August 21, 2009: I succeed in proving a thing I have believed: $\sqrt{2(n+1)} \left[ \sin \frac{\pi ij}{n+1} \right]$

is unitary!

Consider the Fourier matrix of order $2(n+1)$:

$$F_{2(n+1)} = \frac{1}{\sqrt{2(n+1)}} \left[ e^{ij \frac{\pi}{n+1}} \right]^{2(n+1)-1}, \quad \omega_{2(n+1)} = e^{-i \frac{2\pi}{2(n+1)}} = e^{-i \frac{\pi}{n+1}}.$$ 

Note that, if $o_n = \sqrt{2/(n+1)}$, then

$$F_{2(n+1)} = \frac{1}{2} (C - i S), \quad c_{ij} = o_n \cos \frac{ij \pi}{n+1}, \quad s_{ij} = o_n \sin \frac{ij \pi}{n+1}, \quad i, j = 0, \ldots, 2(n+1) - 1.$$ 

Since $S$ and $C$ are real symmetric matrices, we have

$$I = F_{2(n+1)}^* F_{2(n+1)} = \frac{1}{2} (C + i S) \frac{1}{2} (C - i S) = \frac{1}{4} [(C^2 + S^2) + i (SC - CS)],$$

$$Q = F_{2(n+1)}^2 = \frac{1}{2} (C - i S) \frac{1}{2} (C - i S) = \frac{1}{4} [(C^2 - S^2) - i (CS + SC)],$$

being

$$Q = \begin{bmatrix} 1 & J \\ J & 1 \end{bmatrix}, \quad J n \times n \text{ counter-identity.}$$

As a consequence

$$C^2 + S^2 = 4I,$$

$$C^2 - S^2 = 4Q \Rightarrow S^2 = 2(I - Q) = 2 \begin{bmatrix} 0 & I & -J \\ -J & 0 & I \end{bmatrix}.$$ 

Now let $S_{11}, S_{12}, S_{22}$ be the $n \times n$ matrices defined by the equality

$$S = \begin{bmatrix} 0 & S_{11} & S_{12} \\ S_{12}^T & 0 & S_{22} \end{bmatrix},$$

that is,

$$(S_{11})_{rs} = o_n \sin \frac{\pi (1 - r s)}{n+1}, \quad (S_{12})_{rs} = o_n \sin \frac{\pi (1 + r s)}{n+1},$$

$$(S_{22})_{rs} = o_n \sin \frac{\pi (1 + r s)(n+1)}{n+1}, \quad 1 \leq r, s \leq n.$$ 

Observe that $S_{11}$ and $S_{22}$ are real symmetric and related by the identity $S_{22} = JS_{11}^T J$; moreover $S_{12}$ is persymmetric, i.e. $S_{12} = JS_{12}^T$. (Recall that $S_{11}$ is the (sine) transform diagonalizing the algebra $\tau$ of all polynomials in

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \ddots & 1 \\ \vdots & \ddots & 1 \\ 1 & \cdots & 1 \end{bmatrix}$$

1
Since
\[
S^2 = \begin{bmatrix}
0 & S_{12}^2 + S_{12}^T S_{12}^T & S_{11} S_{12} + S_{12} S_{22} \\
S_{12}^T S_{11} + S_{22} S_{12}^T & 0 & S_{12}^T S_{12} + S_{22} S_{22}
\end{bmatrix},
\]
we obtain four identities which in fact reduce to the following only two:
\[
S_{11}^2 + S_{12}^T S_{12}^T = 2I, \quad S_{11} S_{12} + S_{12} J S_{11} = -2I. \tag{1}
\]
The sum of them yields
\[
0 = S_{11} (S_{11} + S_{12} J) + S_{12} J (S_{11} + S_{12} J) = (S_{11} + S_{12} J)^2,
\]
but this can happen only if
\[
S_{11} + S_{12} J = 0, \quad S_{12} = -S_{11} J \tag{2}
\]
(a real symmetric matrix with all the eigenvalues equal to 0 must be null).

Now we are near the thesis. In fact, by (2) the first identity in (1) becomes
\[
2I = S_{11}^2 + (-S_{11} J) (-S_{11} J)^T = 2S_{11}^2, \text{ and so } S_{11}^2 = I.
\]

Remark. From the equality \( F_{2(n+1)} = \frac{1}{i}(C - iS) \) it follows that \( S = i(F_{2(n+1)} - F_{2(n+1)}^*) = i(I - Q)F_{2(n+1)}. \) So, the sine transform of \( z \times 1 \), \( S_{11} z \), can be computed via a discrete Fourier transform of order \( 2(n+1) \):
\[
i(I - Q)F_{2(n+1)} \begin{bmatrix}
0 \\
z \\
0
\end{bmatrix} = \begin{bmatrix}
0 & S_{11} & -S_{11} J \\
S_{11} & 0 & 0 \\
-S_{11} J & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 \\
z \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
S_{11} z \\
0
\end{bmatrix}.
\]

\( \square \) Investigate the four submatrices of \( C \), perhaps they also can be expressed in terms of only one and this one is a transform diagonalizing some algebra of matrices . . .

The matrix
\[
A = \begin{bmatrix}
3 & 2 \\
1 & 2
\end{bmatrix}
\]
does not satisfy the equation \( A^* A = A A^* \), thus there is no unitary matrix diagonalizing \( A \). However, \( T^{-1} A T \) is diagonal for a suitable \( T \):
\[
D^{-1} A D = \begin{bmatrix}
\frac{3}{\sqrt{2}} & 0 \\
\sqrt{2} & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
\frac{\sqrt{2}}{\sqrt{3}} & 0 \\
0 & 1
\end{bmatrix},
\]
\[
\begin{bmatrix}
\alpha & -\beta \\
\beta & \alpha
\end{bmatrix} \begin{bmatrix}
0 & \sqrt{2} \\
\sqrt{2} & 0
\end{bmatrix} \begin{bmatrix}
\alpha & -\beta \\
\beta & \alpha
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 4
\end{bmatrix}, \quad \alpha = \frac{1}{\sqrt{3}}, \quad \beta = \frac{\sqrt{2}}{\sqrt{3}},
\]
\[
T = \frac{1}{\sqrt{3}} \begin{bmatrix}
\sqrt{2} & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\frac{\sqrt{2}}{\sqrt{3}} & 0 \\
0 & 1
\end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix}
\frac{\sqrt{2}}{\sqrt{3}} & 2 \\
-\frac{\sqrt{2}}{\sqrt{3}} & 1
\end{bmatrix}.
\]
The condition number of \( T \) (in the 2-norm), \( \mu_2(T) = \|T\|_2 \|T^{-1}\|_2 \), is greater than 1:
\[
T^* T = \frac{1}{3} \begin{bmatrix}
4 & \sqrt{2} \\
\sqrt{2} & 5
\end{bmatrix} \Rightarrow \|T\|_2 = \sqrt{\rho(T^* T)} = \sqrt{2},
\]
\[2\]
\[ T^{-1} = \frac{\sqrt{3}}{3\sqrt{2}} \begin{bmatrix} 1 & -2 \\ \sqrt{2} & \sqrt{2} \end{bmatrix}, \quad (T^{-1})^*(T^{-1}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \|T^{-1}\|_2 = 1. \]

So, \( \mu_2(T) = \sqrt{3}. \) Since \( \|T\|_\infty = \frac{\sqrt{3} + 2}{\sqrt{2}}, \) \( \|T^{-1}\|_\infty = \frac{\sqrt{3}}{\sqrt{2}}, \) we have \( \mu_\infty(T) = 1 + \sqrt{2}. \)

Can a non-unitary matrix \( T \) have condition number equal to 1?

If yes, then, by the Bauer-Fike theorem, the eigenvalue problem would be optimally conditioned for a class of matrices \( A \) larger than normal (the \( A \) diagonalized by \( T \), \( \mu_2(T) = 1 \)).

A \( n \times n \) matrix \( A \) is said reducible if there exists \( \mathcal{I} \subset \mathcal{N} = \{1, 2, \ldots, n\} \), \( \mathcal{I} \neq \emptyset, \mathcal{N} \), such that \( a_{ik} = 0 \) for all \( i \in \mathcal{I}, k \in \mathcal{N} \setminus \mathcal{I} \). Equivalently, \( A \) is reducible if there exists a permutation matrix \( P \) such that

\[
P^T A P = \begin{bmatrix} \square_{n-I} & * \\ 0 & \sqcup_k \end{bmatrix}, \quad \square_k \text{ } k \times k \text{ matrices, } i \neq 0, n
\]

\( (i = |\mathcal{I}|, n - i = |\mathcal{N} \setminus \mathcal{I}|). \)

Set

\[
C_i = \{ z \in \mathbb{C} : |z - a_{ii}| < \sum_{j=1, j \neq i}^n |a_{ij}| \}.
\]

It is well known that the subset \( \cup_{i=1}^n C_i \) of \( \mathbb{C} \) includes all the eigenvalues of \( A \) (Gershgorin first theorem).

If \( A \) is not reducible then we can say something more:

If \( A \) is a irreducible \( n \times n \) matrix and \( C_i \) are the inner parts of the Gershgorin disks, then the set \( (\cup_{i=1}^n C_i) \cup (\cap_{i=1}^n \partial C_i) \) includes all the eigenvalues of \( A \).

Proof. If \( \lambda \) is an eigenvalue of \( A \), then \( \sum_j a_{ij} x_j = \lambda x_i \), \( \sum_{j,j \neq i} a_{ij} x_j = (\lambda - a_{ii}) x_i \),

\[ |\lambda - a_{ii}| |x_i| \leq \sum_{j,j \neq i} |a_{ij}| |x_j|, \quad \forall \ i. \]

Set \( \mathcal{I} = \{ j : |x_j| = \|x\|_\infty \} \). Assume \( \mathcal{I} \neq \mathcal{N} \) and let \( i \in \mathcal{I} \), \( k \in \mathcal{N} \setminus \mathcal{I} \) such that \( a_{ik} \neq 0 \). Then

\[ |\lambda - a_{ii}| |x_i| \leq \sum_{j,j \neq i} |a_{ij}| |x_j| \]

\[ = \sum_{j \in \mathcal{I}, j \neq i} |a_{ij}| |x_j| + |a_{ik}| |x_k| + \sum_{j \in \mathcal{N} \setminus \mathcal{I}, j \neq k} |a_{ij}| |x_j| \]

\[ < \sum_{j \in \mathcal{I}_j \neq i} |a_{ij}| |x_j| + |a_{ik}| |x_k| + \sum_{j \in \mathcal{N} \setminus \mathcal{I}, j \neq k} |a_{ij}| |x_i| \]

\[ = \sum_{j,j \neq i} |a_{ij}| |x_i|, \]

\[ |\lambda - a_{ii}| < \sum_{j,j \neq i} |a_{ij}|, \text{ i.e. } \lambda \in C_i. \]

Assume now \( \mathcal{I} = \mathcal{N} \), that is all entries of the eigenvector \( x \) have the same absolute value. In this case:

\[ |\lambda - a_{ii}| |x_i| \leq \sum_{j,j \neq i} |a_{ij}| |x_j| = \sum_{j,j \neq i} |a_{ij}| |x_i|, \quad \forall \ i, \]

\[ |\lambda - a_{ii}| \leq \sum_{j,j \neq i} |a_{ij}|, \forall \ i, \text{ therefore either } \lambda \in C_s \text{ for some } s \text{ or } \lambda \in \partial C_i \forall \ i. \]

\[ \Box \] Use the result obtained to prove that any irreducible weakly diagonal dominant \( n \times n \) matrix \( A \) is non singular

\[ \Box \rho(A) \leq \|A\|_\infty. \]
By the Gershgorin first theorem, for any eigenvalue $\lambda$ of $A$ there exists $i$ such that $|\lambda| = |\lambda - a_{ii} + a_{ii}| \leq |\lambda - a_{ii}| + |a_{ii}| \leq \sum_j |a_{ij}| \leq \|A\|_{\infty}$.

\[ \square \]

If $A$ is irreducible and $\sum_j |a_{sj}| < \|A\|_{\infty}$ for some $s$, then $\rho(A) < \|A\|_{\infty}$.

Given an eigenvalue $\lambda$ of $\hat{A}$, the Gershgorin first theorem for irreducible matrices implies either $\exists i \mid |\lambda| = |\lambda - a_{ii} + a_{ii}| \leq |\lambda - a_{ii}| + |a_{ii}| \leq \sum_j |a_{ij}| \leq \|A\|_{\infty}$ or $|\lambda| = |\lambda - a_{ii} + a_{ii}| \leq |\lambda - a_{ii}| + |a_{ii}| = \sum_j |a_{ij}|, \forall i$, also for $i = s$, for which we know that $\sum_j |a_{sj}| < \|A\|_{\infty}$.

(Jacobi method is able to solve linear systems $Ax = b$ with $A$ weakly diagonal dominant because in this case the Jacobi iteration matrix $J$ satisfies the conditions $\exists s \mid \sum_j |J_{sj}| < \|J\|_{\infty}$ and $\|J\|_{\infty} = 1$, thus, by the result of the Exercise, $\rho(J) < 1$.

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**Proof of the existence of the SVD of $A \in \mathbb{C}^{n \times n}$**

$A \in \mathbb{C}^{n \times n} \Rightarrow \exists U, \sigma, V, U, V$ unitary, $\sigma = \text{diag}(|\sigma_i|)$ with $|\sigma_1| \geq |\sigma_2| \geq \cdots \geq |\sigma_n|$ such that $A = U\sigma V^*$.

**Proof.** Let $v_1, \|v_1\|_2 = 1$, be such that $\|A\|_2 = \|A v_1\|_2$ and set $u_1 = A v_1/\|A v_1\|_2$ and set $u_1 = A v_1/\|A v_1\|_2 \|u_1\|_2 = 1$ and $A v_1 = \|A\|_2 u_1$. Let $\tilde{u}_i, \tilde{v}_i \in \mathbb{C}^n$ be such that $U = [u_1 | \tilde{u}_2 | \cdots | \tilde{u}_n]$ and $V = [v_1 | \tilde{v}_2 | \cdots | \tilde{v}_n]$ are unitary. Then

\[
U^* A V = \begin{bmatrix} u_1^* & \tilde{u}_2^* & \cdots & \tilde{u}_n^* \\
\end{bmatrix} A [v_1 | \tilde{v}_2 | \cdots | \tilde{v}_n] = \begin{bmatrix} u_1^* & \tilde{u}_2^* & \cdots & \tilde{u}_n^* \\
\end{bmatrix} \begin{bmatrix} \|A\|_2 u_1 | A \tilde{v}_2 | \cdots | A \tilde{v}_n \end{bmatrix} = \begin{bmatrix} \|A\|_2 & w^* \\
0 & \hat{A} \end{bmatrix}.
\]

\[ ||A||_2 = \|U^* A V\|_2 = \sup_{v \neq 0} \begin{bmatrix} ||A||_2 & w^* \\
0 & \hat{A} \end{bmatrix} v = \begin{bmatrix} ||A||_2 & w^* \\
0 & \hat{A} \end{bmatrix} v = \frac{\|A\|_2}{\|w\|_2} \|w\|_2 \geq \frac{\|A\|_2^2 + \|w\|_2^2}{\sqrt{\|A\|_2^2 + \|w\|_2^2}} = \sqrt{\|A\|_2^2 + \|w\|_2^2} \]

$\Rightarrow \ w = 0$, $\Rightarrow$

\[ ||A||_2 = \|U^* A V\|_2 = \sup_{v \neq 0} \begin{bmatrix} ||A||_2 & w^* \\
0 & \hat{A} \end{bmatrix} v = \begin{bmatrix} ||A||_2 & w^* \\
0 & \hat{A} \end{bmatrix} v = \begin{bmatrix} ||A||_2 & w^* \\
0 & \hat{A} \end{bmatrix} v = \begin{bmatrix} ||A||_2 & \hat{A} \\
0 & \hat{A} \end{bmatrix} v = \|A\|_2 \]

$\Rightarrow \ U^* A V = \begin{bmatrix} ||A||_2 & \hat{A} \\
0 & \hat{A} \end{bmatrix}$ with $\hat{A}$ such that $\|\hat{A}\|_2 \leq ||A||_2$.

The thesis follows if we assume it true for matrices of order $n - 1$.

**On SVD: best rank-$r$ approximation of $A$.**

A $n \times n$, $A = U\sigma V^* = \sum_{r}^{n} \sigma_i u_i v_i^*$, $A_r = \sum_{r}^{r} \sigma_i u_i v_i^*$ $\Rightarrow$

\[ \min\{\|A - B\|_2 : \text{rank}(B) \leq r\} = \|A - A_r\|_2 = \sigma_{r+1} \]
Proof. Let $B$ be a $n \times n$ matrix with complex entries whose rank is no more than $r$ and set $L = \{ v : \ Bv = 0 \}$. Observe that

$$
\| A - B \|_2 = \sup_{v} \frac{\| (A - B)v \|_2}{\| v \|_2} \geq \sup_{v \in L} \frac{\| Av \|_2}{\| v \|_2}.
$$

Set $M = \text{Span} \{ v_1, v_2, \ldots, v_{r+1} \}$. Since $\dim M + \dim L \geq n + 1$, there exists $z \neq 0, z \in M \cap L$, 

$$
\| A - B \|_2 \geq \frac{\| Az \|_2}{\| z \|_2} \geq \sigma_{r+1}
$$

(first: $z \in L$; second: $z \in M \Rightarrow z = \sum_{i=1}^{r+1} \alpha_i v_i \Rightarrow Az = \sum_{i=1}^{r+1} \alpha_i \sigma_i u_i$).

Moreover,

$$
\| A - A_r \|_2 = \| U \text{ diag} (0, \ldots, 0, \sigma_{r+1}, \ldots, \sigma_n) V^* \|_2 = \| \text{ diag} (\ldots) \|_2 = \sigma_{r+1}
$$

and $\text{rank}(A_r) \leq r$.

Remark. We also have:

$$
\min \{ \| A - B \|_F : \text{ rank}(B) \leq r \} = \| A - A_r \|_F = \sqrt{\sum_{j=r+1}^{n} \sigma_j^2}
$$

In functional analysis for compact operators . . . (linear banded operators on Hilbert spaces) use $*$ as a definition of singular values, approximate an object with something of finite dimension

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**On SVD: kernel and image of $A$.**

A $n \times n$, $A = U \sigma V^*$, $\sigma_1 \geq \ldots \geq \sigma_k > 0 = \sigma_{k+1} = \ldots = \sigma_n \Rightarrow$

(1) $\{ x \in \mathbb{C}^n : A x = 0 \} = \text{Span} \{ v_{k+1}, \ldots, v_n \}$

(2) $\{ A x : x \in \mathbb{C}^n \} = \text{Span} \{ u_1, \ldots, u_k \}$

(3) $\text{rank}(A) = k = \# \{ \sigma_i : \sigma_i > 0 \}$

**Proof.** (1): $A x = 0$ iff $\sigma V^* x = 0$ iff $S_k V_k^* x = 0$,

$$
S_k = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{bmatrix}, \quad V_k = \begin{bmatrix} v_1^* \\ \vdots \\ v_k^* \end{bmatrix},
$$

iff $V_k^* x = 0$ iff $x$ is orthogonal to $v_1, \ldots, v_k$ iff $x$ is a linear combination of $v_{k+1}, \ldots, v_n$.

(2):

$$
A x = [U_k \square] \begin{bmatrix} S_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_k^* \\ \square \end{bmatrix} = U_k (S_k V_k^* x), \quad U_k = [u_1 \cdots u_k]
$$

$\Rightarrow A x \in \text{Span} \{ u_1, \ldots, u_k \} \Rightarrow \{ A x : x \in \mathbb{C}^n \} \subset \text{Span} \{ u_1, \ldots, u_k \}$. Now let us show that for any $z \in \mathbb{C}^k$ there exists $x \in \mathbb{C}^n$, $U_k z = A x$:

$$
\exists x | A x = U_k z \text{ iff } \exists x | U_k S_k V_k^* x = U_k z \text{ iff } \exists x | S_k V_k^* x = z \text{ iff } \exists x | V_k^* x = S_k^{-1} z.
$$
Since $\text{rank}(V_k^*) = k$, the latter system admits solution.

**On SVD: exercises**

\[ A = \frac{1}{81} \begin{bmatrix} -65 & 76 & 104 \\ 76 & -206 & 8 \\ 104 & 8 & 109 \end{bmatrix} = U D U^*, \]

\[ U = \frac{1}{9} \begin{bmatrix} -4 & 4 & 7 \\ 8 & 1 & 4 \\ 1 & 8 & -4 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}. \]

Write the SVD of $A$.

\[ \lambda_i \text{ eigenvalues of } A \Rightarrow |\lambda_i| \leq |\sigma_1|, \]

\[ (Ax = \lambda x, A = U \sigma V^* \Rightarrow y^* \sigma^2 y = x^* A^* A x = |\lambda|^2 \|x\|^2 \ldots). \]

**On SVD: how to compute the rank of a matrix, Gram-Schmidt vs SVD**

Let $a_1, a_2, \ldots, a_m, \ldots$ be a sequence of non null $n \times 1$ vectors and set $A_m = [a_1 \ a_2 \ \ldots \ \ a_m], m = 1, 2, \ldots$. There follows an algorithm which computes matrices $Q_m = [q_1 \ q_2 \ \ldots \ q_m], n \times m$, and $R_m$, upper triangular $m \times m$, such that

1. $A_m = Q_m R_m, m = 1, 2, \ldots$
2. $\{q_1\} \cup \{q_k : 2 \leq k \leq m, a_k \notin \text{Span}\{a_1, \ldots, a_{k-1}\}\}$ is an orthonormal basis of the space $\text{Span}\{a_1, \ldots, a_m\}$
3. if $a_k, 2 \leq k \leq m$ is linearly dependent from $a_1, \ldots, a_{k-1}$, then the $k$-row of $R_m$ is null and $q_k$ can be chosen arbitrarily (for instance, $q_k = 0$ or such that $Q_m^* Q_m = I$)
4. The rank of $A_m$ is the number of non null rows of $R_m$

Set $\hat{q}_1 = a_1$ and $q_1 = \hat{q}_1 / \|\hat{q}_1\|_2$. Then $a_1 = \|\hat{q}_1\|_2 q_1$, i.e.

\[ [a_1] = [q_1] [\|\hat{q}_1\|_2]. \]

Set $q_2 = a_2 - r_{12} q_1$, $r_{12}$ such that $\hat{q}_1^* q_2 = 0$ ($r_{12} = q_1^* a_2$) and, if $q_2 \neq 0$, $q_2 = q_2 / \|q_2\|_2$. Then $a_2 = r_{12} q_1 + \|q_2\|_2 q_2$, i.e.

\[ [a_1 \ a_2] = [q_1 \ q_2] \begin{bmatrix} \|q_1\|_2 & r_{12} \\ 0 & \|q_2\|_2 \end{bmatrix}. \]

Else, if $q_2 = 0$, or, equivalently, $a_2 = r_{12} q_1 \in \text{Span}\{a_1\}$, we can write

\[ [a_1 \ a_2] = [q_1 \ q_2] \begin{bmatrix} \|q_1\|_2 & r_{12} \\ 0 & 0 \end{bmatrix}, \quad q_2 := \hat{q}_2 = 0 \text{ or arbitrary}. \]

Assume that the first case occurs. Set $q_3 = a_3 - r_{13} q_1 - r_{23} q_2$, $r_{13}, r_{23}$ such that $q_1^* q_3 = q_2^* q_3 = 0$ ($r_{13} = q_1^* a_3, r_{23} = q_2^* a_3$) and assume $q_3 = 0$, or, equivalently,
\[ a_3 = r_{13} q_1 + r_{23} q_2 \in \text{Span} \{ a_1, a_2 \} \]. Then we can write:

\[
\begin{bmatrix}
  a_1 & a_2 & a_3 \\
\end{bmatrix} = \begin{bmatrix}
  q_1 & q_2 & q_3 \\
\end{bmatrix} \begin{bmatrix}
  \|q_1\|_2 & r_{12} & r_{13} \\
  0 & \|q_2\|_2 & r_{23} \\
  0 & 0 & 0 \\
\end{bmatrix},
\]

\[ q_3 := \hat{q}_3 = 0 \text{ or arbitrary.} \]

Set \( \hat{q}_4 = a_4 - r_{14} q_1 - r_{24} q_2, r_{14}, r_{24} \) such that \( q_4^* \hat{q}_4 = q_2^* \hat{q}_4 = 0 \) \( (r_{14} = q_4^* a_4, r_{24} = q_2^* a_4) \) and assume \( q_4 = 0 \), or, equivalently, \( a_4 = r_{14} q_1 + r_{24} q_2 \in \text{Span} \{ a_1, a_2 \} \). Then we can write:

\[
\begin{bmatrix}
  a_1 & a_2 & a_3 & a_4 \\
\end{bmatrix} = \begin{bmatrix}
  q_1 & q_2 & q_3 & q_4 \\
\end{bmatrix} \begin{bmatrix}
  \|q_1\|_2 & r_{12} & r_{13} & r_{14} \\
  0 & \|q_2\|_2 & r_{23} & r_{24} \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[ q_3 := \hat{q}_3 = 0, q_4 := \hat{q}_4 = 0 \text{ or arbitrary.} \]

Set \( \hat{q}_5 = a_5 - r_{15} q_1 - r_{25} q_2, r_{15}, r_{25} \) such that \( q_5^* \hat{q}_5 = q_2^* \hat{q}_5 = 0 \) \( (r_{15} = q_5^* a_5, r_{25} = q_2^* a_5) \) and assume \( q_5 \neq 0 \). Set \( q_5 = q_5/\|q_5\|_2 \). Then \( a_5 = r_{15} q_1 + r_{25} q_2 + \|q_5\|_2 q_5 \), i.e.

\[
\begin{bmatrix}
  a_1 & a_2 & a_3 & a_4 & a_5 \\
\end{bmatrix} = \begin{bmatrix}
  q_1 & q_2 & q_3 & q_4 & q_5 \\
\end{bmatrix} \begin{bmatrix}
  \|q_1\|_2 & r_{12} & r_{13} & r_{14} & r_{15} \\
  0 & \|q_2\|_2 & r_{23} & r_{24} & r_{25} \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & \|q_5\|_2 \\
\end{bmatrix},
\]

\[ q_3, q_4 \text{ null or arbitrary.} \]

Remark. Since the calculator uses finite arithmetic, the check if \( \hat{q}_k \), \( k \geq 2 \), is zero or nonzero must be replaced with something of type: \( \|\hat{q}_k\| \) is less than \( \varepsilon \) or not? Moreover, take into account that even a very little perturbation in one entry of a triangular matrix can change the value of its rank (see the following example). These facts imply that the (Gram-Schmidt) algorithm illustrated above may generate a numeric rank of \( A_m \) which is different from the rank of \( A_m \).

Example. Let \( R \) be the \( n \times n \) upper triangular matrix

\[
R = \begin{bmatrix}
  1 & -1 & -1 & \cdots & -1 \\
  0 & 1 & -1 & \cdots & -1 \\
  \vdots & \ddots & 1 & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & -1 \\
  0 & \cdots & \cdots & 0 & 1 \\
\end{bmatrix}.
\]

The rank of \( R \) is \( n \), but if the 0 in the \((n, 1)\) entry is replaced with \(-2^{2-n}\) (which for large \( n \) is a very little perturbation), then the rank of \( R \) becomes \( n - 1 \). The SVD of \( R \) predicts this observation. In fact, the singular value \( \sigma_{n-1} \) of \( R \) for \( n = 5, 10, 15 \) has more or less the same value, 1.5, whereas the smallest singular value, \( \sigma_n \), seems to tend to zero:

\[
n = 5: \ \sigma_5 \approx \frac{1}{10}, \quad n = 10: \ \sigma_{10} \approx \frac{1}{100}, \quad n = 15: \ \sigma_{15} \approx \frac{1}{10000}.
\]
So, by examining the singular values of $R$ we see that even if $\det(R) = 1$ (far from zero) for all $n$, greater is $n$, smaller is the distance of $R$ from a singular matrix. (Note that $R$ is not normal, in fact $\mu_2(R) = \sigma_1/\sigma_n \approx 30,200,10^5 > 1 = \max |\lambda_i|/ \min |\lambda_i|$).

It is known that small perturbations on the entries of $A$ imply at most small perturbations on $U, \sigma, V$, $A = U\sigma V^*$ (SVD problem is well conditioned). It follows that the algorithm for the computation of the SVD of $A$ can give accurate approximations of $U, \sigma, V$. Having an accurate approximation of $\sigma$ we can evaluate precisely the rank of $A$; we can even quantify how much $A$ is far from having a smaller rank. Thus it is preferable to compute the rank of a matrix via SVD, instead via Gram-Schmidt.