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XXX

Unirationality of varieties described by families of hypersurfaces and quadratic line complexes

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Introduction

This work concerns two independent subjects which are both fundamental in classical algebraic geometry: unirationality and Grassmann varieties. In particular, we focus here on unirationality of families of hypersurfaces and on proprieties of quadratic line complexes. Both parts contain some interesting results.

An algebraic variety X is *unirational* if we can find a dominant ration map from a projective space to X. If such a map is birational, then X is said *rational*. Lüroth proved in a classical theorem that the two notions coincides for a curve [Lür75]; the same was proved by Castelnuovo and Enriques for surfaces [Cas94], [Enr97]. However, it was not known if the two notions were the same in general until the 1970s, when Clemens and Griffiths [CG72] and Iskovskikh and Manin [IM71] found varieties of dimension three which are unirational but not rational. Then it was clear that it is easier to obtain examples of unirational varieties than rational ones. We work so on the weaker notion of unirationality.

The first part of the thesis is dedicated to prove, under appropriate hypotheses, the unirationality of varieties described by families of hypersurfaces, and we prove it by using two classical ideas. The first idea comes from an article of Conforto [Con41] which proves, under certain condition, the unirationality of a family of quadric hypersurfaces parametrized by a unirational basis. To do it, he finds a rational section: i.e., he finds a rational point on the generic fibre. Then, using a criterion of Roth [Rot50], he obtains the proof. This approach can be found also in an older article by Comessatti [Com40]: he uses the "rational section idea" to prove the rationality of a family of conics, and he does so very nicely working directly with polynomials.

The other idea comes from works proving the unirationality of hypersurfaces of low degree and high dimension. The first of this work can be attributed to Morin [Mor42], and two are the main articles which improves his results: the one by Murre [Mur79] and another by Ciliberto [Cil80]. The methods in these articles are the same: for a generic high dimensional hypersurface of low degree, to find a linear space on it corresponds to proving its unirationality. Then each author gives a lower bound on the dimension of the hypersurfaces for which it is possible to find the aforementioned linear space and to construct the dominant rational map from a projective space which proves the unirationality.

These ideas are put together in the present work and allow us to prove the unirationality of varieties described by suitable families of hypersurfaces of low degree and high dimension: given a family of this kind, we construct the associated Fano family of the appropriate linear spaces contained in them and we find a rational section of it. At the end, we are able to give a numerical criterion for the unirationality of varieties described by suitable families of hypersurfaces.

The second part of the thesis is dedicated to the study of quadratic line complexes, i.e., intersection of Grassmannian of lines with quadrics in the Plücker ambient space. We study the quadratic line complexes in relation with their discriminant loci. Given a quadratic complex of lines, we can see it as assigning to each point of the initial projective space a quadric hypersurface in a hyperplane: the discriminant hypersurface parametrizes the points whose corresponding quadric is singular. Then, studying its proprieties, we construct a double cover of the discriminant of the complex of lines in a four-dimensional space. In the final section, we show the connection between this double cover and O'Grady's double cover of the so called EPW–sextic in the five–dimensional space [OGr06].

Let us summarize the contents of each Chapter. In Chapter 1 we recall some results on basic algebraic geometry subjects, such as quadrics, rationality and osculating spaces.

In Chapter 2, we begin to recall the rationality of Grassmann varieties. Then, in Section 2.2, we recall the concept of a family of hypersurfaces and we show how to find a "rational section". Then we construct the Fano families, and in the final Section of this Chapter, we obtain the final result about unirationality, Corollary 2.24.

In Chapter 3, we introduce and study quadratic line complexes in a projective space of dimension n. We give a description of *special points* and *special lines*, which are strictly related to the concept of *discriminant hypersurface* $D(X)$, as the Proposition 3.17 shows. Then, we study the singularities of this hypersurface, and in the case $n = 4$ (i.e., the lines of the complex lie in a four dimensional projective space), we construct a double cover of $D(X)$. Then in the final section we prove that this is a canonical morphism, and its relation with the *EPW-sextic*.

Chapter 1

Preliminaries

1.1 Quadrics

Let $|\mathscr{O}_{\mathbb{P}^n}(2)|$ be the linear system of quadrics of dimension $n-1$, and let $x = [x_0, \ldots, x_n]$ be the homogeneous coordinates of $\mathbb{P}^n$. A quadric Q in this linear system has equation of the form $x \cdot M \cdot x^t = 0$ for some non-zero symmetric matrix M of order $n+1$ defined up to multiplication by a constant. Thus the matrix entries m_{ij}'s up to scalar multiplication (simply denoted by $[M]$) can be thought as the homogeneous coordinates of the quadric in $|\mathscr{O}_{\mathbb{P}^n}(2)|$. This is a way to compute the dimension of $|\mathscr{O}_{\mathbb{P}^n}(2)|$, which is $\frac{n(n+3)}{2}$.

We define now the *rank* of a quadric as the rank of the associated matrix M.

Remark 1.1. We can geometrically describe the quadrics of $\mathbb{P}^n$ of a given rank k. If a quadric Q has maximal rank, i.e. $k = n+1$, then Q is smooth. Otherwise, if $k \leq n$, it is a cone with vertex a linear space of dimension $n-k$ over a smooth quadric Q' in a $\mathbb{P}^{k-1}$.

Definition 1.2. Let $\mathbb{P}^{mn-1}$ be the projective space associated to the vector space of $m \times n$ matrices. We define the *r-th generic determinantal variety* M_r to be the variety parametrizing matrices of rank at most r.

Similarly, we define the *symmetric determinantal variety* D_k as the variety of symmetric matrices of $|\mathscr{O}_{\mathbb{P}^n}(2)|$ of corank at least k.

We observe that the irreducibility of the objects described above is not obvious. It is a corollary of the following results.

Theorem 1.3. *The generic determinantal variety M_1 inside $\mathbb{P}^{mn-1}$ is the Segre variety* $\mathrm{Seg}(m-1, n-1) = \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$. *The generic determinantal variety M_r for $2 \leq r \leq \min\{m,n\}$ is the $(r-1)$-secant variety of the Segre variety corresponding to M_1.*

The proof is in [Har92, pp. 98-99]. Briefly, a matrix M has rank 1 if and only if it can be expressed as the product $X \cdot Y^t$, where X and Y are non-zero vectors in $\mathbb{K}^m$ and $\mathbb{K}^n$ respectively. Then, a matrix M has rank k or less if and only if the associated linear map between $\mathbb{K}^n \to \mathbb{K}^m$ has rank k if and only if the map can be expressed as a sum of k linear map of rank 1.

Theorem 1.4. *The symmetric determinantal variety D_n inside $|\mathscr{O}_{\mathbb{P}^n}(2)|$ is the Veronese variety $V_{2,n}$. The symmetric determinantal variety D_k for $1 \leq k \leq n$ is the $(n-k)$-secant variety of the Veronese variety corresponding to D_n.*

The proof of the first part is again in [Har92, Ch. 9]. The second statement has a similar proof of Theorem 1.3. A symmetric matrix M has rank 1 if and only if it can be expressed as a product $X \cdot X^t$, where $X \in \mathbb{K}^n \setminus \{0\}$. Then a symmetric matrix has rank less than or equal to k if and only if it can be expressed as the sum of k symmetric rank 1 matrices.

Lemma 1.5. *The following proprieties hold:*

1. *The variety M_k of matrices of rank at most k is irreducible of codimension $(n-k)(m-k)$;*

2. *The variety D_k of symmetric matrices of corank at least k is irreducible of codimension $\binom{k+1}{2}$.*

Proof. We need some basic proprieties of Grassmannian varieties $\mathbb{G}(k,n)$ (respectively, $G(k,n)$) which parametrize all the k-planes of a projective space of dimension n (respectively, all the k-dimensional space of $\mathbb{K}^n$, cf. Remark 2.1). More information about Grassmannians will be discussed in Section 2.1.

1. Let $I \subset \mathbb{P}^{mn-1} \times G(n-k,n)$ be the incidence correspondence given by

$$I = \{(M,\Lambda) | \Lambda \subset \ker(M)\}$$

where we consider M as a linear map (see the remark after Theorem 1.3). Let π_1 and π_2 be the projections on the first and on the second factor respectively. We observe that $M_k = \pi_1(I)$. Then, fixed a point $\Lambda' \in G(n-k,n)$, the fibre of π_2 over Λ' is isomorphic to $\mathbb{P}(V)$, where $V = \mathrm{Hom}(\mathbb{K}^n/\Lambda', \mathbb{K}^m)$. Then we have that I is irreducible and of dimension $\dim(G(n-k,n)) + km - 1 = (n-k)k + km - 1 = k(n+m-k) - 1$. Therefore M_k is irreducible; moreover, since π_1 is generically injective, we have that $\dim(I) = \dim(M_k)$. Thus it follows that $\mathrm{codim}(M_k) = mn - 1 - k(n+m-k) + 1 = (m-k)(n-k)$.

2. We use the description of symmetric matrices as coordinates of quadrics in $\mathbb{P}^n$ and Remark 1.1. The we define the incidence correspondence $J \subset |\mathscr{O}_{\mathbb{P}^n}(2)| \times \mathbb{G}(k-1,n)$ as

$$J = \{(Q,\Lambda) | \Lambda \subset Sing(Q)\}$$

with the two projections π_1 and π_2; thus $\pi_1(J) = D_k$. Any fibre of π_2 is a isomorphic to $|\mathscr{O}_{\mathbb{P}^{n-k}}(2)|$, so J is irreducible of dimension $\dim(\mathbb{G}(k-1,n)) + \frac{(n-k)(n-k+3)}{2} = k(n-k+1) + \frac{(n-k)(n-k+3)}{2} = \frac{n^2+3n+k^2-k}{2}$. Then D_k is irreducible, and, since the first projection is generically injective, D_k has the same dimension of J. Performing a simple calculation we obtain the exact codimension.

□

Definition 1.6. We call D_1 the *discriminant hypersurface.* In particular, its equation is the determinant of a symmetric matrix of order $n+1$, and its degree is $n+1$.

Next result has again a proof in [Har92, Theorem 22.33].

Proposition 1.7. *If $Q_0 \in D_k \backslash D_{k+1}$, then D_k is smooth at Q_0 and its tangent space is given by the linear system of quadrics Q containing $Sing(Q_0)$.*

In addition to this, if $Q_0 \in D_k \setminus D_{k+1}$, then its singular locus $Sing(Q_0)$ is isomorphic to $\mathbb{P}^k$ and its tangent cone to D_1 is given by $\{Q | cork(Q|_{Sing(Q_0)}) \geq 1\}$ and the multiplicity of D_1 at Q_0 is k.

Remark 1.8. We recall that to prove the Proposition 1.7 we use the fact that given a line L passing through $Q_0 \in D_k \setminus D_{k+1}$, the intersection scheme of the vertex $Sing(Q)$ with the base locus scheme of L is either the whole vertex of Q_0 if $L \subset T_{D_1,Q_0}$ or a quadric hypersurface.

1.2 Rationality and Unirationality

Definition 1.9. Let X be an algebraic variety defined over a field $\mathbb{K}$. Then X is said to be *rational* if there exists a birational isomorphism between X and a projective space over $\mathbb{K}$. This amounts to saying that the function field of X is isomorphic to $\mathbb{K}(t_1, \ldots, t_s)$ for some s.

Definition 1.10. Let X be an algebraic variety defined over $\mathbb{K}$. Then X is said to be *unirational* if there exists a dominant rational map

$$\phi : \mathbb{P}^r \dashrightarrow X$$

for some r.

This amounts to saying that there exists an embedding of the function field $K(X)$ in to $\mathbb{K}(t_1, \ldots, t_r)$.

Rationality and unirationality are a fundamental propriety of algebraic varieties, and they will be central topic in Chapter 2. While in general it is not simple to prove the rationality (or even the unirationality) of a variety, there are simple examples of rational and unirational varieties: i.e., Grassmannian varieties in Section 2.1 and smooth quadric hypersurfaces are rational varieties.

Theorem 1.11. *Let X be a smooth quadric hypersurface of $\mathbb{P}^n$ defined over $\mathbb{K}$, with $n \geq 2$. X is unirational if and only if X is rational if and only if X contains a $\mathbb{K}$-rational point (i.e., a point given by an embedding* $\mathrm{Spec}(\mathbb{K}) \to X$*).*

Proof. If there is a rational function $\mathbb{P}^n_{\mathbb{K}} \dashrightarrow X$, then X has a $\mathbb{K}$-rational point.

Conversely, if X has a $\mathbb{K}$-rational point P then we can project X from this point on a $\mathbb{P}^{n-1}_{\mathbb{K}}$ which does not contain P. This map is defined on $X \setminus P$ and it is an isomorphism on this open subset. The last implication is obvious. □

1.3 Some proprieties of osculating spaces

In this section we work over the complex field $\mathbb{C}$.

Definition 1.12. Let X be a k-dimensional variety in $\mathbb{P}^n$ and let P be a smooth point of X. X has a *local analytic parametrization* around P if there exists an analytic neighbourhood U of P and an invertible analytic function $f : V \subset \mathbb{C}^k \to U$ such that $f(0) = P$ and $f(V) = X \cap U$.

Definition 1.13. Let X be a k-dimensional variety in $\mathbb{P}^n$ and let P be a smooth point of X. Suppose there exists an analytic local parametrization of X around P, given by a $(n+1)$-uple $[f_0(u_1, \ldots, u_k), \ldots, f_n(u_1, \ldots, u_k)]$.

We call the *derived point of order r at P* respect to a multi-index $\alpha = (\alpha_1, \ldots, \alpha_k)$, with $|\alpha| = r$, the point P_α whose coordinates are given by

$$P_{\alpha,i} := \frac{\partial^r f_i(u_1, \ldots, u_k)}{\partial u_1^{\alpha_1} \ldots \partial u_k^{\alpha_k}}(0) \tag{1.1}$$

for $i = 0, \ldots, n$, when at least one of the previous derivatives is different from zero. Otherwise, P_α is not defined.

Definition 1.14. We define the *r-osculating space* of X at P as the smallest linear space $\Pi_{r,P}$ containing P and all of its derived points up to order r.

Remark 1.15. From the previous definition, one could believe that the r-osculating space is strictly connected to the given parametrization: this is not true. In fact, from its definition it follows that a hyperplane H contains $\Pi_{r,P}$ if and only if the intersection multiplicity of X with H at P is at least $r+1$. Therefore, the r-osculating space can be alternatively defined as the centre of the projective family of hyperplanes having intersection multiplicity with X at P at least $r+1$.

Definition 1.16. Let C be a curve in $\mathbb{P}^n$ and let Λ be a linear space such that $P \in C \cap \Lambda$. We define the intersection multiplicity of C and Λ at P as the intersection multiplicity of X and H at P, where H is a general hyperplane passing through P

Remark 1.17. If C is a curve contained in X such that P is a smooth point of C, then the intersection multiplicity of C and $\Pi_{r,P}$ at P is at least $r+1$.

Chapter 2

Unirationality of families of hypersurfaces

2.1 Grassmann varieties

In this section we study Grassmann varieties using Semple's approach from [Sem31]. Let $\mathbb{G}(k,n)$ be the Grassmannian of k-planes in $\mathbb{P}^n = \mathbb{P}(V)$, where V is a $\mathbb{K}$-vector space of dimension $n+1$.

Remark 2.1. The Grassmannian $\mathbb{G}(k,n)$ can be also identified with the variety of the $(k+1)$-dimensional subspaces of V. Later this identification will be useful, so we introduce here the notation $G(k,n)$ or $G(k,V)$ to denote the variety containing the subspaces of dimension k of a $\mathbb{K}$-vector space V of dimension n.

Remark 2.2. We observe that there is a one-to-one correspondence between k-dimensional subspaces of V and $(n-k)$ dimensional subspaces of $V^\star$: from this we obtain a canonical isomorphism between $G(k,V)$ and $G(n-k,V^\star)$. Hence there is a duality isomorphism $\mathbb{G}(k,n) \cong \mathbb{G}(n-k-1,n)$. We assume without loss of generality that $k < \frac{n}{2}$.

For any $\Lambda \in \mathbb{G}(k,n)$ we can choose $k+1$ points $P_1, \dots, P_{k+1}$, associated to the vectors $v_1, \dots, v_{k+1}$, such that they span Λ. We can so define a morphism:

$$\phi : \mathbb{G}(k,n) \to \mathbb{P}(\bigwedge^{k+1} V) = \mathbb{P}^N \tag{2.1}$$

$$\phi(\Lambda) = [v_1 \wedge \ldots \wedge v_{k+1}] \tag{2.2}$$

where $N := \binom{n+1}{k+1} - 1$.

This map is well defined: if we choose another $(k+1)$-tuple $v'_1, \dots, v'_{k+1}$ for the same k-space Λ their wedge product differs from the previous one only for the determinant of the change of basis matrix. It is also an injection. This morphism is called *Plücker embedding.*

Explicitly, having fixed a basis $B = \{e_1, \ldots, e_{n+1}\}$ of V we can associate to any k-plane Λ a $(k+1) \times (n+1)$ matrix M_Λ

$$\begin{bmatrix} v_{1,1} & \ldots & v_{1,n+1} \\ \vdots & & \vdots \\ v_{k+1,1} & \ldots & v_{k+1,n+1} \end{bmatrix} \tag{2.3}$$

whose rows are the coordinates of the v_i's with respect to the basis B: two matrices M and M' represent the same k-plane if there exists $A \in GL(k+1)$ such that $M = AM'$.

The homogeneous coordinates of the point in $\mathbb{P}^N$ corresponding to M_Λ are given by the minors of order $k+1$. These are called *Plücker coordinates*, and we denote them by z_I where $I = (i_1, \ldots, i_{k+1})$ is a multi-index with $1 \leq i_1 < i_2 < \ldots < i_{k+1} \leq n+1$.

If we consider the subset U of points of $\mathbb{P}^N$ where the first coordinate $z_{1 \ldots k+1}$ is different from zero, each point $\Lambda \in U \cap \mathbb{G}(k,n)$ represents a k-plane whose associated matrix M_Λ can be written in the form

$$\begin{bmatrix} 1 & 0 & \ldots & 0 & v_{1,k+2} & \ldots & v_{1,n+1} \\ 0 & 1 & \ldots & 0 & v_{2,k+2} & \ldots & v_{2,n+1} \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 & v_{k+1,k+2} & \ldots & v_{k+1,n+1} \end{bmatrix}. \tag{2.4}$$

From this description it follows that $U \cap \mathbb{G}(k,n)$ is isomorphic to $\mathbb{A}^{(k+1)(n-k)}$ and then the Grassmann variety is rational. How can this isomorphism be extended to a rational map from $\mathbb{G}(k,n)$ to $\mathbb{P}^{(k+1)(n-k)}$? We will see that this map is a projection of the Grassmannian from $\mathbb{P}^N$ to $\mathbb{P}^{(k+1)(n-k)}$.

Remark 2.3. We can describe geometrically the open subset U: it is the set of the k-planes of $\mathbb{P}^n$ such that they do not intersect the $(n-k-1)$-plane spanned by the points corresponding to $e_{k+2}, \ldots, e_{n+1}$, the last $n-k$ vectors of the basis B. For any choice of a totally decomposable element of $\wedge^{n-k} V$ (i.e., a vector which can be expressed as $v_1 \wedge \ldots \wedge v_{n-k}$) we can so construct a birational map between $\mathbb{G}(k,n)$ and $\mathbb{P}^{(k+1)(n-k)}$.

Now we consider $\mathbb{P}^M = \mathbb{P}^{(k+1)(n-k)}$ with homogeneous coordinates given by y and $x_{i,j}$ for $i = 1, \ldots, k+1$ and $j = k+2, \ldots, n+1$: in this setting the affine space defined above is the complement of the hyperplane H given by the equation $y = 0$. We define the rational map

$$\psi : \mathbb{P}^M \dashrightarrow \mathbb{P}^N \tag{2.5}$$

such that for a point $[y, x_{1,k+2}, \ldots, x_{k+1,n+1}]$ its image is the point whose coordinates are given by the $k+1$ minors of the matrix

$$\begin{bmatrix} y & 0 & \ldots & 0 & x_{1,k+2} & \ldots & x_{1,n+1} \\ 0 & y & \ldots & 0 & x_{2,k+2} & \ldots & x_{2,n+1} \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & y & x_{k+1,k+2} & \ldots & x_{k+1,n+1} \end{bmatrix}. \tag{2.6}$$

If we consider the open subset $U' = \{y \neq 0\} \subset \mathbb{P}^M$, $\psi|_{U'}$ is the isomorphism described above between $U \cap \mathbb{G}(k, n)$ and $\mathbb{A}^{(k+1)(n-k)}$.

We can describe the linear system $\mathfrak{d}$ of hypersurfaces associated to this map. It corresponds to the vector space $W \subset H^0(\mathbb{P}^M, \mathscr{O}_{\mathbb{P}^M}(k+1))$ spanned by y^{k+1} and the forms

$$y^{k+1-r} D^r_{i_1,\dots,i_r;j_1,\dots,j_r} \tag{2.7}$$

for $r = 1, \dots, k+1$, $k+2 \le i_1 < \dots < i_r \le n+1$ and $1 \le j_1 < \dots j_r \le k+1$, where we denote by $D^r_{i_1,\dots,i_r;j_1,\dots,j_r}$ the minor of order r involving the columns $i_1, \dots, i_r$ and the rows $j_1, \dots, j_r$ of the matrix

$$\begin{bmatrix} x_{1,k+2} & \cdots & x_{1,n+1} \\ x_{2,k+2} & \cdots & x_{2,n+1} \\ \vdots & & \vdots \\ x_{k+1,k+2} & \cdots & x_{k+1,n+1} \end{bmatrix}. \tag{2.8}$$

For a fixed $r = 1, \dots, k+1$, we define m_r as the number of the D^r's. Thus, $m_r = \binom{k+1}{r}\binom{n-k}{r}$.

Lemma 2.4. *The linear system $\mathfrak{d}$ defined above corresponds to hypersurfaces of degree $k+1$ in $\mathbb{P}^M$ passing with multiplicity at least k through a Segre variety* $\mathrm{Seg}(k, n-k-1)$ *contained in the hyperplane H with equation $y = 0$.*

Proof. This linear system $\mathfrak{d}$ has a base locus B_1: observing the basis of W in (2.7), this is the subvariety of H defined by the vanishing of the minors $D^{k+1}_{i_1,\dots,i_{k+1};j_1,\dots,j_{k+1}}$. Then it follows from Theorem 1.3 that B_1 is the $(k-1)$-th secant variety of a Segre variety $\mathrm{Seg}(k, n-k-1)$ inside H.

Inside B_1, moreover, the linear system $\mathfrak{d}$ determines the singular loci B_r for $r = 2, \dots, k$ where every hypersurface of $\mathfrak{d}$ passes through with multiplicity r. Each B_r will be in particular the locus of the points $P \in B_1$ such that P is a point of multiplicity r for B_1. These loci B_r fill up the $(k-r)$-secant variety of $\mathrm{Seg}(k, n-k-1)$ for $r = 2, \dots, k$ (see Section 1.1 and [Har92]). In particular, each hypersurface in the linear system passes through the Segre variety, which is B_k, with multiplicity k.

Now, we show that this last condition implies the others. In fact, if a hypersurface α of degree $k+1$ contains the Segre variety with multiplicity k, each secant line of the Segre must be contained in α with multiplicity $k-1$. Suppose that this line intersects α in a finite number of points: then their intersection multiplicity is at least $2k$, contradiction. So the secant variety B_{k-1} is contained in α, and since the points of B_k have multiplicity 2 inside B_{k-1}, B_{k-1} is contained in α with multiplicity $k-1$.

This process can be repeated and if we take $r+1$ points in general position on $\mathrm{Seg}(k, n-k-1)$, the r-plane they span must be contained in α with multiplicity $k-r$. □

Lemma 2.5. *Let* Seg$(1, k)$ *be a Segre variety in* $\mathbb{P}^n$*. Let* $\phi : \mathbb{P}^1 \to \mathbb{G}(k, n)$ *be the morphism which sends a point* p *to the* k*-plane* $p \times \mathbb{P}^k$*. Then its image is a rational normal curve of degree* $k + 1$ *inside* $\mathbb{G}(k, n)$*.*

Proof. We can assume $n = 2k + 1$. Then if x_0, x_1 are the homogeneous coordinates of $\mathbb{P}^1$ and z_{ij}, $i = 0, 1$ and $j = 0, \dots, k$, are the homogenous coordinates of $\mathbb{P}^{2k+1}$, we can assume that ϕ is the map which sends $[\alpha_0, \alpha_1]$ to the k-plane whose equations in $\mathbb{P}^{2k+1}$ are $\alpha_1 z_{0j} - \alpha_0 z_{1j} = 0$ for $j = 0, \dots, k$. In particular, the image of a point $[x_0, x_1]$ under ϕ is the point of $\mathbb{G}(k, 2k+1)$ whose coordinates are the minors of maximal order of the matrix

$$\begin{bmatrix} x_0 & 0 & \dots & 0 & x_1 & 0 & \dots & 0 \\ 0 & x_0 & \dots & 0 & 0 & x_1 & \dots & 0 \\ \vdots & & \ddots & & & & \ddots & \vdots \\ 0 & 0 & \dots & x_0 & 0 & 0 & \dots & x_1 \end{bmatrix}. \tag{2.9}$$

There are only $k + 2$ non-vanishing Plücker coordinates: these correspond to the monomials of degree $k + 1$ in x_0 and x_1. Then it follows that the linear system associated to the map is complete and equal to $H^0(\mathbb{P}^1, \mathcal{O}(2))$. Therefore the image of ϕ is a rational normal curve of degree $k + 1$. □

We can generalize the result above:

Lemma 2.6. *Let* Seg$(1, r)$ *be a Segre variety in* $\mathbb{P}^n$ *and let* Π *be a* $(k - r - 1)$*-plane, with* $k > r$*, which does not intersect the* $(2r + 1)$*-plane spanned by* Seg$(1, r)$*. Let* $\phi : \mathbb{P}^1 \to \mathbb{G}(k, n)$ *be the morphism which sends a point* p *to the* k*-plane spanned by* Π *and the* r*-plane inside the Segre variety given by* $p \times \mathbb{P}^r$*. Then its image is a rational normal curve of degree* $r + 1$ *inside* $\mathbb{G}(k, n)$*.*

Proof. We can assume $n = k+r+1$: now let Seg$(1, r)$ be a Segre variety inside the $(2r + 1)$-plane L given by the vanishing of the last $k - r$ homogeneous coordinates of $\mathbb{P}^{k+r+1}$, and let M be the $k-r-1$ plane given by the vanishing of the first $2r + 2$ coordinates. Then we can associate to the point $[x_0, x_1]$ of $\mathbb{P}^1$ the matrix whose rows span the join of M and $[x_0, x_1] \times \mathbb{P}^r$. This is the $(k + 1) \times (k + r + 2)$ matrix

$$\left[\begin{array}{c|c} M_{x_0,x_1} & 0_1 \\ \hline 0_2 & I_{k-r} \end{array}\right],$$

where M_{x_0,x_1} is the $(r + 1) \times (2r + 2)$ matrix associated to the r-plane $[x_0, x_1] \times \mathbb{P}^r$ in L, 0_1 is the $(r+1)\times(k-r)$ zero matrix, 0_2 is the $(k-r)\times(2r+2)$ zero matrix and I_{k-r} is the $(k - r)$ identity matrix. Under our assumption, the only non-vanishing Plücker coordinates of the point of the Grassmaniann associated to this matrix are given by the monomials of order $r + 1$ in x_0 and x_1. Thus we have constructed a morphism between $\mathbb{P}^1$ and $\mathbb{G}(k, k + r + 1)$ whose image is a rational normal curve of degree $r + 1$. □

Lemma 2.7. *Let Λ_0 be a point of $\mathbb{G}(k,n)$ and $1 \leq r \leq k$. Then the osculating space Π_{r,Λ_0} is the linear space which is spanned by the Schubert variety W_{r,Λ_0}*

$$\{\Lambda \in \mathbb{G}(k,n) | dim(\Lambda \cap \Lambda_0) \geq k-r\}.$$

Proof. Without loss of generality, we assume that Λ_0 is the point where only the first Plücker coordinate is different from zero. We need a rational parametrization of the Grassmannian in a neighbourhood of Λ_0: we use the rational parametrization from (2.5) given by the map ψ restricted to $\mathbb{A}^M$ to $\mathbb{G}(k,n)$ such that $\psi((0,\ldots,0)) = \Lambda_0$. Then each coordinate function of ψ is given by a minor D^s as above (in the non-homogeneous coordinates of $\mathbb{A}^M$). The derivatives up to order r of the minors D^s with $s \geq r+1$ vanish at $0 \in \mathbb{A}^M$. Then it follows that the r-osculating space Π_{r,Λ_0}, which is the span of the derived points up to order r, has dimension up to $\sum_{i=1}^{r} m_i - 1$, where we recall that m_i is the number of the the D^i's, and it is equal to $\binom{k+1}{i}\binom{n-k}{i}$. Moreover, for each minor D^s with $s \leq r$, there exist a derived point $(\Lambda_0)_\alpha$ with $|\alpha| = s$ such that all its coordinates except the one corresponding to D^s vanish: this implies that the dimension of Π_{r,Λ_0} is exactly $\sum_{i=1}^{r} m_i - 1$.

Now we need to show that given two k-planes L_1 and L_2 in $\mathbb{P}^n$ such that they intersect in a linear space M of dimension $k-r$, then we can construct a Segre variety Seg$(1, r-1)$ such that each L_i intersects it in a different $(r-1)$-plane. To prove this, we project from M in $\mathbb{P}^{n-k+r-1}$ and we obtain two disjoint r-planes L'_1 and L'_2. If we fix an isomorphism $\tau : L'_1 \to L'_2$, the variety defined as the union of the lines joining $p \in L'_1$ to $\tau(p) \in L'_2$ is the desired Segre variety.

Now, from Lemma 2.6, for any k-plane Λ which intersects Λ_0 in a linear space of dimension at least $k-r$, we can construct a rational normal curve of degree r passing through Λ_0 and Λ. From the description of Π_{r,Λ_0} given in Section 1.3, if we take a rational normal curve C of degree r in $\mathbb{G}(k,n)$ passing through Λ_0, then it must be contained in the r-osculating space. Then it follows that W_{r,Λ_0} is contained in Π_{r,Λ_0}.

To show that Π_{r,Λ_0} is exactly the linear space spanned by W_{r,Λ_0} we use a dimension count. In our description, Λ_0 is spanned by the points associated to the vectors $v_1, \ldots, v_{k+1}$. Let Λ be an element of W_{r,Λ_0}: we can assume that is spanned by $k+1$ points and the first $k-r+1$ points lies on Λ_0. Then the Plücker coordinates of Λ are the maximal minors of a matrix $M_\Lambda = [v_{i,j}]_{i=1,\ldots,n+1; j=1,\ldots,k+1}$ where $v_{i,j} = 0$ if $i \in \{k+2,\ldots,n+1\}$ and $j \in \{1,\ldots,k-r+1\}$.

The minors which vanish are those which involve only at least $r+1$ columns among the last $n-k$: so, recalling the definition of the m_i's, the codimension of the linear space spanned by W_{r,Λ_0} is at least $\sum_{i=0}^{k-r} m_{k+1-i}$. Since we can vary the variables $v_{i,j}$'s above, the codimension is exactly $\sum_{i=0}^{k-r} m_{k+1-i}$: in particular the linear space spanned by W_{r,Λ_0} is precisely the r-osculating space. □

Proposition 2.8. *Let* Π *be an element of* $\mathbb{G}(n-k-1,n)$ *and* W_Π *the Schubert variety*

$$\{\Lambda \in \mathbb{G}(k,n) | dim(\Lambda \cap \Pi) \geq 1\}\,. \tag{2.10}$$

Then the projection $\varphi : \mathbb{G}(k,n) \dashrightarrow \mathbb{P}^M$ *from the linear space generated by* W_Π *is the inverse map of a* $\psi : \mathbb{P}^M \dashrightarrow \mathbb{G}(k,n)$ *as in* (2.5)*.*

Proof. We use the notation of Lemma 2.4. First of all we observe that the linear system $\mathfrak{d}$ contains the linear system of hyperplanes of $\mathbb{P}^M$ as a subsystem: this is $|kH+\alpha|$, where α is a hyperplane. From our description above it follows that hypersurfaces of $\mathfrak{d}$ are sent by ψ to hyperplane sections of $\mathbb{G}(k,n)$. Thus we obtain that φ is a projection whose vertex is the intersection of all hyperplanes of $\mathbb{P}^M$ whose intersection with the Grassmann variety contain $\psi(H)$ with multiplicity at least k.

The image of H under ψ is a Grassmannian $\mathbb{G}_0 = \mathbb{G}(k, n-k-1)$: if we impose $y = 0$ in (2.6) this is the Plücker embedding associated to a $(k+1)\times(n-k)$ matrix.

We have that a hyperplane H' contains $\mathbb{G}_0$ with multiplicity r if and only if H' contains $\Pi_{r-1,P}$ for any $P \in \mathbb{G}_0$. Then from Lemma 2.7, the vertex of our projection is spanned by the set

$$\{\Lambda \in \mathbb{G}(k,n) | \forall P \in \mathbb{G}_0, dim(\Lambda \cap P) \geq 1\}$$

which is clearly the same object described in (2.10). In addition to this, from a dimension count similar to the one at the end of previous Lemma, the linear space spanned by W_Π has dimension $N - m_1 - 1 = N - (k+1)(n-k) - 1 = N - M - 1$. □

For what we have already seen here, the birational map between $\mathbb{G}(k,n)$ and $\mathbb{P}^M$ is an isomorphism outside a hyperplane section H of $\mathbb{G}(k,n)$ and a hyperplane H' in $\mathbb{P}^M$. We call these hyperplanes in $\mathbb{P}^N$ *osculating* because they contain the osculating spaces described in the Proposition above.

Lemma 2.9. *Let* $\mathbb{P}^{N\star}$ *be the dual space of hyperplanes of* $\mathbb{P}^N$*. Then the osculating hyperplanes of* $\mathbb{G}(k,n)$ *at a* $\mathbb{G}(k,n-k-1)$ *form a* $\mathbb{G}(n-k-1,n)$ *in* $\mathbb{P}^{N\star}$*.*

Proof. Let $\mathbb{P}^{N\star} = \mathbb{P}[\wedge^{k+1}V]^\star$ be the dual projective space: this is isomorphic to $\mathbb{P}[\wedge^{n-k}V]$. We recall that an osculating hyperplane is a hyperplane whose section with $\mathbb{G}(k,n)$ is the set of k-planes which intersect a fixed $n-k-1$-plane. We want so to show that the set of totally decomposable elements of $\mathbb{P}[\wedge^{n-k}V]$, which corresponds to the set of $(n-k-1)$-planes of $\mathbb{P}^n$, is embedded in $\mathbb{P}^{N\star}$ under a Plücker morphsim. Let H be a $n-k-1$-plane which is spanned by $n-k$ points, which correspond to the vectors $v_1,\dots,v_{n-k}$. A k-plane intersects H if it is spanned by $k+1$ points which corresponds to $w_1,\dots,w_{k+1}$ and the square matrix of order $n+1$ whose rows are $v_1,\dots,v_{n-k},w_1,\dots,w_{k+1}$

does not have maximal rank. Therefore, the set of these k-planes is a variety: this is the section of $\mathbb{G}(k,n)$ with the hyperplane of $\mathbb{P}^N$ of equation

$$\sum_{1\leq i_1<\ldots<i_{n-k}\leq n+1} S(i_1,\ldots,i_{n-k})\alpha_{i_1,\ldots,i_{n-k}} x_{\overline{i_1,\ldots,i_{n-k}}},$$

where the $\alpha_{i_1,\ldots,i_{n-k}}$'s are the Plücker coordinates of H in $\mathbb{G}(n-k-1,n)$, the $x_{\overline{i_1,\ldots,i_{n-k}}}$ are the homogeneous coordinates of $\mathbb{P}^N$, we denote by $\overline{i_1,\ldots,i_{n-k}}$ the $(k+1)$-uple in $n+1$ obtained by subtracting $\{i_1,\ldots,i_{n-k}\}$ from $\{1,\ldots,n+1\}$, and S is the sign of the permutation $(i_1,\ldots,i_{n-k},\overline{i_1,\ldots,i_{n-k}})$. In particular, the variety of osculating hyperplanes is embedded by Plücker inside $\mathbb{P}^{N\star}$. □

Corollary 2.10. *Let $\mathbb{G}(k,n)$ be as above. Then for a general projection ψ, a subvariety V of $\mathbb{G}(k,n)$ is mapped birationally onto its image.*

Proof. A projection is completely determined by the choice of a $(n-k-1)$-dimensional linear space in $\mathbb{P}^n$: these are parametrized by $\mathbb{G}(n-k-1,n)$. Furthermore, the indeterminacy locus and the subvariety contracted by the projection are contained in a hyperplane section of the Grassmannian. Then if V is zero-dimensional, there exists a hyperplane not containing it. If $\dim(V)>0$, then there exists a hyperplane H such that $V\setminus(V\cap H)$ is an open dense subset. □

Lemma 2.11. *Let $\mathbb{G}(k,n)$ be a Grassmannian variety. Then* $\mathrm{Pic}(\mathbb{G}(k,n)) = \mathbb{Z}\cdot[H]$*, where H is an hyperplane section.*

Proof. Let X be a projective variety and let $D\subset X$ be an effective Weil divisor on X. Then we can construct the sequence

$$\mathbb{Z}\to \mathrm{Pic}(X)\to \mathrm{Pic}(U)\to 0$$

where $U=X\setminus D$, and the first map is given by $1\mapsto[D]$. We can apply this construction to $\mathbb{G}(k,n)$ and set D to be the divisor defined by the vanishing of a fixed Plücker coordinate. Then U is an affine space, as we have seen, and $\mathrm{Pic}(U)=0$. In addition to this, $[D]$ is not a torsion element of the Picard group, so the first map is injective. Obviously, $[D]\sim[H]$. □

2.2 Families of hypersurfaces

In this section we discuss about the problem of rationally determining a linear space on a generic fibre of a family of hypersurfaces. In particular, we generalize an approach by Conforto [Con41], which extends a previous work by Comessatti [Com40]. We fix a field $\mathbb{K}$ which is algebraically closed and of characteristic 0. By Lefschetz principle, we can consider $\mathbb{K}=\mathbb{C}$. We recall some general facts, and give the definition of Fano family.

Definition 2.12. Let W be a projective irreducible variety. We call a *family of hypersurfaces (of degree d and dimension $n-1$)* any morphism $\phi : \mathscr{X} \to W$, such that there exists a morphism $g : W \to \mathbb{P}[H^0(\mathbb{P}^n, \mathscr{O}(d))]$ and the following diagram

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\quad f \quad} & \mathscr{H} \\ \downarrow & & \downarrow \\ W & \xrightarrow{\quad g \quad} & \mathbb{P}[H^0(\mathbb{P}^n, \mathscr{O}(d))] \end{array}$$

commutes, where $\mathscr{H} \to \mathbb{P}[H^0(\mathbb{P}^n, \mathscr{O}(d))]$ is the universal family of hypersurfaces.

Definition 2.13. Given two families $\mathscr{X} \to W$ and $\mathscr{Y} \to T$, we say that $\mathscr{X}$ is *birationally equivalent* to $\mathscr{Y}$ (as families) if there exist two birational maps $f : \mathscr{X} \dashrightarrow \mathscr{Y}$ and $g : W \dashrightarrow T$ such that the diagram

$$\begin{array}{ccc} \mathscr{X} & \overset{f}{\dashrightarrow} & \mathscr{Y} \\ \downarrow & & \downarrow \\ W & \overset{g}{\dashrightarrow} & T \end{array}$$

commutes.

Remark 2.14. We observe that it is always possible to obtain a family of hypersurfaces giving a morphism $g : W \to \mathbb{P}[H^0(\mathbb{P}^n, \mathscr{O}(d))]$ and using the pullback construction. With a similar pullback construction, furthermore, we obtain a birationally equivalent family to $\mathscr{X} \to W$ from a birational map $W \dashrightarrow T$.

Lemma 2.15. *Let $\mathscr{X} \to W$ be a family of hypersurfaces. Then we can construct a birationally equivalent family $\mathscr{X}' \to W'$ such that W' is a hypersurface in $\mathbb{P}^{r+1}$ which is birational to W and $\mathscr{X}'$ can be described by the following equation:*

$$\sum_{i_1,\dots,i_d \in \{0,\dots,n\}} a_{i_1\dots i_d}(u_0, \dots, u_{r+1}) \prod_{j=1}^{d} x_{i_j} = 0 \tag{2.11}$$

where $a_{i_1\dots i_d} \in H^0(W', \mathscr{O}_{W'}(\mu))$ for some $\mu \in \mathbb{N}$.

Proof. We observe that giving such a family is equivalent to give a rational map $W \dashrightarrow \mathbb{P}[H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))]$. So, firstly we replace W with a birational equivalent hypersurface $W' \subseteq \mathbb{P}^{r+1}$. Then, giving a rational map $W' \to \mathbb{P}[H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))]$ is equivalent to give a $\binom{n+d}{n}$-uple of elements in $H^0(W', \mathscr{O}'_W(\mu))$ for some μ. □

Given a hypersurface $X \subset \mathbb{P}^n$, the Fano variety of X, denoted by $F_k(X)$, is the subvariety of $\mathbb{G}(k, n)$ whose points represent k-planes contained in X.

Lemma 2.16. *Let X be a hypersurface of degree d in $\mathbb{P}^n$. Let $\varphi : \mathbb{G}(k,n) \dashrightarrow \mathbb{P}^{(k+1)(n-k)}$ be a general projection map as the one described in Lemma 2.8. Then $\varphi|_{F_k(X)}$ is a birational map and $\varphi(F_k(X))$ is described by the vanishing of $\binom{d+k}{k}$ polynomials of degree d.*

Proof. The first assertion follows directly from Corollary 2.10.

Let us consider the incidence variety

$$\Phi = \{(\Lambda, X) | \Lambda \subseteq X\} \subseteq \mathbb{G}(k,n) \times H^0(\mathbb{P}^n, \mathscr{O}(d))$$

and let π_1 and π_2 be the projections on $\mathbb{G}(k,n)$ and $H^0(\mathbb{P}^n, \mathscr{O}(d))$ respectively.

Let f be a homogeneous polynomial of degree d in $n+1$ coordinates; f can be written as

$$f(x_0, \ldots, x_n) = \sum_{d_0+\ldots+d_n=d} \alpha_{d_0\ldots d_n} x_0^{d_0} \ldots x_n^{d_n}$$

where the $\alpha_{d_0\ldots d_n}$ are the homogeneous coordinates of $H^0(\mathbb{P}^n, \mathscr{O}(d))$.

We assume, without loss of generality, that the projection is an isomorphism on the open set of the Grassmannian U where the first Plücker coordinate is different from zero. For every $\Lambda \in U$ we can give a rational parametrization $\phi_\Lambda : \mathbb{P}^k \to \Lambda \subseteq \mathbb{P}^n$ given by

$$[s_0, \ldots, s_k] \mapsto [s_0, \ldots, s_k] \cdot M_\Lambda =$$

$$= [s_0, \ldots, s_k] \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & a_{1,k+2} & a_{1,k+3} & \cdots & a_{1,n+1} \\ 0 & 1 & 0 & \ldots & 0 & a_{2,k+2} & a_{2,k+3} & \cdots & a_{2,n+1} \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & \ldots & 1 & a_{k+1,k+2} & a_{k+1,k+3} & \cdots & a_{k+1,n+1} \end{bmatrix}.$$

If we look at $f(\phi_\Lambda([s_0, \ldots, s_k]))$, this is a form of degree d in $s_0, \ldots, s_k$ with coefficient in $a_{i,j}$ and $\alpha_{d_0\ldots d_n}$.

Therefore, $\pi_1^{-1}(U) \subseteq \Phi$ will be described by the vanishing of these coefficients: their number is $\binom{d+k}{k}$ and these are linear in the $\alpha_{d_0,\ldots,d_n}$'s and of degree d in the $a_{i,j}$.

Hence, if X is a hypersurface corresponding to $x \in H^0(\mathbb{P}^n, \mathscr{O}(d))$, then $F_k(X)$ is given by $\pi_2^{-1}(x)$. Then it follows that $\varphi(F_k(X)|_U) = F_k(X) \cap \pi_1^{-1}(U)$ is given by the vanishing of $\binom{d+k}{k}$ polynomials of degree d in the affine coordinates $a_{i,j}$'s. □

Definition 2.17. Let $\mathscr{X} \to W$ be a family of hypersurfaces. We denote by $F_k(\mathscr{X}) \to W$ the family such that, for a general w in W, the fibre $F_k(\mathscr{X})_w$ is the Fano variety of k-planes in $\mathscr{X}_w$.

Remark 2.18. In this definition we omit the fact that this family really exists and there is a flat morphism between $\mathscr{F}_k(\mathscr{X})$ and W. This fact is proved in the following lemma.

Lemma 2.19. *Let $X \to W$ be a family of hypersurfaces of degree d in $\mathbb{P}^n$ and let $\dim(W) = r$. Then we can find a rational section of $F_k(\mathscr{X})$ if the following condition holds:*

$$n > k + \frac{1}{k+1}\left[\binom{d+k}{k}d^r - 1\right]. \tag{2.12}$$

Proof. Let us assume, as in Lemma 2.15, that W is a hypersurface in $\mathbb{P}^{r+1}$ with homogeneous coordinates $u_0, \dots, u_{r+1}$. W is given by the vanishing of a form $\phi(u_0, \dots, u_{r+1})$ of degree $\overline{m}$.

In addition to this, we assume that $\mathscr{X}$ has equation as in (2.11), i.e.

$$\sum_{i_1,\dots,i_d \in \{0,\dots,n\}} a_{i_1\dots i_d}(u_0, \dots, u_{r+1}) \prod_{j=1}^{d} x_{i_j} = 0 \tag{2.13}$$

where $a_{i_1\dots i_d} \in H^0(W, \mathscr{O}_W(\mu))$ for some $\mu \in \mathbb{N}$, and the $x_0, \dots, x_n$ are the homogeneous coordinates in $\mathbb{P}^n$.

The Fano family $F_k(\mathscr{X}) \to W$ is contained in $W \times \mathbb{G}(k, n)$. Then from Lemma 2.16 there exists a birationally equivalent family $\mathbb{F}_k(\mathscr{X}) \to W$, where, for a general $w \in W$, $\mathbb{F}_k(\mathscr{X})_w$ is birational to $F_k(\mathscr{X})_w$. We observe that $\mathbb{F}_k(\mathscr{X})$ is described by the vanishing of $\binom{d+k}{k}$ equations in $W \times \mathbb{P}^{(k+1)(n-k)}$. In particular, $\mathbb{F}_k(\mathscr{X})$ is described by the following equations

$$\sum_{i_1,\dots,i_d \in \{0,\dots,n\}} b^L_{i_1\dots i_d}(u_0, \dots, u_{r+1}) \prod_{j=1}^{d} y_{i_j} = 0 \tag{2.14}$$

for $L = 1, \dots, \binom{d+k}{k}$, and $b^L_{i_1\dots i_d}(u_0, \dots, u_{r+1}) \in H^0(W, \mathscr{O}_W(\mu))$. The y_{i_j}'s denote here the homogeneous coordinates of $\mathbb{P}^{(k+1)(n-k)}$.

We need now to find a rational section p of $\mathbb{F}_k(\mathscr{X})$, in order to compose the section with the birational map between $\mathbb{F}_k(\mathscr{X})$ and $F_k(\mathscr{X})$; obtaining so a section of the latter. This is possible because, for a general w, we can assume that $p(w)$ is outside the indeterminacy locus of the rational map $\mathbb{F}_k(\mathscr{X}) \dashrightarrow F_k(\mathscr{X})$.

A rational section $p : W \dashrightarrow \mathbb{F}_k(\mathscr{X})$ is given by a $[(k+1)(n-k)+1]$-uple of rational functions of W. Each rational function can be expressed by a homogeneous polynomial in $u_0, \dots, u_{r+1}$ of a fixed degree m that satisfies the condition $\phi(u_0, \dots, u_{r+1}) = 0$.

Supposing $m > \overline{m}$, we can choose M independent elements in $H^0(W, \mathscr{O}_W(m))$, where

$$M = \binom{m+r+1}{r+1} - \binom{m-\overline{m}+r+1}{r+1}. \tag{2.15}$$

We call these forms $\Psi_1, \dots, \Psi_M$.

We want to construct a section p by writing its homogeneous coordinates as linear combinations of the Ψ's, i.e. by writing them as

$$p_i = \sum_{j=1}^{M} \lambda_{i,j} \Psi_j \text{ for } i = 0, \dots, (k+1)(n-k) \tag{2.16}$$

where the $\lambda_{i,j}$'s are new variables. The number of the λ's is

$$[(k+1)(n-k)+1]M = [(k+1)(n-k)+1] \left[\binom{m+r+1}{r+1} - \binom{m-\overline{m}+r+1}{r+1} \right]. \tag{2.17}$$

We need to find the values of these λ's such that p is a section: actually, we will prove that such λ's exist. We replace the y_i's in all the equations (2.14) with the $p_i(u_0, \dots, u_{r+1})$'s and we impose that the result vanishes on W, i.e. it is a form in $\mathbb{K}[u_0, \dots, u_{r+1}]$ which is divisible by $\phi(u_0, \dots, u_{r+1})$.

We make the substitution and for each L we have

$$\sum_{i_1,\dots,i_d \in \{0,\dots,n\}} b^L_{i_1\dots i_d}(u_0, \dots, u_{r+1}) \prod_{j=1}^{d} p_{i_j} = \\ = \sum_{l_1+\dots+l_{r+1}=dm+\mu} F^L_{l_0\cdots l_{r+1}}(\lambda_{i,j}) u_0^{l_0} \cdots u_{r+1}^{l_{r+1}}. \tag{2.18}$$

The homogeneous polynomials that we have after the substitution are of degree $dm+\mu$ with respect to $u_0, \dots, u_{r+1}$ and the coefficients $F^L_{l_0\cdots l_{r+1}}$ are polynomials in the λ's. Thus, for $L = 1, \dots, \binom{d+k}{k}$ we set

$$\sum_{l_1+\dots+l_{r+1}=dm+\mu} F^L_{l_0\cdots l_{r+1}}(\lambda_{i,j}) u_0^{l_0} \cdots u_{r+1}^{l_{r+1}} = \\ = \phi(u_0, \dots, u_{r+1}) \left(\sum_{i_1+\dots+i_{r+1}=dm-\overline{m}+\mu} \alpha^L_{i_0\cdots i_{r+1}} u_0^{i_0} \cdots u_{r+1}^{i_{r+1}} \right) \tag{2.19}$$

where the $\alpha^L_{i_0\cdots i_{r+1}}$ are new variables here introduced. Their number is

$$\binom{d+k}{k} \binom{dm-\overline{m}+\mu+r+1}{r+1}. \tag{2.20}$$

Now to prove the thesis we need to show that, under condition (2.12), there exists an admissible solution of the system of non-homogeneous equations, obtained by equating the coefficients of the monomials of degree $dm+\mu$ in (2.19) for each L. A solution of the system is called *admissible* if it represents a section: in this case, a solution is not admissible if all the λ's are equal to 0. In the system there are

$$\binom{d+k}{k} \binom{dm+\mu+r+1}{r+1} \tag{2.21}$$

equations in the α's and λ's. The total amount of these variables is

$$\begin{gathered}[(k+1)(n-k)+1]\left[\binom{m+r+1}{r+1}-\binom{m-\overline{m}+r+1}{r+1}\right]\\ +\binom{d+k}{k}\binom{dm-\overline{m}+\mu+r+1}{r+1}.\end{gathered}\tag{2.22}$$

If the number of variables is greater than the number of equations, i.e. if the following inequality holds

$$\begin{gathered}[(k+1)(n-k)+1]\left[\binom{m+r+1}{r+1}-\binom{m-\overline{m}+r+1}{r+1}\right]\\ +\binom{d+k}{k}\binom{dm-\overline{m}+\mu+r+1}{r+1}>\binom{d+k}{k}\binom{dm+\mu+r+1}{r+1}\end{gathered}\tag{2.23}$$

our system has admissible solutions: to prove this we use an argument due to Conforto.

In general, given a system of non-homogeneous equations, it is not true that if this is underdeterminate (i.e., the number of equations is lower than the number of the variables) then the set of solutions is non-empty. In the associated affine space we know that the point P given by $\lambda_{i,j}=0$ and $\alpha_{i_0\cdots i_{r+1}}=0$ is a solution of the system, despite the fact that it does not represent an admissible solution: however this implies that the set of solutions has a component of positive dimension which contains P. This set, furthermore, cannot consist only of non admissible solutions: in fact, if all the λ's are equal to 0, from (2.19) it follows that also the α's are 0. So, if the system is underdeterminate, we have a set of solutions of positive dimension.

Therefore, we want to see under which conditions, for m large enough, that inequality (2.23) holds. This can be written as

$$\begin{gathered}[(k+1)(n-k)+1]\left[\binom{m+r+1}{r+1}-\binom{m-\overline{m}+r+1}{r+1}\right]+\\ \binom{d+k}{k}\left[\binom{dm-\overline{m}+\mu+r+1}{r+1}-\binom{dm+\mu+r+1}{r+1}\right]>0\end{gathered}\tag{2.24}$$

The term on the left is a polynomial in m: the condition in order that it is positive for $m\to+\infty$ is that the leading coefficient is positive. The coefficient of the monomial m^{r+1} is equal to zero, so we have to look at m^r. Its coefficient is given by

$$\frac{[(k+1)(n-k)+1]}{(r+1)!}(r+1)\overline{m}+\frac{\binom{d+k}{k}d^r}{(r+1)!}[-(r+1)\overline{m}].$$

After dividing for the positive term $\frac{\overline{m}}{r!}$, we obtain

$$(k+1)(n-k)+1-\binom{d+k}{k}d^r>0$$

that is equivalent to (2.12). □

2.3 Unirationality of families of hypersurfaces

In this section we use the previous results to find out when a family of hypersurfaces is unirational. The main theorem used for this result is from Ciliberto [Cil80], which is a generalization of a work of Morin [Mor42], already studied by Murre [Mur79]. The Criterion in Proposition 2.21 is due to Roth [Rot50].

Definition 2.20. Let X be an algebraic variety defined over $\mathbb{K}$. One says that X is *P-rational* (resp. *P-unirational*) if X is $\mathbb{K}(P)$-rational (resp. $\mathbb{K}(P)$-unirational), where $\mathbb{K}(P)$ is the extension of $\mathbb{K}$ obtaining by adding to $\mathbb{K}$ the coordinates of a point P of X.

Similarly, one says that X is *L-rational* (resp. *L-unirational*) if X is $\mathbb{K}(L)$-rational (resp. $\mathbb{K}(L)$-unirational), where $\mathbb{K}(L)$ is the extension of $\mathbb{K}$ obtaining by adding to $\mathbb{K}$ the Plücker coordinates of a point l (of the right Grassmanian) representing a linear space L contained in X.

These definitions are clearly a generalization of those of Section 1.2.

Proposition 2.21. *Let $\mathscr{X} \to W$ be a family of varieties with W unirational irreducible projective variety. If it is possible to rationally determine on the general fibre $\mathscr{X}_w$ a linear subspace L of dimension k with $k \geq 0$ such that $\mathscr{X}_w$ is $\mathbb{K}(L)$-unirational, then $\mathscr{X}$ is $\mathbb{K}$-unirational.*

In addition, if W is a rational variety and $\mathscr{X}_w$ is $\mathbb{K}(L)$-rational, then $\mathscr{X}$ is a $\mathbb{K}$-rational variety.

Proof. We denote by $s : W \dashrightarrow \mathscr{F}_k(\mathscr{X})$ the rational section of the Fano variety, by $\phi : \mathbb{P}^r_{\mathbb{K}} \dashrightarrow W$ the dominant map which assures the unirationality of W and by $\psi_w : \mathbb{P}^{r'}_{\mathbb{K}(s(w))} \dashrightarrow \mathscr{X}_w$ the dominant map which assures the unirationality of $\mathscr{X}_w$, for a general $w \in W$.

Then we can construct the map

$$\mathbb{P}^r_{\mathbb{K}} \times \mathbb{P}^{r'}_{\mathbb{K}} \dashrightarrow \mathscr{X}$$

such that the pair (t, t') is sent to $\psi_{\phi(t)}(t')$. This is a rational dominant map, and its coefficients are in $\mathbb{K}$.

It follows furthermore that if ϕ and ψ_w are generically finite of degree a and b respectively, then this map is generically finite of degree $a \cdot b$. The second assertion follows. □

Definition 2.22. 1. We define the sequence of the t_d's. These numbers are given by:

$$\begin{cases} t_0 = 0; \\ t_d = t_{d-1} + \binom{d+t_{d-1}}{t_{d-1}+1} \end{cases}$$

2. Let d be an integer such that $d \geq 3$. We define the sequence of the α_d's:
$$\alpha_d = t_{d-3} + 1 + \frac{1}{t_{d-3}+2}\binom{d+t_{d-3}+1}{d}$$
If we set $k := t_{d-3} + 1$, then we have the following theorem.

Theorem 2.23. *Let $n \geq \alpha_d$ and $d \geq 3$. If $X \subset \mathbb{P}^n_{\mathbb{K}}$ is a general hypersurface of degree d then the Fano variety $F_k(X)$ is not empty. For every $L \in F_k(X)$ there exists $M \geq n-1$ and a rational dominant map $\mathbb{P}^M_{\mathbb{K}(L)} \dashrightarrow X$, in other words X is $\mathbb{K}(L)$-unirational.*

The proof is in [Cil80, Theorem 3.2].

This results can be used together with Proposition 2.21 to conclude whether a family is unirational or not.

Corollary 2.24. *Let $\mathscr{X} \to W$ be a family of hypersurfaces of degree d and dimension $n-1$ and W a unirational irreducible variety of dimension r. This family is unirational if the following inequality holds:*
$$n > t_{d-3} + 1 + \frac{1}{t_{d-3}+2}[\binom{d+t_{d-3}+1}{t_{d-3}+1}d^r - 1] \tag{2.25}$$

Proof. If this inequality holds, from Lemma 2.19 that we can rationally determine on a general fibre $\mathscr{X}_w$ of the family a linear space L of dimension $t_{d-3}+1$. If we look at the inequality from Theorem 2.23, we see that the condition $n \geq \alpha_d$ is satisfied. It follows that the generic fibre is $\mathbb{K}$-unirational. Therefore, by Proposition 2.21, the family $\mathscr{X}$ is $\mathbb{K}$-unirational. □

Chapter 3

Segre Complexes

3.1 Quadratic complexes

Let $\mathbb{G}(k,n)$ be a Grassmann variety. We have proved in Lemma 2.11 that $\mathrm{Pic}(\mathbb{G}(k,n)) \cong \mathbb{Z}$, and a generator of this group is $[H] \cong \mathscr{O}_{\mathbb{P}^N}(1)|_{\mathbb{G}(k,n)}$, where again $N = \binom{n+1}{k+1} - 1$ and the inclusion $\mathbb{G}(k,n) \subset \mathbb{P}^N$ is the Plücker embedding. A line bundle H in this class is called the *Plücker hyperplane line bundle*, and we denote it by $\mathscr{O}_{\mathbb{G}(k,n)}(1)$. We set $\mathscr{O}_{\mathbb{G}(k,n)}(d) := \mathscr{O}_{\mathbb{G}(k,n)}(1)^{\otimes d}$ for any $d \in \mathbb{Z}$.

An element in $\mathscr{X}_{d,k,n} := |\mathscr{O}_{\mathbb{G}(k,n)}(d)|$ is called a *d-complex of type* (k,n): geometrically, this is the intersection of the Grassmannian variety with an hypersurface of $\mathbb{P}^N$ of degree d.

3.1.1 Linear line complexes and null polarities

Let $X \in |\mathscr{O}_{\mathbb{G}(1,n)}(1)|$ be a linear line complex. Then X is given by the intersection of $\mathbb{G}(1,n)$ with an hyperplane H of $\mathbb{P}^N$ whose equation is

$$\sum_{0 \leq i < j \leq n} a^X_{i,j} z_{i,j} = 0. \tag{3.1}$$

We can associate to the hyperplane H the antisymmetric matrix A_X of order $n+1$ whose entries in the (i,j) positions for $0 \leq i < j \leq n$ are the $a^X_{i,j}$ of Equation (3.1).

Definition 3.1. Let X be a linear line complex. We define the *null polarity* associated to X the rational map

$$\begin{aligned} \omega_X : \mathbb{P}^n &\dashrightarrow (\mathbb{P}^n)^\star \\ x &\mapsto A_X \cdot x. \end{aligned}$$

In particular, if $\omega_X(x)$ is defined, then this is a hyperplane of $\mathbb{P}^n$ which contains x and all the lines in X passing through x.

3.1.2 Quadratic complexes

In the following section, we will study the quadratic line complexes, i.e. 2-complexes of type $(1, n)$.

Let $X \in \mathscr{X}_{2,1,n}$ be a quadratic line complex, and let s be a smooth point of X. We define the intersections

$$X(s) := \mathbb{G}(1, n) \cap T_{X,s} \qquad T(s) := \mathbb{G}(1, n) \cap T_{\mathbb{G}(1,n),s}.$$

Lemma 3.2. *$T(s)$ is isomorphic to a cone with vertex s over a Segre variety* $\mathrm{Seg}(1, n-2)$*, and $X(s)$ is a cone with vertex s over an hyperplane section of* $\mathrm{Seg}(1, n-2)$*.*

Proof. The first assertion follows from Lemma 2.7: in fact, $T(s) = W_{1,s}$ in the notation of the Lemma. Then $T(s)$ is the subvariety of $\mathbb{G}(1, n)$ containing the lines which intersect s. A line through s in $\mathbb{G}(1, n)$ is given by $\{l | p \in l \subset \Pi\}$, where $p \in s$ and the 2-plane $\Pi \supset s$ are fixed. We obtain $W_{1,s}$ by taking the union of all this lines, which clearly intersect in s, and the base of the cone is the parameter space of the pairs (p, Π) as above, which is a Segre variety $\mathrm{Seg}(1, n-2)$.

Next, since X is smooth at s, we have that $\dim(T_{X,s}) = \dim(T_{\mathbb{G}(1,n),s}) - 1$, $T_{X,s}$ is an hyperplane in $T_{\mathbb{G}(1,n),s}$ and so $X(s)$ is an hyperplane section of $T(s)$ passing through s, and from this we have the thesis. □

Definition 3.3. Let X be a quadratic line complex. An element $s \in \mathbb{G}(1, n)$ is said a *general line* if $X(s)$ is irreducible. Otherwise, s is said to be *special*.

Lemma 3.4. *Let s be a special line of X. Then $X(s)$ is the union of a linear space $\Pi_{X,s}$ of dimension $n - 1$ and a cone $C_{X,s}$ over a* $\mathrm{Seg}(1, n-3)$*.*

Proof. We need to show that a reducible hyperplane section H of $S := \mathrm{Seg}(1, n-2)$ is the union of a $(n-2)$-plane and a $\mathrm{Seg}(1, n-3)$. If H is an hyperplane section of S, then it can be seen as a polynomial of bidegree $(1, 1)$ in two sets of variables. If it is reducible, this is the product of two linear polynomials, which corresponds to the pullbacks of the hyperplane sections on each factor. Then H must be the union of these two pullbacks, which are of the form $p \times \mathbb{P}^{n-2}$ and $\mathbb{P}^1 \times \mathbb{P}^{n-3}$. □

From this description, we observe that if s is a general line, $X(s)$ is an irreducible cone swept-out by a family of $(n-2)$-dimensional spaces parametrized by a one dimensional space corresponding to the points of s. In the proof of Lemma 3.2, we have that the base of the cone $X(s)$ is an hyperplane section S' of the Segre variety $\mathrm{Seg}(1, n-2)$. Then we can restrict the first projection over the project line on S', obtaining a map $\pi' : S' \to s \cong \mathbb{P}^1$. The fibre of π is a linear space of dimension $n - 3$. Then we can define a map $\pi : X(s) \to s$ whose fibre over $z \in s$ is the span of s and $(\pi')^{-1}(z)$.

Since in $\mathbb{G}(1,n)$ these spaces correspond to set of lines passing through a fixed point and contained in a fixed hyperplane, we have that for each $z \in s$, the lines of $X(s)$ containing z lie in a hyperplane $H_X(s,z)$.

If s is special, there is a point $z_s \in s$ such that all the lines passing through z_s lie on the linear space $\Pi_{X,s}$ of Lemma 3.4. For each other $z \in s \setminus \{z_s\}$, the lines lie in a fixed hyperplane $H_{X,s}(z)$. Then z_s and $H_{X,s}(z)$ are called a *special point* and a *special hyperplane* of the complex X respectively.

Remark 3.5. The propriety of special lines and points can be also analytically described. We introduce homogeneous coordinates in $\mathbb{P}^n$ and the corresponding Plücker coordinates $[\mathbf{z}] = [z_{i,j}]$ with $0 \le i < j \le r$ in $\mathbb{P}^N$ (see Section 2.1). The equation of X is of the form $f(\mathbf{z}) = 0$, where f is a quadratic homogeneous polynomial which is defined modulo the ideal of $\mathbb{G}(1,n)$.

Let s, t be incident lines in $\mathbb{P}^n$, so that the pencil joining them corresponds to a line $p_{s,t} \subset \mathbb{G}(1,n)$. Let $[\mathbf{s}] = [s_{i,j}]$ and $[\mathbf{t}] = [t_{i,j}]$ be the Plücker coordinates of the two lines. Then $p_{s,t}$ consists of the ponts of the form $\mathbf{z} = \lambda\mathbf{s} + \mu\mathbf{t}$, with $[\lambda, \mu] \in \mathbb{P}^1$.

The equation of the intersection of $p_{s,t}$ with X can be written explicitly, using this classic Euler's result.

Theorem 3.6. *Let $f(x_1, \dots, x_n)$ be an homogeneous polynomial of degree k over a field $\mathbb{K}$ with* $\mathrm{char}(\mathbb{K}) \nmid k$. *Then the following equality holds:*

$$k \cdot f(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}$$

Therefore, since $f(\mathbf{s}) = 0$, using Euler's formula one has

$$\begin{aligned} f(\lambda\mathbf{s} + \mu\mathbf{t}) =& \frac{1}{2} \sum_{0 \le i < j \le n} (\lambda s_{i,j} + \mu t_{i,j}) \frac{\partial f(\lambda\mathbf{s} + \mu\mathbf{t})}{\partial z_{i,j}} = \\ =& \lambda\mu \sum_{0 \le i < j \le n} t_{i,j} \frac{\partial f(\mathbf{s})}{\partial z_{i,j}} + \mu^2 f(\mathbf{t}) = 0 \end{aligned}$$

where we use that the $\frac{\partial f(\mathbf{z})}{\partial z_{i,j}}$'s are linear polynomials and, since f is a quadratic form,

$$\sum_{0 \le i < j \le n} s_{i,j} \frac{\partial f(t)}{\partial z_{i,j}} = \sum_{0 \le i < j \le n} t_{i,j} \frac{\partial f(s)}{\partial z_{i,j}}.$$

Thus, $p_{s,t} \subset X(s)$ if and only if, for every $[\lambda, \mu] \in \mathbb{P}^1$ the linear part of the equations above vanishes. Then $p_{s,t} \subset X(s)$ if and only if

$$\sum_{0 \le i < j \le n} t_{i,j} \frac{\partial f(s)}{\partial z_{i,j}} = 0. \tag{3.2}$$

This is a linear equation in $\mathbb{P}^N$ which depends on f, but its intersection with $T(s)$ does not depend on the choice of f. This is a linear complex, and we denote the associated null polatity by $\omega_{f,s}$. From what we have said so far we deduce the following Proposition.

Proposition 3.7. *Let s be a smooth point of X. Then:*

1. *the line s is general if and only if ω_s is everywhere defined along s; then for any $z \in s$, one has $H_X(s,z) = \omega_{f,s}(z)$, so that $H_X(s,z)$ varies in a pencil;*

2. *the line s is special if and only if $\omega_{f,s}$ is not defined at a point of s, which coincides with the special point z_s; for all other point $z \in s$, one has that $H_{X,s}(z) = \omega_{f,s}(z)$ does not depend on z.*

Given a point $z \in \mathbb{P}^n$, this defines a linear space Π_z of dimension $n-1$ inside $\mathbb{G}(1,n)$: it is defined as the set of lines of $\mathbb{P}^n$ passing through z. We denote by X_z the scheme theoretical intersection of X with Π_z. Then two cases can occur: either $X_z = \Pi_z$, or X_z is a $(n-2)$-dimensional quadric. We define $Q_X(z)$ as the variety in $\mathbb{P}^n$ described by the lines of X_z. If there are no ambiguities, we omit the X and we denote it simply by $Q(z)$.

Lemma 3.8. *If $X_z \neq \Pi_z$, then $Q(z)$ is a quadric cone in $\mathbb{P}^n$ and $\mathrm{rk}(Q(z)) = \mathrm{rk}(X_z)$, or equivalently $\mathrm{cork}(Q(z)) = \mathrm{cork}(X_z) + 1$.*

Proof. Let $[z_0, \dots, z_n]$ be the homogeneous coordinates of $\mathbb{P}^n$. If we assume $z = [1, 0, \dots, 0]$, then we have an isomorphism between Π_z and the hyperplane given by $z_0 = 0$. Then, since $\mathrm{char}(\mathbb{K}) \neq 2$, we can assume X_z is given by the equation $z_1 + \cdots + z_{\mathrm{rk}(X_z)}$. From our description it is clear that this is the same equation of $Q(z)$. □

Let X be a general quadratic line complex. Then for $z \in \mathbb{P}^n$ general, X_z is smooth and $Q(z)$ has corank equal to 1.

Definition 3.9. Let X be a quadratic line complex. For $k = 0, \dots, n$, we define $D_k(X) \subseteq \mathbb{P}^n$ as the closed subscheme of points z such that $\mathrm{cork}(X_z) \geq k$, with the convention that $\mathrm{cork}(X_z) = n$ means that $X_z = \Pi_z$.

Lemma 3.10. *Let X be a general quadratic line complex. Then either $D_k(X)$ is empty or $D_k(X)$ has dimension*

$$\dim(D_k(X)) = n - \binom{k+1}{2}.$$

In particular, if $D_k(X)$ is non empty, $\binom{k+1}{2} \leq n$ holds.

Proof. Consider the incidence correspondence $I_k \subset \mathbb{P}^n \times \mathscr{X}_{2,1,n}$ defined as

$$I_k = \{(z, X) | \operatorname{cork}(X_z) \geq k\},$$

and let π_1 and π_2 be the two projections. To compute the dimension of $F := \pi_1^{-1}(z) = \{X \in \mathscr{X}_{2,1,n} | \operatorname{cork}(X_z) \geq k\}$ we observe that we can construct a surjection $p : F \twoheadrightarrow D_k$ where $D_k \subset |\mathscr{O}_{\Pi_z}(2)|$ is the variety as in Lemma 1.5. Let Q be a quadric in D_k and let Q' be a quadric hypersurface in $\mathbb{P}^N$ which is a cone over Q. Then $Q' \in F$ and hence $p(Q') = Q$. Then the fibre of p over Q is given by the set $\mathbb{P}[\{Q' + Q'' | Q'' \in H^0(\mathbb{G}(1, n-1), \mathscr{I}_{\Pi_z}(2))\}]$, where $\mathscr{I}_{\Pi_z}$ is the ideal sheaf of Π_z in $\mathbb{G}(k, n)$. Therefore this fibre is a projective space of dimension $h^0(\mathscr{I}_{\Pi_z}(2))$.

In addition to this, we have the short exact sequence of cohomology of sheaves on $\mathbb{G}(1, n)$:

$$0 \to H^0(\mathscr{I}_{\Pi_z}(2)) \to H^0(\mathscr{O}_{\mathbb{G}(1,n)}(2)) \to H^0(\mathscr{O}_{\Pi_z}(2)) \to 0.$$

So $\dim F = h^0(\mathscr{I}_{\Pi_z}(2)) + \dim D_k = h^0(\mathscr{O}_{\mathbb{G}(1,n)}(2)) - h^0(\mathscr{O}_{\Pi_z}(2)) + h^0(\mathscr{O}_{\Pi_z}(2)) - 1 - \binom{k+1}{2}$. Since D_k is irreducible and the fibre of p is a projective space, it follows that also F is irreducible and has dimension $\dim \mathscr{X}_{2,1,n} - \binom{k+1}{2}$. Then $\dim I_k = \dim \mathscr{X}_{2,1,n} + n - \binom{k+1}{2}$. Now, if π_2 is not surjective, we have that for a general X, $D_k(X)$ is empty. Otherwise, we have that $\dim(D_k(X)) = n - \binom{k+1}{2}$ for a general X. □

Definition 3.11. We set $D(X) = D_1(X)$, and we call it the *discriminant locus* of X. From the lemma above, we have that $\dim(D(X))$ is equal to n or $n-1$. If $D(X) = \mathbb{P}^n$, we say that X is *degenerate*. Otherwise, $D(X)$ is a hypersurface. In particular we say that X is *strongly non-degenerate* if there is no $z \in \mathbb{P}^n$ such that $X_z = \Pi_z$.

From Lemma 3.10, if X is a general element in $\mathscr{X}_{2,1,n}$, then it is strongly non-degenerate. To prove it, suppose that X is not strongly non-degenerate: then there exists $z \in D_n(X)$, but for X general $D_n(X)$ is empty.

Remark 3.12. We can construct the loci $D_k(X)$'s as follows. We consider the *universal hyperplane bundle* $\mathbb{P}(\Omega_{\mathbb{P}^n}) \to \mathbb{P}^n$, which is the projectivization of the cotangent bundle $\Omega_{\mathbb{P}^n}$. The fibre of this bundle over a point $z \in \mathbb{P}^n$ is the hyperplane $z^\perp$ in $\mathbb{P}^{n\star}$. Then the bundle $\pi : \mathbb{P}(Sym^2(\Omega_{\mathbb{P}^n})) \to \mathbb{P}^n$ can be intended as the bundle whose fibres over z is the set of quadrics in Π_z. Then a strongly non-degenerate complex corresponds to a section σ_X of π. Conversely, also this bundle can be locally trivialized: each point corresponds to a pair (z, L), where $z \in \mathbb{P}^n$ and $L \in Sym^2(\Omega_{\mathbb{P}^n})_z$ is a 1-dimensional subspace: i.e., L corresponds to a projective quadric cone with z in its vertex. So, for each section of $\mathbb{P}(Sym^2(\Omega_{\mathbb{P}^n}))$ we obtain a strongly non-degenerate complex.

Inside $\mathbb{P}(Sym^2(\Omega_{\mathbb{P}^n}))$ we can define the subschemes $\mathbb{D}_k$ whose points correspond to quadrics of corank at least k. Then $D_k(X)$ is the pullback of $\mathbb{D}_k$ via σ_X.

Example 3.13. A complex $X \in \mathscr{X}_{2,1,2}$ is a conic in $(\mathbb{P}^2)^\star$. We have three possible cases, which correspond to different types of conics:

1. X is irreducible. Then for any $z \in \mathbb{P}^n$, X_z is a 0-dimensional length 2 subscheme. It follows that $z \in D(X)$ if and only if Π_z is tangent to X. Thus, $D(X) = X^\star \subset \mathbb{P}^2$ is the dual conic to X;

2. X consists of two distinct lines X_1, X_2, each line X_i corresponding to a pencil of lines in $\mathbb{P}^2$ with center x_i. Then $z \in D(X)$ if and only if z lies on the line corresponding to $X_1 \cap X_2$. This is the line $l = \langle x_1, x_2 \rangle$. Then, since $L_1 \cap L_2$ is a singular point of X, we have that $D(X)$ is the non-reduced conic consisting of l counted with multiplicity 2;

3. X consists of a multiple line. Then X is degenerate.

Definition 3.14. Let X be a quadratic complex in $\mathscr{X}_{2,1,n}$, with $n \geq 3$, and let π be a plane of $\mathbb{P}^n$. We can consider the scheme theoretical intersection of X with $\mathbb{G}(1,\pi)$. If $X_\pi \neq \mathbb{G}(1,\pi)$, then X_π is a quadratic line complex in $\mathscr{X}_{2,1,2}$, which is a conic, as we have seen in the Example 3.13. If X_π is non-degenerate, X is said to be *non-degenerate on* π. In addiction to this, the conic $D(X_\pi)$ is called the *conic of* X *on* π.

Remark 3.15. Let X be a quadratic complex in $\mathscr{X}_{2,1,n}$, let s be a special point of X and π a plane containing s. We assume that $X_\pi \neq \mathbb{G}(1,\pi)$. Thus we can study the restriction of the null polarity $\omega_{f,s}$ of Proposition 3.7 on the plane π. This is defined on any point $z \in s \setminus \{z_s\}$, unless $\pi \subset H_{X,s}$. This also means that the conic of X on π is irreducible and tangent to s at z_s unless π is contained in the special hyperplane $H_{X,s}$.

Let X be a strongly non-degenerate quadratic line complex. We define a morphism

$$v_X : \mathbb{P}^n \to |\mathscr{O}_{\mathbb{P}^n}(2)| \tag{3.3}$$

given by $v_X(z) := Q(z)$.

Lemma 3.16. *Let X be a general quadratic line complex. Then the image of v_X is isomorphic to the Veronese variety $V_{2,n}$.*

Proof. Let $[x_0, \ldots, x_n]$ be the homogeneous coordinates in $\mathbb{P}^n$ and let $[x_{ij}]$ be the corresponding Plücker coordinates in $\mathbb{P}^{\binom{n+1}{2}-1}$. We assume that $f(x_{ij}) = 0$ is the quadratic equation defining X. If we fix a point $z = [z_0, \ldots, z_n]$, $Q(z)$ is defined by the equation $f(x_i z_j - x_j z_i) = 0$, which is clearly a quadric in the z_i's, and v_X is defined by linear system of quadrics. To show that this is complete, we specialize to a particular X. We assume X is given by the equation $\sum_{0\leq i<j\leq n} x_{ij}^2 = 0$. Then

$$f(x_iz_j - x_jz_i) = \sum_{i=0}^{n}(\sum_{j\neq i} z_j^2)x_i^2 - 2\sum_{0\leq i<j\leq n}(z_iz_j)x_ix_j.$$

Then, varying z, the polynomial above spans the vector space of quadratic forms in the x_i's. Since the linear system associated to a general quadratic complex has dimension greater or equal to a specialized one, it follows that also the general v_X is a Veronese map of degree 2. □

Proposition 3.17. *Let X be a general quadratic complex in $\mathscr{X}_{2,1,n}$, with $n \geq 3$. Then:*

1. *$D(X)$ is the set of special points of X;*
2. *$\mathrm{Sing}(D(X)) = D_2(X)$. In particular if $z \in D(X) \setminus D_2(X)$ and $\mathrm{Sing}(Q(z)) = s$, then s is a special line of X, $z = z_s$ and $T_{D(X),z} = H_{X,s}$. It follows that the dual variety $D(X)^\star \subset \mathbb{P}^{n\star}$ is the set of the special hyperplanes of X;*
3. *if s is a line not lying on $D(X)$ and z is a non special point of s, then $H_X(s,z)$ is the tangent hyperplane to $X(z)$ along s;*
4. *$\deg(D(X)) = 2n - 2$.*

Proof. Since X is general, we can assume it is smooth.

1. Let s be a special line and z its corresponding special point. Then s is a point of X_z, and the definition of special point means that s is a double line for $Q(z)$, and so $z \in D(X)$.

   Conversely, let $z \in D(X)$. Then there is a line s through z which is singular for $Q(z)$: this means that s is a special line and $z = z_s$.

2. We know that $D_2 \subseteq \mathrm{Sing}(D(X))$ by Remark 3.12.

   To prove the other implication, use contraposition: we prove that the points of $D(X) \setminus D_2(X)$ are smooth. Let $z \in D(X) \setminus D_2(X)$, so that $\mathrm{Sing}(Q(z)) = s$ is a line through z. We show that the tangent space to $D(X)$ at z is an hyperplane: in particular, we show that this is exactly $H_{X,s}$.

   Let $l \neq s$ be a line through z. The line l lies on the tangent hyperplane if and only its intersection multiplicity with $D(X)$ is greater than 1. Then, to prove our thesis is equivalent to show that the intersection multiplicity of l with $D(X)$ is equal to 1 except when l lies on $H_{X,s}$. We denote it by m. This number, by Remark 3.12, is equal to the intersection multiplicity at $\sigma_X(z)$ of $\sigma_X(l)$ with $\mathbb{D}$, where σ_X is the section of $\mathbb{P}(Sym^2(\Omega_{\mathbb{P}^n}))$ associated to X.

To compute m, we trivialize $\mathbb{P}(Sym^2(\Omega_{\mathbb{P}^n}))$ locally : let H be a general hyperplane and we denote by p its intersection with s. For each point $w \in l$, we consider the quadric $Q'(w)$ cut by $Q(w)$ on H and we consider the 1-dimensional family of quadrics $\mathscr{Q}'_l = \{Q'(w)|w \in l\}$. Then $Q'(z) \in \mathscr{Q}'_l$, it is singular at p and it lies on the discriminant D_1 of $|\mathscr{O}_H(2)|$. Then in this local trivialization $\sigma_X(z)$ is $Q'(z)$, $\mathscr{Q}'_l$ corresponds to $\sigma_X(l)$ and D_1 corresponds to $\mathbb{D}$. Thus, m is also equal at the intersection of the curve $\mathscr{Q}'_l$ with D_1 at $Q'(z)$ in $|\mathscr{O}_H(2)|$.

By the description of the discriminant locus in Proposition 1.7, we have that $m \geq 2$ if and only if the tangent line to $\mathscr{Q}'_l$ at $Q'(z)$ is a pencil Φ'_l of quadrics with base point p. Since H is general, this happens if and only if the tangent line to the curve $\mathscr{Q}_l := \{Q(w)|w \in l\}$ at $Q(z)$ is a pencil of quadrics Φ_l which have the line s in its base locus.

Now if we consider the plane $\pi = \langle l, s\rangle$, we have a family of reducible conics given by the intersection of the $Q(w)$'s with π. Then $m \geq 2$ if and only if either $\mathbb{G}(1,\pi) \subset X$ or s is the fixed part of the family of conics. This means that the intersection $X_\pi := \mathbb{G}(1,\pi) \cap X$ does not define a conic $D(X_\pi)$ or this is reducible with s as a component. From Remark 3.15, his means that $l \subset H_{X,s}$.

3. Let $l \neq s$ be a line through z and contained in $H_X(s,z)$. The line $p_{s,l} \subset \mathbb{G}(1,n)$ generated by s and l is tangent to X at s by definition, and then it implies that $p_{s,l}$ is tangent to $Q(z)$ at s.

4. Let l be a general line in $\mathbb{P}^n$, and we want to compute the number d of intersection points of l with $D(X)$. The number d is equal also to the number of quadrics of rank $n-1$ inside the family $\mathscr{Q}_l$.

   If H is a general hyperplane, then we can consider the family of quadrics $\mathscr{Q}'_l$ given by the intersection of the elements of $\mathscr{Q}_l$ with H. The generic element of $\mathscr{Q}'_l$ is a quadric of rank n; in the pencil there are d quadrics of rank $n-1$ which comes from the intersection of the $n-1$ rank quadrics in $\mathscr{Q}_l$ with H, and one more rank $n-1$ rank quadric $Q'(z)$, where $z = H \cap l$. Then we have that $d = \delta - \mu$, where δ is the intersection number of $\mathscr{Q}'_l$ with the discriminant hypersurface D and μ is the intersection multiplicity of $\mathscr{Q}'_l$ with D at the point $Q'(z)$.

   By Lemma 3.16, we have that $\mathscr{Q}_l$ is a conic of $|\mathscr{O}_{\mathbb{P}^n}(2)|$, and $\mathscr{Q}'_l$ is also a conic by generality of H. Since $\deg(D) = n$, one has $\delta = 2n$.

   We need now to compute μ. We use again the notation of the proof of the first point of the Proposition. The number μ is equal to the intersection multiplicity of Φ'_l with D at $Q'(z)$, and Φ_l is cut out by $\mathcal{Q}_l$ on H. The general quadric of $\mathscr{Q}_l$ is a cone of rank n with vertex varying in l, then by Bertini's theorem l is in the base locus of Φ_l, because the general element of Φ_l is smooth outside the base locus. Now, if we compute the multiplicity m of a general point of l in this base locus scheme, we have that, for the generality of H, that the general point of l has multiplicity m inside the base locus of Φ_l' and finally that $\mu = 2m$.

To compute m we work locally: we assume that the pencil Φ_l is generated by two quadrics Q_1 and Q_2 of rank n, and the base locus of Φ_l is given by the vanishing of both equations. Then we give local coordinates $x_1, \ldots, x_{n-1}$ inside the first quadric centred at z such that l is given by $x_2 = \ldots = x_{n-1} = 0$. Since l has to lie in B, its equation will be

$$L_1(x_2, \ldots, x_{n-1}) + x_1 L_2(x_2, \ldots, x_{n-1}) + M(x_2, \ldots, x_n) = 0$$

where L_1 and L_2 are linear homogeneous polynomials in $x_2, \ldots, x_{n-1}$ and M is a polynomial of degree 2 in the same variables. Then the multiplicity of z in B is equal to 1, since L_1 cannot identically vanish because Q_2 has rank n.

We have finally proved that $m = 1$, so $\mu = 2$ and $d = 2n - 2$.

□

3.2 Discriminant hypersurface and its singularities

We go on assuming X to be general in $\mathscr{X}_{2,1,n}$, and we study the singular points of $D(X)$ along $D_k(X)$, with k satisfying the condition of Lemma 3.10.

Let $z \in D_k(X) \setminus D_{k+1}(X)$. We know that the multiplicity of $D(X)$ at z is at least k from Remark 3.12 and Proposition 1.7. The quadric $Q(z)$ is a cone with vertex a linear space of dimension k, which will be denoted by $\Sigma(z)$. Let Σ_z be the space of dimension $k-1$ inside the Grassmannian $\mathbb{G}(1, n)$ parametrizing all the lines in $\Sigma(z)$ passing through z, which are special lines. We can therefore construct a morphism

$$\gamma_{X,z} : s \in \Sigma_z \mapsto H_{X,s} \in (\Pi_z)^\star \tag{3.4}$$

whose image is denoted by $\Gamma_{X,z}$. We observe that $H_{X,s}$ defines unambiguously a point of $(\Pi_z)^\star$ (the dual space of Π_z defined after Proposition 3.7) because the lines through z and contained in an hyperplane forms a linear subspace of the Grassmannian of codimension one inside Π_z.

Lemma 3.18. *In this setting, $\Gamma_{X,z}$ is a Veronese variety $V_{2,k-1} \in (\Pi_z)^\star$.*

Proof. To describe the image of $\gamma_{X,z}$ we consider another construction. Take $\pi \subset \Sigma(z)$ a linear subspace of dimension $k-1$ not containing z. Since X is strongly non-degenerate, if $w \in \pi$, then $X_w \neq \Pi_w$ and by definition of $\Sigma(z)$ one has the line $s_w := \langle z, w \rangle \subset Q(w)$. The tangent hyperplane to $Q(w)$ along s_w is the special hyperplane H_{X,s_w}, which will be denoted by H_w. Then we have a morphism

$$\gamma_{X,z,\pi} : w \in \pi \mapsto H_w \in (\Pi_z)^\star \tag{3.5}$$

whose image is $\Gamma_{X,z}$ as above.

We consider the map v_X of Equation (3.3) restricted to the plane π. Then from Lemma 3.16 we have that $v_X(\pi)$ is a Veronese variety $V_{2,k-1}$, all quadrics in $v_X(\pi)$ contain z and $\Gamma_{X,z}$ is the set of the tangent hyperplanes to these quadrics in z. Then also the morphisms of (3.4) and (3.5) are defined by a linear system of quadrics.

We show that the linear system of quadrics associated to $\gamma_{X,z}$ is complete, so also $\Gamma_{X,z}$ is a Veronese $V_{2,k-1}$. This follows from the fact that $v_X(\pi)$ spans a linear space of dimension $\binom{k+1}{2} - 1$ inside $|\mathscr{O}_{\mathbb{P}^n}(2) \otimes \mathscr{I}_{z,\mathbb{P}^n}|$ and no quadric in it has a double point in z: so the tangent hyperplanes at z span a linear space of the same dimension, and so the system is complete. □

We observe that we have supposed the condition on k of Lemma 3.10, so the Veronese variety $\Gamma_{X,z}$ spans a linear space of dimension $\binom{k+1}{2} - 1$: this number is less than or equal to the dimension of $(\Pi_z)^\star$, which is $n-1$.

Proposition 3.19. *Let X be a general quadratic line complex and let $z \in D_k(X) \setminus D_{k+1}(X)$. Then the multiplicity of $D(X)$ at z is exactly k and $D(X)$ has canonical singularities.*

Proof. To prove this results, first we construct a cone $C_{X,z}$ which is an hypersurface of degree k, then we show that this is isomorphic to the cone $C_{D(X),z}$.

We consider the dual of $\Gamma_{X,z}$, $\Gamma^\star_{X,z} \subset \Pi_z$. This is a cone over $V^\star_{2,k-1}$, the latter being isomorphic to the discriminant hypersurface D_1 of $|\mathscr{O}_{\mathbb{P}^{k-1}}(2)|$ as in Theorem 1.4 (for a proof of this fact, see [GKZ08, Ch.1, Example 4.15]). Since $\dim |\mathscr{O}_{\mathbb{P}^{k-1}}(2)| = \binom{k+1}{2} - 1$ then the vertex of $\Gamma^\star_{X,z}$ has dimension $n - \binom{k+1}{2} - 1$ and the degree of the hypersurface (and thus of the cone) is k.

The set of lines in $\Gamma^\star_{X,z}$ then form cone $C_{X,z}$ whose vertex has dimension $n - \binom{k+1}{2}$ over a $V^\star_{2,k-1}$. This has again degree k.

Now let s be a line through z, which corresponds to a point in Π_z and to an hyperplane $s^\star$ in $(\Pi_z)^\star$. We have that $s \subset C_{X,z}$ if and only if $s^\star$ is tangent to $\Gamma_{X,z}$.

Consider the map (3.4). Let Ω_s be the pullback of s via $\gamma_{X,z}$ of the hyperplane $s^\star$. Since this map is a Veronese map, then Ω_s is a quadric hypersurface in $\Sigma_z \cong \mathbb{P}^{k-1}$ or the whole Σ_z (we consider it as a rank 0 quadric). Then $s \subset C_{X,z}$ if and only if Ω_s is singular.

As in (3.3), we consider the map

$$v_X : \mathbb{P}^n \to |\mathscr{O}_{\mathbb{P}^n}(2)|;$$

in particular $v_X(s)$ is a conic passing through the point $Q(z)$. The tangent line to $v_X(s)$ at $Q(z)$ is identified therefore with a pencil of quadric hypersurfaces. The intersection of base locus scheme of the pencil with $\Sigma(z)$ is $\Omega(s)$, which is the quadric cone corresponding to Ω_s in $\Sigma(z)$. We have that $\operatorname{cork}(\Omega(s)) = \operatorname{cork}(\Omega_s) + 1$.

Now we have to locally trivialize $D(X)$ as in Proposition 3.17. Let H be a general hyperplane such that $H \cap s = z$. Then, as in the cited Proposition and from Proposition 1.7, we have that the intersection multiplicity of s with $D(X)$ at z is greater than or equal to $k+1$ if and only if $\Omega'(s)$, the intersection of $\Omega(s)$ with H, is singular. Since $cork(\Omega'(s)) = \operatorname{cork}(\Omega(s)) - 1$, $\Omega'(s)$ is singular if and only if Ω_s is singular. Then we have that $s \subset C_{D(X),z}$ if and only if $s \subset C_{X,z}$. □

This also shows that the tangent space of $D_k(X)$ at z is the vertex of $C_{X,z}$. This implies that $D_k(X)$ is smooth at $z \notin D_{k+1}(X)$.

Remark 3.20. We do not dwell on canonical singularities in this thesis, however we recall that these are the only singularities a canonical variety can have (see Section 3.4). For a hypersurface in $\mathbb{P}^n$, roughly speaking, canonical singularities are ordinary points with multiplicity less or equal to $n-1$, which is satisfied here by condition of Lemma 3.10.

Remark 3.21. The case $n = 3$ is classical: $D(X)$ is in this case a *Kümmer quartic surface* and $D_2(X)$ consists of 16 nodes, i.e. double points with tangent cone of rank 3 (see [GH78, Ch.6, §2]).

3.3 The double cover of the discriminant

Definition 3.22. Let be a $f : X \to Y$ be a finite covering. Then f is *quasi-étale* if the branch locus has codimension larger or equal to 2 in Y.

Remark 3.23. If $n \geq 3$ and odd, then for $z \in D(X) \setminus D_2(X)$ the quadric $X_z \subset \Pi_z$ has corank 1 and hence even rank. Then X_z has two distinct rulings of maximal dimension $\dfrac{n-3}{2}$. So we construct a quasi-étale covering $\delta_X : \mathscr{D}(X) \to D(X)$, where $\delta_X^{-1}(z)$ is identified with the two rulings.

Remark 3.24. If we denote by $\mathscr{D}_k(X)$ the pullback of $D_k(X)$ by δ_X, then from Remark 3.23 it follows that $\mathscr{D}(X)$ is smooth outside $\mathscr{D}_3(X)$.

We want now to construct a double cover of $D(X)$ for any $n \geq 3$, i.e., also for n even. We introduce the following notation. Let X be a general complex in $\mathscr{X}_{2,1,n}$, and let H be a general hyperplane of $\mathbb{P}^n$. Then we denote by $D_H(X)$ the intersection of $D(X)$ with H. We denote by X_H the complex $X \cap \mathbb{G}(1, H)$ which is a general element in $\mathscr{X}_{2,1,n-1}$. Then we can consider the discriminant hypersurface $D(X_H) \subset H$.

Lemma 3.25. *In the above setting one has:*

1. *If $z \in D_H(X) \cap D(X_H)$ is smooth for both hypersurfaces, then they have the same tangent hyperplane in z;*

2. $\mathrm{Sing}(D(X_H)) \subset D_H(X)$ *and* $\mathrm{Sing}(D_H(X)) \subset D(X_H)$.

Proof. Since H is a general hyperplane, $\mathrm{Sing}(D_H(X))$ is the transversal intersection of $\mathrm{Sing}(D(X))$ with H. Furthermore, for any $z \in D_H(X)$, the hyperplane H is not tangent to the tangent cone $C_{D(X),z}$, so $C_{D_H(X),z}$ is exactly the transversal intersection of the latter with H.

Secondly, if we denote by $Q_H(z)$ the intersection of $Q_X(z)$ with H, it is clear that this quadric corresponds to $Q_{X_H}(z)$ for any $z \in H$.

If z is smooth for $D(X_H)$, then $\mathrm{Sing}(Q_H(z))$ is a line s. If z is smooth also for $D_H(X)$, then is smooth for $D(X)$ and $\mathrm{Sing}(Q_X(z)) = s'$ is an other line. From discussion above, we have that $s = s'$. Then from Proposition 3.17, we have that the tangent space $T_{D(X_H),z}$ is the special hyperplane $H_{X_H,s}$, which coincides with $H \cap H_{X,s}$, which is the tangent hyperplane to $D_H(X)$ to z. This proves the first part.

For the second part, let z be a point in $\mathrm{Sing}(D(X_H))$. Then $\dim(\mathrm{Sing}(Q_{X_H}(z))) \geq 2$. Then it follows that $\dim(\mathrm{Sing}(Q_X(x))) \geq 1$ and this prove the first inclusion. The second inclusion follows directly from the first observation of the lemma.

□

Proposition 3.26. *Let $X \in \mathscr{X}_{2,1,n}$, with $n \geq 3$, and assume $D_2(X)$ has dimension $n-3$, then*

$$\deg(D_2(X)) = \frac{4}{3}n(n^2+2) - 4(n^2-2)$$

Proof. It suffices to prove the result for a general complex. In particular, X_H is also general , and from hypothesis $D_2(X_H)$ has the right dimension, i.e. $n-4$. Assume this is not true: this implies that $D_2(X_H)$ is empty by Lemma 3.10, but this is a contradiction because for H general, if $z \in D_k(X) \cap H$, then $z \in D_k(X_H)$. We denote by d_n the degree of $D_2(X)$ with $X \in \mathscr{X}_{2,1,n}$, and we proceed by induction.

For $n = 3$, $d_3 = 16$ and this is true by the classical result cited in Remark 3.21.

Now we consider a general flag $\Sigma \subset H \subset \mathbb{P}^n$, where Σ is a 3-plane and H is a hyperplane. We consider the two surfaces $D_\Sigma(X)$ and $D_\Sigma(X_H)$, which are cut on Σ by $D(X)$ and $D(X_H)$ respectively. Since Σ is general, they are smooth away from the intersection of Σ with $D_2(X)$ and $D_2(X_H)$ respectively. These intersections are two sets of points whose cardinalities are d_n and d_{n-1} respectively. From the description above, these singular points are nodes. The two surfaces intersect in a curve C with multiplicity 2 by the first assertion of Lemma 3.25. Then C has degree $2(n-1)(n-2)$, and it passes through the nodes of both by the second part of Lemma 3.25.

Now we consider the linear system of surfaces of degree $2(n-1)$ in Σ which intersect $D_\Sigma(X_H)$ with multiplicity 2 along C. This system contains $D_\Sigma(X)$ and the general surface in it has hence d_n double points along C. This linear system cuts on C a flat family of divisor on C. Since in the linear system there are the ones consisting of the union of $D_\Sigma(X_H)$ and a general quadric, we have that $d_n = d_{n-1} + 4(n-1)(n-2)$. Using the induction hypothesis we obtain then the result. □

Next we recall the following definition (see [Cat81]):

Definition 3.27. Let S be a surface and let $N = \{x_1, \dots, x_t\}$ be the set of *nodes* of S, or points of type A_1, i.e., isolated double points with tangent cone of rank 3. We say that the set N is *even* if, given a minimal desingularization $p : \tilde{S} \to S$, the sum $\tilde{N}$ of all the irreducible rational curves $\tilde{N}_i = p^{-1}(x_i)$ of self–intersection -2 (also called (-2)-*curves*) is divisible by 2 in $\mathrm{Pic}(\tilde{S})$.

Remark 3.28. We recall a basic construction. Let X be an irreducible smooth projective variety and let $\mathscr{L}$ be an invertible sheaf over X. We suppose moreover that there exists a non zero section $s \in H^0(X, \mathscr{L}^2)$. If $\pi : L \to X$ is the total space of $\mathscr{L}$, we can pull $\mathscr{L}^2$ back over L, obtaining the line bundle $\pi^\star \mathscr{L}^2$, which has a non-zero section $\bar{s}$ which is given by $\pi^\star s - t^2$, where t is the tautological section of $H^0(L, \pi^\star \mathscr{L})$. If we set $Y := \mathrm{div}(\bar{s}) \subset L$, the morphism $\pi|_Y : Y \to X$ is finite and of degree 2, ramified over $R := \mathrm{div}(s)$. See [Cal06, Ch. 6, §1] for details.

The following theorem comes from [Cat81, Proposition 2.11].

Theorem 3.29. *Let S be a reduced surface of degree d in $\mathbb{P}^3$ with an even set of nodes N. Then there exists a quasi–étale double cover $\phi : T \to S$ which is a finite map ramified only at the nodes of N. If S is smooth outside N, then T is smooth.*

Proof. We assume that S is smooth outside N: this will prove also the last statement and does not change the rest of the proof. Using the notation of Definition 3.27 let L be a divisor in $\mathrm{Pic}(\tilde{S})$ such that $2L = \tilde{N} = \sum_{i=1}^t \tilde{N}_i$. Then we can use Theorem 3.29 to construct a double cover of $\tilde{S}$ ramified at the zero locus of the square root of the section of $\mathscr{O}_{\tilde{S}}(2L)$ corresponding to $\tilde{N}$. We denote this map by $p : \tilde{T} \to \tilde{S}$. The curves N_i in $\tilde{S}$ are (-2)-curves and then $B = p^{-1}(\tilde{N}_i)$ are (-1)-curves in $\tilde{T}$. We can now contract these curves obtaining a map π' from $\tilde{T}$ to a smooth surface T and a commutative diagram

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{p} & \tilde{S} \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ T & \xrightarrow{\phi} & S \end{array}$$

and thus the map ϕ is a double cover of S ramified only at the nodes N. □

Remark 3.30. The double cover constructed above is not unique: it depends on the square root of the section associated to $\tilde{N}$. Therefore to achieve the uniqueness we need that $\mathrm{Pic}(\tilde{S})$ has no 2–torsion.

Next we focus on the case $n = 4$.

Lemma 3.31. *Let $X \in \mathscr{X}_{2,1,4}$ be general and let H be a general hyperplane in $\mathbb{P}^4$. Then $D_H(X)$ is a sextic in $\mathbb{P}^3$ with 40 even nodes.*

Proof. The surface $D_H(X)$ has degree 6 and lies in H, which is a $\mathbb{P}^3$. Moreover it has a set N_H of 40 nodes at the intersections of H with the curve $D_2(X)$, which has degree 40 (see Proposition 3.26).

By Lemma 3.25, the surface $D_H(X)$ is tangent to $D(X_H)$, which is a Kummer quartic, along a curve Γ of degree 12. Consider a minimal desingularization Σ of $D_H(X)$. We denote by $\tilde{N}_H$ the sum of the 40 (-2)–curves which map to the nodes of $D_H(X)$. We also denote by $\tilde{\Gamma}$ the strict transform of Γ on Σ. By pulling back the section of $\mathcal{O}_H(4)$ given by $D(X_H)$, we have the linear equivalence relation

$$\tilde{N}_H + 2\tilde{\Gamma} \sim \xi^{\otimes 4} \tag{3.6}$$

where ξ is the pull–back to Σ of $\mathcal{O}_H(1)$. From (3.6) we deduce that the 40 nodes of $D_H(X)$ are even. □

Remark 3.32. (a) By the proof of the previous lemma, we see that $\tilde{N}_H$ is a section of $\mathcal{L}^2$, where $\mathcal{L} = \xi^2 \otimes \mathcal{O}_\Sigma(-\Gamma)$. According to Remark 3.28 and to Theorem 3.29, this defines a smooth quasi–étale double cover $\tilde{D}_H(X)$ of $D_H(X)$ ramified at the 40 nodes of N_H.

(b) If H is no longer generic, but it is simply tangent to $D(H)$ at a point p, but still intersects the curve $D_2(X)$ transversally at 40 points forming a set N_H, then $D_H(X)$ has an additional double point at p. However the previous considerations can be repeated verbatim: in this case there is still a quasi–étale double cover $\tilde{D}_H(X)$ of $D_H(X)$ ramified at the 40 nodes of N_H, the only difference is that $\tilde{D}_H(X)$ is no longer smooth, but it has two nodes mapping to the point p.

(c) Similarly, assume that H is no longer generic, but is simply tangent to the curve $D_2(X)$ at a point q. Then $D_H(X)$ has no longer a set N_H of 40 nodes. The singularities of $D_H(X)$ consist of 38 nodes, the transversal intersection points of H with $D_2(X)$, plus another singularity at q. One may check with a local computation that this singularity is a double point of type A_3, whose resolution consist of a chain $N_1 + N_2 + N_3$ of (-2)–curves, with $N_1 \cdot N_2 = N_2 \cdot N_3 = 1$ and $N_1 \cdot N_3 = 0$. In this case it is also easy to see that the sum N_H of $N_1 + N_3$ plus the 38 (-2)–curves coming from a minimal resolutions of the remaining double points is divisible by 2 in the Picard group of a minimal resolution of $D_H(X)$, so that we still have a double cover as above. Again in this case $\tilde{D}_H(X)$ is no longer smooth but has a node over q.

Theorem 3.33. *If $X \in \mathscr{X}_{2,1,4}$ is general, then there exists a quasi-étale smooth double cover $f : \mathscr{D}(X) \to D(X)$ which is branched along $D_2(X)$.*

Proof. Fix a general plane α in $\mathbb{P}^4$. We denote by Γ the smooth sextic curve cut out by α on $D(X)$ and we consider the pencil $\mathcal{P}$ of hyperplane sections $D_H(X)$ of $D(X)$ cut out by the hyperplanes H through α. The base locus of $\mathcal{P}$ is Γ. By the generality of α, the surfaces $D_H(X)$ in $\mathcal{P}$ all have exactly 40 even nodes at the intersection of H with $D_2(X)$, and no other singularity, except for finitely many surfaces for which cases (b) or (c) in Remark 3.32 occur. In any event, by Remark 3.32, for all surfaces $D_H(X)$ in $\mathcal{P}$, there is a double cover $f_H : \tilde{D}_H(X) \to D_H(X)$, which is smooth, except if either case (b) of Remark 3.32 occurs, in which the double cover has a pair of nodes over the additional node of $D_H(X)$, or case (c) of Remark 3.32 occurs, in which the double cover has a node over the A_3 point of $D_H(X)$.

Consider now the open subset $U = D(X) - \Gamma$. We claim that there is a smooth quasi–étale double cover $f : \tilde{U} \to U$ which is branched over $U \cap D_2(X)$. Indeed, by taking into account Lemma 3.31 and Remark 3.32, (a), we see that the construction of the double cover $f_H : \tilde{D}_H(X) \to D_H(X)$ depends algebraically on the hyperplane H passing through α. Therefore we can consider the algebraic variety

$$\tilde{U} = \bigcup_{\alpha \subset H} (\tilde{D}_H(X) \cap f_H^{-1}(U)),$$

which is double cover $f : \tilde{U} \to U$, where f is the restriction of f_H to $\tilde{D}_H(X) \cup f_H^{-1}(U)$, for all H containing α. Since each $f_H : \tilde{D}_H(X) \to D_H(X)$ is quasi–étale branched over $H \cap D_2(X)$, then also $f : \tilde{U} \to U$ is quasi–étale branched over $U \cap D_2(X)$.

Consider the diagram

$$\begin{array}{ccc} \tilde{U} & \longrightarrow & \mathscr{D}(X) \\ \downarrow & & \downarrow \\ U & \longrightarrow & D(X) \end{array} \tag{3.7}$$

where, abusing notation, we may denote by f both vertical arrows, and $\mathscr{D}(X)$ is the integral closure of $D(X)$ in the field of functions of $\tilde{U}$. The map $f : \mathscr{D}(X) \to D(X)$ is a finite (hence surjective) double cover and $D(X)$ is projective, so also $\mathscr{D}(X)$ is projective. The horizontal arrows in (3.7) are open inclusions. Since $D(X) - U = \Gamma$, the map f is quasi–étale. Moreover, since all points of Γ are smooth for $D(X)$, by the purity of the branch locus we have that f is étale along Γ.

Finally, it is clear that this construction does not depend on α. Indeed, if H is a general hyperplane, the restriction of $f : \mathscr{D}(X) \to D(X)$ to $D_H(X)$ is exactly $f_H : \tilde{D}_H(X) \to D_H(X)$. Since $D_H(X)$ is smooth, we deduce that $\mathscr{D}(X)$ is also smooth. □

Remark 3.34. It is possible that if $X \in \mathscr{X}_{2,1,n}$ is general, and $n \geqslant 6$ is even then there exists a quasi-étale smooth double cover $f : \mathscr{D}(X) \to D(X)$ which is branched along $D_2(X)$. We do not dwell on this here.

3.4 Further remarks on the double cover of the discriminant

In this final section we make some interesting remarks on the double cover of $D(X)$ from Theorem 3.33.

Definition 3.35. A smooth projective variety is called *of general type* if the canonical ring $R(X, K_X) = \oplus_{m\geq 0} H^0(X, mK_x)$ is (finitely generated over the base field $\mathbb{K}$, and) of maximal trascendence degree $n+1$. In addition to this, $Y = \mathrm{Proj}(R(X, K_X))$ is called the *canonical model* of X. The associated morphism is called the *canonical map* and denoted by $\Phi_{K_X} : X \to Y$.

A variety is called *canonical* if it is the canonical model of a certain variety.

Remark 3.36. If C is a smooth curve of genus $g \geq 3$, there are two possible cases about its canonical map:

1. Φ_{K_C} is a canonical embedding in $\mathbb{P}^{g-1}$ or
2. Φ_{K_C} has degree 2 and its image is a rational normal curve (in this case C is called *hyperelliptic*).

Take now S to be a surface with $p_g(S) \geq 3$ and such that Φ_{K_X} is a map of degree 2 whose image is a surface F. Babbage [Bab34] conjectured that $p_g(F) = 0$, similarly to the case of curves. However, it was proved that in the above situation the surface F must satisfy one of the two following condition ([Bea79]):

1. F is a canonical variety or;
2. $p_g(F) = 0$.

A first example of the former case has been found in [Cat81, §3]: one has canonical quintic surfaces F in $\mathbb{P}^3$ with 20 even nodes whose quasi–étale double cover branched at the nodes is a surface X such that Φ_{K_X} coincides with the double cover in question.

The situation in the case $n = 4$ is very similar to Catanese's example mentioned above., i.e., we have the following result.

Proposition 3.37. *Let X be a general complex in $\mathscr{X}_{2,1,4}$. Then $D(X)$ is a canonical sextic threefold and the double cover $f : \mathscr{D}(X) \to D(X)$ is the canonical morphism of $\mathscr{D}(X)$.*

Proof. $D(X)$ has degree 6 and its singular locus is $D_2(X)$. We proved in Proposition 3.19 that $D_2(X)$ is a double locus with canonical singularities. Let $\tilde{D}(X)$ be the desingularization of $D(X)$ obtained by blowing up $D_2(X)$. By adjunction the canonical bundle of $\tilde{D}(X)$ is the pull–back ζ to $\tilde{D}(X)$ of $\mathcal{O}_{\mathbb{P}^4}(1)$, which proves that $D(X)$ is a canonical threefold.

Next, consider the double cover $f : \mathscr{D}(X) \to D(X)$, or rather the associated double cover $\tilde{f} : \tilde{\mathscr{D}}(X) \to \tilde{D}(X)$. This is branched along the divisor E which is a $\mathbb{P}^1$–bundle over $D_2(X)$ contracted to $D_2(X)$ in the resolution $\tilde{D}(X) \to D(X)$. This divisor is therefore divisible by 2 in $\mathrm{Pic}(\tilde{D}(X))$. Let η be such that $\eta^2 \sim \mathcal{O}_{\tilde{D}(X)}(E)$. One has

$$\tilde{f}_\star(K_{\tilde{\mathscr{D}}(X)}) \cong K_{\tilde{D}(X)} \oplus (\eta \otimes K_{\tilde{D}(X)}) = \zeta \oplus (\zeta \otimes \eta). \tag{3.8}$$

Now we claim that $h^0(\zeta \otimes \eta) = 0$. Indeed, if not, one has $h^0(2\zeta \otimes \mathcal{O}_{\tilde{D}(X)}(-E)) > 0$, which implies that $D_2(X)$ is contained in a quadric. This is impossible, as we are going to show. Indeed, if $D_2(X)$ is contained in a quadric, then the 40 nodes N_H of the surface $D_H(X)$, with H a general hyperplane, are also contained in a quadric Q. Let us keep the notation of Lemma 3.31. Since the curve Γ has degree 12, then a quadric Q containing N_H also contains Γ, hence Γ has to be the complete intersection of $D_H(X)$ with Q. From (3.6) we then deduce that $\tilde{N}_H \sim 0$, a contradiction.

In conclusion, since $h^0(\zeta \otimes \eta) = 0$, from (3.8) we deduce that

$$h^0(K_{\tilde{\mathscr{D}}(X)}) = h^0(\zeta) = 5$$

which proves that $f : \mathscr{D}(X) \to D(X)$ is the canonical morphism of $\mathscr{D}(X)$. $\square$

We finish with the following remark: $D(X)$ may be obtained as the section with a general hyperplane H of an EPW-*sextic* Y in $\mathbb{P}^5$, as described in [OGr06]. The 4–fold Y has a surface Z of degree 40 of double points, whose section with H is $D_2(X)$. Arguing as in the proof of Theorem 3.33, one recovers O'Grady's result that there exists a smooth quasi–ètale double cover $X \to Y$, branched along Z, and X is a hyperkäler 4–fold, which is a deformation of the Hilbert scheme of degree 2 of a K3 surface.

Remark 3.38. Given a general $X \in \mathscr{X}_{2,1,n}$ for $n \geq 3$ (the first interesting case is $n = 5$), does there exist a hypersurface F in $\mathbb{P}^{2n-3}$ of degree $2n - 2$, not a cone, whose section with a generic linear space of dimension n is $D(X)$?

If so, and if F has the *right singularities*, by recalling Remark 3.34 we ask: would it be possible to construct a quasi–étale double cover of F which is hyperkäler variety?

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