n)$ and

$$||\dot{\gamma}||_\infty \leq L_0, \quad \forall \gamma \in \Gamma^\eta[x], \tag{4.2.0.1}$$

where $L_0 = L_0(\mu, M', M, \kappa, T, ||G||_\infty, ||DG||_\infty)$.

Proof. Let $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$, $x \in \overline{\Omega}$ and $\gamma \in \Gamma^\eta[x]$. Since $m \in \mathrm{Lip}(0,T;\mathcal{P}(\overline{\Omega}))$, taking $f(t,x,v) = L(x,v) + F(x,m(t))$, one can easly check that all the assumptions of THEOREM 2.3.0.1 are satisfied by f and G. Therefore, we have that $\Gamma^\eta[x] \subset C^{1,1}([0,T];\mathbb{R}^n)$ and, in this case, (2.3.0.3) becomes

$$||\dot{\gamma}||_\infty \leq L_0, \quad \forall \gamma \in \Gamma^\eta[x],$$

where $L_0 = L_0(\mu, M', M, \kappa, T, ||G||_\infty, ||DG||_\infty)$. □

We denote by Γ_{L_0} the set of $\gamma \in \Gamma$ such that (4.2.0.1) holds, i.e.,

$$\Gamma_{L_0} = \{\gamma \in \Gamma : ||\dot{\gamma}||_\infty \leq L_0\}. \tag{4.2.0.2}$$

LEMMA 4.2.0.4. Let $m_0 \in \mathcal{P}(\overline{\Omega})$. Then, $\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$ is a nonempty convex compact subset of $\mathcal{P}_{m_0}(\Gamma)$. Moreover, for every $\eta \in \mathcal{P}_{m_0}(\Gamma_{L_0})$, $m(t) := e_t \sharp \eta$ is Lipschitz continuous of constant L_0, where L_0 is as in PROPOSITION 4.2.0.3.

Proof. Arguing as in REMARK 4.2.0.2, we obtain that $\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$ is a nonempty convex set. Moreover, since Γ_{L_0} is compactly embedded in Γ, one has that $\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$ is compact.
Let $\eta \in \mathcal{P}_{m_0}(\Gamma_{L_0})$ and $m(t) = e_t \sharp \eta$. For any $t_1, t_2 \in [0,T]$, we recall that

$$d_1(m(t_2), m(t_1)) = \sup\left\{ \int_{\overline{\Omega}} \phi(x)(m(t_2,\, dx) - m(t_1,\, dx)) \;\middle|\; \phi : \overline{\Omega} \to \mathbb{R} \text{ is 1-Lipschitz} \right\}.$$

Since ϕ is 1-Lipschitz continuous, one has that

$$\int_{\overline{\Omega}} \phi(x)\,(m(t_2,dx) - m(t_1,dx)) = \int_\Gamma \Big[\phi(e_{t_2}(\gamma)) - \phi(e_{t_1}(\gamma))\Big]\, d\eta(\gamma)$$
$$= \int_\Gamma \Big[\phi(\gamma(t_2)) - \phi(\gamma(t_1))\Big]\, d\eta(\gamma) \leq \int_\Gamma |\gamma(t_2) - \gamma(t_1)|\, d\eta(\gamma).$$

Since $\eta \in \mathcal{P}_{m_0}(\Gamma_{L_0})$, we deduce that

$$\int_\Gamma |\gamma(t_2) - \gamma(t_1)|\, d\eta(\gamma) \leq L_0 \int_\Gamma |t_2 - t_1|\, d\eta(\gamma) = L_0|t_2 - t_1|$$

and so $m(t)$ is Lipschitz continuous of constant L_0. □

In the next result, we deduce the existence of more regular equilibria than those constructed in CHAPTER 3.

THEOREM 4.2.0.5. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary and let $m_0 \in \mathcal{P}(\overline{\Omega})$. Suppose that (L0), (L1), (D1), and (D2) hold true. Then, there exists at least one constrained MFG equilibrium $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$.

Proof. First of all, we recall that for any $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$, there exists a unique Borel measurable family [1] of probabilities $\{\eta_x\}_{x\in\overline{\Omega}}$ on Γ which disintegrates η in the sense that

$$\begin{cases} \eta(d\gamma) = \int_{\overline{\Omega}} \eta_x(d\gamma) m_0(\,dx), \\ supp(\eta_x) \subset \Gamma[x] \;\; m_0 - \text{a.e. } x \in \overline{\Omega}. \end{cases} \tag{4.2.0.3}$$

Proceeding as in SUBSECTION 3.1.3, we introduce the set-valued map

$$E : \mathcal{P}_{m_0}(\Gamma) \rightrightarrows \mathcal{P}_{m_0}(\Gamma),$$

by defining, for any $\eta \in \mathcal{P}_{m_0}(\Gamma)$,

$$E(\eta) = \Big\{ \widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma) : supp(\widehat{\eta}_x) \subseteq \Gamma^{\eta}[x] \;\; m_0 - \text{a.e. } x \in \overline{\Omega} \Big\}. \tag{4.2.0.4}$$

We recall that, by LEMMA 3.1.3.2, the map E has closed graph.
Now, consider the restriction E_0 of E to $\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma)$, i.e.,

$$E_0 : \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0}) \rightrightarrows \mathcal{P}_{m_0}(\Gamma), \quad E_0(\eta) = E(\eta) \;\; \forall \eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0}).$$

We will show that the set-valued map E_0 has a fixed point, i.e., there exists $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$ such that $\eta \in E_0(\eta)$. By LEMMA 3.1.3.1 we have that for any $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$, $E_0(\eta)$ is a nonempty convex set. Moreover, we have that

$$E_0(\mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})) \subseteq \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0}). \tag{4.2.0.5}$$

Indeed, let $\eta \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$ and $\hat{\eta} \in E_0(\eta)$. Since, by PROPOSITION 4.2.0.3 one has that

$$\Gamma^{\eta}[x] \subset \Gamma_{L_0} \quad \forall x \in \overline{\Omega},$$

and by definition of E_0 we deduce that

$$supp(\widehat{\eta}) \subset \Gamma_{L_0}.$$

So, $\widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma_{L_0})$. By LEMMA 4.2.0.4, $\widehat{\eta} \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$.
Since E has closed graph, by LEMMA 4.2.0.4 and (4.2.0.5) we have that E_0 has closed graph as well. Then, the assumptions of Kakutani's Theorem [55] are satisfied and so, there exists $\overline{\eta} \in \mathcal{P}^{\mathrm{Lip}}_{m_0}(\Gamma_{L_0})$ such that $\overline{\eta} \in E_0(\overline{\eta})$. □

[1] We say that $\{\eta_x\}_{x\in\overline{\Omega}}$ is a Borel family (of probability measures) if $x \in \overline{\Omega} \longmapsto \eta_x(B) \in \mathbb{R}$ is Borel for any Borel set $B \subset \Gamma$.

4.3 REGULARITY OF MILD SOLUTIONS

In this section, we deduce the existence of more regular mild solutions than those constructed in CHAPTER 3.

A direct consequence of PROPOSITION 2.4.0.3 is the existence of Lipschitz continuous mild solutions.

THEOREM 4.3.0.1. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Suppose that (L0), (L1), (D1) and (D2) hold true. There exists at least one mild solution (u, m) of the constrained MFG problem in $\overline{\Omega}$. Moreover,

(i) u is Lipschitz continuous in $(0, T) \times \overline{\Omega}$;

(ii) $m \in \text{Lip}(0, T; \mathcal{P}(\overline{\Omega}))$ and $\text{Lip}(m) = L_0$, where L_0 is the constant in (4.2.0.1).

Proof. Let $m_0 \in \mathcal{P}(\overline{\Omega})$ and let $\eta \in \mathcal{P}^{\text{Lip}}_{m_0}(\Gamma)$ be a constrained MFG equilibrium for m_0. Then, by THEOREM 4.2.0.5 there exists at least one mild solution (u, m) of the constrained MFG problem in $\overline{\Omega}$. Moreover, by THEOREM 4.2.0.5 one has that $m \in \text{Lip}(0, T; \mathcal{P}(\overline{\Omega}))$ and $\text{Lip}(m) = L_0$, where L_0 is the constant in (4.2.0.1). Finally, by PROPOSITION 2.4.0.3 we conclude that u is Lipschitz continuous in $(0, T) \times \overline{\Omega}$. □

Applying LEMMA 2.5.1.1 and PROPOSITION 2.4.0.4, respectively, we obtain the following results.

COROLLARY 4.3.0.2. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Suppose that (L0)-(L3), (D1)-(D3) hold true. Let (u, m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$. Then, u is locally semiconcave with modulus $\omega(r) = Cr^{\frac{1}{2}}$ in $(0, T) \times \overline{\Omega}$.

COROLLARY 4.3.0.3. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Suppose that $(L0)$-$(L3)$ and $(D1)$-$(D3)$ hold true. Let (u, m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$. Then, u is locally semiconcave with linear modulus in Ω.

CHAPTER 5

MEAN FIELD GAMES SYSTEM

This Chapter is devoted to prove that Lipschitz continuous mild solutions (u, m) of the constrained MFG problem in $\overline{\Omega}$ solve a MFG system. More precisely, we show that

(i) u is a constrained viscosity solution of

$$\begin{cases} -\partial_t u + H(x, Du) = F(x, m(t)) & \text{in } (0, T) \times \overline{\Omega} \\ u(x, T) = G(x, m(T)) & \text{in } \overline{\Omega}; \end{cases}$$

(ii) there exists $V : [0, T] \times \overline{\Omega} \to \mathbb{R}^n$ Borel measurable vector field such that m is a solution in the sense of distribution of

$$\begin{cases} \partial_t m + div(V\, m) = 0, & \text{in } [0, T] \times \overline{\Omega} \\ m(0, x) = m_0(x) & \text{in } \overline{\Omega} \end{cases}$$

that is, for all $\phi \in C^1([0, T] \times \overline{\Omega})$ one has that

$$\int_{\overline{\Omega}} \phi(0, x) m_0(\,dx) - \int_{\overline{\Omega}} \phi(T, x) m(T, dx) = \int_0^T \int_{\overline{\Omega}} \Big[\partial_t \phi(t, x) + \langle \nabla \phi(t, x), V(t, x)\rangle\Big] m(t, dx)\, dt.$$

Moreover, we will give an explicit form of V. More precisely, for all $(t, x) \in (0, T) \times \Omega$ with x in the support of $m(t)$ one has that

$$V(t, x) = -D_p H(x, Du(t, x)),$$

and

$$V(t, x) = -D_p H(x, p^\tau(t, x) + \lambda_+(t, x)\nu(x)),$$

for all $(t, x) \in (0, T) \times \partial\Omega$ with x in the support of $m(t)$.

5.1 HAMILTON-JACOBY-BELLMAN EQUATION

Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Assume that H, F and G satisfy the assumptions in SECTION 4.1. Let $m \in \text{Lip}(0,T;\mathcal{P}(\overline{\Omega}))$. Consider the following equation

$$-\partial_t u + H(x, Du) = F(x, m(t)) \quad \text{in } (0,T)\times\overline{\Omega}. \tag{5.1.0.1}$$

We recall the definition of constrained viscosity solution.

DEFINITION 5.1.0.1. Let $u \in C((0,T)\times\overline{\Omega})$. We say that:

(i) u is a viscosity supersolution of (5.1.0.1) in $(0,T)\times\overline{\Omega}$ if

$$-\partial_t\phi(t,x) + H(x, D\phi(t,x)) \geq F(x, m(t)),$$

for any $\phi \in C^1(\mathbb{R}^{n+1})$ such that $u-\phi$ has a local minimum, relative to $(0,T)\times\overline{\Omega}$, at $(t,x)\in(0,T)\times\overline{\Omega}$;

(ii) u is a viscosity subsolution of (5.1.0.1) in $(0,T)\times\Omega$ if

$$-\partial_t\phi(t,x) + H(x, D\phi(t,x)) \leq F(x, m(t)),$$

for any $\phi \in C^1(\mathbb{R}^{n+1})$ such that $u-\phi$ has a local maximum, relative to $(0,T)\times\Omega$, at $(t,x)\in(0,T)\times\Omega$;

(iii) u is constrained viscosity solution of (5.1.0.1) in $(0,T)\times\overline{\Omega}$ if it is a subsolution in $(0,T)\times\Omega$ and a supersolution in $(0,T)\times\overline{\Omega}$.

REMARK 5.1.0.2. Owing to PROPOSITION 1.3.0.4, DEFINITION 5.1.0.1 can be expressed in terms of subdifferential and superdifferential, i.e.,

$$\begin{aligned}
&-p_1 + H(x,p_2) \leq F(x,m(t)) \quad \forall\,(t,x)\in(0,T)\times\Omega,\ \forall\,(p_1,p_2)\in D^+u(t,x),\\
&-p_1 + H(x,p_2) \geq F(x,m(t)) \quad \forall\,(t,x)\in(0,T)\times\overline{\Omega},\ \forall\,(p_1,p_2)\in D^-u(t,x).
\end{aligned}$$

A direct consequence of the definition of mild solution is the following result.

PROPOSITION 5.1.0.3. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let H and F satisfy hypotheses $(H0)$-$(H3)$ and $(D1)$-$(D3)$, respectively. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$. Then, u is a constrained viscosity solution of (5.1.0.1) in $(0,T)\times\overline{\Omega}$.

REMARK 5.1.0.4. Given $m \in \text{Lip}(0,T;\mathcal{P}(\overline{\Omega}))$, it is known that u is the unique constrained viscosity solution of (5.1.0.1) in $\overline{\Omega}$.

For simplicity, we set

$$Q_m = \{(t,x) \in (0,T) \times \Omega : x \in supp(m(t))\}, \quad \partial Q_m = \{(t,x) \in (0,T) \times \partial\Omega : x \in supp(m(t))\}.$$

THEOREM 5.1.0.5. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let H and F satisfy hypotheses $(H0)$-$(H3)$ and $(D1)$-$(D3)$, respectively. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$ and let $(t,x) \in Q_m$. Then,

$$-p_1 + H(x,p_2) = F(x,m(t)), \quad \forall\,(p_1,p_2) \in D^+u(t,x). \tag{5.1.0.2}$$

Proof. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$. Since u is a constrained viscosity solution of (5.1.0.1) in $\overline{\Omega}$, we know that

$$-p_1 + H(x,p_2) \leq F(x,m(t)) \quad \forall\,(t,x) \in (0,T) \times \Omega, \ \ \forall\,(p_1,p_2) \in D^+u(t,x).$$

So, it suffices to prove that the converse inequality also holds. Let us take $(t,x) \in Q_m$ and $(p_1,p_2) \in D^+u(t,x)$. Since $(t,x) \in Q_m$, then there exists an optimal trajectory $\gamma : [0,T] \to \overline{\Omega}$ such that $\gamma(t) = x$. Let $r \in \mathbb{R}$ be small enough and such that $0 \leq t-r \leq t$. Since $(p_1,p_2) \in D^+u(t,x)$ one has that

$$u(t-r,\gamma(t-r)) - u(t,x) \leq -p_1 r - \langle p_2, x-\gamma(t-r)\rangle + o(r).$$

Since

$$x - \gamma(t-r) = \int_{t-r}^{t} \dot\gamma(s)\,ds,$$

we get

$$\langle p_2, x-\gamma(t-r)\rangle = \int_{t-r}^{t} \langle p_2, \dot\gamma(s)\rangle\,ds. \tag{5.1.0.3}$$

By the dynamic programming principle and (5.1.0.3) one has that

$$\begin{aligned}\int_{t-r}^{t} \Big[L(\gamma(s),\dot\gamma(s)) + F(\gamma(s),m(s))\Big]\,ds &= u(t-r,\gamma(t-r)) - u(t,x) \\ &\leq -\int_{t-r}^{t} \langle p_2,\dot\gamma(s)\rangle\,ds - p_1 r + o(r).\end{aligned}$$

By our assumptions on L and F and by THEOREM 2.3.0.1, one has that

$$\begin{aligned} &L(\gamma(s),\dot\gamma(s)) = L(x,\dot\gamma(t)) + o(r), \\ &F(\gamma(s),m(s)) = F(x,m(t)) + o(r), \\ &\langle p_2,\dot\gamma(s)\rangle = \langle p_2,\dot\gamma(t)\rangle + o(r), \end{aligned} \tag{5.1.0.4}$$

for all $s \in [t-r,t]$. Hence,

$$-p_1 - \langle p_2,\dot\gamma(t)\rangle - L(x,\dot\gamma(t)) \geq F(x,m(t)),$$

and so by the definition of H we conclude that

$$-p_1 + H(x,p_2) = -p_1 + \sup_{v\in\mathbb{R}^n}\{-\langle p,v\rangle - L(x,v)\} \geq -p_1 - \langle p_2, \dot\gamma(t)\rangle - L(x,\dot\gamma(t)) \geq F(x,m(t)).$$

This completes the proof. □

PROPOSITION 5.1.0.6. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let H and F satisfy the hypotheses $(H0)$-$(H3)$ and $(D1)$-$(D3)$, respectively. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$ and let $(t,x)\in Q_m$. Then, u is differentiable at (t,x).

Proof. By THEOREM 5.1.0.5 one has that

$$-p_1 + H(x,p_2) = F(x,m(t)) \;\; \forall\,(t,x)\in Q_m,\; \forall\,(p_1,p_2)\in D^+u(t,x).$$

Since $H(x,\cdot)$ is strictly convex and $D^+u(t,x)$ is a convex set, the above equality implies that $D^+u(t,x)$ is a singleton. By COROLLARY 4.3.0.2, and by [24, Proposition 3.3.4] one has that u is differentiable at (t,x). □

Let $x\in\partial\Omega$. We denote by $H^\tau : \partial\Omega\times\mathbb{R}^n \to \mathbb{R}$ the tangential Hamiltonian

$$H^\tau(x,p) = \sup_{\substack{v\,\in\,\mathbb{R}^n\\ \langle v,\nu(x)\rangle\,=\,0}} \{-\langle p,v\rangle - L(x,v)\}, \tag{5.1.0.5}$$

where $\nu(x)$ is the outward unit normal vector to $\partial\Omega$ in x.

THEOREM 5.1.0.7. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let H and F satisfy hypotheses $(H0)$-$(H3)$ and $(D1)$-$(D3)$, respectively. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$ and let $(t,x)\in\partial Q_m$. Then,

$$-p_1 + H^\tau(x,p_2) = F(x,m(t)),\quad \forall\,(p_1,p_2)\in D^+u(t,x). \tag{5.1.0.6}$$

Before giving the proof, let us prove a technical lemma.

LEMMA 5.1.0.8. Let $(t,x)\in(0,T)\times\partial\Omega$ and let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. Let $v\in\mathbb{R}^n$ be such that $\langle v,\nu(x)\rangle = 0$. Then, there exists $\widehat{\gamma}\in\Gamma_t[x]$ such that $\dot{\widehat{\gamma}}(t) = v$.

Proof. Let $(t,x)\in(0,T)\times\partial\Omega$ and let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. Let $v\in\mathbb{R}^n$ be such that $\langle v,\nu(x)\rangle = 0$. Let $R>0$ be small enough and let γ be the trajectory defined by

$$\gamma(s) = x + (s-t)v,$$

for all s such that $|s-t|<R$. We denote by $\widehat{\gamma}$ the projection of γ on $\overline{\Omega}$, i.e.,

$$\widehat{\gamma}(s) = \gamma(s) - d_\Omega(\gamma(s))Db_\Omega(\gamma(s)),$$

for all s such that $|s-t|<R$. By construction we observe that $\widehat{\gamma}\in\Gamma_t[x]$. We only have to prove that $\dot{\widehat{\gamma}}(t)=v$. Hence, recalling that $d_\Omega(\gamma(t))=0$ one has that

$$\frac{\widehat{\gamma}(s)-x}{s-t}=v-\frac{d_\Omega(\gamma(s))Db_\Omega(\gamma(s))}{s-t}=v-\left(\frac{d_\Omega(\gamma(s))-d_\Omega(\gamma(t))}{s-t}\right)Db_\Omega(\gamma(s)).$$

By LEMMA 2.1.0.1, and by the definition of γ we have that

$$\left|\frac{d_\Omega(\gamma(s))-d_\Omega(\gamma(t))}{s-t}\right|=\left|\fint_t^s\langle Db_\Omega(\gamma(r)),\dot{\gamma}(r)\rangle\mathbf{1}_{\Omega^c}\,dr\right|\leq\fint_t^s|\langle Db_\Omega(\gamma(r)),\dot{\gamma}(r)\rangle|\,dr.$$

Since $r\longmapsto\langle Db_\Omega(\gamma(r)),\gamma(r)\rangle$ is a continuous function, for $r\to 0$ one has that

$$\fint_t^s|\langle Db_\Omega(\gamma(r)),\gamma(r)\rangle|\,dr\to 0.$$

Hence,

$$\left|\frac{d_\Omega(\gamma(s))-d_\Omega(\gamma(t))}{s-t}\right|\to 0,$$

and so $\dot{\widehat{\gamma}}(t)=v$. This completes the proof. □

Proof of THEOREM 5.1.0.7. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$. Let us take $(t,x)\in\partial Q_m$ and $(p_1,p_2)\in D^+u(t,x)$. Let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. Let $v\in\mathbb{R}^n$ be such that $\langle v,\nu(x)\rangle=0$. Let $r>0$ be small enough and such that $0<t<t+r<T$. By LEMMA 5.1.0.8 there exists $\gamma\in\Gamma_t[x]$ such that $\dot{\gamma}(t)=v$. Since $(p_1,p_2)\in D^+u(t,x)$ one has that

$$u(t+r,\gamma(t+r))-u(t,x)\leq\langle p_2,\gamma(t+r)-x\rangle+rp_1+o(r). \tag{5.1.0.7}$$

By the dynamic programming principle we have that

$$u(t+r,\gamma(t+r))-u(t,x)\geq-\int_t^{t+r}\Big[L(\gamma(s),\dot{\gamma}(s))+F(\gamma(s),m(s))\Big]\,ds. \tag{5.1.0.8}$$

Moreover,

$$\langle p_2,\gamma(t+r)-x\rangle=\int_t^{t+r}\langle p_2,\dot{\gamma}(s)\rangle\,ds. \tag{5.1.0.9}$$

Using (5.1.0.8) and (5.1.0.9) in (5.1.0.7), we deduce that

$$-\int_t^{t+r}\Big[L(\gamma(s),\dot{\gamma}(s))+F(\gamma(s),m(s))+\langle p_2,\dot{\gamma}(s)\rangle\Big]\,ds-rp_1\leq o(r).$$

By our assumptions on L and F and by THEOREM 2.3.0.1, one has that

$$\begin{aligned}L(\gamma(s),\dot{\gamma}(s))&=L(x,\dot{\gamma}(t))+o(r),\\F(\gamma(s),m(s))&=F(x,m(t))+o(r),\\\langle p_2,\dot{\gamma}(s)\rangle&=\langle p_2,\dot{\gamma}(t)\rangle+o(r),\end{aligned} \tag{5.1.0.10}$$

for all $s \in [t, t+r]$. Using (5.1.0.10), dividing by r, and passing to the limit for $r \to 0$ we obtain that

$$-p_1 - \langle p_2, v\rangle - L(x,v) - F(x, m(t)) \leq 0. \tag{5.1.0.11}$$

By the arbitrariness of v, and by the definition of H^τ (5.1.0.11) implies that

$$-p_1 + H^\tau(x, p_2) \leq F(x, m(t)).$$

Now, we prove that the converse inequality also holds. Let $\gamma : [0,T] \to \overline{\Omega}$ be an optimal trajectory and such that $\gamma(t) = x$. Since $\gamma(t) \in \partial\Omega$, and $\gamma(s) \in \overline{\Omega}$ for all $s \in [0,T]$ one has that $\langle \dot\gamma(t), \nu(x)\rangle = 0$. Let $r > 0$ be small enough and such that $0 < t - r \leq t$. Since $(p_1, p_2) \in D^+u(t,x)$, and by the dynamic programming principle one has that

$$\begin{aligned}&\int_{t-r}^{t} \Big[L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m(s))\Big]\, ds = u(t-r, \gamma(t-r)) - u(t, \gamma(t))\\ &\leq -\langle p_2, \gamma(t) - \gamma(t-r)\rangle - rp_1 + o(r).\end{aligned}$$

Hence, we obtain that

$$\int_{t-r}^{t} \Big[L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m(s)) + \langle p_2, \dot\gamma(s)\rangle\Big]\, ds + rp_1 \leq o(r).$$

Arguing as above we deduce that

$$-p_1 - [\langle p_2, \dot\gamma(t)\rangle + L(x, \dot\gamma(t))] \geq F(x, m(t)).$$

Since $\langle \dot\gamma(t), \nu(x)\rangle = 0$, and by the definition of H^τ we conclude that

$$\begin{aligned}-p_1 + H^\tau(x, p_2) = -p_1 + &\sup_{\substack{v \in \mathbb{R}^n\\ \langle v, \nu(x)\rangle = 0}} \{-\langle p_2, v\rangle - L(x,v)\}\\ &\geq -p_1 - \langle p_2, \dot\gamma(t)\rangle - L(x, \dot\gamma(t)) \geq F(x, m(t)).\end{aligned}$$

This completes the proof. □

REMARK 5.1.0.9. Let $(t,x) \in \partial Q_m$. By the definition of H^τ for all $p \in D_x^+u(t,x)$ one has that

$$H^\tau(x,p) = H^\tau(x, p^\tau), \tag{5.1.0.12}$$

where p^τ is the tangential component of p.

In the next result, we give a full description of $D^+u(t,x)$ at $(t,x) \in \partial Q_m$.

PROPOSITION 5.1.0.10. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$ and let $(t,x) \in \partial Q_m$. The following holds true.

(a) The partial derivative of u with respect to t, denoted by $\partial_t u(t,x)$, does exist and

$$D^+u(t,x) = \{\partial_t u(t,x)\} \times D_x^+ u(t,x).$$

(b) All $p \in D_x^+u(t,x)$ have the same tangential component, which will be denoted by $p^\tau(t,x)$, that is,

$$\left\{p^\tau \in \mathbb{R}^n : p \in D_x^+u(t,x)\right\} = \{p^\tau(t,x)\}. \tag{5.1.0.13}$$

(c) For all $\theta \in \mathbb{R}^n$ such that $|\theta| = 1$ and $\langle \theta, \nu(x)\rangle = 0$ one has that

$$\partial_\theta^+ u(t,x) = \langle p^\tau(t,x), \theta\rangle. \tag{5.1.0.14}$$

Moreover,

$$\partial_{-\nu}^+ u(t,x) = -\lambda_+(t,x) := -\max\{\lambda_p(t,x) : p \in D_x^+u(t,x)\}, \tag{5.1.0.15}$$

where

$$\lambda_p(t,x) = \max\{\lambda \in \mathbb{R} : p^\tau(t,x) + \lambda\nu(x) \in D_x^+u(t,x)\}, \ \ \forall p \in D_x^+u(t,x). \tag{5.1.0.16}$$

(d) $D_x^+u(t,x) = \{p \in \mathbb{R}^n : p = p^\tau(t,x) + \lambda\nu(x),\ \lambda \in (-\infty, \lambda_+(t,x)]\}$.

Proof. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$. Let $(t,x) \in \partial Q_m$. By THEOREM 5.1.0.7, and by REMARK 5.1.0.9 we know that

$$-p_1 + H^\tau(x, p_2^\tau) = F(x, m(t)), \quad \forall (p_1,p_2) \in D^+u(t,x). \tag{5.1.0.17}$$

We argue by contradiction. Let $p = (p_1,p_2)$, $q = (q_1,q_2) \in D^+u(t,x)$ be such that $p_2^\tau \neq q_2^\tau$. Let $\lambda \in [0,1]$. Since $D^+u(t,x)$ is a convex set, we have that $p_\lambda = (p_{1,\lambda}, p_{2,\lambda}) = (\lambda p_1 + (1-\lambda)q_1, \lambda p_2 + (1-\lambda)q_2) \in D^+u(t,x)$. Moreover, we observe that

$$\begin{aligned}\lambda(p_2^\tau + p_2^\nu) + (1-\lambda)(q_2^\tau + q_2^\nu) = [\lambda p_2^\tau + (1-\lambda)q_2^\tau] \\ + [\lambda p_2^\nu + (1-\lambda)q_2^\nu] = p_{2,\lambda}^\tau + p_{2,\lambda}^\nu.\end{aligned}$$

Since $p_\lambda \in D^+u(t,x)$ then it satisfies (5.1.0.17) and

$$\begin{aligned}H^\tau(x, p_{2,\lambda}^\tau) &= p_{1,\lambda} + F(x,m(t)) = \lambda p_1 + (1-\lambda)q_1 + F(x,m(t)) \\ &= \lambda[p_1 + F(x,m(t))] + (1-\lambda)[q_1 + F(x,m(t))].\end{aligned}$$

Since H^τ is strictly convex, and recalling that p and q satisfy (5.1.0.17) we have that

$$\lambda H^\tau(x,p_2^\tau) + (1-\lambda)H^\tau(x,q_2^\tau) > H^\tau(x,p_{2,\lambda}^\tau) = \lambda H^\tau(x,p_2^\tau) + (1-\lambda)H^\tau(x,q_2^\tau),$$

and so we conclude that $p_1 = q_1$ and $p_2^\tau = q_2^\tau$. Thus, (a) and (b) hold true.
Let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. Let $\theta \in \mathbb{R}^n$ be such that $|\theta| = 1$ and $\langle \theta, \nu(x)\rangle = 0$. By LEMMA 1.3.1.4, and by (b) we deduce that

$$\partial_\theta^+ u(t,x) = \min_{p \in D_x^+ u(t,x)} \langle p, \theta\rangle = \min_{p \in D_x^+ u(t,x)} \langle p^\tau(t,x), \theta\rangle = \langle p^\tau(t,x), \theta\rangle,$$

and so (5.1.0.14) holds. Since u is locally semiconcave in $(0,T)\times\overline{\Omega}$, and by PROPOSITION 1.3.1.6 we have that

$$-\partial_{-\nu}^+ u(t,x) = \max\{\lambda_p(t,x) : p \in D_x^+ u(t,x)\} =: \lambda_+(t,x),$$

where

$$\lambda_p(t,x) = \max\{\lambda \in \mathbb{R} : p^\tau + \lambda\nu(x) \in D_x^+ u(t,x)\}.$$

Using PROPOSITION 1.3.0.3, and (c) we have that (d) holds. This completes the proof. □

THEOREM 5.1.0.11. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$. The following holds true.

(i) Let $(t,x) \in (0,T)\times\Omega$. Then, one has that

$$\limsup_{\substack{(s,y)\in(0,T)\times\Omega \\ (s,y)\to(t,x)}} D^+u(s,y) \subset D^+u(t,x). \tag{5.1.0.18}$$

In particular, for all $(t,x) \in Q_m$ we have that

$$\limsup_{\substack{(s,y)\in(0,T)\times\Omega \\ (s,y)\to(t,x)}} D^+u(s,y) = \Big\{\Big(\partial_t u(t,x), D_x u(t,x)\Big)\Big\}. \tag{5.1.0.19}$$

(ii) Let $(t,x) \in \partial Q_m$. Then,

$$\limsup_{\substack{(s,y)\in Q_m \\ (s,y)\to(t,x)}} D^+u(s,y) = \Big\{\Big(\partial_t u(t,x),\ p^\tau(t,x) + \lambda_+(t,x)\nu(x)\Big)\Big\}, \tag{5.1.0.20}$$

where $p^\tau(t,x)$ and $\lambda_+(t,x)$ are given in (5.1.0.13) and (5.1.0.15), respectively.

Proof. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$. By COROLLARY 4.3.0.2, by PROPOSITION 5.1.0.6, and by [24, Proposition 3.3.4] we deduce that (i) holds. Hence, we only need to analyze the point (ii).

Step 1

First, we prove that there exists a unique $\lambda \in (-\infty, \lambda_+(t,x)]$ such that (5.1.0.20) holds. Let $(t,x) \in \partial Q_m$ and let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. Let (t_k,x_k), $(s_k,y_k) \in Q_m$ be such that

$$(t_k,x_k) \xrightarrow{k\to\infty} (t,x), \quad Du(t_k,x_k) \xrightarrow{k\to\infty} p^\tau(t,x) + \lambda_0\nu(x),$$
$$(s_k,y_k) \xrightarrow{k\to\infty} (t,x), \quad Du(s_k,y_k) \xrightarrow{k\to\infty} p^\tau(t,x) + \lambda_1\nu(x),$$

where $\lambda_0 < \lambda_1 \leq \lambda_+(t,x)$. By THEOREM 5.1.0.5, and PROPOSITION 5.1.0.6 we have that

$$-\partial_t u(t_k,x_k) + H(x_k, Du(t_k,x_k)) = F(x_k, m(t_k)), \tag{5.1.0.21}$$
$$-\partial_t u(s_k,y_k) + H(y_k, Du(s_k,y_k)) = F(y_k, m(s_k)). \tag{5.1.0.22}$$

Passing to the limit in (5.1.0.21) and (5.1.0.22) we obtain that

$$-\partial_t u(t,x) + H(x, p^\tau(t,x) + \lambda_0\nu(x)) = F(x, m(t)),$$
$$-\partial_t u(t,x) + H(x, p^\tau(t,x) + \lambda_1\nu(x)) = F(x, m(t)).$$

Recalling that $(t,x) \in \partial Q_m$, and by THEOREM 5.1.0.7 one has that

$$H(x, p^\tau(t,x) + \lambda_0\nu(x)) = H^\tau(x, p^\tau(t,x)),$$
$$H(x, p^\tau(t,x) + \lambda_1\nu(x)) = H^\tau(x, p^\tau(t,x)).$$

Since H is strictly convex with respect to the second variable, then there exists $\lambda \in (\lambda_0, \lambda_1)$ such that

$$H(x, p^\tau(t,x) + \lambda\nu(x)) < H(x, p^\tau(t,x) + \lambda_i\nu(x)) \quad \text{for } i = 0, 1. \tag{5.1.0.23}$$

By the definition of H and H^τ, and by (5.1.0.23) one has that

$$\begin{aligned} H^\tau(x, p^\tau(t,x)) &= H(x, p^\tau(t,x) + \lambda_0\nu(x)) > H(x, p^\tau(t,x) + \lambda\nu(x)) \\ &= \sup_{v\in\mathbb{R}^n} \{-\langle p^\tau(t,x) + \lambda\nu(x), v\rangle - L(x,v)\} \\ &\geq \sup_{\substack{v \in \mathbb{R}^n \\ \langle v, \nu(x)\rangle = 0}} \{-\langle p^\tau(t,x), v\rangle - L(x,v)\} = H^\tau(x, p^\tau(t,x)). \end{aligned}$$

Therefore, $\lambda_0 = \lambda_1$ and so there exists a unique $\overline{\lambda} \in (-\infty, \lambda_+(t,x)]$ such that

$$\limsup_{\substack{(s,y) \in Q_m \\ (s,y)\to(t,x)}} D^+u(s,y) = \left\{ \left(\partial_t u(t,x),\ p^\tau(t,x) + \overline{\lambda}\nu(x) \right) \right\}. \tag{5.1.0.24}$$

Step 2

Now, we want to show that $\lambda_+(t,x) = \overline{\lambda}$. Let $\overline{p} \in \mathbb{R}^n$ be such that $\overline{p} = p^\tau + \overline{\lambda}\nu(x)$. Suppose that there exists two sequences $\{x_k\} \subset \Omega$, $\{p_k\} \subset \mathbb{R}^n$ such that:

(a) u is differentiable in (t, x_k);

(b) $\lim_{k\to+\infty} x_k = x$ and $\lim_{k\to+\infty} \frac{x_k-x}{|x_k-x|} = -\nu(x)$;

(c) $p_k \in D_x^+u(t,x_k)$ and $\lim_{k\to+\infty} p_k = \overline{p}$.

We want to prove that

$$\overline{p} \in \{p \in D_x^+u(t,x) : \langle p, -\nu(x)\rangle \le \langle q, -\nu(x)\rangle \,\forall\, q \in D_x^+u(t,x)\}. \tag{5.1.0.25}$$

By PROPOSITION 1.3.0.7 we have that $\overline{p} \in D_x^+u(t,x)$. Moreover, u is differentiable at (t,x_k) then $\{p_k\} = Du(t,x_k)$. Hence,

$$\frac{u(t,x_k)-u(t,x)}{|x_k-x|} \ge \langle Du(t,x_k), \frac{x_k-x}{|x_k-x|}\rangle + o(|x-x_k|), \quad \forall k \ge 1.$$

Thus we have that

$$\partial^+_{-\nu}u(t,x) \ge \limsup_{k\to+\infty} \frac{u(t,x_k)-u(t,x)}{|x_k-x|} \ge -\langle \overline{p}, \nu(x)\rangle.$$

By PROPOSITION 1.3.0.4 we conclude that

$$\langle \overline{p}, -\nu(x)\rangle = \partial^+_{-\nu}u(t,x) = \min_{p\in D_u^+(t,x)} \langle p, -\nu(x)\rangle. \tag{5.1.0.26}$$

So $\overline{p}$ satisfies (5.1.0.25). Moreover, by PROPOSITION 5.1.0.10, and by (5.1.0.26) one has that

$$-\lambda_+(t,x) = \partial^+_{-\nu}u(t,x) = \langle p^\tau + \overline{\lambda}\nu(x), -\nu(x)\rangle = -\overline{\lambda},$$

and this completes the proof. □

A direct consequence of the results of this section is the following theorem.

THEOREM 5.1.0.12. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let H, F and G satisfy hypotheses $(H0)$-$(H3)$ and $(D1)$-$(D3)$, respectively. Then, u is a constrained viscosity solution of

$$\begin{cases} -\partial_t u + H(x,Du) = F(x,m(t)) & \text{in } (0,T)\times\overline{\Omega} \\ u(x,T) = G(x,m(T)) & \text{in } \overline{\Omega}. \end{cases}$$

Moreover, u is solution in point-wise sense of

$$-\partial_t u + H(x,Du) = F(x,m(t)) \quad \text{in } Q_m,$$

and

$$-\partial_t u + H^\tau(x,p^\tau(t,x)) = F(x,m(t)) \quad \text{in } \partial Q_m.$$

5.2 THE CONTINUITY EQUATION

Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Assume that H, F and G satisfy the assumptions in SECTION 4.1. The main result of this section is the following theorem.

THEOREM 5.2.0.1. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let $m_0 \in \mathcal{P}(\overline{\Omega})$ and let (u, m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$. Then, there exists $V : [0,T] \times \overline{\Omega} \to \mathbb{R}^n$ Borel measurable vector field such that m is a solution in the sense of distribution of the continuity equation

$$\begin{cases} \partial_t m + div(V\, m) = 0, & \text{in } [0,T] \times \overline{\Omega}, \\ m(0,x) = m_0(x), & \text{in } \overline{\Omega}, \end{cases} \tag{5.2.0.1}$$

that is, for all $\phi \in C^1([0,T] \times \overline{\Omega})$ one has that

$$\int_{\overline{\Omega}} \phi(0,x) m_0(dx) - \int_{\overline{\Omega}} \phi(T,x) m(T,dx) = \int_0^T \int_{\overline{\Omega}} \Big[\partial_t \phi(t,x) + \langle \nabla\phi(t,x), V(t,x)\rangle\Big] m(t,dx)\, dt.$$

Moreover, V is continuous on Q_m and

$$V(t,x) = \begin{cases} -D_pH\big(x, Du(t,x)\big) & \text{if } (t,x) \in Q_m, \\ -D_pH\big(x, p^\tau(t,x) + \lambda_+(t,x)\nu(x)\big) & \text{if } (t,x) \in \partial Q_m, \end{cases} \tag{5.2.0.2}$$

where $\lambda_+(t,x)$ is given by (5.1.0.15) and $\nu(x)$ is the outward unit normal vector to $\partial\Omega$ in x.

It is convenient to divide the proof of THEOREM 5.2.0.1 in several lemmas.

LEMMA 5.2.0.2. Let $m_0 \in \mathcal{P}(\overline{\Omega})$ and let $\eta \in \mathcal{P}^{\text{Lip}}_{m_0}(\Gamma)$ be a constrained MFG equilibrium for m_0. Let $e_t : \Gamma \to \overline{\Omega}$ be the evaluation map defined in (3.1.2.1). Let $\{\eta^t_x\}_{x\in\overline{\Omega}}$ be a unique Borel measurable family of probabilities on Γ which disintegrates η with respect to e_t. Then, $\{\eta^t_x\}_{x\in\overline{\Omega}}$ is Borel measurable with respect to (t,x).

Proof. Let $m_0 \in \mathcal{P}(\overline{\Omega})$ and let $\eta \in \mathcal{P}^{\text{Lip}}_{m_0}(\Gamma)$ be a constrained MFG equilibrium for m_0. Let $e_t : \Gamma \to \overline{\Omega}$ be the evaluation map defined in (3.1.2.1) and let m be the probability measure on $\overline{\Omega}$ defined by (3.1.2.2) for all $t \in [0,T]$. Let $\mathcal{L}^1$ be Lebesgue measure and let π be a continuous map defined by

$$\pi := (I_d, e_t) : [0,T] \times \Gamma \to [0,T] \times \overline{\Omega},$$
$$(t,\gamma) \longmapsto (t, \gamma(t)),$$

where $I_d : [0,T] \to [0,T]$. Let $\overline{\eta}$ be the product probability measure on $[0,T] \times \Gamma$ defined by

$$\overline{\eta} = \frac{1}{T}\mathcal{L}^1 \otimes \eta.$$

We define w the probability measure on $[0,T]\times\overline{\Omega}$ as

$$w := \pi\sharp\overline{\eta}.$$

Step 1

We observe that $dw(t,x) = m(t,dx)\frac{dt}{T}$. Indeed, let $[a,b]\subset[0,T]$ and $B\in\mathcal{B}(\overline{\Omega})$. By the defintion of w and m one has that

$$\begin{aligned} w([a,b]\times B) &= \overline{\eta}\Big(\pi^{-1}([a,b]\times B)\Big) = \overline{\eta}\Big(\Big\{(t,\gamma): t\in[a,b], \gamma(t)\in B\Big\}\Big) \\ &= \overline{\eta}\Big(\Big\{(t,\gamma): t\in[a,b], \gamma\in e_t^{-1}(B)\Big\}\Big) = \int_a^b \eta(e_t^{-1}(B))\frac{dt}{T} = \int_a^b m(t,B)\frac{dt}{T}. \end{aligned}$$

Hence, passing to the sigma algebra on the product $[0,T]\times\overline{\Omega}$, we obtain that $dw(t,x) = m(t,dx)\frac{dt}{T}$.

Step 2

Let $\{\eta_x^t\}_{x\in\overline{\Omega}}\subset\mathcal{P}_{m_0}(\Gamma)$ be the disintegration of η with respect to e_t and let $\{\overline{\eta}_{t,x}\}_{(t,x)\in[0,T]\times\overline{\Omega}}$ be a unique Borel measurable family of probabilities on $[0,T]\times\Gamma$ which disintegrates $\overline{\eta}$. Now, we show that for any $\mathcal{B}\in\mathcal{B}(\Gamma)$ one has that

$$\eta_x^t(\mathcal{B}) = \overline{\eta}_{t,x}(\{t\}\times\mathcal{B}) \quad \forall t\in[0,T]. \tag{5.2.0.3}$$

Let us take $f\in C([0,T])$ and $g\in C(\Gamma)$. Let $\phi\in C([0,T]\times\Gamma)$ be defined by $\phi(t,\gamma) = f(t)g(\gamma)$. Since $\{\overline{\eta}_{t,x}\}_{(t,x)\in[0,T]\times\overline{\Omega}}$ disintegrates $\overline{\eta}$, and by the definition of w we have that

$$\int_{[0,T]\times\Gamma} f(t)g(\gamma)\,d\overline{\eta}(t,\gamma) = \int_{[0,T]\times\overline{\Omega}}\left(\int_{\pi^{-1}(t,x)} f(s)g(\gamma)\,d\overline{\eta}_{t,x}(s,\gamma)\right)dw(t,x).$$

Moreover, by the definition of π we have that $\pi^{-1}(t,x) = \{t\}\times e_t^{-1}(x)$ and so

$$\int_{[0,T]\times\Gamma} f(t)g(\gamma)\,d\overline{\eta}(t,\gamma) = \int_{[0,T]\times\overline{\Omega}}\left(\int_{\{t\}\times e_t^{-1}(x)} f(s)g(\gamma)\,d\overline{\eta}_{t,x}(s,\gamma)\right)dw(t,x). \tag{5.2.0.4}$$

Recalling that $m(t) := e_t\sharp\eta$ and $\{\eta_x^t\}_{x\in\overline{\Omega}}$ disintegrates η, one has that

$$\int_\Gamma g(\gamma)\eta(d\gamma) = \int_{\overline{\Omega}}\left(\int_{e_t^{-1}(x)} g(\gamma)\eta_x^t(d\gamma)\right)m(t,dx), \quad \forall t\in[0,T]. \tag{5.2.0.5}$$

By the definition of $\overline{\eta}$, and (5.2.0.5) we deduce that

$$\begin{aligned} \int_{[0,T]\times\Gamma} f(t)g(\gamma)\,d\overline{\eta}(t,\gamma) &= \frac{1}{T}\int_0^T f(t)\,dt\int_\Gamma g(\gamma)\eta(d\gamma) \\ &= \frac{1}{T}\int_0^T f(t)\,dt\int_{\overline{\Omega}}\left(\int_{e_t^{-1}(x)} g(\gamma)\eta_x^t(d\gamma)\right)m(t,dx) \\ &= \int_{[0,T]\times\overline{\Omega}} f(t)\left(\int_{e_t^{-1}(x)} g(\gamma)\eta_x^t(d\gamma)\right)\frac{m(t,dx)}{T}\,dt. \end{aligned}$$

By Step 1, and by (5.2.0.4) we conclude that (5.2.0.3) holds. Thus, $\{\eta_x^t\}_{x\in\overline{\Omega}}$ is Borel measurable with respect to (t,x). □

In the next lemma we show that there exists a measurable vector field such that m is a solution in the sense of distribution of (5.2.0.1).

LEMMA 5.2.0.3. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let $m_0 \in \mathcal{P}(\overline{\Omega})$. Let $\eta \in \mathcal{P}^{\text{Lip}}_{m_0}(\Gamma)$ be a constrained MFG equilibrium for m_0 and let $\{\eta^t_x\}_{x\in\overline{\Omega}} \subset \mathcal{P}_{m_0}(\Gamma)$ be the disintegration of η with respect to the evaluation map e_t. Let m be the probability measure on $\overline{\Omega}$ defined in (3.1.2.2). Then, there exists a Borel measurable vector field

$$V : [0,T] \times \overline{\Omega} \longrightarrow \mathbb{R}^n$$
$$(t,x) \longmapsto V(t,x) = \int_{e_t^{-1}(x)} \dot\gamma(t)\eta^t_x(\,d\gamma)$$

such that m is a solution in the sense of distribution of the continuity equation

$$\begin{cases} \partial_t m + div(V\,m) = 0, & \text{in } [0,T]\times\overline{\Omega}, \\ m(0,x) = m_0(x), & \text{in } \overline{\Omega}, \end{cases} \tag{5.2.0.6}$$

that is, for all $\phi \in C^1([0,T]\times\overline{\Omega})$ one has that

$$\int_{\overline{\Omega}} \phi(0,x)m_0(\,dx) - \int_{\overline{\Omega}} \phi(T,x)m(T,dx) = \int_0^T \int_{\overline{\Omega}} \Big[\partial_t\phi(t,x) + \langle \nabla\phi(t,x), V(t,x)\rangle\Big] m(t,dx)\,dt.$$

Proof. The idea of the proof is based on [33, Theorem 1]. Let $\eta \in \mathcal{P}^{\text{Lip}}_{m_0}(\Gamma)$ be a constrained MFG equilibrium for m_0 and let $\{\eta^t_x\}_{x\in\overline{\Omega}} \subset \mathcal{P}_{m_0}(\Gamma)$ be the disintegration of η with respect to the evaluation map e_t. Set $m(t) = e_t\sharp\eta$ for all $t\in[0,T]$. Let $\phi \in C^1([0,T]\times\overline{\Omega})$. By the definition of m we have that

$$\frac{d}{dt}\left[\int_{\overline{\Omega}} \phi(t,x)m(t,dx)\right] = \frac{d}{dt}\left[\int_\Gamma \phi(t,\gamma(t))\eta(\,d\gamma)\right] = \int_\Gamma \partial_t\phi(t,\gamma(t))\eta(\,d\gamma)$$
$$+ \int_\Gamma \langle\nabla\phi(t,\gamma(t)),\dot\gamma(t)\rangle\eta(\,d\gamma) = \int_{\overline{\Omega}} \partial_t\phi(t,x)m(t,dx) + \int_\Gamma \langle\nabla\phi(t,\gamma(t)),\dot\gamma(t)\rangle\eta(\,d\gamma).$$

Since $\{\eta^t_x\}_{x\in\overline{\Omega}}$ is the disintegration of η one has that

$$\int_\Gamma \langle\nabla\phi(t,\gamma(t)),\dot\gamma(t)\rangle\eta(\,d\gamma) = \int_{\overline{\Omega}} \left(\int_{e_t^{-1}(x)} \langle\nabla\phi(t,x),\dot\gamma(t)\rangle\eta^t_x(\,d\gamma)\right) m(t,dx)$$
$$= \int_{\overline{\Omega}} \Big\langle \nabla\phi(t,x), \int_{e_t^{-1}(x)} \dot\gamma(t)\eta^t_x(\,d\gamma)\Big\rangle m(t,dx).$$

Integrating on $[0,T]$ one has that

$$\int_{\overline{\Omega}} \phi(0,x)m(0,\,dx) - \int_{\overline{\Omega}} \phi(T,x)m(T,\,dx)$$
$$= \int_0^T \int_{\overline{\Omega}} \partial_t\phi(t,x)m(t,dx)\,dt + \int_0^T \int_{\overline{\Omega}} \Big\langle \nabla\phi(t,x), \int_{e_t^{-1}(x)} \dot\gamma(t)\eta^t_x(\,d\gamma)\Big\rangle m(t,dx)\,dt$$

$$= \int_0^T \int_{\overline{\Omega}} \Big[\partial_t \phi(t,x) + \langle \nabla\phi(t,x), V(t,x)\rangle\Big] m(t,dx)\, dt,$$

which implies that m is a solution in the sense of distribution of (5.2.0.6). Moreover, by LEMMA 5.2.0.2 we note that the vector field V defined by

$$V(t,x) = \int_{e_t^{-1}(x)} \dot\gamma(t)\eta_x^t(d\gamma), \quad \forall (t,x) \in [0,T]\times\overline{\Omega}$$

is measurable with respect to (t,x). □

REMARK 5.2.0.4. By definition of V we observe that $V(t,x)$ is tangential to $\partial\Omega$ for all $(t,x)\in\partial Q_m$.

Proof of THEOREM 5.2.0.1. Let (u,m) be a mild solution of the constrained MFG problem in $\overline{\Omega}$. By LEMMA 5.2.0.3 there exists a Borel measurable vector field

$$\begin{aligned} V : [0,T]\times\overline{\Omega} &\longrightarrow \mathbb{R}^n \\ (t,x) &\longmapsto V(t,x) = \int_{e_t^{-1}(x)} \dot\gamma(t)\eta_x^t(d\gamma) \end{aligned}$$

such that m is a solution in the sense of distribution of (5.2.0.1). Let us take $(t,x)\in Q_m$. Recalling the definition of $V(t,x)$, by THEOREM 2.3.0.1, and THEOREM 5.1.0.11 one has that

$$V(t,x) = -D_pH(x, Du(t,x)).$$

Let $(t,x)\in\partial Q_m$ and let $\nu(x)$ be the outward unit normal vector to $\partial\Omega$ in x. We consider the set

$$\Lambda(t,x) = \Big\{\lambda\in(-\infty,\lambda_+(t,x)] : \dot\gamma(t) = -D_pH(x, p^\tau(t,x)+\lambda\nu(x))\Big\}.$$

By THEOREM 2.3.0.1, and THEOREM 5.1.0.11 one has that

$$\dot\gamma(t) \in \Big\{ -D_pH(x,p^\tau(t,x)+\lambda\nu(x)) : \lambda\in(-\infty,\lambda_+(t,x)]\Big\}.$$

Therefore, we have that $\Lambda(t,x)$ is nonempty. Moreover, we observe that

$$\langle\dot\gamma(t),\nu(x)\rangle = 0, \tag{5.2.0.7}$$

for η_x^t-almost every $\gamma\in e_t^{-1}(x)$. Hence one has that

$$\Lambda(t,x) \subset \Big\{\lambda\in(-\infty,\lambda_+(t,x)] : \langle -D_pH(x,p^\tau(t,x)+\lambda\nu(x)),\nu(x)\rangle = 0\Big\}.$$

Since H is strictly convex with respect to the second variable we have that

$$\langle D_{pp}H(x,p^\tau(t,x)+\lambda\nu(x))\nu(x),\nu(x)\rangle > 0,$$

and in particular $\lambda \longmapsto \langle D_pH(x, p^\tau(t,x) + \lambda\nu(x)), \nu(x)\rangle$ is a nondecresing map. Thus, there exists at most one $\widetilde{\lambda}$ belongs to $\Lambda(t,x)$ and so

$$\Lambda(t,x) = \left\{\widetilde{\lambda}\right\}. \tag{5.2.0.8}$$

Before proving that $\lambda_+(t,x) = \widetilde{\lambda}$, we show that $\lambda_+(t,x)$ satisfies the following inequality

$$\langle D_pH(x, p^\tau(t,x) + \lambda_+(t,x)\nu(x)), \nu(x)\rangle \leq 0. \tag{5.2.0.9}$$

We argue by contradiction and we assume that

$$\langle D_pH(x, p^\tau(t,x) + \lambda_+(t,x)\nu(x)), \nu(x)\rangle > 0.$$

Since H is strictly convex with respect to the second variable, there exists $\lambda_0 \in (-\infty, \lambda_+(t,x)]$ such that

$$H(x, p^\tau(t,x) + \lambda_0\nu(x)) < H(x, p^\tau(t,x) + \lambda_+(t,x)\nu(x)).$$

Recalling that

$$H(x, p^\tau(t,x) + \lambda_+(t,x)\nu(x)) = \partial_t u(t,x) + F(x, m(t)),$$

one has that

$$\begin{aligned}\partial_t u(t,x) + F(x,m(t)) > H(x, p^\tau(t,x) + \lambda_0\nu(x)) \geq \sup_{\substack{v \,\in\, \mathbb{R}^n \\ \langle v,\, \nu(x)\rangle \,=\, 0}} \{-\langle p, v\rangle - L(x,v)\} \\ = H^\tau(x, p^\tau(t,x)) = \partial_t u(t,x) + F(x, m(t)).\end{aligned}$$

Therefore, (5.2.0.9) holds.
Now we only have to prove that $\widetilde{\lambda} = \lambda_+(t,x)$. We argue by contradiction and we suppose that $\widetilde{\lambda} \neq \lambda_+(t,x)$. Since $\lambda \longmapsto \langle D_pH(x, p^\tau(t,x) + \lambda\nu(x)), \nu(x)\rangle$ is a nondecresing map, by (5.2.0.9), and by (5.2.0.8), we conclude that

$$0 = \langle D_pH(x, p^\tau(t,x) + \widetilde{\lambda}\nu(x)), \nu(x)\rangle < \langle D_pH(x, p^\tau(t,x) + \lambda_+(t,x)\nu(x)), \nu(x)\rangle \leq 0.$$

Thus, $\widetilde{\lambda} = \lambda_+(t,x)$. Therefore, (5.2.0.2) holds, and by our assumptions on H, V is continuous on Q_m. This completes the proof.

□

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