2 Novembre 2011

Example (Jessica). Let A and A' be the following 3×3 matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A' = \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1 \\ 1/4 & 0 & 0 \end{bmatrix}.$$

Observe that $[111]A = [111]$, $[111]A' = [111]$, thus both A and A' have 1 as eigenvalue; moreover $1 = \rho(A)$ (use first Gershgorin theorem). Note also that A^k, $k = 1, 2, \ldots$, is never a positive matrix (it remains of course a permutation matrix for all k), whereas there is an s such that $(A')^s$ is positive, $s = 5$. This corresponds to the fact that the first matrix A has, besides 1, also the remaining two eigenvalues of absolute value 1, whereas for the second matrix A' the eigenvalue $1 = \rho(A)$ dominates in absolute value the remaining two eigenvalues.

Question: when in a non negative stochastic by columns matrix a perturbation of a zero to nonzero makes its powers positive, or equivalently makes 1 dominant with respect all the remaining $n-1$ eigenvalues? This happens for example when the perturbation makes some power of the matrix both irreducible and non negative with at least a positive diagonal entry (so, first of all, the perturbation must make, or maintain, the matrix irreducible ...) ... (See the Theorem, Corollary and Note below).

In the following take into account that when we say that a matrix $n \times n$ A is stochastic by columns (stochbycol) we mean only that $\sum_i a_{ij} = 1$, $\forall j$, i.e. the entries a_{ij} can be arbitrary complex numbers (they are not necessarily non negative). Also recall that for a $n \times n$ matrix M, $\rho(M)$ denotes the non negative number $\max_i |\lambda_i(M)|$, where $\lambda_i(M)$, $i = 1, \ldots, n$, are the eigenvalues of M. The following Lemma is fundamental in our arguments.

Lemma (Perron-Frobenius)

Let M be a non negative ($M_{ij} \geq 0$), irreducible $n \times n$ matrix. Then $\rho(M)$ is positive, $\rho(M)$ is a simple eigenvalue of M, and there exists a unique positive vector $\mathbf{z}$ (z_i positive for all i) such that $\|\mathbf{z}\|_1 = 1$ and $M\mathbf{z} = \rho(M)\mathbf{z}$. If M is also stochbycol then $1 = \rho(M)$.

proof: see Varga, Matrix Iterative Analysis.

Lemma

If λ is an eigenvalue of a $n \times n$ matrix M then $|\lambda| \leq \|M\|_1$.

proof: Let $\mathbf{x} \neq \mathbf{0}$ be such that $M^T\mathbf{x} = \lambda\mathbf{x}$, and r such that $|x_r| =$

$\max_j |x_j|$. Note that $|x_r|$ is positive.

$$\lambda x_i = \sum_{j=1}^{n} [M^T]_{ij} x_j = \sum_{j=1}^{n} [M]_{ji} x_j, \ |\lambda||x_i| \le \sum_{j=1}^{n} |[M]_{ji}||x_j| \le |x_r| \sum_{j=1}^{n} |[M]_{ji}|, \ \forall\, i,$$

$$|\lambda||x_r| \le |x_r| \sum_{j=1}^{n} |[M]_{jr}|, \ |\lambda| \le \sum_{j=1}^{n} |[M]_{jr}| \le \max_i \sum_{j=1}^{n} |[M]_{ji}| =: \|M\|_1.$$

Theorem.

Let A be a stochbycol $n \times n$ matrix, i.e. $\sum_{i=1}^{n} a_{ij} = 1, \forall\, j$. Assume that there exists $k \ge 1$ such that $A^k \ge 0$. Then

i) 1 is an eigenvalue of A

ii) $1 = \rho(A)$

iii) If A^k is irreducible (note that this implies that A is irreducible), then 1 is a simple eigenvalue of A ($1 \le m_g^A(1) \le m_a^A(1) = 1$) and $\exists !\ \mathbf{z}$ positive such that $\|\mathbf{z}\|_1 = 1$ and $A\mathbf{z} = \mathbf{z}$. If, moreover, $[A^k]_{ii}$ is positive for some i, then the remaining $n-1$ eigenvalues of A have absolute value less than 1.

Examples:

$$A = \begin{bmatrix} 1 & 1+\varepsilon \\ 0 & -\varepsilon \end{bmatrix}, \ 0 < \varepsilon < 1, \ A^2 \ge 0; \quad A = \begin{bmatrix} -\frac{1}{2} & b \\ \frac{3}{2} & 1-b \end{bmatrix}, \ 0 < b < \frac{1}{2}, \ A^2 > 0;$$

Eigenvalues of this second matrix A: $1, -b - \frac{1}{2}$;
eigenvector corresponding to 1: $[\frac{2}{3}by \mid y]^T$, $y \in \mathbb{C}$, it can be positive!

Exercise: Find a, b such that, if $A = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$, then A^2 is non negative, irreducible and with some zero entry.

Remark: A irreducible does not imply A^k irreducible ($A = \begin{bmatrix} \beta-1 & \beta \\ 2-\beta & 1-\beta \end{bmatrix}$)

proof of the Theorem.

i) Since A is stochastic by columns, we have $A^T\mathbf{e} = \mathbf{e}$, $\mathbf{e} = [1\ 1\ \cdots\ 1]^T$, so 1 is eigenvalue of A^T, and therefore of A (a matrix and its transpose have the same eigenvalues).

ii) If λ is an eigenvalue of A then λ^k is an eigenvalue of A^k. Then, by the Lemma,

$$|\lambda^k| \le \|A^k\|_1 = \max_j \sum_i |[A^k]_{ij}| = \max_j \sum_i [A^k]_{ij} = \max_j 1 = 1$$

(recall that A stochbycol implies A^j stochbycol for all j). Thus, $|\lambda|^k = |\lambda^k| \le 1$, which implies $|\lambda| \le 1$. So the absolute value of any eigenvalue of A

is bounded by 1, and, by i), there is at least one eigenvalue whose absolute value is 1, i.e. 1.

iii) Since A^k is a non negative, irreducible, stochbycol $n \times n$ matrix, by the Perron-Frobenius theorem $1 = \rho(A^k)$ is a simple eigenvalue of A^k and there exists a unique positive vector $\mathbf{z}$ such that $\|\mathbf{z}\|_1 = \mathbf{e}^T\mathbf{z} = 1$, $A^k\mathbf{z} = \mathbf{z}$. Let $\mathbf{y} \neq \mathbf{0}$ be an eigenvector of A corresponding to its eigenvalue 1, thus $A\mathbf{y} = \mathbf{y}$. Note that then $\mathbf{y}$ also satisfies the identities $A^j\mathbf{y} = \mathbf{y}$, $\forall j$, and in particular the identity $A^k\mathbf{y} = \mathbf{y}$. Since $m_g^{A^k}(1) = 1$, this implies $\mathbf{y} = \alpha\mathbf{z}$, for some $\alpha \in \mathbb{C}$. So we have $A\mathbf{z} = \mathbf{z}$ and $m_a^A(1) \geq m_g^A(1) = 1$ (Stefano). Finally note that $m_a^A(1) \leq m_a^{A^k}(1) = 1$, thus $m_a^A(1) = 1$, and the first assertion is proved.

Note that all the Gershgorin circles $\mathcal{G}_j$ of $(A^k)^T$ are in the set $\mathcal{B} = \{z \in \mathbb{C} : |z| \leq 1\}$ and their borders pass through 1. More precisely, they coincide with $\mathcal{B}$ if $[A^k]_{jj} = 0$, otherwise they touch the circle $|z| = 1$ only in 1. Since A^k is irreducible, we can apply the third Gershgorin theorem and say that a complex number z, $|z| = 1$, not being inside any circle, can be an eigenvalue of A^k only if $z = 1$. This and the fact that 1 is a simple eigenvalue of A^k imply that all the remaining $n-1$ eigenvalues of A^k must have absolute value smaller than 1, and thus, that exactly $n-1$ eigenvalues of A must have absolute value smaller than 1.

Corollary (Francesco).

Let A be a stochbycol $n \times n$ matrix, and assume that there exists $s \geq 1$ such that A^s is positive. Then 1 is an eigenvalue of A and the remaining $n-1$ eigenvalues of A have absolute value less than 1.

proof. Apply the Theorem for $k = s$.

Note

Let A be a stochbycol $n \times n$ matrix, and assume that there exists $k \geq 1$ such that A^k is non negative and irreducible. If the remaining $n-1$ eigenvalues of A have absolute value less than 1, then there exists s such that A^s is positive.

proof of the Note.

Let A be a $n \times n$ stochbycol matrix such that A^k is non negative and irreducible for some $k \geq 1$. Let J be the Jordan form of A. Then, by the Theorem, there is a non singular matrix S such that

$$S^{-1}AS = J = \begin{bmatrix} 1 & & \\ & \big[|\lambda| = 1,\ \lambda \neq 1\big] & \\ & & \big[|\lambda| < 1\big] \end{bmatrix}.$$

Moreover, we can assume that the first column of S is exactly the vector $\mathbf{z}$ of the Theorem. If so, then $\mathbf{e}_1^T S^{-1} A\mathbf{z} = \mathbf{e}_1^T S^{-1}\mathbf{z}$ must be equal to 1. But $\mathbf{e}_1^T S^{-1} A = \mathbf{e}_1^T S^{-1}$ and in the same time, of course, $\mathbf{e}^T A = \mathbf{e}^T$, thus $\mathbf{e}_1^T S^{-1}$ must be equal to $\alpha\mathbf{e}^T$ for some $\alpha \in \mathbb{C}$ ($m_g^{A^T}(1) = m_g^A(1) = 1$). Then we have $\alpha\mathbf{e}^T\mathbf{z} = 1$, that implies $\alpha = 1$. In other words, if we assume that the first column of S is exactly the vector $\mathbf{z}$ of the Theorem, then the first row of S^{-1} is exactly the vector $\mathbf{e}^T$.

Now partition S and S^{-1} according to the form of J:

$$S = \begin{bmatrix} \mathbf{z} & X & \tilde{X} \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} \mathbf{e}^T \\ Y \\ \tilde{Y} \end{bmatrix}$$

(note that X and $\tilde{X}$ must satisfy the identities $\mathbf{e}^T X = \mathbf{0}^T$, $\mathbf{e}^T\tilde{X} = \mathbf{0}^T$; note also that in case A is also stochastic by rows then $\mathbf{z} = \frac{1}{n}\mathbf{e}$ and $Y\mathbf{e} = \mathbf{0}$, $\tilde{Y}\mathbf{e} = \mathbf{0}$). Then

$$A^j = SJ^jS^{-1} = \begin{bmatrix} \mathbf{z} & X & \tilde{X} \end{bmatrix} \begin{bmatrix} 1 & & \\ & \Big[|\lambda| = 1,\, \lambda \neq 1\Big]^j & \\ & & \Big[|\lambda| < 1\Big]^j \end{bmatrix} \begin{bmatrix} \mathbf{e}^T \\ Y \\ \tilde{Y} \end{bmatrix},$$

$$A^j = \mathbf{z}\mathbf{e}^T + X\Big[|\lambda| = 1,\, \lambda \neq 1\Big]^j Y + \tilde{X}\Big[|\lambda| < 1\Big]^j \tilde{Y}.$$

If there is no eigenvalue λ of A such that $|\lambda| = 1$, $\lambda \neq 1$, then the last formula implies that A^j tends to the rank one matrix $\mathbf{z}\mathbf{e}^T$, which is positive. Thus there must exists an s such that A^s is positive.

Problem

Let M be $n \times n$, stochbycol, non negative, irreducible, with $M_{ii} = 0$, $\forall\, i$. Assume also that 1 is not the only eigenvalue of M whose absolute value is 1 (or, equivalently, that M^s is not positive, $\forall\, s$).

(Examples of such M: irreducible permutations, . . ., see below).

Set $M' = M + \varepsilon\mathbf{e}_r\mathbf{e}_s^T - \varepsilon\mathbf{e}_k\mathbf{e}_s^T$, with r, s, k such that $r \neq s$, $M_{rs} = 0$, M_{ks} positive, and ε positive, $\varepsilon < M_{ks}$ ($\leq$?).

(Note that also M' is $n \times n$, stochbycol, non negative, irreducible, with $M'_{ii} = 0$, $\forall\, i$).

Look for r, s such that the remaining $n-1$ eigenvalues of M' have absolute value less than 1 (which is equivalent to say that there exists s such that $(M')^s$ is positive).

(Note that M in the Problem has the role of the matrix A^k in the Theorem).

Example: irreducible permutations, a conjecture

Any $n \times n$ permutation matrix is stochbycol, non negative. Moreover, all its eigenvalues have absolute value equal to 1. So, *irreducible* permutation matrices satisfy the assumption on M in the Problem. Note that the diagonal entries of an irreducible permutation matrix are all zeros.

Conjecture (Riccardo): Given a generic $n \times n$ irreducible permutation matrix P, if we perturb one of its zeros, say the one in position (i, j) $(i \neq j)$, to a positive value ε, and the 1 in the same column, say column j, to the positive value $1 - \varepsilon$ (thus $0 < \varepsilon < 1$), and call P' the resulting matrix (P' is $n \times n$, stochbycol, non negative, irreducible), then the remaining eigenvalues of P' have absolute value less than 1 ($\exists\, s$ such that $(P')^s > 0$) if and only if either the diagonal $(i + t, j + t)$, $t \in \mathbb{Z}$, or the antidiagonal $(i + t, j - t)$, $t \in \mathbb{Z}$, of P contains only zero entries.

We now see that the conjecture is true for the following particular irreducible $n \times n$ permutation matrix

$$P = \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{bmatrix}.$$

Set $X = P + \varepsilon \mathbf{e}_i \mathbf{e}_j^T$, $1 \leq i, j \leq n$. Then (Paolo)

$$X^n = \begin{cases} I + \varepsilon X^{n-i+j-1} & i \geq j \quad \text{(checked for } X = P + \varepsilon \mathbf{e}_{i-j+1} \mathbf{e}_1^T \text{)} \\ I + \varepsilon X^{j-i-1} & i < j \quad \text{(checked for } X = P + \varepsilon \mathbf{e}_1 \mathbf{e}_{j-i+1} \text{)} \end{cases}$$

(prove this formula for X^n !). Note that in case $i < j$ the matrix X^n is reducible iff $j - i = 1$, and in case $i \geq j$ the matrix X^n is reducible iff $i - j = n - 1$.

Now set

$$P' = \begin{cases} P + \varepsilon \mathbf{e}_i \mathbf{e}_j^T - \varepsilon \mathbf{e}_{j-1} \mathbf{e}_j^T & j > 1 \\ P + \varepsilon \mathbf{e}_i \mathbf{e}_1^T - \varepsilon \mathbf{e}_n \mathbf{e}_1^T & j = 1 \end{cases}.$$

The matrix $(P')^k$ has the nonzero entries exactly in the same places where X^k, $X = P + \varepsilon \mathbf{e}_i \mathbf{e}_j^T$, has the nonzero entries ($k = 0, 1, 2, \ldots$). As a consequence, for all i, j such that $[P]_{ij} = 0$ $((i, j) \notin \{(n, 1), (t, t + 1), t = 1, \ldots, n - 1\})$, the matrix P' satisfies all the hypothesis on A of the Theorem at least for

$k = n$, and thus, for such i, j, we can say that 1 is an eigenvalue of P' and the remaining $n - 1$ eigenvalues have absolute value less than 1.

By a direct calculation one can see that the conjecture is also true for the following particular 4×4 irreducible permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

In fact, the matrix P' obtained by perturbing the zeros of P in positions $(2, 1)$, $(1, 3)$, $(3, 4)$, $(4, 2)$ to ε, and the 1s in the same columns to $1 - \varepsilon$, never satisfies the inequality $(P')^s > 0$. For example,

$$P' = \begin{bmatrix} 0 & 1-\varepsilon & & 0 \\ 0 & 0 & 0 & \\ & 0 & 0 & 0 \\ 0 & \varepsilon & & 0 \end{bmatrix}, \quad (P')^k = \begin{bmatrix} & 0 & 0 & \\ 0 & & & 0 \\ 0 & & & 0 \\ & 0 & 0 & \end{bmatrix}, \quad (P')^{k+1} = \begin{bmatrix} 0 & & & 0 \\ & 0 & 0 & \\ & 0 & 0 & \\ 0 & & & 0 \end{bmatrix}$$

(note that the eigenvalues of such P' are $1, -1, \pm\mathbf{i}\sqrt{1-\varepsilon}$). The inequality $(P')^s > 0$ is instead obtained by perturbing all remaining zeros of P.

Exercise (Valerio). Check the conjecture for

$$P = \begin{bmatrix} 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 \\ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & & 0 & 0 \end{bmatrix}.$$

For example, if we perturb the $(1, 3)$ zero entry:

$$P' = \begin{bmatrix} 0 & & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & & 0 \\ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 1-\varepsilon & 0 & 0 \end{bmatrix}, \quad (P')^k = \begin{bmatrix} & & & & \\ & 0 & 0 & & 0 \\ 0 & & & 0 & \\ 0 & & & 0 & \\ & 0 & 0 & & 0 \end{bmatrix}, \quad (P')^{k+1} = \begin{bmatrix} & & & & \\ 0 & & & 0 & \\ & 0 & 0 & & 0 \\ & 0 & 0 & & 0 \\ 0 & & & 0 & \end{bmatrix},$$

...

Note that in the conjecture the assumption $P=$ permutation is essential. Consider, for instance, the following 5×5 non negative stochbycol irreducible matrix.

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-\varepsilon & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \varepsilon & 0 & 0 \end{bmatrix}, \quad 0 < \varepsilon < 1.$$

It is simple to observe that the matrix P^s is not positive, $\forall\, s$, and thus there is at least an eigenvalue λ of P such that $\lambda \neq 1$, $|\lambda| = 1$. So, all assumptions in the conjecture are satisfied by P except the permutation assumption. Now, perturbing the $(4, 2)$ zero entry of P does not yield $(P')^s > 0$ (instead, perturbing any other zero entry yields $(P')^s > 0$).

Other matrices: other conjectures.

If P is any irreducible matrix with $[P]_{i,i+1} \neq 0$, $\forall\, i$, not satisfying $P^s > 0$, then $(P')^s > 0$ whenever P' is obtained by perturbing a zero in a diagonal of zeros of P. Is this assertion true?

For example check the assertion for the following matrices P

$$P = \begin{bmatrix} 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1-\varepsilon & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-\varepsilon & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \varepsilon & 0 & 0 \end{bmatrix}, \quad 0 < \varepsilon < 1,$$

$$P = \begin{bmatrix} 0 & b_1 & & & & \\ 1 & 0 & b_2 & & & \\ & 1-b_1 & 0 & & & \\ & & 1-b_2 & & & \\ & & & b_{n-2} & & \\ & & & 0 & 1 \\ & & & 1-b_{n-2} & 0 \end{bmatrix}, \quad b_i \in (0, 1)$$

[characteristic polynomial of the first: $\lambda(\lambda^3 - 1)$; -1 is eigenvalue of the third]. For example, in the third 3×3 case, by perturbing the $(3, 1)$ zero entry, we obtain

$$P' = \begin{bmatrix} 0 & b & 0 \\ 1-\varepsilon & 0 & 1 \\ \varepsilon & 1-b & 0 \end{bmatrix}$$

whose characteristic polynomial is $(\lambda - 1)(\lambda^2 + \lambda + b\varepsilon)$ ...

If P' is any irreducible matrix with $[P']_{i,i+1} \neq 0$, $\forall\, i$, such that at least two of its diagonals contain each at least one nonzero entry, then $(P')^s > 0$ for some s. Is this assertion true ?

The assertion can be verified on the matrices

$$P' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1-\varepsilon & 0 \end{bmatrix}, \quad P' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1-\varepsilon \\ 0 & 0 & 0 & \varepsilon & 0 \end{bmatrix}, \quad 0 < \varepsilon < 1$$

[characteristic polynomial of the first: $\lambda(\lambda - 1)(\lambda^2 + \lambda + \varepsilon)$].

Exercise.

If we start from a reducible permutation matrix P, then at least two zeros of P must be perturbed in order to obtain a matrix P' with the property $(P')^s > 0$. Prove it! (hint: P is the direct sum of two permutations)

What zero entry in P to perturb in order to reduce as more as possible the absolute value of the remaining $n-1$ eigenvalues?

Case of the particular permutation P:

Eigenvalues of

$$P' = \begin{bmatrix} 0 & 1 & 0 \\ \varepsilon & 0 & 1 \\ 1-\varepsilon & 0 & 0 \end{bmatrix} \quad : \quad 1,\ \frac{1}{2}(-1 \pm \sqrt{1-4(1-\varepsilon)}),$$

so, for ε going from 0 to 1, they are first complex and coniugate with real part $-\frac{1}{2}$, then real and coincident (equal to $-\frac{1}{2}$), and lastly real and distinct, one going to -1 and the other one going to 0.

Eigenvalues of

$$P' = \begin{bmatrix} \varepsilon & 1 & 0 \\ 0 & 0 & 1 \\ 1-\varepsilon & 0 & 0 \end{bmatrix} \quad : \quad 1,\ \frac{1}{2}(-(1-\varepsilon) \pm \mathbf{i}\sqrt{(1-\varepsilon)(\varepsilon+3)}),$$

so, for ε going from 0 to 1, they are complex and coniugate and go both to coincide to 0.

Eigenvalues of

$$P' = \begin{bmatrix} 0 & 1 & & \\ & & 1 & \\ \varepsilon & & & 1 \\ 1-\varepsilon & & & 0 \end{bmatrix} \quad : \quad 1,\ ???$$

When ε goes to 1, only one of the remaining three eigenvalues goes to 0. The other two go to $e^{\mathbf{i}2\pi/3}$ and $e^{\mathbf{i}4\pi/3}$.

Eigenvalues of

$$P' = \begin{bmatrix} 0 & 1 & & \\ \varepsilon & & 1 & \\ & & & 1 \\ 1-\varepsilon & & & 0 \end{bmatrix} \quad : \quad 1,\ ??????$$

When ε goes to 1, two of the remaining three eigenvalues goes to 0. The third goes to -1.

In the $n \times n$ case, if in

$$P' = P + \varepsilon \mathbf{e}_i \mathbf{e}_1^T - \varepsilon \mathbf{e}_n \mathbf{e}_1^T = \begin{bmatrix} & & 1 & & \\ & & & 1 & \\ \varepsilon & & & & \ddots \\ & & & & & 1 \\ 1-\varepsilon & & & & \end{bmatrix},$$

the parameter ε tends to 1, then, of the eigenvalues of P' different from 1, $n-i$ eigenvalues become 0, and the remaining $i-1$ tend to $e^{\mathbf{i}2\pi k/i}$, $k = 1, \dots, i-1$.

Let a_{ij} be real numbers. There exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

if $a_{11}a_{22} - a_{12}a_{21} \geq 0$. Formulas: compute β from the equality

$$\beta^2 = \frac{a_{12}^2}{a_{11} + a_{22} \pm 2\sqrt{a_{11}a_{22} - a_{12}a_{21}}},$$

and γ, α, δ from the identities

$$\gamma = \frac{a_{21}}{a_{12}}\beta, \ \alpha = \frac{a_{12}}{2\beta} - \frac{(a_{22} - a_{11})\beta}{2a_{12}}, \ \delta = \frac{(a_{22} - a_{11})\beta}{2a_{12}} + \frac{a_{12}}{2\beta}.$$

Result (Francesco). A $n \times n$ non negative, irreducible, $\rho(A) = 1$. Then there exists a diagonal matrix D with positive entries such that DAD^{-1} is stochastic by columns, non negative, and has the same pattern of A (the positive entries of DAD^{-1} are exactly where are the positive entries of A).

proof: use the Perron-Frobenius lemma.

Exercise. A $n \times n$, $1 = \rho(A)$ is eigenvalue of A, and there exists $k \geq 1$ such that A^k is non negative and irreducible. Then there exists a diagonal matrix D with positive entries such that DAD^{-1} is stochastic by columns, and has the same pattern of A (the nonzero entries of DAD^{-1} are exactly where are the nonzero entries of A, and have the same sign).

Is this assertion true ? Yes! (see (ii) of the next Theorem)

Questions: A non negative and unitary implies A permutation ? Yes! (if one of the columns of A has at least two nonzero entries, then another column of A must have two nonzero entries in the same positions and one of them must be negative, by the unitary condition). A irreducible with all eigenvalues on the circle $\{z : |z| = 1\}$ implies A unitary ? Fra: no! see

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} = \frac{1}{2} \begin{bmatrix} x & 2-x \\ x+2 & -x \end{bmatrix}$$

Remark: $A^T\mathbf{x} = \mathbf{x}$, $x_i \neq 0 \;\Rightarrow\; (D^{-1}A^TD)(D^{-1}\mathbf{x}) = D^{-1}\mathbf{x}$, $(D^{-1}\mathbf{x})_i > 0$.

Example: $A = QP^TQ$*, P web transition matrix*

Let P be the transition matrix of the web (which is quasi-stochastic by rows). Then there exists a permutation Q such that

$$QPQ^T = \begin{bmatrix} [\,0\,] & & & & & & \\ \vdots & \ddots & & N_1^T & & & \\ [\,0\,] & \cdots & [\,0\,] & & & & \\ [\,0\,] & \cdots & [\,0\,] & M^T & & N_2^T & \\ & & & & [\,0\,] & & \\ & O & & O & \vdots & \ddots & \\ & & & & [\,0\,] & \cdots & [\,0\,] \end{bmatrix}$$

where the columns are all non null except the first ones which clearly are null, the rows are all non null except the last ones which clearly are null, no row or column of M is null, N_1^T may have null columns, N_2^T may have null rows, $[M]_{ii} = 0 \;\forall\, i$.

Of course, also QPQ^T is quasi-stochastic by rows. So, $0 \leq \rho(M) \leq 1$. $\rho(M)$ is certainly equal to 1 if $N_2 = 0$; if M is irreducible then $\rho(M)$ is certainly less than 1 if $N_2 \neq 0$. $\rho(M) = 0$ when ?

QP^TQ^T has the structure of A in the Theorem. Note that $\rho(L_1) = \rho(L_2) = 0 < \rho(M)$.

Moreover, there exists a permutation Q such that

$$QPQ^T = \begin{bmatrix} M^T & [\,0\,] & \cdots & [\,0\,] & & & \\ & [\,0\,] & \ddots & \vdots & & & \\ & & \ddots & [\,0\,] & & & \\ & & & [\,0\,] & & & \\ [\,0\,] & \cdots & \cdots & [\,0\,] & [\,0\,] & & \\ \vdots & & & & \ddots & \ddots & \\ [\,0\,] & \cdots & \cdots & \cdots & \cdots & [\,0\,] & [\,0\,] \end{bmatrix}$$

where each row and each column of M is non null.

Yet, there exists a permutation Q such that

$$QPQ^T = \begin{bmatrix} M^T & [0] & & [0] & & [0] & \\ & [0] & & \vdots & & \vdots & \\ [0] & [0] & [0] & [0] & & \vdots & \\ & & & [0] & & \vdots & \\ [0] & \cdots & \cdots & [0] & [0] & [0] & \\ & & & & & [0] & \\ [0] & \cdots & \cdots & \cdots & \cdots & [0] & [0] \end{bmatrix}$$

where each row and each column of M is non null.

The procedure generating QPQ^T

$$P = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

Move the r_1 null rows of the upper-left $n \times n$ submatrix of P down:

$$R_1PR_1^T = \begin{bmatrix} \\ \\ \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix}$$

Move the r_2 null rows of the upper-left $(n-r_1)\times(n-r_1)$ submatrix of $R_1PR_1^T$ down:

$$R_2R_1PR_1^TR_2^T = \begin{bmatrix} \\ \\ \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

Move the r_3 null rows of the upper-left $(n-r_1-r_2)\times(n-r_1-r_2)$ submatrix of $R_2R_1PR_1^TR_2^T$ down:

$$R_3R_2R_1PR_1^TR_2^TR_3^T = \begin{bmatrix} &&&&&&& \\ &&&&&&& \\ &&&&&&& \\ &&&&&&& \\ 0 & \cdot & \cdot & \cdot & 0 &&& \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 && \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \end{bmatrix}$$

In the upper-left $(n-r_1-r_2-r_3)\times(n-r_1-r_2-r_3)$ submatrix of $R_3R_2R_1PR_1^TR_2^TR_3^T$ there is no null row.

Move the c_1 null columns of the upper-left $(n-r_1-r_2-r_3)\times(n-r_1-r_2-r_3)$ submatrix of $R_3R_2R_1PR_1^TR_2^TR_3^T$ on the right:

$$C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^T = \begin{bmatrix} &&&0&&&& \\ &&&\cdot&&&& \\ &&&\cdot&&&& \\ &&&0&&&& \\ 0 & \cdot & \cdot & \cdot & 0 &&& \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 && \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \end{bmatrix}$$

Move the c_2 null columns of the upper-left $(n-r_1-r_2-r_3-c_1)\times(n-r_1-r_2-r_3-c_1)$ submatrix of $C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^T$ on the right:

$$C_2C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^TC_2^T = \begin{bmatrix} &&0&0&&&&& \\ &&\cdot&\cdot&&&&& \\ &&0&\cdot&&&&& \\ &&&0&&&&& \\ 0 & \cdot & \cdot & \cdot & 0 &&&& \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 &&& \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 && \end{bmatrix}$$

Move the c_3 null columns of the upper-left $(n-r_1-r_2-r_3-c_1-c_2)\times(n-r_1-r_2-r_3-c_1-c_2)$ submatrix of $C_2C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^TC_2^T$ on the right:

$$C_3C_2C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^TC_2^TC_3^T =$$

$$\begin{bmatrix} M^T & 0 & 0 & 0 & & & \\ & 0 & \cdot & \cdot & & & \\ & & 0 & \cdot & & & \\ & & & 0 & & & \\ 0 & \cdot & \cdot & \cdot & 0 & & \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} = \begin{bmatrix} M^T & 0 & N_2^T \\ N_1^T & U_1^T & S^T \\ 0 & 0 & L_2^T \end{bmatrix}$$

In the the upper-left $(n - r_1 - r_2 - r_3 - c_1 - c_2 - c_3) \times (n - r_1 - r_2 - r_3 - c_1 - c_2 - c_3)$ submatrix of $C_3C_2C_1R_3R_2R_1PR_1^TR_2^TR_3^TC_1^TC_2^TC_3^T$ there is no null column (besides no null row). Call it M^T.

$$A = \begin{bmatrix} M & N_1 & 0 \\ 0 & U_1 & 0 \\ N_2 & S & L_2 \end{bmatrix}$$