On lower Hessenberg Toeplitz matrices

Set

$$H_n = \begin{bmatrix} a_1 & \gamma & & \\ a_2 & a_1 & & \\ & & & \gamma \\ a_n & & a_2 & a_1 \end{bmatrix}, \quad H_n^T = \begin{bmatrix} a_1 & a_2 & & a_n \\ \gamma & a_1 & & \\ & & & a_2 \\ & & \gamma & a_1 \end{bmatrix}.$$

Recall that

$$AX - XA = \sum \mathbf{x}_m \mathbf{y}_m^T \quad \Rightarrow \quad A = -\sum \tau(J\mathbf{x}_m) \begin{bmatrix} \tau(I_1^{n-1}\mathbf{y}_m) & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} + \tau(JA\mathbf{e}_n)$$

(the converse is true if $\sum \tau(\mathbf{x}_m)\mathbf{y}_m = \mathbf{0}$). Note that

$$H_nX - XH_n = -\begin{bmatrix} a_2-\gamma \\ a_3 \\ \\ a_n \\ \rho \end{bmatrix} \mathbf{e}_1^T + \mathbf{e}_n[\rho \ a_n \quad a_3 \ a_2-\gamma]$$

Thus, since $(H_n^{-1})^T J = JH_n^{-1}$ (H_n^{-1} is persymmetric because H_n is),

$$\begin{aligned} H_n^{-1}X - XH_n^{-1} &= H_n^{-1}\begin{bmatrix} a_2-\gamma \\ a_3 \\ \\ a_n \\ \rho \end{bmatrix} \mathbf{e}_1^T H_n^{-1} - H_n^{-1}\mathbf{e}_n[\rho \ a_n \quad a_3 \ a_2-\gamma]H_n^{-1} \\ &= H_n^{-1}\begin{bmatrix} a_2-\gamma \\ a_3 \\ \\ a_n \\ \rho \end{bmatrix} (JH_n^{-1}\mathbf{e}_n)^T - H_n^{-1}\mathbf{e}_n(JH_n^{-1}\begin{bmatrix} a_2-\gamma \\ a_3 \\ \\ a_n \\ \rho \end{bmatrix})^T, \end{aligned}$$

$$H_n = (\tau(\begin{bmatrix} \rho \\ a_n \\ \\ a_2 \end{bmatrix}) - \gamma J)\begin{bmatrix} J & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} - \begin{bmatrix} \tau(\begin{bmatrix} a_3 \\ \\ a_n \\ \rho \end{bmatrix}) & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} + \tau(\begin{bmatrix} a_1 \\ \gamma \\ \end{bmatrix}),$$

$$\begin{aligned} H_n^{-1} &= -\tau(JH_n^{-1}\begin{bmatrix} a_2-\gamma \\ a_3 \\ \\ a_n \\ \rho \end{bmatrix})\begin{bmatrix} \tau(I_1^{n-1}JH_n^{-1}\mathbf{e}_n) & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} \\ &\quad +\tau(JH_n^{-1}\mathbf{e}_n)(\begin{bmatrix} \tau(I_1^{n-1}JH_n^{-1}\begin{bmatrix} a_2-\gamma \\ a_3 \\ \\ a_n \\ \rho \end{bmatrix}) & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} + I). \end{aligned}$$

Note that if $|[H_n^{-1}]_{1i}| = |[H_n^{-1}]_{n+1-i,n}| = |\det A_{n-i}/\det A_n| \neq 0$ then

$$H_n^{-1}\begin{bmatrix} a_2 \\ \\ a_n \\ \rho \end{bmatrix} = \frac{1}{[H_n^{-1}]_{1i}}(H_n^{-1}\mathbf{e}_{i-1} - Z^T H_n^{-1}\mathbf{e}_i),\ \rho = -\frac{1}{[H_n^{-1}]_{1i}}[0\ a_n \quad a_2]H_n^{-1}\mathbf{e}_i$$

So, if $\det A_{n-1} \neq 0$ then the first and the last columns of H_n^{-1} define H_n^{-1}. How to compute such columns ?

We now assume $\gamma = 1$:

$$E_1 = \begin{bmatrix} 1 & & \\ x_1 & 1 & \\ & & \\ & & 1 \end{bmatrix},\ x_1 = -\tfrac{1}{a_1} = -\tfrac{1}{u_{11}}, \ldots,$$

$$E_{n-1} = \begin{bmatrix} 1 & & \\ & & \\ & 1 & \\ & x_{n-1} & 1 \end{bmatrix},\ x_{n-1} = -\tfrac{1}{a_{n-1}^{(n-1)}} = -\tfrac{1}{u_{n-1n-1}}$$

$$E_{n-1}\cdots E_1 H_n^T = U,\ \ H_n E_1^T \cdots E_{n-1}^T = U^T,\ \ (E_{n-1}^T)^{-1}\cdots(E_1^T)^{-1}H_n^{-1} = (U^T)^{-1},$$

$$u_{11} = a_1,\ u_{22} = \frac{a_1^2 - a_2}{a_1} = \frac{\det A_2}{\det A_1},\ u_{33} = \frac{\det A_3}{\det A_2},\ u_{nn} = \frac{\det A_n}{\det A_{n-1}}$$

$$H_n^{-1} = E_1^T \cdots E_{n-1}^T (U^T)^{-1},\quad H_n^{-1}\mathbf{e}_1 = E_1^T \cdots E_{n-1}^T (U^T)^{-1}\mathbf{e}_1$$

$$H_n^{-1}\mathbf{e}_n = E_1^T \cdots E_{n-1}^T (U^T)^{-1}\mathbf{e}_n = \frac{1}{U_{nn}}E_1^T \cdots E_{n-1}^T \mathbf{e}_n$$

$$H_n^{-1}\mathbf{e}_n = \frac{1}{u_{nn}}[(x_{n-1}\cdots x_2 x_1)\ (x_{n-1}\cdots x_2)\ldots\ (x_{n-1}x_{n-2})\ (x_{n-1})\ 1]^T$$

$$H_n^{-1}\mathbf{e}_n = [(-1)^{n-1}\frac{1}{u_{nn}\cdots u_{11}}\ \ (-1)^{n-2}\frac{1}{u_{nn}\cdots u_{22}}\ \ \ldots\ -\frac{1}{u_{nn}u_{n-1n-1}}\ \ \frac{1}{u_{nn}}]^T$$

$$H_n^{-1}\mathbf{e}_n = \frac{1}{\det A_n}[(-1)^{n-1}\ \ (-1)^{n-2}\det A_1\ \ \ldots\ -\det A_{n-2}\ \ \det A_{n-1}]^T$$

An alternative way to compute $H_n^{-1}\mathbf{e}_1$ and $H_n^{-1}\mathbf{e}_n$ ...

Assume n even. So, H_n, $\gamma = 1$, can be written as follows

$$H_n = \begin{bmatrix} H_{\frac{n}{2}} & (\mathbf{e}_{\frac{n}{2}}^m)(\mathbf{e}_1^m)^T \\ T_{\frac{n}{2}} & H_{\frac{n}{2}} \end{bmatrix} = A_n + \mathbf{e}_{\frac{n}{2}}\mathbf{e}_{\frac{n}{2}+1}^T,\ A_n = \begin{bmatrix} H_{\frac{n}{2}} & 0 \\ T_{\frac{n}{2}} & H_{\frac{n}{2}} \end{bmatrix},\quad m = n/2.$$

Note that $T_{\frac{n}{2}}$ is a generic (non symmetric) Toeplitz matrix. Then, by the Sherman-Morrison formula, since $\mathbf{e}_1^T H_n^{-1} = \mathbf{e}_n^T (JH_n^{-1})^T = (H_n^{-1}\mathbf{e}_n)^T J$, we have the following formula for H_n^{-1}:

$$\begin{aligned} H_n^{-1} &= A_n^{-1} - \tfrac{1}{1+\mathbf{e}_{m+1}^T A_n^{-1}\mathbf{e}_m} A_n^{-1}\mathbf{e}_m\mathbf{e}_{m+1}^T A_n^{-1} \\ &= \begin{bmatrix} H_{\frac{n}{2}}^{-1} & 0 \\ -H_{\frac{n}{2}}^{-1}T_{\frac{n}{2}}H_{\frac{n}{2}}^{-1} & H_{\frac{n}{2}}^{-1} \end{bmatrix} \\ &\quad - \tfrac{1}{1-(H_m^{-1}\mathbf{e}_m^{(m)})^T JT_m H_m^{-1}\mathbf{e}_m} \begin{bmatrix} H_m^{-1}e_m^{(m)} \\ -H_m^{-1}T_m H_m^{-1}\mathbf{e}_m^{(m)} \end{bmatrix} \Big[-(H_m^{-1}\mathbf{e}_m^{(m)})^T JT_m H_m^{-1}\ |\ (H_m^{-1}\mathbf{e}_m^{(m)})^T J \Big]. \end{aligned}$$

As a consequence (for simplicity set $\mathbf{e}_m = \mathbf{e}_m^{(m)}$), we have a formula for $H_n^{-1}\mathbf{e}_n$:

$$H_n^{-1}\mathbf{e}_n = \begin{bmatrix} \mathbf{0} \\ H_m^{-1}\mathbf{e}_m \end{bmatrix} - \frac{(H_m^{-1}\mathbf{e}_m)_1}{1-(H_m^{-1}\mathbf{e}_m)^T J T_m H_m^{-1}\mathbf{e}_m} \begin{bmatrix} H_m^{-1}\mathbf{e}_m \\ -H_m^{-1}T_m H_m^{-1}\mathbf{e}_m \end{bmatrix}$$

(computations: $H_m^{-1}\mathbf{e}_m$, $T_m(H_m^{-1}\mathbf{e}_m)$, $H_m^{-1}(T_m H_m^{-1}\mathbf{e}_m)$, and $O(n)$ a.o.), and a formula for $H_n^{-1}\mathbf{b}$, in fact, if $\mathbf{b}_1$ and $\mathbf{b}_2$ denote the first and the second half of $\mathbf{b}$, then

$$H_n^{-1}\mathbf{b} = \begin{bmatrix} H_m^{-1}\mathbf{b}_1 \\ -H_m^{-1}T_m H_m^{-1}\mathbf{b}_1 + H_m^{-1}\mathbf{b}_2 \end{bmatrix} - \frac{(H_m^{-1}\mathbf{e}_m)^T J(\mathbf{b}_2 - T_m H_m^{-1}\mathbf{b}_1)}{1-(H_m^{-1}\mathbf{e}_m)^T J T_m H_m^{-1}\mathbf{e}_m} \begin{bmatrix} H_m^{-1}\mathbf{e}_m \\ -H_m^{-1}T_m H_m^{-1}\mathbf{e}_m \end{bmatrix}$$

(computations: $H_m^{-1}\mathbf{e}_m$, $T_m(H_m^{-1}\mathbf{e}_m)$, $H_m^{-1}(T_m H_m^{-1}\mathbf{e}_m)$, $H_m^{-1}\mathbf{b}_1$, $T_m(H_m^{-1}\mathbf{b}_1)$, $H_m^{-1}(\mathbf{b}_2 - T_m H_m^{-1}\mathbf{b}_1)$)

From now on, we set $\gamma = 1$ in the matrix H_n.

Computing $H_n^{-1}\mathbf{e}_n$ from $H_{n-1}^{-1}\mathbf{e}_{n-1}^{(n-1)}$

$$H_n = A_n + \mathbf{e}_{n-1}\mathbf{e}_n^T, \quad A_n = \begin{bmatrix} H_{n-1} & \mathbf{0} \\ [\, a_n \quad a_2 \,] & a_1 \end{bmatrix},$$

$$\begin{aligned} H_n^{-1} &= A_n^{-1} - \tfrac{1}{1+\mathbf{e}_n^T A_n^{-1}\mathbf{e}_{n-1}} A_n^{-1}\mathbf{e}_{n-1}\mathbf{e}_n^T A_n^{-1} \\ &= \begin{bmatrix} H_{n-1}^{-1} & \mathbf{0} \\ -\frac{1}{a_1}[\, a_n \quad a_2 \,] H_{n-1}^{-1} & a_1^{-1} \end{bmatrix} \\ &\quad - \tfrac{1}{1-\frac{1}{a_1}[a_n\cdots a_2]H_{n-1}^{-1}\mathbf{e}_{n-1}^{(n-1)}} \begin{bmatrix} H_{n-1}^{-1}\mathbf{e}_{n-1}^{(n-1)} \\ -\frac{1}{a_1}[a_n\cdots a_2]H_{n-1}^{-1}\mathbf{e}_{n-1}^{(n-1)} \end{bmatrix} \begin{bmatrix} -\frac{1}{a_1}[a_n\cdots a_2]H_{n-1}^{-1} & \frac{1}{a_1} \end{bmatrix}, \end{aligned}$$

$$H_n^{-1}\mathbf{e}_n = \begin{bmatrix} \mathbf{0} \\ a_1^{-1} \end{bmatrix} - \frac{1/a_1}{1-\frac{1}{a_1}[a_n\cdots a_2]H_{n-1}^{-1}\mathbf{e}_{n-1}^{(n-1)}} \begin{bmatrix} H_{n-1}^{-1}\mathbf{e}_{n-1}^{(n-1)} \\ -\frac{1}{a_1}[a_n\cdots a_2]H_{n-1}^{-1}\mathbf{e}_{n-1}^{(n-1)} \end{bmatrix}.$$

Thus, assuming H_k non singular, $k = 1, \ldots, n$, and starting from $H_1^{-1}\mathbf{e}_1 = \frac{1}{a_1}$, one can compute $H_n^{-1}\mathbf{e}_n$ with $n^2 + O(n)$ a.o. .

Computing $H_n^{-1}\mathbf{e}_1$

$$H_n = A_n + \mathbf{e}_1\mathbf{e}_2^T, \quad A_n = \begin{bmatrix} a_1 & \mathbf{0}^T \\ \begin{bmatrix} a_2 \\ \\ a_n \end{bmatrix} & H_{n-1} \end{bmatrix},$$

$$\begin{aligned} H_n^{-1} &= A_n^{-1} - \tfrac{1}{1+\mathbf{e}_2^T A_n^{-1}\mathbf{e}_1} A_n^{-1}\mathbf{e}_1\mathbf{e}_2^T A_n^{-1} \\ &= \begin{bmatrix} a_1^{-1} & \mathbf{0}^T \\ -\frac{1}{a_1}H_{n-1}^{-1}\begin{bmatrix} a_2 \\ \\ a_n \end{bmatrix} & H_{n-1}^{-1} \end{bmatrix} \\ &\quad - \frac{1}{1-\frac{1}{a_1}\mathbf{e}_1^T H_{n-1}^{-1}\begin{bmatrix} a_2 \\ \\ a_n \end{bmatrix}} \begin{bmatrix} a_1^{-1} \\ -\frac{1}{a_1}H_{n-1}^{-1}\begin{bmatrix} a_2 \\ \\ a_n \end{bmatrix} \end{bmatrix} \begin{bmatrix} -\frac{1}{a_1}\mathbf{e}_1^T H_{n-1}^{-1}\begin{bmatrix} a_2 \\ \\ a_n \end{bmatrix} & \mathbf{e}_1^T H_{n-1}^{-1} \end{bmatrix}, \end{aligned}$$

$$H_n^{-1}\mathbf{e}_1 = \begin{bmatrix} a_1^{-1} \\ -\frac{1}{a_1}H_{n-1}^{-1}\begin{bmatrix} a_2 \\ \\ a_n \end{bmatrix} \end{bmatrix} - \frac{-\frac{1}{a_1}\mathbf{e}_1^T H_{n-1}^{-1}\begin{bmatrix} a_2 \\ \\ a_n \end{bmatrix}}{1-\frac{1}{a_1}\mathbf{e}_1^T H_{n-1}^{-1}\begin{bmatrix} a_2 \\ \\ a_n \end{bmatrix}} \begin{bmatrix} a_1^{-1} \\ -\frac{1}{a_1}H_{n-1}^{-1}\begin{bmatrix} a_2 \\ \\ a_n \end{bmatrix} \end{bmatrix}$$

$$= \frac{1}{1-\frac{1}{a_1}\mathbf{e}_1^T H_{n-1}^{-1}\begin{bmatrix} a_2 \\ \\ a_n \end{bmatrix}} \begin{bmatrix} a_1^{-1} \\ -\frac{1}{a_1}H_{n-1}^{-1}\begin{bmatrix} a_2 \\ \\ a_n \end{bmatrix} \end{bmatrix}.$$

...

Arrow matrices.

$$(T+H)X-X(T+H) = \begin{bmatrix} -t_1+t_{-1} & t_{-2}+h_0 & \cdots & t_{-n+1}+h_{n-3} & -h_n+h_{n-2} \\ -t_2-h_0 & & & & -t_{-n+1}-h_{n+1} \\ \vdots & & & & \vdots \\ -t_{n-1}-h_{n-3} & & & & -t_{-2}-h_{2n-2} \\ -h_{n-2}+h_n & t_{n-1}+h_{n+1} & \cdots & t_2+h_{2n-2} & -t_{-1}+t_1 \end{bmatrix}$$

When the rank of the above $(T+H)X-X(T+H)$ is equal to 2? For instance, if

$$T = \begin{bmatrix} \square & \square & 0 & 0 \\ \square & & \square & \\ & & & 0 \\ & & & \square \\ \square & \square & \square & \square \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & \square & \square \\ & & & \square \\ 0 & & & 0 \\ \square & & & \\ \square & \square & 0 & 0 \end{bmatrix}$$

then $A = T+H, (T+H)^T, J(T+H), J(T+H)^T$ are such that $AX - XA$ has rank 2. ...

$$\begin{bmatrix} & & & & & \bullet & 0 \\ & & & & \bullet & 0 & \\ & & & \bullet & 0 & & \\ & & \bullet & 0 & & & \\ & \bullet & 0 & & & & \\ \bullet & 0 & & & & & \\ 0 & & & & & & \end{bmatrix}, \begin{bmatrix} & & & & \bullet & 0 & \\ & & & & & \bullet & 0 \\ & & & & \bullet & 0 & \\ & & & \bullet & 0 & & \\ & & \bullet & 0 & & & \\ & \bullet & 0 & & & & \\ \bullet & 0 & & & & & \end{bmatrix}, \begin{bmatrix} & & \bullet & 0 & & \\ & & & \bullet & 0 & \\ & & & & \bullet & 0 \\ & & & \bullet & 0 & \\ & & \bullet & 0 & & \\ & \bullet & 0 & & & \\ \bullet & 0 & & & & \end{bmatrix}, \begin{bmatrix} \bullet & 0 & & & \\ & \bullet & 0 & & \\ & & \bullet & 0 & \\ & & & \bullet & 0 \\ & & \bullet & 0 & \\ & \bullet & 0 & & \\ \bullet & 0 & & & \end{bmatrix},$$

Evaluation of $\zeta(s) = \sum_{r=1}^{+\infty} \frac{1}{r^s}$

Assume $s > 1$. Then the Euler-Maclaurin formula for $f(x) = \frac{1}{x^s}$ and $n \to +\infty$ becomes:

$$\sum_{r=m}^{+\infty} \frac{1}{r^s} = \frac{1}{2m^s} + \frac{1}{(s-1)m^{s-1}} + \sum_{j=1}^{k} \frac{B_{2j}(0)}{(2j)!} \frac{s(s+1)\cdots(s+2j-2)}{m^{s+2j-1}} + u_{k+1},$$

$$|u_{k+1}| \leq 2\frac{|B_{2k+2}(0)|}{(2k+2)!}\frac{s(s+1)\cdots(s+2k)}{m^{s+2k+1}}. \qquad (*)$$

(proof: $\int_m^n \frac{1}{x^s}\,dx = \frac{1}{s-1}(\frac{1}{m^{s-1}} - \frac{1}{n^{s-1}})$; $f^{(j)}(x) = (-1)^j s(s+1)\cdots(s+j-1)x^{-s-j}$; $f^{(2j-1)}(x) = -s(s+1)\cdots(s+2j-2)x^{-s-2j+1}$)

Exercise. Assume m fixed (f.i. $m=2$). Find the value of k that minimizes the upper bound in (*).

$$\begin{array}{ll} k=0: & 2\frac{1}{6\cdot 2}\frac{s}{m^{s+1}}, \\ k=1: & 2\frac{1}{30\cdot 4!}\frac{s(s+1)(s+2)}{m^{s+3}}, \\ k=2: & 2\frac{1}{42\cdot 6!}\frac{s(s+1)(s+2)(s+3)(s+4)}{m^{s+5}}, \\ k=3: & 2\frac{1}{30\cdot 8!}\frac{s(s+1)(s+2)(s+3)(s+4)(s+5)(s+6)}{m^{s+7}}, \ldots \end{array}$$

Note. For k large (how much?) there is a good estimate of $|B_{2k+2}(0)|$ (see below).

Estimates of $\zeta(3/2)$. Assume $s = \frac{3}{2}$.

$$\sum_{r=m}^{+\infty}\frac{1}{r\sqrt{r}} = \frac{1}{2m\sqrt{m}} + \frac{2}{\sqrt{m}} + \sum_{j=1}^{k}\frac{B_{2j}(0)}{(2j)!}\frac{3(5)\cdots(4j-1)}{2^{2j-1}m^{2j+\frac{1}{2}}} + u_{k+1},$$

$$|u_{k+1}| \leq \frac{|B_{2k+2}(0)|}{(2k+2)!}\frac{3(5)\cdots(4k+3)}{2^{2k}m^{2k+2+\frac{1}{2}}}. \qquad (*)$$

A rational approximation . It is clear that the only way to avoid the computation of radicals is to set $m=1$:

$$\sum_{r=1}^{+\infty}\frac{1}{r\sqrt{r}} = \frac{1}{2} + 2 + \sum_{j=1}^{k}\frac{B_{2j}(0)}{(2j)!}\frac{3(5)\cdots(4j-1)}{2^{2j-1}} + u_{k+1},$$

$$|u_{k+1}| \leq \frac{|B_{2k+2}(0)|}{(2k+2)!}\frac{3(5)\cdots(4k+3)}{2^{2k}}. \qquad (*)$$

$$\begin{array}{l} k=0: \ \frac{1}{2}+2,\ |u_1| \leq \frac{1}{4}; \\ k=1: \ (\frac{5}{2})+\frac{1}{8},\ |u_2| \leq \frac{7}{192} \approx 0.036; \\ k=2: \ (\frac{21}{8})-\frac{7}{384},\ |u_3| \leq \frac{11}{512} \approx 0.0214; \\ k=3: \ (\frac{1001}{384})+?,\ |u_4| \leq \frac{13\cdot 33}{2^{14}} \approx 0.026. \end{array}$$

So, $\frac{1001}{384}$ is the better rational estimate of $\zeta(\frac{3}{2})$ which can be obtained (at least by our knowledges). The error is bounded by $0.025 = \frac{1}{40}$.

An approximation in $\mathbb{Q}[\sqrt{2}]$. It is clear that the only way to avoid the computation of radicals greater than $\sqrt{2}$ is to set $m=2$:

$$\sum_{r=2}^{+\infty}\frac{1}{r\sqrt{r}} = \frac{1}{4\sqrt{2}} + \frac{2}{\sqrt{2}} + \sum_{j=1}^{k}\frac{B_{2j}(0)}{(2j)!}\frac{3(5)\cdots(4j-1)}{2^{2j-1}2^{2j}\sqrt{2}} + u_{k+1},$$

$$|u_{k+1}| \leq \left(\frac{|B_{2k+2}(0)|}{(2k+2)!}\frac{3(5)\cdots(4k+3)}{2^{2k}}\right)\frac{1}{2^{2k+2}\sqrt{2}}. \qquad (*)$$

$$\begin{array}{l}
k=0: \ \frac{1}{4\sqrt{2}}+\frac{2}{\sqrt{2}}, \ |u_1| \le \frac{1}{4}\frac{1}{2^2\sqrt{2}} \approx 0.044;\\
k=1: \ (\frac{9}{4\sqrt{2}})+\frac{1}{32\sqrt{2}}, \ |u_2| \le \frac{7}{192}\frac{1}{2^4\sqrt{2}} \approx 0.0015;\\
k=2: \ (\frac{73}{32\sqrt{2}})-\frac{7}{(3)2^{11}\sqrt{2}}, \ |u_3| \le \frac{11}{512}\frac{1}{2^6\sqrt{2}} \approx 0.00023;\\
k=3: \ (\frac{14009}{6144\sqrt{2}})+?, \ |u_4| \le \frac{13\cdot 33}{2^{14}}\frac{1}{2^8\sqrt{2}} \approx 0.000071.
\end{array}$$

So, $1+\frac{14009}{6144\sqrt{2}} = 2.6122817\ldots$ approximates $\zeta(\frac{3}{2})$ with an error bounded by $0.00025 = \frac{1}{4000}$.

An approximation in $\mathbb{Q}[\sqrt{2},\sqrt{3}]$. One can avoid the computation of radicals greater than $\sqrt{3}$ by setting $m=3$, or, better, $m=4$:

$$\sum_{r=4}^{+\infty}\frac{1}{r\sqrt{r}} = \frac{1}{16}+1+\sum_{j=1}^{k}\frac{B_{2j}(0)}{(2j)!}\frac{3(5)\cdots(4j-1)}{2^{2j}4^{2j}}+u_{k+1},$$

$$|u_{k+1}| \le \Big(\ \frac{|B_{2k+2}(0)|}{(2k+2)!}\frac{3(5)\cdots(4k+3)}{2^{2k}}\ \Big)\frac{1}{4^{2k+2+\frac{1}{2}}}.$$

Let us check the approximations proposed, for small values of k:

$$\begin{array}{l}
k=0: \ \frac{1}{16}+1, \ |u_1| \le \frac{1}{4}\frac{1}{4^{2+\frac{1}{2}}} = \frac{1}{128};\\
k=1: \ (\frac{17}{16})+\frac{1}{256}, \ |u_2| \le \frac{7}{192}\frac{1}{4^{4+\frac{1}{2}}} = \frac{7}{(3)2^{15}} \approx 0.0000712;\\
k=2: \ (\frac{273}{256})-\frac{7}{(3)2^{16}}, \ |u_3| \le \frac{11}{512}\frac{1}{4^{6+\frac{1}{2}}} = \frac{11}{2^{22}} \approx 0.0000026;\\
k=3: \ (???)+\frac{11}{2^{23}}, \ |u_4| \le \frac{13\cdot 33}{2^{14}}\frac{1}{4^{8+\frac{1}{2}}} \approx 0.0000002.
\end{array}$$

So, $(1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}})+\frac{\cdots}{\cdots}$ approximates $\zeta(\frac{3}{2})$ with an error bounded by $\ldots$.

Estimates of $\zeta(5/2)$. Assume $s=\frac{5}{2}$.

$$\sum_{r=m}^{+\infty}\frac{1}{r^2\sqrt{r}} = \frac{1}{2m^2\sqrt{m}}+\frac{2}{3m\sqrt{m}}+\sum_{j=1}^{k}\frac{B_{2j}(0)}{(2j)!}\frac{5(7)\cdots(4j+1)}{2^{2j-1}m^{2j+1+\frac{1}{2}}}+u_{k+1},$$

$$|u_{k+1}| \le \frac{|B_{2k+2}(0)|}{(2k+2)!}\frac{5(7)\cdots(4k+5)}{2^{2k}m^{2k+3+\frac{1}{2}}} = \frac{4k+5}{3m}\cdot(\text{bound for } s=\frac{3}{2}).$$

Thus, for $m=1$, we have $|u_{k+1}| \le \frac{4k+5}{3}\cdot(\text{bound for } s=\frac{3}{2})$: $\frac{5}{3}0.25 = 0.416$ ($k=0$), $3\cdot 0.036 = 0.108$ ($k=1$), $\frac{13}{3}0.0214 = 0.0927$ ($k=2$); and, for $m=2$, we have $|u_{k+1}| \le \frac{4k+5}{6}\cdot(\text{bound for } s=\frac{3}{2})$: $\frac{5}{6}0.044 = 0.036$ ($k=0$), $\frac{3}{2}0.00159 = 0.0023$ ($k=1$), $\frac{13}{6}0.000236 = 0.00051$ ($k=2$), $\frac{17}{6}0.000071 = 0.000201$ ($k=3$).

So, by setting $m=2$ and $k=3$ we obtain an approximation of $\zeta(\frac{5}{2})$ involving the only radical $\sqrt{2}$ and with an error bounded by 0.000201:

$$\begin{array}{l}
\hat{\zeta(\frac{5}{2})} = 1+\frac{1}{2^3\sqrt{2}}+\frac{1}{3\sqrt{2}}+\frac{5}{(3)2^6\sqrt{2}}-\frac{7}{2^{12}\sqrt{2}}+\frac{11\cdot 13}{(3)2^{17}\sqrt{2}} = 1.34155356\ldots,\\
|\hat{\zeta(\frac{5}{2})}-\zeta(\frac{5}{2})| \le 0.000201
\end{array}$$

We know that

$$\begin{array}{l}
\zeta(\frac{3}{2}) \approx \hat{\zeta(\frac{3}{2})} = 2.6122817\ldots, \ \zeta(2) = \frac{\pi^2}{6} = 1.644934067\ldots,\\
\zeta(\frac{5}{2}) \approx \hat{\zeta(\frac{5}{2})} = 1.34155356\ldots
\end{array}$$

where the error for the first and the third value is bounded, respectively, by 0.000236 and 0.000201.

We can use the above estimates to obtain an approximation of the following integral:

$$I = \int_{\frac{3}{2}}^{\frac{5}{2}} \zeta(s)\, ds = 1 + \sum_{r=2}^{+\infty} \int_{\frac{3}{2}}^{\frac{5}{2}} \frac{1}{r^s}\, ds,$$
$$\int_{\frac{3}{2}}^{\frac{5}{2}} (\tfrac{1}{r})^s\, ds = \int_{\frac{3}{2}}^{\frac{5}{2}} e^{s \log_e \frac{1}{r}}\, ds = \frac{1}{(\log_e r) r\sqrt{r}}(1 - \tfrac{1}{r}).$$

In fact, by the Romberg extrapolation method,

$$I_1 = 1[\tfrac{1}{2}\zeta(\tfrac{3}{2}) + \tfrac{1}{2}\zeta(\tfrac{5}{2})],\ I_{\frac{1}{2}} = \tfrac{1}{2}[\tfrac{1}{2}\zeta(\tfrac{3}{2}) + \zeta(2) + \tfrac{1}{2}\zeta(\tfrac{5}{2})],$$
$$\tilde{I} = \frac{2^2 I_{\frac{1}{2}} - I_1}{2^2 - 1} = \tfrac{1}{3}[\tfrac{1}{2}\zeta(\tfrac{3}{2}) + 2\zeta(2) + \tfrac{1}{2}\zeta(\tfrac{5}{2})] \approx 1.7555\ldots \approx I.$$

Let ε be in $(0,1]$, and set $f(x) = x^{\varepsilon-1}$. Then $f'(x) = (\varepsilon-1)x^{\varepsilon-2}$, $f^{(s)}(x) = (\varepsilon-1)(\varepsilon-2)\cdots(\varepsilon-s)x^{\varepsilon-s-1}$, $f^{(2j-1)}(x) = (\varepsilon-1)(\varepsilon-2)\cdots(\varepsilon-2j+1)x^{\varepsilon-2j}$ $\Rightarrow$

$$\sum_{r=m}^{n} \frac{1}{r^{1-\varepsilon}} - \frac{n^\varepsilon}{\varepsilon} = \frac{1}{2}(\frac{1}{m^{1-\varepsilon}} + \frac{1}{n^{1-\varepsilon}}) - \frac{m^\varepsilon}{\varepsilon}$$
$$+ \sum_{j=1}^{k} \frac{B_{2j}(0)}{(2j)!}(\varepsilon-1)(\varepsilon-2)\cdots(\varepsilon-2j+1)\big(\frac{1}{n^{2j-\varepsilon}} - \frac{1}{m^{2j-\varepsilon}}\big) + u_{k+1},$$

$$|u_{k+1}| \leq 2\frac{|B_{2k+2}(0)|}{(2k+2)!}|(\varepsilon-1)(\varepsilon-2)\cdots(\varepsilon-2k-1)|\big|\frac{1}{n^{2k+2-\varepsilon}} - \frac{1}{m^{2k+2-\varepsilon}}\big|.$$

Set $\gamma_\varepsilon = \lim_{n\to+\infty}(\sum_{r=1}^{n} \frac{1}{r^{1-\varepsilon}} - n^\varepsilon/\varepsilon)$. Then

$$\begin{aligned}\gamma_\varepsilon &= \sum_{r=1}^{m-1} \frac{1}{r^{1-\varepsilon}} + \lim_{n\to+\infty}\big(\sum_{r=m}^{n} \frac{1}{r^{1-\varepsilon}} - \frac{n^\varepsilon}{\varepsilon}\big)\\ &= \sum_{r=1}^{m-1} \frac{1}{r^{1-\varepsilon}} + \frac{1}{2m^{1-\varepsilon}} - \frac{m^\varepsilon}{\varepsilon}\\ &\quad - \sum_{j=1}^{k} \frac{B_{2j}(0)}{(2j)!}\frac{(\varepsilon-1)(\varepsilon-2)\cdots(\varepsilon-2j+1)}{m^{2j-\varepsilon}} + u_{k+1}(\infty),\end{aligned}$$

$$|u_{k+1}(\infty)| \leq 2\frac{|B_{2k+2}(0)|}{(2k+2)!}\frac{|(\varepsilon-1)(\varepsilon-2)\cdots(\varepsilon-2k-1)|}{m^{2k+2-\varepsilon}}.$$

Exercise. Set $\varepsilon = \frac{1}{2}$.

Remark. The number γ_ε seems well defined for any $\varepsilon \in (-\infty, 0) \cup \{0\} \cup (0,1]$ (casio PB200):

$\varepsilon = 1: \quad \gamma_\varepsilon = 0;$
$\varepsilon = 0.75: \quad \gamma_\varepsilon = -0.724;$
$\varepsilon = 0.5: \quad \gamma_\varepsilon = -1.44917;$
$\varepsilon = 0.25: \quad \gamma_\varepsilon = -3.439613;$
$\varepsilon = 0.1: \quad \gamma_\varepsilon = -9.429;$
$\varepsilon = 0.01: \quad \gamma_\varepsilon = -99.42;$
$\varepsilon = 0: \quad \gamma_\varepsilon = 0.577\ldots;$
$\varepsilon = -0.5: \quad \gamma_\varepsilon = 2.61\ldots;$
$\varepsilon = -1.0: \quad \gamma_\varepsilon = 1.64\ldots;$
$\varepsilon = -1.5: \quad \gamma_\varepsilon = 1.34\ldots.$

```
5 E = 0.75
10 I = 1 : S = 0 : N = 2000
20 S = S + 1/(I^(1-E)) : I = I + 1 : IF I ≤ N; GOTO20
30 Y = S - (N^E)/E
40 PRINT Y
```

```
10 I = 1 : S = 0 : N = 800
20 S = S + 1/I : I = I + 1 : IF I ≤ N; GOTO20
30 Y = S - LN N
40 PRINT Y
```

One guesses that γ_ε is equal to the Riemann zeta function when ε is negative; is well defined and increasing (from $-\infty$ to 0) when $\varepsilon \in (0,1]$; is equal to the Eulero-Mascheroni constant when $\varepsilon = 0$ (that is, is equal to $\lim_{n\to\infty}(\sum_{r=1}^{n} 1/r - \log_e n)$).

Computing the Bernoulli numbers
Assume we know that $\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{+\infty} \frac{B_n(x)}{n!} t^n$ (yet, however, we have no proof of this equality). In particular, for $x = 0$ we have:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{+\infty} \frac{B_n(0)}{n!} t^n = -\frac{1}{2}t + \sum_{k=0}^{+\infty} \frac{B_{2k}(0)}{(2k)!} t^{2k}.$$

Then

$$\begin{aligned} t &= (-\tfrac{1}{2}t + \textstyle\sum_{k=0}^{+\infty} \frac{B_{2k}(0)}{(2k)!} t^{2k}) t (\sum_{r=0}^{+\infty} \frac{t^r}{(r+1)!}) \\ &= -\tfrac{1}{2}t^2 (\textstyle\sum_{r=0}^{+\infty} \frac{t^r}{(r+1)!}) + t(\sum_{k=0}^{+\infty} \frac{B_{2k}(0)}{(2k)!} t^{2k})(\sum_{r=0}^{+\infty} \frac{t^r}{(r+1)!}) \\ &= -\tfrac{1}{2} \textstyle\sum_{j=2}^{+\infty} \frac{t^j}{(j-1)!} + t \sum_{k,r=0}^{+\infty} \frac{B_{2k}(0) t^{2k+r}}{(2k)!(r+1)!}. \end{aligned}$$

By setting $2k + r = j - 1$ in the latter equality, one realizes that the coefficient of t^j, $j = 2, 3, \ldots$, on the right is

$$-\frac{1}{2}\frac{1}{(j-1)!} + \sum_{k=0}^{[\frac{j-1}{2}]} \frac{B_{2k}(0)}{(2k)!(j-2k)!} \qquad (*)$$

and it must be zero, like the coefficient of t^j on the left (note that the coefficient of t on the right is 1, like the coefficient of t on the left).

So we have the conditions:

$$-\frac{1}{2}j + \sum_{k=0}^{[\frac{j-1}{2}]} \binom{j}{2k} B_{2k}(0) = 0, \; j = 2, 3, 4, \ldots \; . \qquad (**)$$

(A) By considering separately the cases j even and j odd, we have

$$j = 2s, \; s = 1, 2, 3, \ldots : \quad -s + \textstyle\sum_{k=0}^{s-1} \binom{2s}{2k} B_{2k}(0) = 0,$$
$$j = 2s+1, \; s = 1, 2, 3, \ldots : \quad -\tfrac{2s+1}{2} + \textstyle\sum_{k=0}^{s} \binom{2s+1}{2k} B_{2k}(0) = 0.$$

It follows that

$$\begin{bmatrix} \binom{2}{0} & & & & \\ \binom{4}{0} & \binom{4}{2} & & & \\ \binom{6}{0} & \binom{6}{2} & \binom{6}{4} & & \\ \binom{8}{0} & \binom{8}{2} & \binom{8}{4} & \binom{8}{6} & \\ \vdots & & & & \ddots \end{bmatrix} \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ B_6(0) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{bmatrix},$$

$$\begin{bmatrix} 1 & & & & & \\ 1 & 6 & & & & \\ 1 & 15 & 15 & & & \\ 1 & 28 & 70 & 28 & & \\ 1 & 45 & 210 & 210 & 45 & \\ \vdots & & & & & \ddots \end{bmatrix} \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ B_6(0) \\ B_8(0) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{bmatrix},$$

and

$$\begin{bmatrix} \binom{1}{0} & & & & \\ \binom{3}{0} & \binom{3}{2} & & & \\ \binom{5}{0} & \binom{5}{2} & \binom{5}{4} & & \\ \binom{7}{0} & \binom{7}{2} & \binom{7}{4} & \binom{7}{6} & \\ \vdots & & & & \ddots \end{bmatrix} \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ B_6(0) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 5/2 \\ 7/2 \\ \vdots \end{bmatrix},$$

$$\begin{bmatrix} 1 & & & & & \\ 1 & 3 & & & & \\ 1 & 10 & 5 & & & \\ 1 & 21 & 35 & 7 & & \\ 1 & 36 & 126 & 84 & 9 & \\ \vdots & & & & & \ddots \end{bmatrix} \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ B_6(0) \\ B_8(0) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 5/2 \\ 7/2 \\ 9/2 \\ \vdots \end{bmatrix}.$$

In other words, the Bernoulli numbers can be obtained by solving (by forward substitution) a lower triangular linear system (one of the above two). For example, by forward solving the first system, I have obtained the first Bernoulli numbers:

$$\begin{gathered} B_0(0) = 1,\ B_2(0) = \tfrac{1}{6},\ B_4(0) = -\tfrac{1}{30},\ B_6(0) = \tfrac{1}{42}, \\ B_8(0) = -\tfrac{1}{30},\ B_{10}(0) = \tfrac{5}{66},\ B_{12}(0) = -\tfrac{691}{2730}, \\ B_{14}(0) = \tfrac{7}{6} \approx 1.16,\ B_{16}(0) = -\tfrac{47021}{6630} \approx -7.09, \\ \ldots,\ B_{2k}(0) \approx (-1)^{k+1} 4 \left(\tfrac{k}{\pi e}\right)^{2k} \sqrt{\pi k}\ \text{ (see below).} \end{gathered}$$

[... An algorithm bi-diagonalizing a lower triangular matrix ... α, β such that $\alpha T_1 + \beta T_2$ has a sparse lower triangular inverse...].

A final remark. Let us consider the finite versions of the above two linear systems. If $\mathbf{b}$ denotes the vector $[B_0(0)\, B_2(0)\, B_4(0)\, \cdots\, B_{2s}(0)]^T$, then such finite systems can be rewritten as follows

$$T_1\mathbf{b} = \mathbf{e},\ T_2\mathbf{b} = \mathbf{e}, \quad \mathbf{e} = [1\,1\,\cdots\,1]^T$$

where T_1, T_2 are lower triangular $(s+1)\times(s+1)$ matrices, with $[T_1]_{jj} = 2j-1$, $j = 1, \ldots, s+1$, $[T_2]_{11} = 1$, $[T_2]_{jj} = 2$, $j = 2, \ldots, s+1$. It follows that $\mathbf{b}$ is left unchanged by the transformation $T_1^{-1}T_2$, which is a lower triangular matrix with diagonal entries $1, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \ldots, \frac{2}{2s+1}$, i.e. $\mathbf{b}$ is the eigenvector of the dominant eigenvalue 1 of the matrix $T_1^{-1}T_2$. So, in principle, $\mathbf{b}$ could be computed by means of the inverse power iterations. Write $T_1^{-1}T_2$:

$$T_1^{-1}T_2 = \begin{bmatrix} 1 & \\ \frac{1}{2} & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \\ \frac{2}{3} & 2 \end{bmatrix} = \begin{bmatrix} 1 & \\ \frac{1}{18} & \frac{2}{3} \end{bmatrix},$$

$$T_1^{-1}T_2 = \begin{bmatrix} 1 & & \\ \frac{1}{2} & 3 & \\ \frac{1}{3} & 5 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & & \\ \frac{2}{3} & 2 & \\ \frac{2}{5} & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \frac{1}{18} & \frac{2}{3} & \\ -\frac{19}{450} & \frac{2}{15} & \frac{2}{5} \end{bmatrix}, \ \ldots .$$

(B) Instead, by considering the equations obtained by equating (*) to zero for $j = 3, 4, 7, 8, 11, 12, \ldots, 4s+3, 4s+4, \ldots$, we obtain a lower block triangular system whose blocks are 2×2 matrices. More specifically, the first block-row condition:

$$\begin{bmatrix} \frac{1}{0!3!} & \frac{1}{2!1!} \\ \frac{1}{0!4!} & \frac{1}{2!2!} \end{bmatrix} \begin{bmatrix} B_0(0) \\ B_2(0) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2!} \\ \frac{1}{3!} \end{bmatrix}, \begin{bmatrix} B_0(0) \\ B_2(0) \end{bmatrix} = 2!4! \begin{bmatrix} \frac{1}{2!2!} & -\frac{1}{2!1!} \\ -\frac{1}{0!4!} & \frac{1}{0!3!} \end{bmatrix} \frac{1}{2} \begin{bmatrix} \frac{1}{2!} \\ \frac{1}{3!} \end{bmatrix}.$$

The second block-row condition:

$$\begin{bmatrix} \frac{1}{0!7!} & \frac{1}{2!5!} \\ \frac{1}{0!8!} & \frac{1}{2!6!} \end{bmatrix} \begin{bmatrix} B_0(0) \\ B_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{4!3!} & \frac{1}{6!1!} \\ \frac{1}{4!4!} & \frac{1}{6!2!} \end{bmatrix} \begin{bmatrix} B_4(0) \\ B_6(0) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{6!} \\ \frac{1}{7!} \end{bmatrix},$$

$$\begin{bmatrix} B_4(0) \\ B_6(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{4!3!} & \frac{1}{6!1!} \\ \frac{1}{4!4!} & \frac{1}{6!2!} \end{bmatrix}^{-1} \left(\frac{1}{2} \begin{bmatrix} \frac{1}{6!} \\ \frac{1}{7!} \end{bmatrix} - \begin{bmatrix} \frac{1}{0!7!} & \frac{1}{2!5!} \\ \frac{1}{0!8!} & \frac{1}{2!6!} \end{bmatrix} \begin{bmatrix} B_0(0) \\ B_2(0) \end{bmatrix} \right).$$

The generic block-row condition:

$$\begin{bmatrix} \frac{1}{0!(4s+3)!} & \frac{1}{2!(4s+1)!} \\ \frac{1}{0!(4s+4)!} & \frac{1}{2!(4s+2)!} \end{bmatrix} \begin{bmatrix} B_0(0) \\ B_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{4!(4s-1)!} & \frac{1}{6!(4s-3)!} \\ \frac{1}{4!(4s)!} & \frac{1}{6!(4s-2)!} \end{bmatrix} \begin{bmatrix} B_4(0) \\ B_6(0) \end{bmatrix} + \ldots +$$
$$\begin{bmatrix} \frac{1}{(4s)!3!} & \frac{1}{(4s+2)!1!} \\ \frac{1}{(4s)!4!} & \frac{1}{(4s+2)!2!} \end{bmatrix} \begin{bmatrix} B_{4s}(0) \\ B_{4s+2}(0) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{(4s+2)!} \\ \frac{1}{(4s+3)!} \end{bmatrix}, \ s = 0, 1, 2, \ldots,$$

from which we have the following formula:

$$\begin{bmatrix} B_{4s}(0) \\ B_{4s+2}(0) \end{bmatrix} = (4s)!(4s+2)!4! \begin{bmatrix} \frac{1}{(4s+2)!2!} & -\frac{1}{(4s+2)!1!} \\ -\frac{1}{(4s)!4!} & \frac{1}{(4s)!3!} \end{bmatrix} \left(\frac{1}{2} \begin{bmatrix} \frac{1}{(4s+2)!} \\ \frac{1}{(4s+3)!} \end{bmatrix} \right.$$
$$\left. - \sum_{j=0}^{s-1} \begin{bmatrix} \frac{1}{(4j)!(4s+3-4j)!} & \frac{1}{(4j+2)!(4s+1-4j)!} \\ \frac{1}{(4j)!(4s+4-4j)!} & \frac{1}{(4j+2)!(4s+2-4j)!} \end{bmatrix} \begin{bmatrix} B_{4j}(0) \\ B_{4j+2}(0) \end{bmatrix} \right),$$

$s = 0, 1, 2, \ldots$, or , alternatively, multiplying by $\begin{bmatrix} 1 & 0 \\ -\frac{1}{4} & 1 \end{bmatrix}$,

$$\begin{bmatrix} \frac{1}{0!(4s+3)!} & \frac{1}{2!(4s+1)!} \\ \frac{1}{0!(4s+4)!}\frac{-4s}{4} & \frac{1}{2!(4s+2)!}\frac{2-4s}{4} \end{bmatrix} \begin{bmatrix} B_0(0) \\ B_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{4!(4s-1)!} & \frac{1}{6!(4s-3)!} \\ \frac{1}{4!(4s)!}\frac{4-4s}{4} & \frac{1}{6!(4s-2)!}\frac{6-4s}{4} \end{bmatrix} \begin{bmatrix} B_4(0) \\ B_6(0) \end{bmatrix} + \ldots +$$

$$+ \begin{bmatrix} & \\ & \frac{1}{(4s-2)!6!}\frac{-2}{4} \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} + \begin{bmatrix} \frac{1}{(4s)!3!} & \frac{1}{(4s+2)!1!} \\ 0 & \frac{1}{(4s+2)!2!}\frac{2}{4} \end{bmatrix} \begin{bmatrix} B_{4s}(0) \\ B_{4s+2}(0) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{(4s+2)!} \\ \frac{1}{(4s+3)!}\frac{1-4s}{4} \end{bmatrix}, \; s = 0, 1, 2, \ldots.$$

Here below are three formulas for the generic Bernoulli number in terms of the previous Bernoulli numbers:

$$B_{2s}(0) = \frac{1}{2} - \frac{1}{2s+1} \sum_{k=0}^{s-1} \binom{2s+1}{2k} B_{2k}(0), \; s = 1, 2, 3, \ldots \qquad (odd)$$

$$\frac{2s+1}{2} B_{2s}(0) = \frac{1}{2} - \frac{1}{2s+2} \sum_{k=0}^{s-1} \binom{2s+2}{2k} B_{2k}(0), \; s = 1, 2, 3, \ldots, \qquad (even)$$

$$(2s+1)B_{2s}(0) = \frac{3-2s}{2} + \sum_{k=0}^{s-2} \frac{s-k+1-2}{s-k+1} \binom{2s+1}{2k} B_{2k}(0), \; s = 2, 3, 4, \ldots \qquad (third)$$

(the latter formula is obtained by combining the previous two).

Exercise. $B_{2k}(0)B_{2k}(\frac{1}{2}) < 0 \; \forall \, k > 0$.

Assume $B_{2k}(0) = 0$ for some k. Then, since $B_{2k}(x) = B_{2k}(1-x)$, we also have $B_{2k}(1) = 0$. Moreover, we also have $B_{2k}(\frac{1}{2}) = 0$, otherwise, in order to have $\int_0^1 B_{2k} = 0$ it should be $2kB_{2k-1}(\xi) = B'_{2k}(\xi) = 0$ for some $\xi \in (0, \frac{1}{2})$ which is not possible. So, $B_{2k}(0) = B_{2k}(\frac{1}{2}) = B_{2k}(1) = 0$. But this implies again $2kB_{2k-1}(\xi) = B'_{2k}(\xi) = 0$ for some $\xi \in (0, \frac{1}{2})$, which is not possible.

Now assume $B_{2k}(0) > 0$. Then $B_{2k}(\frac{1}{2}) < 0$ otherwise in order to have $\int_0^1 B_{2k} = 0$ it should be $2kB_{2k-1}(\xi) = B'_{2k}(\xi) = 0$ for some $\xi \in (0, \frac{1}{2})$, which is not possible.

Exercise. $B_{2k+2}(0)B_{2k}(0) < 0$

Since, for k large, $\zeta(2k) \approx 1$ and $(2k)! \approx (2k)^{2k} e^{-2k} \sqrt{4\pi k}$ (Stirling's formula), we have

$$|B_{2k}(0)| = \frac{2(2k)!}{(2\pi)^{2k}} \zeta(2k) \approx 4 \Big(\frac{k}{\pi e}\Big)^{2k} \sqrt{\pi k}$$

/mathworld.wolfram.com/BernoulliNumber.html
http://numbers.computation.free.fr/Constants/Miscellaneous/bernoulli.html

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Proof of the identity $te^{xt}/(e^t - 1) = \sum_{n=0}^{+\infty} B_n(x) t^n / n!$

Set $F(x,t) = \sum_{n=0}^{+\infty} \frac{B_n(x)}{n!} t^n$ (the function F is well defined for any $x \in \mathbb{R}$ provided that $|t| \leq ?$). Note that F solves the following differential equation:

$$\frac{\partial}{\partial x} F(x,t) = \frac{\partial}{\partial x}(1 + \sum_{n=1}^{+\infty} \frac{B_n(x)}{n!} t^n) = \sum_{n=1}^{+\infty} \frac{B_{n-1}(x)}{(n-1)!} t^n = t \sum_{n=1}^{+\infty} \frac{B_{n-1}(x)}{(n-1)!} t^{n-1} = tF(x,t),$$

and $F(0,t)+t=F(1,t)$. So, $F(x,t)=T(t)e^{xt}$ where $T(t)+t=T(t)e^t$. (alternatively, $\int_0^1 F(x,t)\,dx = T(t)\int_0^1 e^{xt}\,dx = T(t)(e^t-1)/t$. $\int_0^1 \sum_{n=0}^{+\infty} \frac{B_n(x)}{n!}t^n\,dx = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\int_0^1 B_n(x)\,dx = 1$).

Discussing with Francesco

Set

$$Z = \begin{bmatrix} 0 & & & & \\ a_2 & & & & \\ & a_3 & & & \\ & & & & \\ & & & a_n & 0 \end{bmatrix}.$$

Then $Z^0 = I$,

$$Z^2 = \begin{bmatrix} 0 & & & & & \\ 0 & & & & & \\ a_2a_3 & & & & & \\ & a_3a_4 & & & & \\ & & & & & \\ & & & a_{n-1}a_n & 0 & 0 \end{bmatrix},\ Z^3 = \begin{bmatrix} 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ a_2a_3a_4 & & & & & & \\ & a_3a_4a_5 & & & & & \\ & & & & & & \\ & & & a_{n-2}a_{n-1}a_n & 0 & 0 & 0 \end{bmatrix}, \ldots$$

Note that $(I-Z)^{-1} = I + Z + Z^2 + \ldots + Z^{n-1} = \sum_{k=0}^{+\infty} Z^k$. In fact,

$$A_n = I - Z = \begin{bmatrix} 1 & & & & \\ -a_2 & 1 & & & \\ & -a_3 & 1 & & \\ & & \ddots & \ddots & \\ & & & -a_n & 1 \end{bmatrix} = \begin{bmatrix} A_{n-1} & \mathbf{0} \\ -a_n\mathbf{e}_{n-1}^T & 1 \end{bmatrix},$$

$$A_n^{-1} = \begin{bmatrix} A_{n-1}^{-1} & \mathbf{0} \\ a_n\mathbf{e}_{n-1}^T A_{n-1}^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & & & \\ a_2 & 1 & & & & & \\ a_2a_3 & a_3 & 1 & & & & \\ a_2a_3a_4 & a_3a_4 & a_4 & 1 & & & \\ & & & & & & \\ & & & & a_{n-1} & 1 & \\ & & & & a_{n-1}a_n & a_n & 1 \end{bmatrix}.$$

Define $X = \sum_{k=0}^{+\infty} \frac{1}{k!}Z^k = \sum_{k=0}^{n-1} \frac{1}{k!}Z^k$. Since, for $i \geq j$,

$$Z^{i-j} = \begin{bmatrix} & & \\ a_2a_3\cdots a_{i-j+1} & & \\ & a_3a_4\cdots a_{i-j+2} & \\ & & \\ & & a_{n-(i-j)+1}\cdots a_{n-1}a_n \end{bmatrix}$$

where the non zero entries are in positions $i-j+s, s$, $s = 1,\ldots,n-(i-j)$, the entry i,j $(1 \leq j \leq i \leq n)$ of X is

$$[X]_{ij} = \frac{1}{(i-j)!}[Z^{i-j}]_{ij} = \frac{1}{(i-j)!}a_{j+1}\cdots a_{i-1}a_i.$$

In particular, if $a_s = s-1$, then

$$[X]_{ij} = \frac{1}{(i-j)!} j \cdots (i-2)(i-1) = \begin{pmatrix} i-1 \\ j-1 \end{pmatrix}, \quad 1 \le j \le i \le n.$$

The lower triangular matrices appearing in the linear systems defining the Bernoulli numbers (see the sections below) are submatrices of X, $a_s = s-1$:

$$X = \begin{bmatrix} \binom{0}{0} & & & & & & \\ \binom{1}{0} & \binom{1}{1} & & & & & \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & & & \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & & & \\ \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & & \\ \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} & \\ \binom{6}{0} & \binom{6}{1} & \binom{6}{2} & \binom{6}{3} & \binom{6}{4} & \binom{6}{5} & \binom{6}{6} \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}.$$

Remark:

$$Z = \begin{bmatrix} 0 & & & & \\ 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & \end{bmatrix}, \; Z^2 = \begin{bmatrix} 0 & & & & & & \\ 0 & & & & & & \\ 2 & & & & & & \\ & 6 & & & & & \\ & & 12 & & & & \\ & & & 20 & & & \\ & & & & 30 & & \\ & & & & & 42 & \end{bmatrix} = \phi + \psi,$$

$$\phi = \begin{bmatrix} 0 & & & & & & \\ 0 & & & & & & \\ 2 & & & & & & \\ & 0 & & & & & \\ & & 12 & & & & \\ & & & 0 & & & \\ & & & & 30 & & \\ & & & & & 0 & \\ & & & & & & \end{bmatrix}, \; \psi = \begin{bmatrix} 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ & 6 & & & & & \\ & & 0 & & & & \\ & & & 20 & & & \\ & & & & 0 & & \\ & & & & & 42 & \\ & & & & & & \end{bmatrix},$$

$$\sum_{k=1}^{+\infty} \frac{1}{(2k)!} \phi^k = \begin{bmatrix} 0 & & & & & & \\ 0 & & & & & & \\ \binom{2}{0} & & & & & & \\ 0 & 0 & & & & & \\ \binom{4}{0} & 0 & \binom{4}{2} & & & & \\ 0 & 0 & 0 & 0 & & & \\ \binom{6}{0} & 0 & \binom{6}{2} & 0 & \binom{6}{4} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ \binom{8}{0} & 0 & \binom{8}{2} & 0 & \binom{8}{4} & 0 & \binom{8}{6} \end{bmatrix}$$

and

$$\left(\sum_{k=0}^{+\infty}\frac{1}{(2k+1)!}\psi^k\right)\begin{bmatrix} 0 & & & & & \\ 1 & & & & & \\ & 0 & & & & \\ & & 3 & & & \\ & & & 0 & & \\ & & & & 5 & \end{bmatrix} = \begin{bmatrix} 0 & & & & & & & \\ \binom{1}{0} & & & & & & & \\ 0 & 0 & & & & & & \\ \binom{3}{0} & 0 & \binom{3}{2} & & & & & \\ 0 & 0 & 0 & 0 & & & & \\ \binom{5}{0} & 0 & \binom{5}{2} & 0 & \binom{5}{4} & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \\ \binom{7}{0} & 0 & \binom{7}{2} & 0 & \binom{7}{4} & 0 & \binom{7}{6} & \end{bmatrix}.$$

Compact representations of the (even and odd) Bernoulli triangular matrices

$$\phi = \begin{bmatrix} 0 & & & & & \\ 2 & & & & & \\ & 12 & & & & \\ & & 30 & & & \\ & & & 56 & & \\ & & & & \ddots & \end{bmatrix}$$

(Note: $2 = 1 \cdot 2$, $12 = 3 \cdot 4$, $30 = 5 \cdot 6$, $56 = 7 \cdot 8$, $\dots$)

$$\begin{bmatrix} 2 & & & & \\ & 12 & & & \\ & & 30 & & \\ & & & 56 & \\ & & & & \ddots \end{bmatrix}\sum_{k=0}^{+\infty}\frac{1}{(2k+2)!}\phi^k = \begin{bmatrix} \binom{2}{0} & & & & \\ \binom{4}{0} & \binom{4}{2} & & & \\ \binom{6}{0} & \binom{6}{2} & \binom{6}{4} & & \\ \binom{8}{0} & \binom{8}{2} & \binom{8}{4} & \binom{8}{6} & \\ \vdots & & & & \ddots \end{bmatrix}.$$

$$\psi = \begin{bmatrix} 0 & & & & & \\ 6 & & & & & \\ & 20 & & & & \\ & & 42 & & & \\ & & & 72 & & \\ & & & & \ddots & \end{bmatrix}$$

(Note: $6 = 2 \cdot 3$, $20 = 4 \cdot 5$, $42 = 6 \cdot 7$, $72 = 8 \cdot 9$, $\dots$)

$$\left(\sum_{k=0}^{+\infty}\frac{1}{(2k+1)!}\psi^k\right)\begin{bmatrix} 1 & & & & \\ & 3 & & & \\ & & 5 & & \\ & & & 7 & \\ & & & & \ddots \end{bmatrix} = \begin{bmatrix} \binom{1}{0} & & & & \\ \binom{3}{0} & \binom{3}{2} & & & \\ \binom{5}{0} & \binom{5}{2} & \binom{5}{4} & & \\ \binom{7}{0} & \binom{7}{2} & \binom{7}{4} & \binom{7}{6} & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Given

$$\phi = \begin{bmatrix} 0 & & & & & & \\ 2 & & & & & & \\ & 12 & & & & & \\ & & 30 & & & & \\ & & & 56 & & & \\ & & & & 90 & & \\ & & & & & \ddots & \end{bmatrix},$$

the Bernoulli triangular even and odd linear systems can be easily reduced to the systems

$$\sum_{k=0}^{+\infty} \frac{1}{(2k+2)!}\phi^k \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ B_6(0) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1/2 \\ 2/12 \\ 3/30 \\ 4/56 \\ 5/90 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/6 \\ 1/10 \\ 1/14 \\ 1/18 \\ \vdots \end{bmatrix} = \mathbf{q}_e, \qquad (even)$$

$$\sum_{k=0}^{+\infty} \frac{1}{(2k+1)!}\phi^k \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ B_6(0) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1/1 \\ (3/2)/3 \\ (5/2)/5 \\ (7/2)/7 \\ (9/2)/9 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ \vdots \end{bmatrix} = \mathbf{q}_o, \qquad (odd)$$

In other words, we can use the same matrix ϕ in order to define them.

Question.
There exists M sparse non singular such that the matrices

$$M(\textstyle\sum_{k=0}^{+\infty} \frac{1}{(2k+2)!}\phi^k)M^{-1} = \sum_{k=0}^{+\infty} \frac{1}{(2k+2)!}(M\phi M^{-1})^k,$$
$$M(\textstyle\sum_{k=0}^{+\infty} \frac{1}{(2k+1)!}\phi^k)M^{-1} = \sum_{k=0}^{+\infty} \frac{1}{(2k+1)!}(M\phi M^{-1})^k$$

are much more sparse than $\sum_{k=0}^{+\infty} \frac{1}{(2k+2)!}\phi^k$ and $\sum_{k=0}^{+\infty} \frac{1}{(2k+1)!}\phi^k$?

If yes, then ...

Remark.
Introduce the ε-circulant-type matrices C_e and C_o with the same lower triangular part of the matrices $\sum_{k=0}^{+\infty} \frac{1}{(2k+2)!}\phi^k$ and $\sum_{k=0}^{+\infty} \frac{1}{(2k+1)!}\phi^k$. Then the solutions of the systems $C_e z = \mathbf{q}_e$ and $C_o z = \mathbf{q}_o$ could be satisfactory approximations of the Bernoulli numbers; smaller ε better the approximations.

An ε-circulant-type matrix is any polynomial in the following matrix

$$\begin{bmatrix} 0 & & & \varepsilon a_1 \\ a_2 & & & \\ & a_3 & & \\ & & a_n & 0 \end{bmatrix}, \ (a_1 = 1\ ?).$$

Set

$$e(z)=\sum_{k=0}^{+\infty}\frac{1}{(2k+2)!}z^k,\quad o(z)=\sum_{k=0}^{+\infty}\frac{1}{(2k+1)!}z^k.$$

Note that $2x^2e(x^2)=e^x+e^{-x}-2$ and $2xo(x^2)=e^x-e^{-x}$ (Roberto Peirone).

Thus, for any $n\times n$ matrix A such that there exists B for which $B^2=A$ ($B=\sqrt{A}$), we have the equalities

$$2A\sum_{k=0}^{+\infty}\frac{1}{(2k+2)!}A^k=e^B+e^{-B}-2I,\quad 2B\sum_{k=0}^{+\infty}\frac{1}{(2k+1)!}A^k=e^B-e^{-B}.$$

Unfortunately, for our matrix ϕ there is no matrix B such that $B^2=\phi$.

Perhaps, the matrix $\sum_{k=0}^{+\infty}\frac{1}{(2k+1)!}\phi^k$ is similar to a lower triangular Toeplitz matrix:

$$\begin{bmatrix} a &&&& \\ & b &&& \\ && c && \\ &&& d & \\ &&&& e \end{bmatrix}\Big(\sum_{k=0}^{4}\frac{1}{(2k+1)!}\begin{bmatrix} 0 &&&& \\ 2 &&&& \\ & 12 &&& \\ && 30 && \\ &&& 56 & 0 \end{bmatrix}^k\Big)\begin{bmatrix} a^{-1} &&&& \\ & b^{-1} &&& \\ && c^{-1} && \\ &&& d^{-1} & \\ &&&& e^{-1} \end{bmatrix}=$$

$$\sum_{k=0}^{4}\frac{1}{(2k+1)!}\begin{bmatrix} a &&&& \\ & b &&& \\ && c && \\ &&& d & \\ &&&& e \end{bmatrix}\begin{bmatrix} 0 &&&& \\ 2 &&&& \\ & 12 &&& \\ && 30 && \\ &&& 56 & 0 \end{bmatrix}^k\begin{bmatrix} a^{-1} &&&& \\ & b^{-1} &&& \\ && c^{-1} && \\ &&& d^{-1} & \\ &&&& e^{-1} \end{bmatrix}=$$

$$\sum_{k=0}^{4}\frac{1}{(2k+1)!}\Big(\begin{bmatrix} a &&&& \\ & b &&& \\ && c && \\ &&& d & \\ &&&& e \end{bmatrix}\begin{bmatrix} 0 &&&& \\ 2 &&&& \\ & 12 &&& \\ && 30 && \\ &&& 56 & 0 \end{bmatrix}\begin{bmatrix} a^{-1} &&&& \\ & b^{-1} &&& \\ && c^{-1} && \\ &&& d^{-1} & \\ &&&& e^{-1} \end{bmatrix}\Big)^k=$$

$$\sum_{k=0}^{4}\frac{1}{(2k+1)!}\Big(\begin{bmatrix} 0 &&&& \\ 2b &&&& \\ & 12c &&& \\ && 30d && \\ &&& 56e & 0 \end{bmatrix}\begin{bmatrix} a^{-1} &&&& \\ & b^{-1} &&& \\ && c^{-1} && \\ &&& d^{-1} & \\ &&&& e^{-1} \end{bmatrix}\Big)^k=$$

$$\sum_{k=0}^{4}\frac{1}{(2k+1)!}\Big(\begin{bmatrix} 0 &&&& \\ 2ba^{-1} &&&& \\ & 12cb^{-1} &&& \\ && 30dc^{-1} && \\ &&& 56ed^{-1} & 0 \end{bmatrix}\Big)^k.$$

Choose a,b,c,d,e such that

$$2ba^{-1}=12cb^{-1}=30dc^{-1}=56ed^{-1}=cost\neq 0$$

If it is possible, then one can compute Bernoulli numbers by solving a lower triangular Toeplitz linear system.

Perhaps generalized-Toeplitz (generalized circulants) simply reduce to Toeplitz (circulants). Perhaps it is useless to introduce such generalized matrices ...

Solving infinite Toeplitz lower triangular linear systems

Assume we want to compute the solution of the system $T_n \mathbf{x}_n = \mathbf{c}_n$, n-section of the infinite lower triangular Toeplitz linear system

$$T\mathbf{x} = \mathbf{c}, \quad T = \begin{bmatrix} 1 & & & \\ a_2 & 1 & & \\ a_3 & a_2 & 1 & \\ \vdots & & & \ddots \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix},$$

i.e. T_n is the $n \times n$ upper left submatrix of T and $\mathbf{c}_n = I_n^1 \mathbf{c}$ (thus $\mathbf{x}_n = I_n^1 \mathbf{x}$).

Note that

$$T_{2k}\mathbf{x}_{2k} = \begin{bmatrix} T_k & 0 \\ M_k & T_k \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ I_{2k}^{k+1}\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_k \\ I_{2k}^{k+1}\mathbf{c} \end{bmatrix} = \mathbf{c}_{2k},$$

so, if we have solved $T_k \mathbf{x}_k = \mathbf{c}_k$ (we have $\mathbf{x}_k$) then $\mathbf{x}_{2k}$ ($I_{2k}^{k+1}\mathbf{x}$) is obtained by calculating the matrix-vector product $M_k \mathbf{x}_k$ (note that M_k is Toeplitz!) and by solving the linear system

$$T_k I_{2k}^{k+1}\mathbf{x} = I_{2k}^{k+1}\mathbf{c} - M_k \mathbf{x}_k.$$

In other words,

$s(2k) \leq t(k) + 2s(k)$,
$s(k) =$ multiplicative complexity of solving a system $T_k \mathbf{z} = \mathbf{w}$,
$t(k) =$ multiplicative complexity of a matrix-vector multiplication $M_k \mathbf{z}$.

An obvious algorithm solving $T_{2^r}\mathbf{x}_{2^r} = \mathbf{c}_{2^r}$ consists in solving the first equation (no operation is required, i.e. $s(1) = 0$), solving the second equation (by one multiplication, i.e. $t(1) = 1$, $s(1) = 0$), solving the third and fourth equations simultaneously (doing $t(2)$ and $s(2) \leq t(1) + 2s(1)$ operations), solving the 5th-8th equations simultaneously (doing $t(4)$ and $s(4) \leq t(2) + 2s(2)$ operations), and so on. It is clear that

$$\begin{aligned} s(2^r) &\leq s(1) + t(1) + s(1) + t(2) + s(2) + t(4) + s(4) + t(8) + s(8) + \ldots + t(2^{r-1}) + s(2^{r-1}), \\ &= 2s(2^{r-1}) + t(2^{r-1}) = \ldots = 2^{r-1}s(2) + \textstyle\sum_{j=1}^{r-1} 2^{j-1} t(2^{r-j}). \end{aligned}$$

($s(1) = 0$, $t(1) = 1$, $s(2) = 1$). Noting that $t(k) = O(k \log_2 k)$ one easily proves that $s(2^r) = O(2^r r^2) = O(n(\log_2 n)^2)$ (if $n = 2^r$).

Bernoulli numbers computation by solving a triangular Toeplitz linear system

$$D\phi D^{-1} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 0 & & & & & \\ 2 & & & & & \\ & 12 & & & & \\ & & 30 & & & \\ & & & \ddots & & \\ & & & & (2n-3)(2n-2) & 0 \end{bmatrix} \begin{bmatrix} d_1^{-1} & & & \\ & d_2^{-1} & & \\ & & \ddots & \\ & & & d_n^{-1} \end{bmatrix}$$

$$
= \begin{bmatrix} 0 & & & & & \\ 2\frac{d_2}{d_1} & & & & & \\ & 12\frac{d_3}{d_2} & & & & \\ & & 30\frac{d_4}{d_3} & & & \\ & & & \ddots & & \\ & & & & (2n-3)(2n-2)\frac{d_n}{d_{n-1}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & & & & & \\ x & & & & & \\ & x & & & & \\ & & x & & & \\ & & & \ddots & & \\ & & & & x & 0 \end{bmatrix},
$$

where the latter identity holds if

$$
d_k = \frac{x^{k-1} d_1}{(2k-2)!}, \quad k = 1, 2, 3, \dots, n,
$$

or, equivalently, if

$$
D = d_1 D_x, \quad D_x = \begin{bmatrix} 1 & & & & \\ & \frac{x}{2!} & & & \\ & & \frac{x^2}{4!} & & \\ & & & \ddots & \\ & & & & \frac{x^{n-1}}{(2n-2)!} \end{bmatrix}.
$$

It follows that the Bernoulli triangular linear systems $\sum_{k=0}^{n-1} \frac{1}{(2k+1)!}\phi^k \mathbf{b} = \mathbf{q}_o$ and $\sum_{k=0}^{n-1} \frac{1}{(2k+2)!}\phi^k \mathbf{b} = \mathbf{q}_e$ ($\mathbf{b}$ is the vector of the first n Bernoulli numbers) are equivalent to the Toeplitz triangular systems $\sum_{k=0}^{n-1} \frac{1}{(2k+1)!}(D_x \phi D_x^{-1})^k (D_x \mathbf{b}) = D_x \mathbf{q}_o$, and $\sum_{k=0}^{n-1} \frac{1}{(2k+2)!}(D_y \phi D_y^{-1})^k (D_y \mathbf{b}) = D_y \mathbf{q}_e$ (for any x, $y \neq 0$).

Let us write them explicitly:

$$
\sum_{k=0}^{n-1} \frac{x^k}{(2k+1)!} \quad k+1 \begin{bmatrix} & & & \\ 1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & 1 \end{bmatrix} (D_x \mathbf{b}) = D_x \mathbf{q}_o.
$$

$$
\sum_{k=0}^{n-1} \frac{y^k}{(2k+2)!} \quad k+1 \begin{bmatrix} & & & \\ 1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & 1 \end{bmatrix} (D_y \mathbf{b}) = D_y \mathbf{q}_e.
$$

The Toeplitz triangular linear system solver (what is the optimal? The one we know of cost $O(n(\log_2 n)^2)$?) gives the following vector $\mathbf{z}$:

$$
\mathbf{z} = D_x \mathbf{b} = \begin{bmatrix} 1 \cdot B_0(0) \\ \frac{x}{2!} B_2(0) \\ \frac{x^2}{4!} B_4(0) \\ \\ \frac{x^s}{(2s)!} B_{2s}(0) \\ \\ \frac{x^{n-1}}{(2n-2)!} B_{2n-2}(0) \end{bmatrix},
$$

from which one computes the vector of the first n Bernoulli numbers $\mathbf{b} = D_x^{-1}\mathbf{z}$.

Choice of x (y) ? So that the entries of the solution vector $\mathbf{z}$ have magnitude smaller than 1:

$\frac{x^0}{(0)!}B_0(0) = 1$
$|\frac{x^1}{(2)!}B_2(0)| \leq 1$ iff $|x| \leq 12$
$|\frac{x^2}{(4)!}B_4(0)| \leq 1$ iff $|x| \leq 26.84$
$|\frac{x^3}{(6)!}B_6(0)| \leq 1$ iff $|x| \leq 31.2$
$|\frac{x^4}{(8)!}B_8(0)| \leq 1$ iff $|x| \leq 33.2$
$|\frac{x^5}{(10)!}B_{10}(0)| \leq 1$ iff $|x| \leq 34.37$
$|\frac{x^6}{(12)!}B_{12}(0)| \leq 1$ iff $|x| \leq 35.2$
$|\frac{x^7}{(14)!}B_{14}(0)| \leq 1$ iff $|x| \leq 35.8$
$|\frac{x^8}{(16)!}B_{16}(0)| \leq 1$ iff $|x| \leq 36.2$
$|\frac{x^s}{(2s)!}B_{2s}(0)| \leq 1$ about iff $|\frac{x^s}{(2s)!}4\sqrt{\pi s}\frac{s^{2s}}{(\pi e)^{2s}}| \leq 1$ iff $|x|^s \leq \frac{(2s)!}{s^{2s}}\frac{(\pi e)^{2s}}{4\sqrt{\pi s}}$
$|\frac{x^{16}}{(32)!}B_{32}(0)| \leq 1$ about iff $|x|^{16} \leq \frac{1}{1293}\frac{(8.54)^{32}}{4 \cdot 7.09}$ iff $|x| \leq 37.82$

$(\frac{x^s}{(2s)!}B_{2s}(0) \approx (-1)^{s+1}\frac{x^s}{(2s)!}4\sqrt{\pi s}\frac{s^{2s}}{(\pi e)^{2s}})$.
Roberto Peirone: $a_s = \frac{(2s)!}{s^{2s}}$, $\frac{a_{s+1}}{a_s} = \frac{(2n+1)(2n+2)}{(n+1)^2}\frac{1}{((1+\frac{1}{n})^n)^2} \rightarrow \frac{4}{e^2}$, which is less than one, so $a_s \rightarrow 0$.
$a_s = \frac{x^s}{(2s)!}4\sqrt{\pi s}\frac{s^{2s}}{(\pi e)^{2s}}$, x positive $\Rightarrow \frac{a_{s+1}}{a_s} \rightarrow \frac{x}{4\pi^2}$, thus if $x = 4\pi^2 \approx 39.4$ we can conclude neither $a_s \rightarrow 0$ nor $a_s \rightarrow +\infty$.
$a_s = \frac{x^s}{(2s)!}B_{2s}(0)$, x positive $\Rightarrow \frac{a_{s+1}}{a_s} = \frac{xB_{2s+2}(0)}{(2s+2)(2s+1)B_{2s}(0)} \rightarrow ?$

Other criteria for the choice of x: the entries of the coefficient matrix less than 1 but great as possible; the diagonal entries of D^{-1} less big as possible; ...

Remark.

$$
\begin{array}{c} k+1 \\ \\ \\ \end{array}
\begin{bmatrix} & & & \\ 1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & & \\ & \frac{x}{2!} & & & \\ & & \frac{x^2}{4!} & & \\ & & & \ddots & \\ & & & & \frac{x^{n-1}}{(2n-2)!} \end{bmatrix}
$$

$$
= \begin{array}{c} k+1 \\ \\ \\ \end{array}
\begin{bmatrix} & & & \\ 1 & & & \\ & \frac{x}{2!} & & \\ & & \ddots & \\ & & & \frac{x^{n-k-1}}{(2(n-k-1))!} \end{bmatrix}
$$

$$
= \begin{array}{c} k+1 \\ \\ \\ \end{array}
\begin{bmatrix} \cdot & & & & & \\ & \cdot & & & & \\ & & 1 & & & \\ & & & \frac{x}{2!} & & \\ & & & & \ddots & \\ & & & & & \frac{x^{n-k-1}}{(2(n-k-1))!} \end{bmatrix}
\begin{array}{c} k+1 \\ \\ \\ \end{array}
\begin{bmatrix} & & & \\ 1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & 1 \end{bmatrix}.
$$

Set $B_0(0) = 1$ and $k := 1$. Choose x. Then REPEAT {

$$\text{solve}\quad \begin{bmatrix} 1 & & & \\ \frac{x}{3!} & \cdot & & \\ \vdots & \cdot & \cdot & \\ \frac{x^{k-1}}{(2k-1)!} & \cdot & \frac{x}{3!} & 1 \end{bmatrix} \begin{bmatrix} \frac{x^k}{(2k)!} B_{2k}(0) \\ \frac{x^{k+1}}{(2k+2)!} B_{2k+2}(0) \\ \vdots \\ \frac{x^{2k-1}}{(4k-2)!} B_{4k-2}(0) \end{bmatrix}$$

$$= \tfrac{1}{2} \begin{bmatrix} \frac{x^k}{(2k)!} \\ \frac{x^{k+1}}{(2k+2)!} \\ \vdots \\ \frac{x^{2k-1}}{(4k-2)!} \end{bmatrix} - \begin{bmatrix} \frac{x^k}{(2k+1)!} & \cdot & \frac{x^2}{5!} & \frac{x}{3!} \\ \cdot & & & \frac{x^2}{5!} \\ \cdot & & & \cdot \\ \frac{x^{2k-1}}{(4k-1)!} & \cdot & \cdot & \frac{x^k}{(2k+1)!} \end{bmatrix} \begin{bmatrix} B_0(0) \\ \frac{x}{2!} B_2(0) \\ \vdots \\ \frac{x^{k-1}}{(2k-2)!} B_{2k-2}(0) \end{bmatrix}$$

IF ??: {compute $B_{2k}(0)$, $B_{2k+2}(0)$, …, $B_{4k-2}(0)$, and, if necessary, update x}
Set $k := 2k$. }

$$k = 1: \quad 1 \cdot \frac{x}{2!} B_2(0) = \frac{1}{2} \frac{x}{2!} - \frac{x}{3!} B_0(0) = \frac{1}{12} x,$$

$$k = 2: \quad \begin{bmatrix} 1 & 0 \\ \frac{x}{3!} & 1 \end{bmatrix} \begin{bmatrix} \frac{x^2}{4!} B_4(0) \\ \frac{x^3}{6!} B_6(0) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{x^2}{4!} \\ \frac{x^3}{6!} \end{bmatrix} - \begin{bmatrix} \frac{x^2}{5!} & \frac{x}{3!} \\ \frac{x^3}{7!} & \frac{x^2}{5!} \end{bmatrix} \begin{bmatrix} B_0(0) \\ (\frac{x}{2!} B_2(0)) \end{bmatrix},$$

$$k = 4: \quad \begin{bmatrix} 1 & & & \\ \frac{x}{3!} & 1 & & \\ \frac{x^2}{5!} & \frac{x}{3!} & 1 & \\ \frac{x^3}{7!} & \frac{x^2}{5!} & \frac{x}{3!} & 1 \end{bmatrix} \begin{bmatrix} \frac{x^4}{8!} B_8(0) \\ \frac{x^5}{10!} B_{10}(0) \\ \frac{x^6}{12!} B_{12}(0) \\ \frac{x^7}{14!} B_{14}(0) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{x^4}{8!} \\ \frac{x^5}{10!} \\ \frac{x^6}{12!} \\ \frac{x^7}{14!} \end{bmatrix} - \begin{bmatrix} \frac{x^4}{9!} & \frac{x^3}{7!} & \frac{x^2}{5!} & \frac{x}{3!} \\ \frac{x^5}{11!} & \frac{x^4}{9!} & \frac{x^3}{7!} & \frac{x^2}{5!} \\ \frac{x^6}{13!} & \frac{x^5}{11!} & \frac{x^4}{9!} & \frac{x^3}{7!} \\ \frac{x^7}{15!} & \frac{x^6}{13!} & \frac{x^5}{11!} & \frac{x^4}{9!} \end{bmatrix} \begin{bmatrix} B_0(0) \\ \frac{x}{2!} B_2(0) \\ (\frac{x^2}{4!} B_4(0)) \\ (\frac{x^3}{6!} B_6(0)) \end{bmatrix}, \ldots$$

Set $B_0(0) = 1$ and $k := 1$. Choose y. Then REPEAT {

$$\text{solve}\quad \begin{bmatrix} \frac{1}{2} & & & \\ \frac{y}{4!} & \cdot & & \\ \vdots & \cdot & \cdot & \\ \frac{y^{k-1}}{(2k)!} & \cdot & \frac{y}{4!} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{y^k}{(2k)!} B_{2k}(0) \\ \frac{y^{k+1}}{(2k+2)!} B_{2k+2}(0) \\ \vdots \\ \frac{y^{2k-1}}{(4k-2)!} B_{4k-2}(0) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4k+2} \frac{y^k}{(2k)!} \\ \frac{1}{4k+6} \frac{y^{k+1}}{(2k+2)!} \\ \vdots \\ \frac{1}{8k-2} \frac{y^{2k-1}}{(4k-2)!} \end{bmatrix} - \begin{bmatrix} \frac{y^k}{(2k+2)!} & \cdot & \frac{y^2}{6!} & \frac{y}{4!} \\ \cdot & & & \frac{y^2}{6!} \\ \cdot & & & \cdot \\ \frac{y^{2k-1}}{(4k)!} & \cdot & \cdot & \frac{y^k}{(2k+2)!} \end{bmatrix} \begin{bmatrix} B_0(0) \\ \frac{y}{2!} B_2(0) \\ \vdots \\ \frac{y^{k-1}}{(2k-2)!} B_{2k-2}(0) \end{bmatrix}$$

IF ??: {compute $B_{2k}(0)$, $B_{2k+2}(0)$, …, $B_{4k-2}(0)$, and, if necessary, update y}
Set $k := 2k$. }

$$k = 1: \quad \frac{1}{2} \cdot \frac{y}{2!} B_2(0) = \frac{1}{6} \frac{y}{2!} - \frac{y}{4!} B_0(0) = \frac{1}{2} \frac{1}{12} y,$$

$$k = 2: \quad \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{y}{4!} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{y^2}{4!} B_4(0) \\ \frac{y^3}{6!} B_6(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{10} \frac{y^2}{4!} \\ \frac{1}{14} \frac{y^3}{6!} \end{bmatrix} - \begin{bmatrix} \frac{y^2}{6!} & \frac{y}{4!} \\ \frac{y^3}{8!} & \frac{y^2}{6!} \end{bmatrix} \begin{bmatrix} B_0(0) \\ (\frac{y}{2!} B_2(0)) \end{bmatrix},$$

$$k=4: \quad \begin{bmatrix} \frac{1}{2} & & & \\ \frac{y}{4!} & \frac{1}{2} & & \\ \frac{y^2}{6!} & \frac{y}{4!} & \frac{1}{2} & \\ \frac{y^3}{8!} & \frac{y^2}{6!} & \frac{y}{4!} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{y^4}{8!}B_8(0) \\ \frac{y^5}{10!}B_{10}(0) \\ \frac{y^6}{12!}B_{12}(0) \\ \frac{y^7}{14!}B_{14}(0) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{18}\frac{y^4}{8!} \\ \frac{1}{22}\frac{y^5}{10!} \\ \frac{1}{26}\frac{y^6}{12!} \\ \frac{1}{30}\frac{y^7}{14!} \end{bmatrix} - \begin{bmatrix} \frac{y^4}{10!} & \frac{y^3}{8!} & \frac{y^2}{6!} & \frac{y}{4!} \\ \frac{y^5}{12!} & \frac{y^4}{10!} & \frac{y^3}{8!} & \frac{y^2}{6!} \\ \frac{y^6}{14!} & \frac{y^5}{12!} & \frac{y^4}{10!} & \frac{y^3}{8!} \\ \frac{y^7}{16!} & \frac{y^6}{14!} & \frac{y^5}{12!} & \frac{y^4}{10!} \end{bmatrix} \begin{bmatrix} B_0(0) \\ \frac{y}{2!}B_2(0) \\ (\frac{y^2}{4!}B_4(0)) \\ (\frac{y^3}{6!}B_6(0)) \end{bmatrix}, \ldots$$

A lower triangular Toeplitz system whose solution is $B_2(0), B_4(0), \ldots$

$$\sum_{k=0}^{n-1} \frac{x^k}{(2k+1)!} Z_n^k (D_x \mathbf{b}) = D_x \mathbf{q}_o$$

$$\sum_{k=0}^{n-1} \frac{x^k}{(2k+1)!} (P_1 Z_n P_1^T)^k (P_1 D_x \mathbf{b}) = P_1 D_x \mathbf{q}_o$$

$$P_1 Z_n P_1^T = \begin{bmatrix} Z_{n-1} & \mathbf{e}_1 \\ \mathbf{0}^T & 0 \end{bmatrix}, \ (P_1 Z_n P_1^T)^2 = \begin{bmatrix} Z_{n-1}^2 & \mathbf{e}_2 \\ \mathbf{0}^T & 0 \end{bmatrix}, \ldots$$

$$\sum_{k=0}^{n-1} \frac{x^k}{(2k+1)!} \begin{bmatrix} Z_{n-1}^k & \mathbf{e}_k \\ \mathbf{0}^T & 0 \end{bmatrix} (P_1 D_x \mathbf{b}) = P_1 D_x \mathbf{q}_o$$

$$\sum_{k=0}^{n-1} \frac{x^k}{(2k+1)!} (Z_{n-1}^k I_{n-1}^1 (P_1 D_x \mathbf{b}) + (D_x \mathbf{b})_1 \mathbf{e}_k^{(n-1)}) = I_{n-1}^1 P_1 D_x \mathbf{q}_o$$

$$\sum_{k=0}^{n-1} \frac{x^k}{(2k+1)!} Z_{n-1}^k I_n^2 (D_x \mathbf{b}) = I_n^2 D_x \mathbf{q}_o - (D_x \mathbf{b})_1 \sum_{k=0}^{n-1} \frac{x^k}{(2k+1)!} \mathbf{e}_k^{(n-1)}$$

Extern algorithm:

For each step (computing $B_{2k}, \ldots, B_{4k-2}$ from $B_0, \ldots, B_{2k-2}$) we have to multiply a $k \times k$ Toeplitz matrix by a vector, and to solve a $k \times k$ lower triangular Toeplitz linear system, $k = 1, 2, 4, 8, \ldots$, i.e. for each step we have to call two functions. The first function use FFT. The second function is identical to the Extern algorithm.

A lower triangular sparse system whose solution is $B_2(0), B_4(0), \ldots$ (Ramanujan)

Ramanujan in his first paper states that the Bernoulli numbers satisfy the following equations

$$\begin{bmatrix} 1 \\ 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & 1 \\ 0 & \frac{5}{2} & 0 & 0 & 1 \\ 0 & 0 & 11 & 0 & 0 & 1 \\ \frac{1}{5} & 0 & 0 & \frac{143}{4} & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & \frac{286}{3} & 0 & 0 & 1 \\ 0 & 0 & \frac{204}{5} & 0 & 0 & 221 & 0 & 0 & 1 \\ \frac{1}{7} & 0 & 0 & \frac{1938}{7} & 0 & 0 & \frac{3230}{7} & 0 & 0 & 1 \\ 0 & \frac{11}{2} & 0 & 0 & \frac{7106}{5} & 0 & 0 & \frac{3553}{4} & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} B_2(0) \\ B_4(0) \\ B_6(0) \\ B_8(0) \\ B_{10}(0) \\ B_{12}(0) \\ B_{14}(0) \\ B_{16}(0) \\ B_{18}(0) \\ B_{20}(0) \\ B_{22}(0) \\ \cdot \end{bmatrix} = \begin{bmatrix} 1/6 \\ -1/30 \\ 1/42 \\ 1/45 \\ -1/132 \\ 4/455 \\ 1/120 \\ -1/306 \\ 3/665 \\ 1/231 \\ -1/552 \\ \cdot \end{bmatrix}$$

(actually, since Ramanujan Bernoulli numbers are the moduli of ours, his equations are a bit different).

So, for example, from the above equations I have easily computed $B_{18}(0), B_{20}(0), B_{22}(0)$ (from the ones already computed):

$$B_{18}(0) = \frac{43867}{798} \approx 54.97,\ \ B_{20}(0) = -\frac{174611}{330} \approx -529.12,\ \ B_{22}(0) = \frac{854513}{138} \approx 6192.12.$$

Problem. Is it possible to obtain such Ramanujan sparse equations by our arguments ? Is it possible to obtain other sparse equations (hopefully more sparse than Ramanujan ones) defining the Bernoulli numbers ?

Note that the Ramanujan matrix, say R, has nonzero entries exactly in the places where the matrix $\sum_k \gamma_k (Z^3)^k$ has nonzero entries, but it is not a polynomial in the matrix $d(\mathbf{v})Z^3$ whatever vector $\mathbf{v}$ is chosen. ...

$$\sum_{i=0}^{+\infty} d(\mathbf{v}_i)Z^i \sum_{j=0}^{+\infty} \alpha_j Z^j = \sum_{k=0}^{+\infty} (\sum_{i,j:\, i+j=k} \alpha_j d(\mathbf{v}_i))Z^k$$

$k=0:\ \ \alpha_0 d(\mathbf{v}_0)$
$k=1:\ \ \alpha_1 d(\mathbf{v}_0) + \alpha_0 d(\mathbf{v}_1)$
$k=2:\ \ \alpha_2 d(\mathbf{v}_0) + \alpha_1 d(\mathbf{v}_1) + \alpha_0 d(\mathbf{v}_2)$
$k=3:\ \ \alpha_3 d(\mathbf{v}_0) + \alpha_2 d(\mathbf{v}_1) + \alpha_1 d(\mathbf{v}_2) + \alpha_0 d(\mathbf{v}_3)$
$k=4:\ \ \alpha_4 d(\mathbf{v}_0) + \alpha_3 d(\mathbf{v}_1) + \alpha_2 d(\mathbf{v}_2) + \alpha_1 d(\mathbf{v}_3) + \alpha_0 d(\mathbf{v}_4)$
$k=5:\ \ \alpha_5 d(\mathbf{v}_0) + \alpha_4 d(\mathbf{v}_1) + \alpha_3 d(\mathbf{v}_2) + \alpha_2 d(\mathbf{v}_3) + \alpha_1 d(\mathbf{v}_4) + \alpha_0 d(\mathbf{v}_5)$
$k=6:\ \ \alpha_6 d(\mathbf{v}_0) + \alpha_5 d(\mathbf{v}_1) + \alpha_4 d(\mathbf{v}_2) + \alpha_3 d(\mathbf{v}_3) + \alpha_2 d(\mathbf{v}_4) + \alpha_1 d(\mathbf{v}_5) + \alpha_0 d(\mathbf{v}_6)$

Discussing with Fra

Given $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots \in \mathbb{R}^{+\infty}$ (in the simplest case $\mathbf{w}_k = w_k \mathbf{e}$, $\mathbf{e} = [1\ 1\ 1\ \cdots]^T$), look for $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots \in \mathbb{R}^{+\infty}$ such that

$$\sum_{i=0}^{+\infty} d(\mathbf{v}_i)Z^i \sum_{j=0}^{+\infty} \alpha_j Z^j = \sum_{k=0}^{+\infty} d(\mathbf{w}_k) Z^k \qquad (*)$$

i.e. such that

$$\sum_{k=0}^{+\infty} \Big(\sum_{i,j:\, i+j=k} \alpha_j d(\mathbf{v}_i) \Big) Z^k = \sum_{k=0}^{+\infty} d(\mathbf{w}_k) Z^k.$$

For such equality to be valid it is enough to require

$$I_{+\infty}^{k+1} \Big(\sum_{i,j:\, i+j=k} \alpha_j \mathbf{v}_i \Big) = I_{+\infty}^{k+1} \mathbf{w}_k$$

i.e.

$$I_{+\infty}^{k+1} \mathbf{v}_k = \frac{1}{\alpha_0} \Big(I_{+\infty}^{k+1} \mathbf{w}_k - \sum_{j=1}^{k} \alpha_j I_{+\infty}^{k+1} \mathbf{v}_{k-j} \Big).$$

The above general arguments (i.e. the above arguments with $\mathbf{w}_k \neq w_k \mathbf{e}$) are *necessary* in order to obtain the Ramanujan lower triangular system (defining Bernoulli numbers $B_2(0), B_4(0), \ldots$) starting from our lower triangular Toeplitz systems (defining Bernoulli numbers), because the coefficient matrix of the Ramanujan system is of the type $\sum_{k=0}^{+\infty} d(\mathbf{w}_k)Z^k$, $\mathbf{w}_0 = \mathbf{e}$, $\mathbf{w}_{3j+1} = \mathbf{w}_{3j+2} = \mathbf{0}$, $\mathbf{w}_{3j+3}$ suitable, $j = 0, 1, 2, \ldots$, but is not a polynomial in $d(\mathbf{v})Z^3$. [I have yet to verify that such general arguments are also *sufficient*; in particular, it is important to check if there's an explicit formula for the infinite Ramanujan lower triangular matrix].

In the simplest case, $\mathbf{w}_k = w_k\mathbf{e}$, one can look for $\mathbf{v}_k = v_k\mathbf{e}$ solving (*), and thus (*) reduces to

$$\sum_{i=0}^{+\infty} v_i Z^i \sum_{j=0}^{+\infty} \alpha_j Z^j = \sum_{k=0}^{+\infty} w_k Z^k, \quad L(\mathbf{v})L(\alpha) = L(\mathbf{w}).$$

This is equivalent to solve the system $L(\alpha)\mathbf{v} = \mathbf{w}$. By choosing $\mathbf{w}$ sparse (for instance $\mathbf{w} = [1\ 0\ 0\ 1\ 0\ 0\ 1 \ldots]^T$) and solving $L(\alpha)\mathbf{v} = \mathbf{w}$ we will have $\mathbf{v}$ such that the system

$$L(\mathbf{w})(D_x\mathbf{b}) = L(\mathbf{v})L(\alpha)(D_x\mathbf{b}) = L(\mathbf{v})D_x\mathbf{q}_o$$

is more sparse than our system $L(\alpha)D_x\mathbf{b} = D_x\mathbf{q}_o$ (or $L(\alpha)D_x\mathbf{b} = D_x\mathbf{q}_e$), and thus easier to be solved. But, in order to obtain a sparse system easy to solve we have to solve a full system, like the original !

Problem. There exists a sparse $\mathbf{w}$ for which $L(\alpha)\mathbf{v} = \mathbf{w}$, in spite of its fullness, is much easier to solve than our original system $L(\alpha)D_x\mathbf{b} = D_x\mathbf{q}_o$?