The Atiyah-Singer index theorem

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Description

In this course we give an introduction to the heat kernel proof of the Atiyah-Singer index theorem for Dirac-type operators.
Prerequisites: this approach to the index formula requires only basic notions of differential geometry and functional analysis. Interested students from every mathematical backgroud are welcome to attend.

Program

1. Presentation, statement of the Hirzebruch signature formula, the Chern-Gauss-Bonnet formula, the Riemann-Roch-Grothendieck formula, vector bundles and connections.

2. Characteristic classes: basic Chern-Weil theory, Chern classes, Pontryagin classes, L-class, $\widehat{A}$-class, Todd class.

3. Differential operators: locality, order, principal symbol, adjoints, ellipticity, generalised Laplacians, Fredholm operators.

4. Clifford modules and Dirac operators: basics of Clifford algebras and Spin geometry, Dirac operators

5. Analysis of Dirac operators: the Bochner-Lichnerowicz-Weitzenboch formula for the square of a Dirac operator, application to positive scalar curvature obstructions on Spin manifolds, Sobolev spaces, elliptic regularity and Fredholmness of elliptic operators.

6. Heat Kernel: spectral theorem and functional calculus, heat equation and heat kernel, Duhamel formula, asymptotic expansion.

7. Traces: Hilbert-Schmidt operators, trace-class operators, Lidski's theorem, gradings on Clifford modules, the canonical grading, basic Hodge theory (signature operator).

8. Traces and the index: supertraces and the McKean-Singer formula, relation of the index with the heat kernel expansion.

9. The local formula: the idea of Getzler and the Harmonic oscillator, local computation for Spin manifolds, Hirzebruch signature formula, Riemann-Roch-Grothendieck formula, Chern-Gauss-Bonnet formula

Selected references

1. N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators*, volume 298 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

2. H. B. Lawson and M-L. Michelsohn Spin geometry. Princeton Mathematical Series, 38. Princeton University Press, Princeton

3. J. Roe. Elliptic operators, topology and asymptotic methods (II edition). *Pitman research notes in Mathematics*, 395 1999.