Theorem HYP.
Let $A$ be such that $\rho(A)$ is eigenvalue of $A$ and there exists $k \geq 1$ such that $A^k$ is non negative and irreducible (HYP) (NOTE: if $A$ is non negative, then HYP on $A$ is equivalent to IRREDUCIBILITY of $A$). Then

(i) $\rho(A)$ is positive, is a simple eigenvalue of $A$ and of $A^T$, and $\exists! \mathbf{z}, \mathbf{w}$ both positive vectors such that $\|\mathbf{z}\|_1 = \|\mathbf{w}\|_1 = 1$ and $A\mathbf{z} = \rho(A)\mathbf{z}$, $A^T\mathbf{w} = \rho(A)\mathbf{w}$.

(ii) There exists a diagonal matrix $D$ with positive diagonal entries such that $DAD^{-1}$ is $\rho(A)$-stochastic by columns (or by rows). Note that $(DAD^{-1})_{ij} \neq 0$ iff $(A)_{ij} \neq 0$, and $(DAD^{-1})_{ij}$ and $(A)_{ij}$ have the same argument [A and $DAD^{-1}$ have the same pattern].

(iii) If $(A^k)_{ii}$ is positive for some $i$, then the remaining $n-1$ eigenvalues of $A$ have absolute value less than $\rho(A)$.

(iv) If $A^k$ is positive, then the remaining $n-1$ eigenvalues of $A$ have absolute value less than $\rho(A)$.

(v) If the remaining $n-1$ eigenvalues of $A$ have absolute value less than $\rho(A)$, then $\frac{1}{\rho(A)}A^j \to \frac{\mathbf{z}\mathbf{w}^T}{\mathbf{w}^T\mathbf{w}}$, and therefore there exists $s$ such that $A^s$ is positive.

(vi) $A$ is similar to a $\rho(A)$-stochastic by rows and by columns matrix.

PROOF. SEE Appendix.

NOTE. Let $A$ be a non negative $n \times n$ matrix. Then $A$ is primitive ($A \geq 0$, $A$ irreducible, $\rho(A)$ dominates the remaining eigenvalues of $A$) iff there exists $m$ such that $A^m$ is positive.

EXERCISE (by Fra). Let $A$ be irreducible, with hermitian pattern (in the sense that $a_{ij} \neq 0$ iff $a_{ji} \neq 0$, and, in such case, $a_{ij}a_{ji} \in \mathbb{R}^+$), and such that $A^2$ is non negative and irreducible. Prove that then there exists $m$ such that $A^m$ is positive.

Corollary HYP. Let $A$ be a stochastic by columns $n \times n$ matrix, i.e. $\sum_i a_{ij} = 1 \forall j$ ($a_{ij} \in \mathbb{C}$). Assume that there exists $k \geq 1$ such that $A^k \geq 0$. Then 1 is eigenvalue of $A$ and $1 = \rho(A)$ (SEE Appendix). If, moreover, $A^k$ is irreducible, then all assertions (i)-(vi) hold with $\rho(A) = 1$ and $\mathbf{w}^T = \mathbf{e}^T = [1 \ 1 \ \cdots \ 1]$.

The result stated in the latter Corollary justifies the researches of Riccardo.

At the end of the Appendix, are reported some considerations on $n \times n$ stochastic by columns matrices $A$ (with complex entries), from which one deduces:

- If $m^A_\lambda(1) = m^A_\mu(1)$, then there exists $\mathbf{z}$ such that $A\mathbf{z} = \mathbf{z}$, $\mathbf{z}^T\mathbf{e} \neq 0$.
- If $m^A_\lambda(1) > m^A_\mu(1)$ and $\exists k$ such that $A^k \geq 0$, then $(1 = \rho(A)$ is eigenvalue of $A$, see above, and) $A$ must have an eigenvalue $\lambda \neq 1$ such that $|\lambda| = 1$. 

1
2 × 2 THEOREM. Let $A$ be a square $n \times n$ matrix that can be partitioned as follows

$$A = \begin{bmatrix} M & 0 \\ N & L \end{bmatrix}$$

where $M$ ($L$) is square and the number of its columns (rows) is equal to the number of columns (rows) of $N$. $M$ can have complex entries, $N \geq 0$, $L \geq 0$.

Assume that $M$ satisfies HYP (this implies $\rho(M)$ positive). Assume also that $\rho(L) < \rho(M)$ (this implies $(\rho(M)L)^{-1}N \geq 0$).

Then $\rho(A)$ ($= \rho(M)$) is positive, is a simple eigenvalue of $A$ and of $A^T$, and exists a unique $z \geq 0$ such that $\|z\|_1 = 1$ and $Az = \rho(A)z$:

$$z = \begin{bmatrix} \tilde{z} \\ (\rho(M)I - L)^{-1}N\tilde{z} \end{bmatrix}, \quad \tilde{z} > 0, \quad M\tilde{z} = \rho(M)\tilde{z},$$

and a unique $w \geq 0$ such that $\|w\|_1 = 1$ and $A^Tw = \rho(A)w$:

$$w = \begin{bmatrix} \tilde{w} \\ 0 \end{bmatrix}, \quad \tilde{w} > 0, \quad M^T\tilde{w} = \rho(M)\tilde{w}.$$ 

Moreover, there exists a diagonal matrix $D$ with positive diagonal entries such that $DMD^{-1}$ is $\rho(M)$-stochastic by columns. As a consequence, by the third Gershgorin theorem, if $[M^k]_{ii} > 0$ for some $i$, then the remaining order $\rho(M) - 1$ eigenvalues of $M$ (the remaining $n - 1$ eigenvalues of $A$) have absolute value smaller than $\rho(M)$ ($= \rho(A)$), and thus, if $j \to +\infty$,

$$\frac{1}{\rho(M)^j}A^j = \frac{zw^T}{w^Tz} \begin{bmatrix} M^j \\ \sum_{i=0}^{j-1} L^i NM^{j-i} \\ L^j \end{bmatrix}.$$ 

Proof. See the Appendix.
3 × 3 THEOREM. Let \( A \) be a square \( n \times n \) matrix that can be partitioned as follows

\[
A = \begin{bmatrix}
L_1 & 0 & 0 \\
N_1 & M & 0 \\
S & N_2 & L_2
\end{bmatrix}
\]

with \( M, L_1, L_2 \) square. \( M \) and \( S \) can have complex entries, \( L_1 \geq 0, N_1 \geq 0, L_2 \geq 0, N_2 \geq 0. \)

Assume that \( M \) satisfies HYP (note that this implies \( \rho(M) \) positive). Assume also that \( \rho(L_1) < \rho(M), \rho(L_2) < \rho(M) \) (note that this implies \( (\rho(M)I - L_2)^{-1}N_2 \geq 0, (\rho(M)I - L_2^T)^{-1}N_2^T \geq 0) \).

Then \( \rho(A) (= \rho(M)) \) is positive, is a simple eigenvalue of \( A \) and of \( A^T \), and exists a unique \( z \geq 0 \) such that \( \|z\|_1 = 1 \) and \( Az = \rho(A)z \):

\[
z = \begin{bmatrix}
0 \\
\tilde{z} \\
(\rho(M)I - L_2)^{-1}N_2\tilde{z}
\end{bmatrix}, \text{ } \tilde{z} > 0, \text{ } M\tilde{z} = \rho(M)\tilde{z},
\]

and a unique \( w \geq 0 \) such that \( \|w\|_1 = 1 \) and \( A^Tw = \rho(A)w \):

\[
w = \begin{bmatrix}
(\rho(M)I - L_1^T)^{-1}N_1^T\tilde{w} \\
w \\
0
\end{bmatrix}, \text{ } \tilde{w} > 0, \text{ } M^T\tilde{w} = \rho(M)\tilde{w}.
\]

Moreover, there exists a diagonal matrix \( D \) with positive diagonal entries such that \( DMD^{-1} \) is \( \rho(M) \)-stochastic by columns. As a consequence, by the third Gershgorin theorem, if \( |M^j|_{ii} > 0 \) for some \( i \), then the remaining \( \text{order}(M) - 1 \) eigenvalues of \( M \) (the remaining \( n - 1 \) eigenvalues of \( A \)) have absolute value smaller than \( \rho(M) (= \rho(A)) \), and thus, if \( j \to +\infty \),

\[
\frac{1}{\rho(M)^j} A^j \to \frac{zw^T}{w^Tz} \begin{bmatrix}
L_1^j & 0 & 0 \\
M^j & 0 \\
L_2^j
\end{bmatrix}.
\]

Proof. Left to the reader.
Consider a $n \times n$ matrix $A$ of the form

$$
A = \begin{bmatrix} M & 0 \\ N & L \end{bmatrix}
$$

with $M, L$ square, non-negative, $N$ non-negative, $M$ and $N$ with the same number of columns, $M_{ii} = 0 \ \forall \ i$, $\sum_i M_{ij} + \sum_k N_{kj} = 1 \ \forall \ j$, and $L$ with the structure

$$
L = \begin{bmatrix} 0 & \cdots & 0 \\ L_{21} & 0 & \cdots \\ \vdots & \ddots & \ddots \\ L_{r1} & \cdots & 0 \\
\end{bmatrix}
$$

where the diagonal zeros are null square matrices not necessarily of the same order, and $\sum_{s=t+1} \sum_i |L_{st}|_{ij} = 1 \ \forall \ j \ \forall \ t = 1, \ldots, r - 1$. Then

$$
\rho(A) = \rho(M) = \begin{cases}
1 & \text{if } N = 0 \ (\Rightarrow \ M \ \text{stochbycol}) \\
\leq 1 & \text{if } N \neq 0, \ M \ \text{reducible} \\
> 1 & \text{if } N \neq 0, \ M \ \text{irreducible}
\end{cases}
$$

Assume also that no column of $M$ is null.

These assumptions are satisfied by $QP^TQ^T$ where $P$ is the transition matrix of the web and $Q$ is the permutation putting together and down all null rows and null sub-rows of $P$, in the sense that

$$
QP^TQ^T = \begin{bmatrix} M^T & N^T \\ 0 & L^T \end{bmatrix}
$$

with $M, L$ square, each row of $M^T$ non-null, and $L$ with a strictly lower triangular block structure (see below for a precise definition of $Q$). By the 2 by 2 Theorem, if the square matrix $M$ satisfies HYP (iff $M$ irreducible ($M \geq 0$)), since $\rho(L) = 0 < \rho(M) = \rho(P)$, then it is uniquely defined $z, z \geq 0$, $\|z\|_1 = 1$, such that $QP^TQ^Tz = \rho(P)z$, $P^T(Q^Tz) = \rho(P)(Q^Tz)$ with $\rho(P) < 1$, unless $N = 0$ in which case $\rho(P) = 1$, i.e., if we set $p = Q^Tz$, we have $p_j = \sum_{i: p_i > 0} \frac{1}{\rho(P)}$, $\|p\|_1 = 1$, $p_i \geq 0$.

Note that $p_i = (Q^Tz)_i = z_q_i$ is null whenever $q_i = \text{order}(M) + s$ where the $s$th row of $(I - \frac{1}{\rho(M)L})^{-1}N$ is null. If we want $p_i > 0 \ \forall \ i$, then it is enough to perturb one zero entry of each null row of $(I - \frac{1}{\rho(M)L})^{-1}N$. In order to do this, it is sufficient to perturb one zero entry of each null row of $N$

\[ \text{[in fact,} \]

$$
(I - \frac{1}{\rho(M)L})^{-1}N = N + \frac{1}{\rho(M)}LN + \ldots + \frac{1}{\rho(M)^{r-1}}L^{r-1}N
$$

where $L^{r}N \geq 0$

(f.i. $0 \rightarrow \frac{1}{\deg(\cdot)+1}$), say the one in position $r$, and maintain non null but reduce (f.i. $\frac{1}{\deg(\cdot)} \rightarrow \frac{1}{\deg(\cdot)+1}$) the nonzero entries in the $r$ column of $N$ and of $M$ so that the resulting $M'$ and $N'$ yet satisfy $\sum_i M'_{ij} + \sum_k N'_{kj} = 1 \ \forall \ j$, and all other assumptions [$M' \geq 0, N' \geq 0, M'$ satisfies HYP iff $M'$ irreducible].
Observe also that $0 < \rho(P') = \rho(M') < \rho(P) = \rho(M) \leq 1$ ($= 1$ iff $N = 0$), where $P'$ is defined by the following equality

$$Q(P')^T Q^T = \begin{bmatrix} M' & 0 \\ N' & L \end{bmatrix}$$

(since $M' \leq M$, $M' \neq M$, $M$ is irreducible, we have $\rho(M') < \rho(M)$) and it is uniquely defined $z'$, $z' > 0$, $\|z'\|_1 = 1$, such that $Q(P')^T Q^T z' = \rho(P')z'$, $(P')^T (Q^T z') = \rho(P')(Q^T z')$ with $\rho(P') < 1$, i.e., if we set $p' = Q^T z'$, we have $p'_j = \sum_{i: i \rightarrow j} \frac{(1/\rho(P'))}{\deg(i)} p'_i$, $\|p'\|_1 = 1$, $p' > 0$. 
Consider a $n \times n$ matrix $A$ of the form

$$A = \begin{bmatrix} L_1 & 0 & 0 \\ N_1 & M & 0 \\ S & N_2 & L_2 \end{bmatrix}$$

with $M, L_1, L_2$ square, non-negative, $N_1, N_2, S$ non-negative, $M$ and $N_2$ with the same number of columns, $L_1, N_1, S$ with the same number of columns, $\sum_k [L_1]_{kj} + \sum_i [N_1]_{ij} + \sum_k [S]_{kj} = 1 \forall j$, $\sum_i M_{ij} + \sum_k [N_2]_{kj} = 1 \forall j$, $M_{ii} = 0 \forall i$, and $L_2$ with the structure

$$L_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ (L_2)_{21} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (L_2)_{r1} & \cdots & (L_2)_{rr} & 0 \end{bmatrix}$$

where the diagonal zeros are null square matrices not necessarily of the same order, and $\sum_{s=t+1}^i (L_2)_{st} = 1 \forall j \forall t = 1, \ldots, r - 1$, and $L_1$ with the structure

$$L_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ (L_1)_{21} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (L_1)_{s1} & \cdots & (L_1)_{ss} & 0 \end{bmatrix}$$

where the diagonal zeros are null square matrices not necessarily of the same order.

Then

$$\rho(A) = \rho(M) = \begin{cases} 1 & N_2 = 0 \Rightarrow \text{M stochbycol} \\ \leq 1 & N_2 \neq 0, \text{M reducible} \\ < 1 & N_2 \neq 0, \text{M irreducible} \end{cases}$$

Assume $M$ with no null row and no null column.

These assumptions are satisfied by $QP^TQ^T$ where $P$ is the transition matrix of the web and $Q$ is the permutation putting together and down (together and on left) all null rows and null sub-rows (null columns and null sub-columns) of $P$, in the sense that

$$QPQ^T = \begin{bmatrix} L_1^T & N_1^T & S^T \\ M^T & N_2^T & 0 \\ 0 & 0 & L_2^T \end{bmatrix}$$

with $M, L_1, L_2$ square, each row and column of $M^T$ non-null, and $L_1, L_2$ with a strictly lower triangular block structure (see below for a precise definition of $Q$). By the $3 \times 3$ Theorem, if the square matrix $M$ satisfies HYP (iff $M$ irreducible ($M \geq 0$)), since $\rho(L_2) = \rho(L_1) = 0 < \rho(M) = \rho(P)$, then it is uniquely defined $z, z \geq 0, \|z\|_1 = 1$, such that $QP^TQ^Tz = \rho(P)z$, $P^T(Q^Tz) = \rho(P)(Q^Tz)$ with $\rho(P) < 1$, unless $N_2 = 0$ in which case $\rho(P) = 1$, i.e., if we set $p = Q^Tz$, we have $p_j = \sum_{i: i \rightarrow j} \frac{(1/\rho(P))^{\deg(i)} p_i, \|p\|_1 = 1, p_i \geq 0}$. Note that $p_i = (Q^Tz)_i = z_{q_i}$ is null whenever $q_i \leq \text{order}(L_1)$ or $q_i = \text{order}(L_1) + \text{order}(M) + s$ where the sth row of $(I - \frac{1}{\rho(M)}L_2)^{-1}N_2$ is null. If
we want $p_i > 0 \forall i : q_i > \text{order}(L_1)$, then it is enough to perturb one zero entry of each null row of $(I - \frac{1}{\rho(M)}L_2)^{-1}N_2$. In order to do this, it is sufficient to perturb one zero entry of each null row of $N_2$.

[ in fact ]

$$(I - \frac{1}{\rho(M)}L_2)^{-1}N_2 = N_2 + \frac{1}{\rho(M)}L_2N_2 + \ldots + \frac{1}{\rho(M)^r-1}L_2^rN_2$$

where $L_2^rN_2 \geq 0$

(f.i. $0 \rightarrow \frac{1}{\text{deg}(\cdot)+1}$), say the one in position $r$, and maintain non null but reduce (f.i. $\frac{1}{\text{deg}(\cdot)} \rightarrow \frac{1}{\text{deg}(\cdot)+1}$) the nonzero entries in the $r$ column of $N_2$ and of $M$ so that the resulting $M'$ and $N'_2$ yet satisfy $\sum_i M'_{ij} + \sum_k (N'_2)_{kj} = 1 \forall j$, and all other assumptions [$M' \geq 0$, $N'_2 \geq 0$, $M'$ satisfies HYP iff $M'$ irreducible].

Observe also that $0 < \rho(P') = \rho(M') < \rho(P) = \rho(M) \leq 1$ (= 1 iff $N_2 = 0$), where $P'$ is defined by the following equality

$${Q(P')}^TQ = \begin{bmatrix} L_1 & 0 & 0 \\ N_1 & M' & 0 \\ S & N' & L_2 \end{bmatrix}$$

(since $M' \leq M$, $M' \neq M$, $M$ is irreducible, we have $\rho(M') < \rho(M)$) and it is uniquely defined $z'$, $z' \geq 0$ ($z'_i = 0$ iff $i \leq \text{order}(L_1)$), $\|z'\|_1 = 1$, such that $Q(P')^TQ^Tz' = \rho(P'z'$, $(P')^TQ^Tz' = \rho(P'((Q^Tz')$ with $\rho(P') < 1$, i.e., if we set $p' = Q^Tz'$, we have $p'_j = \sum_i : i \rightarrow \frac{(1/\rho(P'))}{\text{deg}(i)}p'_i$, $\|p'\|_1 = 1$, $p' \geq 0$ ($p'_i = 0$ iff $q_i \leq \text{order}(L_1)$).
Example: $A = QP^TQ^T$, $P$ web transition matrix

Let $P$ be the $n \times n$ transition matrix of an oriented graph with $n$ vertices (i.e. the web graph), i.e. $P_{ij} = \frac{1}{\text{deg}(i)}$ if there is a link from $i$ to $j$ ($\text{deg}(i) = \# \text{ links from } i$), and $P_{ij} = 0$ otherwise. Note that $P$ is a non negative matrix such that $\sum_j P_{ij} = 1$ if $\text{deg}(i) > 0$, and $\sum_j P_{ij} = 0$ if $\text{deg}(i) = 0$, i.e. $P$ is a non negative quasi-stochastic by rows matrix.

Here below is the procedure generating $QPQ^T$

\[ P = \begin{bmatrix} \end{bmatrix} \]

Move the $r_1$ null rows of the upper-left $n \times n$ submatrix of $P$ down:

\[ R_1PR_1^T = \begin{bmatrix} \end{bmatrix} \]

Move the $r_2$ null rows of the upper-left $(n - r_1) \times (n - r_1)$ submatrix of $R_1PR_1^T$ down:

\[ R_2R_1PR_1^T R_2^T = \begin{bmatrix} \end{bmatrix} \]

Move the $r_3$ null rows of the upper-left $(n - r_1 - r_2) \times (n - r_1 - r_2)$ submatrix of $R_2R_1PR_1^T R_2^T$ down:

\[ R_3R_2R_1PR_1^T R_2^T R_3^T = \begin{bmatrix} \end{bmatrix} \]

In the upper-left $(n - r_1 - r_2 - r_3) \times (n - r_1 - r_2 - r_3)$ submatrix of $R_3R_2R_1PR_1^T R_2^T R_3^T$ there is no null row. Call it $M^T$ and try to apply the $2 \times 2$ THEOREM.

Move the $c_1$ null columns of the upper-left $(n - r_1 - r_2 - r_3) \times (n - r_1 - r_2 - r_3)$ submatrix of $R_3R_2R_1PR_1^T R_2^T R_3^T$ on the left:

\[ C_1R_3R_2R_1PR_1^T R_2^T R_3^T C_1^T = \begin{bmatrix} \end{bmatrix} \]

Move the $c_2$ null columns of the almost upper-left $(n - r_1 - r_2 - r_3 - c_1) \times (n - r_1 - r_2 - r_3 - c_1)$
submatrix of $C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^T$ on the left:

$$C_2C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^T = L_T \begin{bmatrix} 0 & 0 & 0 & M^T & L_T^T \end{bmatrix}$$

Move the $c_3$ null columns of the almost almost upper-left $(n - r_1 - r_2 - r_3 - c_1 - c_2) \times (n - r_1 - r_2 - r_3 - c_1 - c_2)$ submatrix of $C_2C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^T$ on the left:

$$C_3C_2C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^T = L_T \begin{bmatrix} 0 & 0 & 0 & M^T & L_T^T \end{bmatrix}$$

In the almost almost upper-left $(n - r_1 - r_2 - r_3 - c_1 - c_2 - c_3) \times (n - r_1 - r_2 - r_3 - c_1 - c_2 - c_3)$ submatrix of $C_3C_2C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^T$ there is no null column (besides no null row). Call it $M^T$ and try to apply the $3 \times 3$ THEOREM.
APPENDIX

Perron-Frobenius theorem. Let \( M \) be a non negative \((M_{ij} \geq 0)\), irreducible \( n \times n \) matrix. Then \( \rho(M) \) is positive, \( \rho(M) \) is a simple eigenvalue of \( M \) (this implies that \( \rho(A) \) is positive!) and there exists a unique positive vector \( z \) (\( z_i \) positive for all \( i \)) such that \( \|z\|_1 = 1 \) and \( Mz = \rho(M)z \). If \( M \) is also stochastic by columns, then \( 1 = \rho(M) \).

PROOF of THEOREM HYP

(i) Since \( A^k \) is a non negative, irreducible \( n \times n \) matrix, by the Perron-Frobenius theorem \( \rho(A^k) \) is a positive simple eigenvalue of \( A^k \) (this implies that \( \rho(A) \) is positive!) and there exists a unique positive vector \( z \) such that \( \|z\|_1 = 1 \), \( A^kz = \rho(A^k)z = \rho(A)^kz \). Let \( y \neq 0 \) be an eigenvector of \( A \) corresponding to its eigenvalue \( \rho(A) \), thus \( Ay = \rho(A)y \). Note that then \( y \) also satisfies the identities \( A^ky = \rho(A)^ky \), \( \forall j \), and in particular the identity \( A^ky = \rho(A)^ky \). Since \( m_8^k(\rho(A)^k) = 1 \), this implies \( y = \alpha z \), for some \( \alpha \in \mathbb{C} \). So we have \( Az = \rho(A)z \) and \( m_8^k(\rho(A)) \geq m_8^k(\rho(A)) = 1 \) (Stefano). Finally note that \( m_8^k(\rho(A)) \leq m_8^k(\rho(A)^k) = 1 \), thus \( m_8^k(\rho(A)) = 1 \). The assertion on \( w \) follows by observing that \( A \) satisfies HYP iff \( A^T \) satisfies HYP.

(ii) By (i) we know that there exists a unique positive vector \( w \) such that \( \|w\|_1 = 1 \), \( A^T w = \rho(A)w \). It follows that \( \sum_i [A^T]_{ji} w_i = \rho(A)w_j \), and thus \( \sum_i w_i[A]_{ij}w_j^{-1} = \rho(A) \) (\( \forall j \)). Now observe that the latter identity can be rewritten as follows \( \sum_i [DAD^{-1}]_{ij} = \rho(A) \), \( \forall j \), where \( D = d(w) \) is a diagonal matrix with positive diagonal entries.

(iii) Let \( D \) be the matrix introduced in (ii). Then \( (A^k)_{ii} \) positive implies \( (DA^kD^{-1})_{ii} = [(DAD^{-1})^k]_{ii} \), positive. \( A^k \) non negative implies \( DA^kD^{-1} = (DAD^{-1})^k \), non negative. \( A^k \) irreducible implies \( DA^kD^{-1} = (DAD^{-1})^k \), irreducible.

Note also that, since \( DAD^{-1} \) is \( \rho(A) \)-stochastic by columns, i.e. \( (DAD^{-1})^Te = \rho(A)e \), we have that \( (DAD^{-1})^k \) is \( \rho(A)^k \)-stochastic by columns, i.e. \( ((DAD^{-1})^k)^Te = \rho(A)^ke \).

Note that then all the Gershgorin circles \( \mathcal{G}_j \) of \( (DAD^{-1})^k \) are in the set \( \mathcal{B} = \{z \in \mathbb{C} : |z| \leq \rho(A)^k\} \) and their borders pass through the point \( \rho(A)^k \). Moreover, the \( \mathcal{G}_j \) coincide with \( \mathcal{B} \) if \( [(DAD^{-1})^k]_{ij} = 0 \), otherwise they touch the circle \( |z| = \rho(A)^k \) only in \( \rho(A)^k \). So, we can apply the third Gershgorin theorem to the matrix \( (DAD^{-1})^k \) and say that a complex number \( z \), \( |z| = \rho(A)^k \), not being inside any circle, can be an eigenvalue of \( (DAD^{-1})^k \) only if \( z = \rho(A)^k \), since \( \rho(A)^k \) is the only point in \( \alpha \partial K_j \). This and the fact that \( \rho(A)^k \) is a simple eigenvalue of \( (DAD^{-1})^k \) imply that the remaining \( n-1 \) eigenvalues of \( (DAD^{-1})^k \) must have absolute value smaller than \( \rho(A)^k \), and thus, that exactly \( n-1 \) eigenvalues of \( DAD^{-1} \) must have absolute value smaller than \( \rho(A) \).

(iv) It follows from (iii)

(v) Assume that \( A \) satisfies HYP. Let \( J \) be the Jordan form of \( A \). Then there is a non singular matrix \( S \) such that

\[
S^{-1} AS = J = \begin{bmatrix} \rho(A) & & \\ & |\lambda| = \rho(A), \lambda \neq \rho(A) & \\ & |\lambda| < \rho(A) & \end{bmatrix}
\]
Moreover, we can assume that the first column of $S$ is exactly the vector $z$ introduced in (i). Note that $e_1^T S^{-1} A = \rho(A) e_1^T S^{-1}$ and in the same time, of course, $w^T A = \rho(A) w^T$, where $w$ is the other vector introduced in (i). Thus $e_1^T S^{-1}$ must be equal to $\alpha w^T$ for some $\alpha \in \mathbb{C}$ ($m_1^A(\rho(A)) = m_0^A(\rho(A)) = 1$).

Then, since $e_1^T S^{-1} z = 1$, we must have $\alpha w^T z = 1$, that implies $\alpha = 1/w^T z$. In other words, if we assume that the first column of $S$ is exactly the vector $z$, then the first row of $S^{-1}$ is exactly the vector $1/w^T z$. Now consider a partition of $S$ and $S^{-1}$ according to the form of $J$:

$$
S = \begin{bmatrix} z & X & \tilde{X} \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} \frac{1}{w^T z} w^T & Y & \tilde{Y} \end{bmatrix}
$$

(note that $X$, $\tilde{X}$, $Y$, $\tilde{Y}$ must satisfy the identities $w^T X = 0^T$, $w^T \tilde{X} = 0^T$, $Y z = 0$, $\tilde{Y} z = 0$). Then

$$
\frac{1}{\rho(A)^j} A^j = \frac{1}{\rho(A)^j} S J S^{-1} = \begin{bmatrix} z & X & \tilde{X} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{\rho(A)^j} [\lambda = \rho(A), \lambda \neq \rho(A)] & \frac{1}{\rho(A)^j} [\lambda = \rho(A), \lambda \neq \rho(A)]^T \end{bmatrix} \begin{bmatrix} \frac{1}{w^T z} w^T & Y & \tilde{Y} \end{bmatrix},
$$

$$
\frac{1}{\rho(A)^j} A^j = \frac{1}{w^T z} z w^T + X \frac{1}{\rho(A)^j} [\lambda = \rho(A), \lambda \neq \rho(A)] Y + \tilde{X} \frac{1}{\rho(A)^j} [\lambda = \rho(A), \lambda \neq \rho(A)] \tilde{Y}.
$$

If there is no eigenvalue $\lambda$ of $A$ such that $|\lambda| = \rho(A)$, $\lambda \neq \rho(A)$, then the last formula implies that, as $j \to +\infty$, the matrix $\frac{1}{\rho(A)^j} A^j$ tends to the rank one matrix $\frac{1}{w^T z} z w^T$, which is positive. Thus there must exists an $s$ such that $A^s$ is positive.

(vi) Let $S$ be a non singular matrix. First notice that $S A S^{-1}$ is $\rho(A)$-stochastic by columns and by rows iff $(S A S^{-1})^T e = \rho(A) e$, $(S A S^{-1}) e = \rho(A) e$ iff $A^T (S^T e) = \rho(A) (S^T e)$, $A (S^{-1} e) = \rho(A) (S^{-1} e)$. Since $A$ satisfies HYP, there exist positive vectors $z$ and $w'$ such that $A^T w' = \rho(A) w'$, $A z = \rho(A) z$, $\|z\|_1 = 1$, $\sum_i w'_i z_i = n$. Now the problem is reduced to find $S$ such that $S^T e = w'$ (1), $S z = e$ (2). The matrix $S = M + (e - M z) e^T$ satisfies (2) for all $M$, so it is enough to choose $M$ such that (1) holds:

$$
S^T e = M^T e + ((e - M z)^T e) e = w'.
$$

The latter equality is satisfied in particular by $M = d(w')$, and such choice of $M$ makes $S$ non singular (check it!).

PROOF of COROLLARY HYP

Since $A$ is stochastic by columns, we have $A^T e = e$, $e = [1 \ 1 \ \cdots \ 1]^T$, so 1 is eigenvalue of $A^T$, and therefore of $A$ (a matrix and its transpose have the same eigenvalues). If $\lambda$ is an eigenvalue of $A$ then $\lambda^k$ is an eigenvalue of $A^k$. Then

$$
|\lambda|^k \leq \|A^k\|_1 = \max_j \sum_i |[A^k]_{ij}| = \max_j \sum_i |[A^k]_{ij}| = \max_j = 1 = 1
$$

(recall that $\mu$ eig of $M$ implies $|\mu| \leq \|M\|_1$, and that $A$ stochbycol implies $A^j$ stochbycol for all $j$). Thus, $|\lambda|^k = |\lambda|^k \leq 1$, which implies $|\lambda| \leq 1$. So the absolute value of any eigenvalue of $A$ is bounded by 1, and at least one of them (i.e. 1) has absolute value one.
PROOF of the $2 \times 2$ THEOREM

By the Perron-Frobenius theorem, $\rho(M^k)$ is a positive simple eigenvalue of $M^k$ and there exists $\tilde{z} > 0$ such that $M^k \tilde{z} = \rho(M^k) \tilde{z}$. $\rho(M)$ is positive since $\rho(M)^k = \rho(M^k) > 0$. Observe that $M y = \rho(M) y$, $y \neq 0$, implies $M^k y = \rho(M)^k y = \rho(M) y$, thus $y = \alpha \tilde{z}$, $M \tilde{z} = \rho(M) \tilde{z}$, and $m_a^M(\rho(M)) = 1$. Moreover, $m_a^M(\rho(M)) \leq m_a^{M^k}(\rho(M)^k) = m_a^{M^k}(\rho(M^k)) = 1$. So, $\rho(M)$ is positive, is a simple eigenvalue of $M$, and thus $\rho(A) (= \rho(M))$ is positive, and is a simple eigenvalue of $A$. In fact,

$$
\begin{bmatrix}
M & 0 \\
N & L
\end{bmatrix}
\begin{bmatrix}
\tilde{z} \\
x
\end{bmatrix} =
\begin{bmatrix}
M \tilde{z} \\
N \tilde{z} + L x
\end{bmatrix} = \rho(M)
\begin{bmatrix}
\tilde{z} \\
x
\end{bmatrix}
$$

implies $x = (\rho(M) I - L)^{-1} N \tilde{z}$. Finally, of course, $\tilde{z}$ can be chosen so that $\|z\|_1 = 1$ where

$$
z = \begin{bmatrix}
\tilde{z} \\
(\rho(M) I - L)^{-1} N \tilde{z}
\end{bmatrix}.
$$

Analogously, $\rho(M)$ is a simple eigenvalue of $M^T$ and there exists $\tilde{w} > 0$ such that $M^T \tilde{w} = \rho(M) \tilde{w}$, and thus $\rho(A) (= \rho(M))$ is a simple eigenvalue of $A^T$. In fact,

$$
\begin{bmatrix}
M^T & N^T \\
0 & L^T
\end{bmatrix}
\begin{bmatrix}
\tilde{w} \\
x
\end{bmatrix} =
\begin{bmatrix}
M^T \tilde{w} + N^T x \\
L^T x
\end{bmatrix} = \rho(M)
\begin{bmatrix}
\tilde{w} \\
x
\end{bmatrix}
$$

implies $x = 0$. Finally, of course, $\tilde{w}$ can be chosen so that $\|w\|_1 = 1$ where

$$
w = \begin{bmatrix}
\tilde{w} \\
0
\end{bmatrix}.
$$

The proof of the remaining assertions is left to the reader (proceed as in the proof of (ii),(iii),(v) of Theorem HYP).

PROOF of the $3 \times 3$ THEOREM

Left to the reader.
In the following \( A \) is stochastic by columns and in \( \mathbb{C}^{n \times n} \)

Assume \( m_a(1) = m_g(1) = q \) \((\Rightarrow q = m_a^{A^T}(1) = m_g^{A^T}(1))\). Let \( z_i \neq 0 \) be linearly independent and such that \( Az_i = z_i, i = 1, \ldots, q \), and consider the Jordan canonical form of \( A \):

\[
S = \begin{bmatrix} Z & X & \bar{X} \end{bmatrix}, \quad Z = \begin{bmatrix} z_1 & \cdots & z_q \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} E & \cdot & \cdot \\ Y & \cdot & \cdot \\ Y & \cdot & \cdot \end{bmatrix},
\]

\[
S^{-1}AS = \begin{bmatrix} I_q & |\lambda| \geq 1, \lambda \neq 1 \\ |\lambda| = 1 & |\lambda| < 1 \end{bmatrix}.
\]

Observe that the equalities \((e_i^T S^{-1})A = (e_i^T S^{-1})\), \(r = 1, \ldots, q\), and \(e_i^T A = e_i^T\) imply \(e_i^T = \sum_{r=1}^{q}\beta_i e_i^T S^{-1}\). Moreover, \(E\) must be such that \(I_q = EAZ = EZ\). Thus \(e_i^T Z = \sum_{r=1}^{q}\beta_i(e_i^T S^{-1})Z = \sum_{r=1}^{q}\beta_i e_i^T\), and therefore \(\beta_i = e_i^T z_i\). In other words, the following formula must hold:

\[
e_i^T = \sum_{r=1}^{q}(e_i^T z_i)(e_i^T S^{-1}) \quad \text{if } q = 1 : e_i^T = (e_i^T z_i)(e_i^T S^{-1})].
\]

Note: the latter formula proves that at least one \(e_i^T z_i\) must be nonzero \((\text{if } q = 1: e_i^T z_i \text{ must be nonzero})\) !

Then we have the following representation of \(A^r\):

\[
A^r = ZE + X \begin{bmatrix} |\lambda| \geq 1, \lambda \neq 1 \\ |\lambda| = 1 & |\lambda| < 1 \end{bmatrix} Y + \bar{X} \begin{bmatrix} |\lambda| \geq 1, \lambda \neq 1 \\ |\lambda| = 1 & |\lambda| < 1 \end{bmatrix} \bar{Y}
\]

Such formula let us conclude that if the eigenvalues \(\lambda\) of \(A\) different from 1 are such that \(|\lambda| < 1\), then \(A^r \to ZE\), and, in particular, \(A^r v\) converges to a linear combination of the eigenvectors of 1 \((\text{if } q = 1: A^r \to \frac{z_i^T}{e_i^T z} \text{ and } A^r v\) converges to a multiple of \(z\); as a consequence, if \(z > 0\) (as in the example \(A = \begin{bmatrix} -\frac{1}{2} & b \\ \frac{1}{2} & 1 - b \end{bmatrix}; 0 < b < \frac{3}{2}\) then \(\exists r\) such that \(A^r > 0 (A^2 > 0)\)).

Example. For both the following matrices

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},
\]

we have \(m_a(1) = m_g(1) = 2\). For the first matrix 1 is dominant, whereas for the second one 1 is not dominant.

Assume now \(m_a(1) = 2 > m_g(1) = 1\). In this case we shall do the following remark: if \(A^k\) is non negative for some \(k\), then \(A\) must have an eigenvalue \(\lambda \neq 1\) such that \(|\lambda| = 1\). We conjecture that the latter remark holds in the more general case \(m_a(1) > m_g(1)\) (it is not true if \(m_a(1) = m_g(1)\), see the above Example).

Let \(z_1, z_2 \neq 0\) and linearly independent be such that \(Az_1 = z_1, Az_2 = z_1 + z_2\). Consider the Jordan canonical form of \(A\):

\[
S = \begin{bmatrix} Z & X & \bar{X} \end{bmatrix}, \quad Z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} E & \cdot & \cdot \\ Y & \cdot & \cdot \\ Y & \cdot & \cdot \end{bmatrix},
\]

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\[ S^{-1}AS = \begin{bmatrix}
1 & 1 \\
0 & 1 \\
|\lambda| \geq 1, \lambda \neq 1 & |\lambda| < 1
\end{bmatrix}. \]

Then we have the following representation of \( A^r \):

\[
A^r = Z \begin{bmatrix}
1 & r \\
0 & 1
\end{bmatrix} E + X \begin{bmatrix}
|\lambda| \geq 1, \lambda \neq 1 & |\lambda| < 1
\end{bmatrix} Y + \tilde{X} \begin{bmatrix}
|\lambda| \geq 1, \lambda \neq 1 & |\lambda| < 1
\end{bmatrix} \tilde{Y}
\]

(prove it!).

Now let us prove the remark. So, assume \( A^k \geq 0 \) \( \Rightarrow 1 = \rho(A) \) is eigenvalue of \( A \) (by Corollary HYP). If all \( \lambda \neq 1 \) are such that \( |\lambda| < 1 \), then, chosen \( p \) such that \((z_1)_p \neq 0\), we would have \( \|(A^{km})_{p,j}\| \to +\infty \) as \( m \to +\infty \), and this is not possible since \( A^{km} \) is non negative and stochastic by columns for all \( m \).

Note: If \( A \in \mathbb{C}^{n \times n} \) is stochastic by columns and \( S \) is the matrix transforming \( A \) in Jordan form, i.e.

\[ S^{-1}AS = \begin{bmatrix}
[1] & [\lambda] \geq 1, \lambda \neq 1 \\
[\lambda] = 1
\end{bmatrix}, \quad [1] = \begin{bmatrix}
U_{q_1} & \cdots & U_{q_g}
\end{bmatrix}, \quad U_s = \begin{bmatrix}
1 & 1 \\
\vdots & \ddots & \vdots \\
1 & 1
\end{bmatrix} \]

then \( e^T(Se_j) = 0 \), if \( j \neq q_1, q_1 + q_2, \ldots, q_1 + q_2 + \cdots + q_g \) (prove it!).