Eli Maor, e the story of a number, among references: George F. Simmons, Calculus with Analytic Geometry, NY, McGraw Hill, 1985, pp. 734–739

How Chebycev polynomials arise

Set $y(x) = x^n - p_{n-1}^{ott}(x)$ where p_{n-1}^{ott} is the unique polynomial of degree at most $n-1$ solving the minimum problem

$$\min_{p\in\mathbb{P}_{n-1}} \max_{[-1,1]} |x^n - p(x)|$$

where $\mathbb{P}_{n-1}$ is the set of all polynomials of degree less than or equal to $n-1$. Moreover, set $\mu = \max_{[-1,1]} |y(x)|$.

Graphical considerations let us find such p_{n-1}^{ott}, for $n = 1, 2, 3$:

For $n = 1$ the given problem $\min_{p\in\mathbb{P}_0} \max_{[-1,1]} |x - p(x)|$ has the obvious solution $p_0^{ott}(x) = 0$. Moreover, observe that $y(x_i) = (-1)^i\mu$, $\mu = 1$, $x_1 = -1$, $x_0 = 1$.

For $n = 2$ the problem $\min_{p\in\mathbb{P}_1} \max_{[-1,1]} |x^2 - p(x)|$ is solved by $p_1^{ott}(x) = \frac{1}{2}$. Moreover, $y(x_i) = (-1)^i\mu$, $\mu = \frac{1}{2}$, $x_2 = -1$, $x_1 = 0$, $x_0 = 1$.

For $n = 3$ the solution p_2^{ott} of the problem $\min_{p\in\mathbb{P}_2} \max_{[-1,1]} |x^3 - p(x)|$ must be a straight line with positive slope which intersects x^3 in three distinct points whose abscissas are $\xi_2 \in (-1, 0)$, $\xi_1 = 0$, $\xi_0 = -\xi_2 \in (0, 1)$, i.e. $p_2^{ott} = \alpha x$ with $0 < \alpha < 1$. Consider the function $g(x) = x^3 - \alpha x$ in the interval $[-1, 1]$ and notice that $g'(x) = 3x^2 - \alpha$ and thus $g'(\pm\sqrt{\frac{\alpha}{3}}) = 0$, $g(\pm\sqrt{\frac{\alpha}{3}}) = \pm\frac{\alpha}{3}\sqrt{\frac{\alpha}{3}} - \alpha(\pm\sqrt{\frac{\alpha}{3}}) = \mp\frac{2}{3}\alpha\sqrt{\frac{\alpha}{3}}$. Moreover, $g(\pm 1) = \pm 1 - \alpha(\pm 1)$. So, we have to choose $\alpha \in (0, 1)$ so that $\max\{1 - \alpha, \frac{2}{3}\alpha\sqrt{\frac{\alpha}{3}}\}$ is minimum, i.e. $\alpha \in (0, 1)$ such that

$$1 - \alpha = \frac{2}{3\sqrt{3}}\alpha\sqrt{\alpha} \;\Rightarrow\; \alpha = \frac{3}{4}.$$

Thus, $p_2^{ott}(x) = \frac{3}{4}x$. Moreover, observe that $y(x_i) = (-1)^i\mu$, $\mu = \frac{1}{4}$, $x_3 = -1$, $x_2 = -\frac{1}{2}$, $x_1 = \frac{1}{2}$, $x_0 = 1$.

In general, Chebycev-Tonelli theory states that $y(x) = x^n - p_{n-1}^{ott}(x)$ must assume the values μ and $-\mu$ alternately in $n + 1$ points x_j of $[-1, 1]$, $-1 \leq x_n < x_{n-1} < \ldots < x_2 < x_1 < x_0 \leq 1$: $y(x_j) = (-1)^j\mu$. Obviously $y'(x_i) = 0$, $i = 1, \ldots, n-1$, whereas $y'(x_0)y'(x_n) \neq 0$ since $y'(x)$ is a polynomial of degree $n - 1$. Thus $x_n = -1$, $x_0 = 1$. Consider now the function $y(x)^2 - \mu^2$. It is zero in all the x_i and its derivative, $2y(x)y'(x)$, is zero in $x_1, x_2, \ldots, x_{n-1}$. It follows that $y(x)^2 - \mu^2 = c(x^2 - 1)y'(x)^2$ for some real constant c. Noting that the coefficient of x^{2n} is on the left 1 and on the right cn^2, we conclude that

$$\frac{n^2}{1 - x^2} = \frac{y'(x)^2}{\mu^2 - y(x)^2}, \quad \frac{n}{\sqrt{1 - x^2}} = \pm\frac{y'(x)}{\sqrt{\mu^2 - y(x)^2}}.$$

The latter equality is solved by $y(x) = \mu\cos(n \arccos x + c)$, $c \in \mathbb{R}$. Then the identity $y(1) = \mu$ implies $c = 2k\pi$, and thus

$$y(x) = x^n - p_{n-1}^{ott}(x) = \mu\cos(n \arccos x), \quad 1 - x^2 \geq 0.$$

Finally, observe that

$$\begin{aligned}
&\cos(0\arccos x) = 1|_{[-1,1]} =: T_0(x)|_{[-1,1]},\\
&\cos(\arccos x) = x|_{[-1,1]} =: T_1(x)|_{[-1,1]},\\
&\cos(2\arccos x) = 2\cos(\arccos x)\cos(\arccos x) - \cos(0\arccos x)\\
&\qquad\qquad = 2x^2 - 1|_{[-1,1]} =: T_2(x)|_{[-1,1]},\\
&\cos((j+1)\arccos x) = 2\cos(\arccos x)\cos(j\arccos x) - \cos((j-1)\arccos x)\\
&\qquad\qquad = 2xT_j(x) - T_{j-1}(x)|_{[-1,1]} =: T_{j+1}(x)|_{[-1,1]}.
\end{aligned}$$

Thus, $\mu = \frac{1}{2^{n-1}}$ because $T_n(x) = 2^{n-1}x^n + \cdots$. So, we have the important result:

$$y(x) = x^n - p_{n-1}^{ott}(x) = \frac{1}{2^{n-1}}\cos(n\arccos x) = \frac{1}{2^{n-1}}T_n(x), \quad 1 - x^2 \geq 0.$$

Let us see two examples. The already studied specific case $n = 3$ is now immediately obtained:

$$\begin{aligned}
&y(x) = x^3 - p_2^{ott}(x) = \tfrac{1}{4}\cos(3\arccos x) = \tfrac{1}{4}(4x^3 - 3x) = x^3 - \tfrac{3}{4}x,\\
&y(x_j) = (-1)^j\tfrac{1}{4}, \quad p_2^{ott}(x) = x^3 - (x^3 - \tfrac{3}{4}x) = \tfrac{3}{4}x.
\end{aligned}$$

The cases $n > 3$ are analogously easily solved. In particular, for $n = 4$ we have

$$\begin{aligned}
&y(x) = x^4 - p_3^{ott}(x) = \tfrac{1}{8}\cos(4\arccos x) = \tfrac{1}{8}(8x^4 - 8x^2 + 1) = x^4 - x^2 + \tfrac{1}{8},\\
&y(x_j) = (-1)^j\tfrac{1}{8}, \quad p_3^{ott}(x) = x^4 - (x^4 - x^2 + \tfrac{1}{8}) = x^2 - \tfrac{1}{8}.
\end{aligned}$$

Deflation

Le A be a $n \times n$ matrix. Denote by λ_i, $i = 1, \ldots, n$, the eigenvalues of A and by $\mathbf{y}_i$ the corresponding eigenvectors. So, we have $A\mathbf{y}_i = \lambda_i\mathbf{y}_i$, $i = 1, \ldots, n$.

Assume that $\lambda_1, \mathbf{y}_1$ are given and that $\lambda_1 \neq 0$. Choose $\mathbf{w} \in \mathbb{C}^n$ such that $\mathbf{w}^*\mathbf{y}_1 \neq 0$ (given $\mathbf{y}_1$ choose $\mathbf{w}$ not orthogonal to $\mathbf{y}_1$) and set

$$W = A - \frac{\lambda_1}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1\mathbf{w}^*.$$

It is known that the eigenvalues of W are

$$0,\ \lambda_2,\ \ldots,\ \lambda_j,\ \ldots,\ \lambda_n$$

i.e. they are the same of A except λ_1 which is replaced with 0. Let us prove this fact. Consider a matrix S whose first column is $\mathbf{y}_1$ and whose remaining columns $\mathbf{x}_2, \ldots, \mathbf{x}_n$ are chosen such that S is non singular. Observe that

$$S^{-1}AS = S^{-1}[A\mathbf{y}_1\ A\mathbf{x}_2\ \cdots\ A\mathbf{x}_n] = [\lambda_1\mathbf{e}_1\ S^{-1}A\mathbf{x}_2 \cdots S^{-1}A\mathbf{x}_n].$$

So, if we call B the $(n-1) \times (n-1)$ lower right submatrix of $S^{-1}AS$, then $p_A(\lambda) = (\lambda - \lambda_1)p_B(\lambda)$. But we also have

$$\begin{aligned}
S^{-1}WS &= S^{-1}AS - S^{-1}\tfrac{\lambda_1}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1\mathbf{w}^*S\\
&= \begin{bmatrix} \lambda_1 & \mathbf{c}^T \\ \mathbf{0} & B \end{bmatrix} - \tfrac{\lambda_1}{\mathbf{w}^*\mathbf{y}_1}\mathbf{e}_1[\mathbf{w}^*\mathbf{y}_1\ \mathbf{w}^*\mathbf{x}_2 \cdots \mathbf{w}^*\mathbf{x}_n]\\
&= \begin{bmatrix} \lambda_1 & \mathbf{c}^T \\ \mathbf{0} & B \end{bmatrix} - \begin{bmatrix} \lambda_1 & \mathbf{d}^T \\ \mathbf{0} & O \end{bmatrix}\\
&= \begin{bmatrix} 0 & \mathbf{c}^T - \mathbf{d}^T \\ \mathbf{0} & B \end{bmatrix},
\end{aligned}$$

and thus the identity $p_W(\lambda) = \lambda p_B(\lambda)$, from which the thesis.

Let $\mathbf{w}_1$, $\mathbf{w}_2$, …, $\mathbf{w}_j$, …, $\mathbf{w}_n$ be the corresponding eigenvectors ($W\mathbf{w}_1 = \mathbf{0}$, $W\mathbf{w}_j = \lambda_j \mathbf{w}_j$ $j = 2, \dots, n$). Is it possible to obtain the $\mathbf{w}_j$ from the $\mathbf{y}_j$?

First observe that

$$A\mathbf{y}_1 = \lambda_1 \mathbf{y}_1 \;\Rightarrow\; W\mathbf{y}_1 = \mathbf{0}: \;\mathbf{w}_1 = \mathbf{y}_1. \qquad (a)$$

Then, for $j = 2, \dots, n$,

$$W\mathbf{y}_j = A\mathbf{y}_j - \frac{\lambda_1}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1\mathbf{w}^*\mathbf{y}_j = \lambda_j\mathbf{y}_j - \lambda_1\frac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1. \qquad (1)$$

If we impose $\mathbf{y}_j = \mathbf{w}_j + c\mathbf{y}_1$, $j = 2, \dots, n$, then (1) becomes,

$$\begin{aligned} W\mathbf{w}_j + cW\mathbf{y}_1 &= \lambda_j\mathbf{w}_j + c\lambda_j\mathbf{y}_1 - \lambda_1\tfrac{\mathbf{w}^*\mathbf{w}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1 - c\lambda_1\mathbf{y}_1 \\ &= \lambda_j\mathbf{w}_j + \mathbf{y}_1[c\lambda_j - \lambda_1\tfrac{\mathbf{w}^*\mathbf{w}_j}{\mathbf{w}^*\mathbf{y}_1} - \lambda_1 c] \end{aligned}$$

So, if $\lambda_j \neq \lambda_1$ and

$$\mathbf{w}_j = \mathbf{y}_j - \frac{\lambda_1}{\lambda_j - \lambda_1}\frac{\mathbf{w}^*\mathbf{w}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1, \qquad (2)$$

then $W\mathbf{w}_j = \lambda_j\mathbf{w}_j$. If, moreover, $\lambda_j \neq 0$, then $\mathbf{w}^*\mathbf{y}_j = \mathbf{w}^*\mathbf{w}_j + \frac{\lambda_1}{\lambda_j - \lambda_1}\mathbf{w}^*\mathbf{w}_j \Rightarrow \mathbf{w}^*\mathbf{y}_j = \mathbf{w}^*\mathbf{w}_j\frac{\lambda_j}{\lambda_j - \lambda_1} \Rightarrow \mathbf{w}^*\mathbf{w}_j = \frac{\lambda_j - \lambda_1}{\lambda_j}\mathbf{w}^*\mathbf{y}_j$. So, by (2),

$$\begin{aligned} &\text{for all } j \in \{2 \dots n\} \mid \lambda_j \neq \lambda_1, 0: \\ &A\mathbf{y}_j = \lambda_j\mathbf{y}_j \;\Rightarrow \\ &W(\mathbf{y}_j - \tfrac{\lambda_1}{\lambda_j}\tfrac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1) = \lambda_j(\mathbf{y}_j - \tfrac{\lambda_1}{\lambda_j}\tfrac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1): \;\mathbf{w}_j = \mathbf{y}_j - \tfrac{\lambda_1}{\lambda_j}\tfrac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1. \end{aligned} \qquad (b)$$

Note that a formula for $\mathbf{y}_j$ in terms of $\mathbf{w}_j$ holds: see (2).

As regards the case $\lambda_j = \lambda_1$, it is simple to show that

$$\begin{aligned} &\text{for all } j \in \{2 \dots n\} \mid \lambda_j = \lambda_1: \\ &A\mathbf{y}_j = \lambda_j\mathbf{y}_j \;\Rightarrow \\ &W(\mathbf{y}_j - \tfrac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1) = \lambda_j(\mathbf{y}_j - \tfrac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1): \;\mathbf{w}_j = \mathbf{y}_j - \tfrac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1. \end{aligned} \qquad (c)$$

Note that the vectors $\mathbf{y}_j - \frac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1$ are orthogonal to $\mathbf{w}$. Is it possible to find from (c) an expression of $\mathbf{y}_j$ in terms of $\mathbf{w}_j$?

It remains the case $\lambda_j = 0$: find ? in

$$\begin{aligned} &\text{for all } j \in \{2 \dots n\} \mid \lambda_j = 0: \\ &A\mathbf{y}_j = \lambda_j\mathbf{y}_j = \mathbf{0} \;\Rightarrow\; W(?) = \lambda_j(?) = \mathbf{0}: \;\mathbf{w}_j = ? \end{aligned} \qquad (d?)$$

$(\mathbf{y}_j = \mathbf{w}_j - \frac{\mathbf{w}^*\mathbf{w}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1 \Rightarrow \mathbf{w}^*\mathbf{y}_j = 0)$ …

Choices of $\mathbf{w}$. Since $\mathbf{y}_1^*\mathbf{y}_1 \neq 0$ one can set $\mathbf{w} = \mathbf{y}_1$. In this way, if A is hermitian also W is hermitian. … . If i is such that $(\mathbf{y}_1)_i \neq 0$ then $\mathbf{e}_i^T A\mathbf{y}_1 = \lambda_1(\mathbf{y}_1)_i \neq 0$. So one can set $\mathbf{w}^* = \mathbf{e}_i^T A = \text{row } i \text{ of } A$. In this way the row i of W is null and therefore we can introduce a matrix of order $n-1$ whose eigenvalues are λ_2, …, λ_n (the unknown eigenvalues of A).

Exercise on deflation

The matrix

$$G = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{8} & \frac{1}{8} \\ \frac{11}{16} & \frac{1}{4} & \frac{1}{16} \end{bmatrix}$$

satisfies the identity $G\mathbf{e} = \mathbf{e}$, $\mathbf{e} = [1\ 1\ 1]^T$. So, G has the eigenvalue 1 with corresponding eigenvector $\mathbf{e}$. Moreover, since $\rho(G) \leq \|G\|_\infty = 1$, all the eigenvalues of G have modulus less than or equal to 1.

Let 1, λ_2, λ_3 be the eigenvalues of G. It is known that the matrix

$$W = G - \frac{1}{\mathbf{w}^*\mathbf{e}}\mathbf{e}\mathbf{w}^* = G - \frac{1}{\mathbf{e}_i^T G\mathbf{e}}\mathbf{e}\mathbf{e}_i^T G = G - \mathbf{e}\mathbf{e}_i^T G$$

for any $i = 1, 2, 3$ has 0, λ_2, λ_3 as eigenvalues. For $i = 1$ we obtain

$$W = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{8} & -\frac{3}{8} \\ \frac{7}{16} & 0 & -\frac{7}{16} \end{bmatrix},$$

thus the remaining eigenvalues of G are $-\frac{1}{8}$ and $-\frac{7}{16}$.

Now observe that 1, $\lambda_2 = -\frac{1}{8}$, $\lambda_3 = -\frac{7}{16}$ are eigenvalues also of G^T. In particular, there exists $\mathbf{p}$ such that $G^T\mathbf{p} = \mathbf{p}$, but $\mathbf{p}$ has to be computed. The following inverse power iterations

$$\mathbf{v}_0,\ \|\mathbf{v}_0\|_1 = 1,\ \mathbf{a}_k = (G^T - (1+\varepsilon)I)^{-1}\mathbf{v}_k, \quad \mathbf{v}_{k+1} = a_k/\|\mathbf{a}_k\|_1, \ldots$$

generate $\mathbf{v}_k$ convergent to $\mathbf{p}$, $\|\mathbf{p}\|_1 = 1$, with a convergence rate $O(\frac{1+\varepsilon-1}{1+\varepsilon+\frac{1}{8}})$.

One eigenvalue at a time with power iterations

Assume A diagonalizable with eigenvalues λ_j such that $|\lambda_1| > |\lambda_k|$, $k = 2, \ldots, n$. Let $\mathbf{v} \neq \mathbf{0}$ be a vector. Then

$$A^k\mathbf{v} = \sum_j \alpha_j A^k \mathbf{x}_j = \sum_j \alpha_j \lambda_j^k \mathbf{x}_j,\ \ \frac{1}{\lambda_1^k}A^k\mathbf{v} = \alpha_1\mathbf{x}_1 + \sum_{j\neq 1}\alpha_j\frac{\lambda_j^k}{\lambda_1^k}\mathbf{x}_j.$$

Thus

$$\frac{1}{\lambda_1^k}\mathbf{z}^*A^k\mathbf{v} = \alpha_1\mathbf{z}^*\mathbf{x}_1 + \sum_{j\neq 1}\alpha_j\frac{\lambda_j^k}{\lambda_1^k}\mathbf{z}^*\mathbf{x}_j,$$

$$\frac{1}{\lambda_1^{k+1}}\mathbf{z}^*A^{k+1}\mathbf{v} = \alpha_1\mathbf{z}^*\mathbf{x}_1 + \sum_{j\neq 1}\alpha_j\frac{\lambda_j^{k+1}}{\lambda_1^{k+1}}\mathbf{z}^*\mathbf{x}_j,$$

$$\frac{\mathbf{z}^*A^{k+1}\mathbf{v}}{\mathbf{z}^*A^k\mathbf{v}} \ \to\ \lambda_1, \quad k \to \infty.$$

So, if an eigenvalue dominates the other eigenvalues, then such eigenvalue can be approximated better and better by computing the quantities:

$$A\mathbf{v},\ \frac{\mathbf{z}^*A\mathbf{v}}{\mathbf{z}^*\mathbf{v}},\ A^2\mathbf{v} = A(A\mathbf{v}),\ \frac{\mathbf{z}^*A^2\mathbf{v}}{\mathbf{z}^*A\mathbf{v}},\ A^3\mathbf{v} = A(A^2\mathbf{v}),\ \frac{\mathbf{z}^*A^3\mathbf{v}}{\mathbf{z}^*A^2\mathbf{v}},\ \ldots$$

It is clear that each new approximation requires a multiplication $A\mathbf{w}$.

Positive definite matrices and the choice $\mathbf{w} = \mathbf{y}_1^*$

Let A be a positive definite $n \times n$ matrix and let λ_j, $\mathbf{y}_j$ be such that $A\mathbf{y}_j = \lambda_j \mathbf{y}_j$. Assume that $0 < \lambda_n < \lambda_{n-1} < \cdots < \lambda_2 < \lambda_1$. Then compute λ_1 via power iterations, and $\mathbf{y}_1$ from a weak approximation λ_1^* of λ_1 via inverse power iterations, both applied to A. Then the eigenvalues of

$$A - \frac{\lambda_1}{\|\mathbf{y}_1\|^2}\mathbf{y}_1\mathbf{y}_1^* \text{ are } 0,\, \lambda_n,\, \lambda_{n-1}, \cdots,\, \lambda_2.$$

Compute λ_2 via power iterations, and $\mathbf{y}_2$ from a weak approximation λ_2^* of λ_2 via inverse power iterations, both applied to $A - \frac{\lambda_1}{\|\mathbf{y}_1\|^2}\mathbf{y}_1\mathbf{y}_1^*$. Then the eigenvalues of

$$(A - \frac{\lambda_1}{\|\mathbf{y}_1\|^2}\mathbf{y}_1\mathbf{y}_1^*) - \frac{\lambda_2}{\|\mathbf{y}_2\|^2}\mathbf{y}_2\mathbf{y}_2^* \text{ are } 0,\, 0,\, \lambda_n, \cdots,\, \lambda_3.$$

...

$$(\cdots(A - \frac{\lambda_1}{\|\mathbf{y}_1\|^2}\mathbf{y}_1\mathbf{y}_1^*) - \frac{\lambda_2}{\|\mathbf{y}_2\|^2}\mathbf{y}_2\mathbf{y}_2^* \cdots) - \frac{\lambda_n}{\|\mathbf{y}_n\|^2}\mathbf{y}_n\mathbf{y}_n^* \text{ are } 0,\, 0,\, \ldots,\, 0.$$

It follows that $A = \sum_{j=1}^n \frac{\lambda_j}{\|\mathbf{y}_j\|^2}\mathbf{y}_j\mathbf{y}_j^* = QDQ^*$, $Q = [\frac{1}{\|\mathbf{y}_1\|_2}\mathbf{y}_1 \; \frac{1}{\|\mathbf{y}_2\|_2}\mathbf{y}_2 \; \cdots \; \frac{1}{\|\mathbf{y}_n\|_2}\mathbf{y}_n]$. Note that the matrix Q is unitary (eigenvectors corresponding to distinct eigenvalues of a hermitian matrix must be orthogonal).

The QR method for 2×2 *matrices*

Set

$$A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}, \quad x, y, w, z \in \mathbb{R}.$$

Choose $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 = 1$ and $[Q_1 A]_{21} = 0$, where

$$Q_1 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix},$$

i.e. $\alpha = \frac{x}{\sqrt{x^2+z^2}}$, $\beta = \frac{-z}{\sqrt{x^2+z^2}}$. Then

$$Q_1 A = \begin{bmatrix} \sqrt{x^2+z^2} & \frac{xy+zw}{\sqrt{x^2+z^2}} \\ 0 & \frac{-zy+xw}{\sqrt{x^2+z^2}} \end{bmatrix} =: R.$$

Now define the matrix $B = RQ_1^T$:

$$B = \begin{bmatrix} x + \frac{z(xy+zw)}{x^2+z^2} & -z + \frac{x(xy+zw)}{x^2+z^2} \\ \frac{z(xw-zy)}{x^2+z^2} & \frac{x(xw-zy)}{x^2+z^2} \end{bmatrix}.$$

Note that $B = Q_1 A Q_1^T$, $Q_1^T = Q_1^{-1}$, so that B has the same eigenvalues of A. (Moreover, B is real symmetric if A is real symmetric).

So, by setting $x_0 = x$, $y_0 = y$, $w_0 = w$, $z_0 = z$ we can define the four sequences

$$\begin{aligned} &x_{k+1} = x_k + \tfrac{z(x_k y_k + z_k w_k)}{x_k^2+z_k^2}, \quad y_{k+1} = -z_k + \tfrac{x_k(x_k y_k + z_k w_k)}{x_k^2+z_k^2}, \\ &z_{k+1} = \tfrac{z(x_k w_k - z_k y_k)}{x_k^2+z_k^2}, \quad w_{k+1} = \tfrac{x_k(x_k w_k - z_k y_k)}{x_k^2+z_k^2}, \\ &k = 0, 1, 2, \ldots, \end{aligned}$$

which satisfy (by the theory on QR method) the properties:

$$z_k \ \to\ 0, \quad x_k,\, w_k \ \to \ \text{ eigenvalues of } A, \quad k \ \to\ +\infty$$

provided the eigenvalues of A are distinct in modulus (try to prove this assertion). For example, if $x = w = 2$ and $y = z = -1$, then $x_1 = \frac{14}{5}$, $y_1 = -\frac{3}{5}$, $w_1 = \frac{6}{5}$, $z_1 = -\frac{3}{5}$, $x_2 = \frac{122}{41}$, $y_2 = -\frac{9}{41}$, $w_2 = \frac{42}{41}$, $z_2 = -\frac{9}{41}$, ... It is clear that x_k and w_k tend to 3 and 1, the eigenvalues of A.

Some results on matrix algebras

Given a $n \times n$ matrix X, set

$$K_X = \{A :\ AX - XA = 0\}, \quad \mathcal{P}(X) = \{p(X) :\ p \text{ polynomials}\}.$$

Note that $\mathcal{P}(X) \subset K_X$, and

$$\mathcal{P}(X) = K_X \text{ iff } \dim \mathcal{P}(X) = \dim K_X = n.$$

Let Z denote the $n \times n$ shift-forward matrix, i.e. $[Z]_{ij} = 1$ if $i = j + 1$, and $[Z]_{ij} = 0$ otherwise. Note that

$$\begin{aligned}
&K_Z = \mathcal{P}(Z) = \{\text{lower triangular Toeplitz matrices}\},\\
&K_{Z^T} = \mathcal{P}(Z^T) = \{\text{upper triangular Toeplitz matrices}\},\\
&K_{Z^T+\varepsilon \mathbf{e}_n\mathbf{e}_1^T} = \mathcal{P}(Z^T + \varepsilon \mathbf{e}_n\mathbf{e}_1^T) = \{\varepsilon \text{ circulant matrices}\},\\
&K_{Z^T+Z} = \mathcal{P}(Z^T + Z) = \{\tau \text{ matrices}\},\\
&\{\text{symmetric circulant matrices}\} = \mathcal{P}(Z^T + Z + \mathbf{e}_n\mathbf{e}_1^T + \mathbf{e}_1\mathbf{e}_n^T)\\
&\qquad \subset K_{Z^T+Z+\mathbf{e}_n\mathbf{e}_1^T+\mathbf{e}_1\mathbf{e}_n^T} = \{A + JB :\ A, B \text{ circulant matrices }\}
\end{aligned}$$

($\mathbf{e}_i^T J = \mathbf{e}_{n-i+1}$ $i = 1, \ldots, n$, J =counteridentity).

Set $X = Z+Z^T$. Then the condition $AX = XA$, $A = (a_{ij})_{i,j=1}^n$, is equivalent to the n^2 conditions:

$$a_{i,j-1} + a_{i,j+1} = a_{i-1,j} + a_{i+1,j}, \quad 1 \le i, j \le n,$$

$a_{i,0} = a_{i,n+1} = a_{0,j} = a_{n+1,j} = 0$. Thus a generic matrix of τ has the form (in the case $n = 5$):

$$\begin{bmatrix} a & b & c & d & e \\ b & a+c & b+d & c+e & d \\ c & b+d & a+c+e & b+d & c \\ d & c+e & b+d & a+c & b \\ e & d & c & b & a \end{bmatrix}.$$

Since $XS = SD$, $S_{ij} = \sqrt{\frac{2}{n+1}} \sin \frac{ij\pi}{n+1}$ ($S^2 = I$), $D =$ diag $(2 \cos \frac{j\pi}{n+1})$, and matrices from τ are determined from their first row $\mathbf{z}^T$, we have the representation:

$$\tau(\mathbf{z}) = Sd(S\mathbf{z})d(S\mathbf{e}_1)^{-1}S$$

($\tau(\mathbf{z})$ = matrix of τ whose first row is $\mathbf{z}^T$).

Given a generic non singular matrix M, we have the representation

$$\{Md(\mathbf{z})M^{-1} :\ \mathbf{z} \in \mathbb{C}^n\} = \{Md(\mathbf{z})d(M^T\mathbf{v})^{-1}M^{-1} :\ \mathbf{z} \in \mathbb{C}^n\}$$

for any vector $\mathbf{v}$ such that $(M^T\mathbf{v})_j \neq 0, \forall\, j$ (note that $\mathbf{v}^T Md(\mathbf{z})d(M^T\mathbf{v})^{-1}M^{-1} = \mathbf{z}^T$). For M =Fourier, sine matrices, one can choose $\mathbf{v} = \mathbf{e}_1$ (so circulants and

τ matrices are determined by their first row). But there are significant matrices M (associated to fast discrete transforms) for which $\mathbf{v}$ cannot be chosen equal to $\mathbf{e}_1$ (i.e. matrices diagonalized by M are not determined by their first row).

An example of matrix algebra which is not commutative is $\mathcal{L} = \{A + JB : A, B \text{ circulants}\}$. The best approximation (in the Frobenius norm) in $\mathcal{L}$ of a given matrix A, call it $\mathcal{L}_A$, is well defined. It is known that $\mathcal{L}_A$ is hermitian any time A is hermitian. But it is not known if (in case A hermitian) $\mathbf{z}^* A\mathbf{z} > 0$ $\forall\, \mathbf{z} \neq \mathbf{0}$ implies $\mathbf{z}^*\mathcal{L}_A\mathbf{z} > 0\ \forall\, \mathbf{z} \neq \mathbf{0}$.

Assume $\{t_k\}_{k=0}^{+\infty}$, $t_k \in \mathbb{R}$, such that

$$\sum_{k=0}^{+\infty} |t_k| < +\infty. \tag{1}$$

Set $t(\theta) = \sum_{k=-\infty}^{+\infty} t_{|k|} e^{\mathbf{i}k\theta}$, $t_{\min} = \min t(\theta)$, $t_{\max} = \max t(\theta)$. Then the eigenvalues of $T^{(n)} = (t_{|i-j|})_{j,j=1}^n$ are in the interval $[t_{\min}, t_{\max}]$ for all n (proof omitted).

Let $C_{T^{(n)}}$ be the best circulant approximation of $T^{(n)}$. Since

$$C_{T^{(n)}} = F\, \mathrm{diag}\,((F^* T^{(n)} F)_{ii}) F^*,\ F_{ij} = \frac{1}{\sqrt{n}} \omega_n^{(i-1)(j-1)},\ \omega_n = e^{-\mathbf{i}2\pi/n},$$

we have

$$t_{\min} \le \min \lambda(T^{(n)}) \le \min \lambda(C_{T^{(n)}}),\ \max \lambda(C_{T^{(n)}}) \le \max \lambda(T^{(n)}) \le t_{\max}.$$

In particular, if

$$t_{\min} > 0, \tag{2}$$

then the $T^{(n)}$ and the $C_{T^{(n)}}$ are positive definite, and $\mu_2(C_{T^{(n)}}) \le \mu_2(T^{(n)}) \le \frac{t_{\max}}{t_{\min}}$; moreover, if $E_n E_n^T = C_{T^{(n)}}$, and $\alpha_j^{(n)}$ and $\beta_j^{(n)}$ are the eigenvalues, respectively, of $I - E_n^{-1} T^{(n)} E_n^{-T}$ and $C_{T^{(n)}} - T^{(n)}$ in nondecreasing order, then

$$\frac{1}{t_{\max}} |\beta_j^{(n)}| \le \frac{1}{\max \lambda(C_{T^{(n)}})} |\beta_j^{(n)}| \le |\alpha_j^{(n)}| \le \frac{1}{\min \lambda(C_{T^{(n)}})} |\beta_j^{(n)}| \le \frac{1}{t_{\min}} |\beta_j^{(n)}| \tag{2.5}$$

(apply the Courant-Fisher minimax characterization of the eigenvalues of a real symmetric matrix to $I - E_n^{-1} T^{(n)} E_n^{-T}$).

Theorem. If (1) holds, then the eigenvalues of $C_{T^{(n)}} - T^{(n)}$ are clustered around 0. If (1) and (2) hold, then the eigenvalues of $I - C_{T^{(n)}}^{-1} T^{(n)}$ are clustered around 0.

Proof. For the sake of simplicity, set $T = T^{(n)}$. Fix a number N, $n > 2N$, and let $W^{(N)}$ and $E^{(N)}$ be the $n \times n$ matrices defined by

$$[W^{(N)}]_{ij} = \begin{cases} [C_T - T]_{ij} & i, j \le n - N \\ 0 & \text{otherwise} \end{cases}$$

and

$$C_T - T = E^{(N)} + W^{(N)}. \tag{3}$$

Note that $[C_T]_{1j} = ((n - j + 1)t_{j-1} + (j - 1)t_{n-j+1})/n$, $j = 1, \ldots, n$, and thus, for $i, j = 1, \ldots, n$, we have

$$[C_T - T]_{ij} = -\frac{s_{|i-j|} |i - j|}{n},\ s_k = t_k - t_{n-k}.$$

Now observe that the rank of $E^{(N)}$ is less than or equal to $2N$, so $E^{(N)}$ has at least $n-2N$ null eigenvalues. Also observe that $C_T - T$, $E^{(N)}$ and $W^{(N)}$ are all real symmetric matrices. In the following we prove that, for any fixed $\varepsilon > 0$, there exist N_ε and $\nu_\varepsilon \geq 2N_\varepsilon$ such that

$$\|W^{(N_\varepsilon)}\|_1 < \varepsilon \quad \forall\, n > \nu_\varepsilon. \tag{4}$$

As a consequence of this fact and of the identity (3) for $N = N_\varepsilon$, we shall have that for all $n > \nu_\varepsilon$ at least $n - 2N_\varepsilon$ eigenvalues of $C_T - T$ are in $(-\varepsilon, \varepsilon)$. Moreover, if $t_{\min} > 0$, then, by (2.5), we shall also obtain the clustering around 0 of the eigenvalues of $I - C_T^{-1}T$.

So, let us prove (4). First we have

$$\|W^{(N)}\|_1 \leq \frac{2}{n}\sum_{j=1}^{n-N-1} j|s_j| \leq 2\sum_{j=N+1}^{n-1} |t_j| + \frac{2}{n}\sum_{j=1}^{N} j|t_j|. \tag{5}$$

Then, for any $\varepsilon > 0$ choose N_ε such that $2\sum_{j=N_\varepsilon+1}^{+\infty}|t_j| < \frac{\varepsilon}{2}$ and set $N = N_\varepsilon$ in (5) and in the previous arguments. If ν_ε, $\nu_\varepsilon \geq 2N_\varepsilon$, is such that, $\forall\, n > \nu_\varepsilon$, $\frac{2}{n}\sum_{j=1}^{N_\varepsilon} j|t_j| < \frac{\varepsilon}{2}$ (the sequence $\frac{1}{n}\sum_{j=1}^{n-1} j|t_j|$ tends to 0 if (1) holds), then by (5) we have the thesis (4).

Stai usando il seguente algoritmo (il primo a p.18 dell'articolo) che calcola direttamente una successione di e vettori $\mathbf{x}_k$ convergente a $\mathbf{x}$ tale che $\mathbf{p} = \frac{1}{\|\mathbf{x}\|_1}\mathbf{x}$? Se non lo stai usando, allora leggilo attentamente ed implementalo accuratamente, rispondendomi alle domande che troverai.

$$\mathbf{x}_{k+1} = \mathbf{x}_k + F(I - \alpha\sqrt{n}d(\overline{Fc}))^{-1}F^*(\mathbf{v} - A^T\mathbf{x}_k)$$

$$F_{s,j} = \frac{1}{\sqrt{n}}\omega_n^{(s-1)(j-1)},\ s,j = 1,\dots,n,\ \omega_n = e^{-\mathbf{i}2\pi/n}$$

$$d(\mathbf{z}) = \begin{bmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{bmatrix}$$

$$\mathbf{c} = [c_0\ c_1\ \cdots\ c_{n-1}]^T,\quad c_0 = s_0/n = 0,\quad c_i = (s_i + s_{-n+i})/n,\ i = 1,\dots,n-1$$

$$s_1 = \sum_{i=1}^{n-1}[P]_{i,i+1},\ s_{-1} = \sum_{i=1}^{n-1}[P]_{i+1,i},\dots$$

Quindi, ogni volta che n e' una potenza di 2: calcolo dei c_j, $j = 0,\dots,n-1$ ($c_0 = 0$); calcolo di $F\mathbf{c}$; calcolo dei $\overline{F\mathbf{c}}$ (il vettore coniugato di $F\mathbf{c}$); calcolo della matrice diagonale $D = (I - \alpha\sqrt{n}d(\overline{F\mathbf{c}}))^{-1}$. Poi, per ogni $k = 0, 1, \dots$, calcolo di

$$\mathbf{x}_{k+1} = \mathbf{x}_k + FDF^*(\mathbf{v} - (I - \alpha P^T)\mathbf{x}_k)$$

(scegliendo $\mathbf{x}_0 = \mathbf{v} = [1/n\ \cdots\ 1/n]^T$).

Nota che esiste una matrice di permutazione Q tale che $F^* = QF$, $F = QF^*$, hai usato questo fatto per calcolare $\mathbf{x}_{k+1}$? Quindi $F^*\mathbf{z}$ e' semplicemente una permutazione di $F\mathbf{z}$ (e viceversa); la FFT che hai tu calcola $F\mathbf{z}$ o $F^*\mathbf{z}$?

I vettori $\mathbf{x}_k$ dovrebbero convergere a un vettore $\mathbf{x}$ che una volta normalizzato dovrebbe coincidere con il vettore page-rank $\mathbf{p}$, cioe' $\mathbf{p} = \frac{1}{\|\mathbf{x}\|_1}\mathbf{x}$.

Mi scrivi dettagliatamente i tre criteri di arresto che usi? Quello per potenze dovrebbe differire da quelli usati per RE e RE precondizionato perche' i vettori generati dal metodo delle potenze sono gia' normalizzati.

$$y'(t) = -\frac{1}{2y(t)}, \quad y(0) = 1 \quad (y(t) = \sqrt{1-t})$$

$$\sqrt{1-t} = \frac{1}{\sqrt{p}} \ iff \ t = 1 - \frac{1}{p}$$

Integrate in $[0, 1 - \frac{1}{p}]$ the Cauchy problem to obtain an approximation of $\frac{1}{\sqrt{p}}$.

$p = 3$: Eulero for $h = \frac{1}{3}$, two steps; for $h = \frac{1}{6}$, four steps.

$$\eta(x_i + h) = \eta(x_i) + hf(x_i, \eta(x_i)) = \eta(x_i) - h\frac{1}{2\eta(x_i)}$$

$$\eta(0 + \frac{1}{3}) = \eta(0) - \frac{1}{3}\frac{1}{2\eta(0)} = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\eta(\frac{1}{3} + \frac{1}{3}) = \eta(\frac{1}{3}) - \frac{1}{3}\frac{1}{2\eta(\frac{1}{3})} = \frac{5}{6} - \frac{1}{5} = \frac{19}{30}$$

Idem, implicit Euler: $h = \frac{1}{3}$ not ok; $h = \frac{1}{6}$ ok?.

$$\eta(x_i + h) = \eta(x_i) + hf(x_i + h, \eta(x_i + h)) = \eta(x_i) - h\frac{1}{2\eta(x_i + h)}$$

$$\eta(x_i + h)^2 - \eta(x_i + h)\eta(x_i) + h\frac{1}{2} = 0$$

$$\eta(x_i + h) = \frac{1}{2}(\eta(x_i) \pm \sqrt{\eta(x_i)^2 - 2h})$$

$$\eta(0 + \frac{1}{3}) = \frac{1}{2}(\eta(0) \pm \sqrt{\eta(0)^2 - \frac{2}{3}}) = \frac{1}{2}(1 \pm \sqrt{1 - \frac{2}{3}}) = \frac{1}{2} \pm \frac{1}{2}\frac{1}{\sqrt{3}}\frac{\sqrt{3}+1}{2\sqrt{3}}$$

$$\eta(\frac{1}{3} + \frac{1}{3}) = \frac{1}{2}(\eta(\frac{1}{3}) \pm \sqrt{\eta(\frac{1}{3})^2 - \frac{2}{3}})$$

not real!

The given matrix is non negative and stochastic by columns

$$\lambda^3 - \lambda^2(1 - a - b) - \lambda b - a = (\lambda - 1)(\lambda^2 + (a+b)\lambda + a)$$

Eigenvalues:

$$1, \ -\frac{a+b}{2} \pm \frac{\sqrt{(a+b)^2 - 4a}}{2}$$

We know that their absolute value is less than or equal to 1. Question: when is it equal to 1?

Assume they are real. Then question becomes:

$$-\frac{a+b}{2} + \frac{\sqrt{(a+b)^2 - 4a}}{2} = 1$$

$$-\frac{a+b}{2}-\frac{\sqrt{(a+b)^2-4a}}{2}=-1$$

Assume they are not real. Then they can be rewritten as follows:

$$-\frac{a+b}{2}\pm\mathbf{i}\frac{\sqrt{4a-(a+b)^2}}{2}$$

Thus, question becomes:

$$\frac{(a+b)^2}{4}+\frac{4a-(a+b)^2}{4}=1$$

equality which is satisfied iff $a=1$

An equivalent definition of Bernoulli polynomials

The degree n Bernoulli polynomial $B_n(x)$ is uniquely determined by the conditions

$$B_n(x+1)-B_n(x)=nx^{n-1}, \quad \int_0^1 B_n(x)\,dx=0. \tag{1}$$

Note that the first condition in (1) implies:

$$\int_t^{t+1} B_n(x+1)dx-\int_t^{t+1} B_n(x)dx=n\left[\frac{x^n}{n}\right]_t^{t+1},$$

$$\int_{t+1}^{t+2} B_n(y)dy-\int_t^{t+1} B_n(x)dx=(t+1)^n-t^n.$$

By writing the latter identity for $t=0,1,\dots,x-1$, taking into account the second condition in (1), and summing, we obtain:

$$\int_x^{x+1} B_n(y)dy=x^n,\ \forall\, x\in\mathbb{R}. \tag{2}$$

So, (1) implies (2). Of course, (2) implies the second condition in (1) (choose $x=0$). It can be shown that (2) implies also that B_n must be a polynomial of degree at least n and must satisfy the first condition in (1).

Assume that we know that (2) implies that B_n must be a polynomial. Let us show that then its degree is at least n. If, on the contrary, $B_n(y)=a_0y^{n-1}+\dots$ then $\int_x^{x+1} B_n(y)dy=[\frac{a_0}{n}y^n+\dots]_x^{x+1}=\frac{a_0}{n}[(x+1)^n-x^n]+\dots$ is a degree $n-1$ polynomial, and thus cannot be equal to x^n.

Finally, the fact that (2) implies the first condition in (1) can be shown by deriving (2) with respect to x, and remembering the rule:

$$\frac{d}{dx}\int_{f(x)}^{g(x)} h(x,y)dy=h(x,g(x))g'(x)-h(x,f(x))f'(x)+\int_{f(x)}^{g(x)}\frac{\partial}{\partial x}h(x,y)\,dy.$$