An example of preconditioning

Let A and E be the $n \times n$ matrices

$$A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}.$$

We have

$$\tilde{A} = E^{-1}AE^{-T} = I + \mathbf{e}\mathbf{e}^T, \quad \mathbf{e} = [1\ 1\ \ldots\ 1]^T.$$

The eigenvalues of the matrix $\tilde{A}$ are: 1 $n-1$ times, and $1 + \mathbf{e}^T\mathbf{e} = n+1$. So, the condition number of $\tilde{A}$ (in norm 2) is $n+1$.

Let us compute the condition number of A. The eigenvalues of A are known in explicit form: $2 - 2\cos\frac{j\pi}{n+1}$, $j = 1, \ldots, n$. Thus,

$$\mu_2(A) = \frac{2 - 2\cos\frac{n\pi}{n+1}}{2 - 2\cos\frac{\pi}{n+1}} = \frac{1 + \cos\frac{\pi}{n+1}}{1 - \cos\frac{\pi}{n+1}} = \frac{1 + \cos(2\frac{\pi}{2(n+1)})}{1 - \cos(2\frac{\pi}{2(n+1)})} = \frac{2\cos^2\frac{\pi}{2(n+1)}}{2\sin^2\frac{\pi}{2(n+1)}} = \frac{1}{\operatorname{tg}^2\frac{\pi}{2(n+1)}}.$$

Since $\lim_{n\to+\infty}(\frac{\pi}{2(n+1)})^2/\operatorname{tg}^2\frac{\pi}{2(n+1)} = 1$, we can conclude that $\mu_2(A) = O(n^2)$.

It follows that, in order to solve a system $A\mathbf{x} = \mathbf{b}$ where the coefficient matrix A is as above, it is convenient to apply the linear systems solver at disposal to the equivalent system $E^{-1}AE^{-T}E^T\mathbf{x} = E^{-1}\mathbf{b}$, i.e. compute $\tilde{\mathbf{x}} = E^T\mathbf{x}$.

Proof that a DFT of order n can be reduced to two DFT of order $n/2$

$$\mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix} = W\mathbf{z}$$

($\omega = \omega_n$, $W = W_n$). Then, for $i = 0, \ldots, n-1$,

$$\begin{aligned} y_i &= \sum_{j=0}^{n-1} \omega^{ij} z_j = \sum_{p=0}^{m-1} \omega^{i(2p)} z_{2p} + \sum_{p=0}^{m-1} \omega^{i(2p+1)} z_{2p+1} \\ &= \sum_{p=0}^{m-1} (\omega^2)^{ip} z_{2p} + \omega^i \sum_{p=0}^{m-1} (\omega^2)^{ip} z_{2p+1} = \sum_{p=0}^{m-1} \omega_m^{ip} z_{2p} + \omega_n^i \sum_{p=0}^{m-1} \omega_m^{ip} z_{2p+1}, \end{aligned} \quad (1)$$

($\omega_n^2 = \omega_m$, $m = n/2$) and, for $i = 0, \ldots, m-1$,

$$\begin{aligned} y_{m+i} &= \sum_{p=0}^{m-1} \omega_m^{(m+i)p} z_{2p} + \omega_n^m \omega_n^i \sum_{p=0}^{m-1} \omega_m^{(m+i)p} z_{2p+1} \\ &= \sum_{p=0}^{m-1} \omega_m^{ip} z_{2p} - \omega_n^i \sum_{p=0}^{m-1} \omega_m^{ip} z_{2p+1}. \quad (2) \end{aligned}$$

Formulas (1), $i = 0, \ldots, m-1$, and (2) in matrix form become:

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{bmatrix} = W_m \begin{bmatrix} z_0 \\ z_2 \\ \vdots \\ z_{n-2} \end{bmatrix} + \begin{bmatrix} 1 & & & \\ & \omega_n & & \\ & & \ddots & \\ & & & \omega_n^{m-1} \end{bmatrix} W_m \begin{bmatrix} z_1 \\ z_3 \\ \vdots \\ z_{n-1} \end{bmatrix},$$

$$\begin{bmatrix} y_m \\ y_{m+1} \\ \vdots \\ y_{n-1} \end{bmatrix} = W_m \begin{bmatrix} z_0 \\ z_2 \\ \vdots \\ z_{n-2} \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & \omega_n & & \\ & & \ddots & \\ & & & \omega_n^{m-1} \end{bmatrix} W_m \begin{bmatrix} z_1 \\ z_3 \\ \vdots \\ z_{n-1} \end{bmatrix}.$$

It follows that

$$\mathbf{y} = W_n \mathbf{z} = \begin{bmatrix} W_m & DW_m \\ W_m & -DW_m \end{bmatrix} \begin{bmatrix} z_0 \\ z_2 \\ \vdots \\ z_{n-2} \\ z_1 \\ z_3 \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} W_m & 0 \\ 0 & W_m \end{bmatrix} Q\mathbf{z}$$

where the permutation matrix Q is defined in an obvious way.

The real part of $\lambda(A)) > 0$ vs $\lambda(A_h) > 0$, also for A real

$A \in \mathbb{C}^{n\times n}$, $A_h = \frac{1}{2}(A + A^*)$, $A_{ah} = \frac{1}{2}(A - A^*)$.
Definition: A_h is p.d. iff $\mathbf{z}^* A_h \mathbf{z} > 0$, $\forall\, \mathbf{z} \in \mathbb{C}^n$, $\mathbf{z} \neq \mathbf{0}$.
Then

$$\begin{aligned} \mathbf{z}^* A\mathbf{z} &= \mathbf{z}^* A_h \mathbf{z} + \mathbf{z}^* A_{ah}\mathbf{z}, \\ \mathbf{z}^* A\mathbf{z} &= (\mathbf{z}_R - \mathbf{i}\mathbf{z}_I)^T (A_R + \mathbf{i}A_I)(\mathbf{z}_R + \mathbf{i}\mathbf{z}_I) \\ &= \mathbf{z}_R^T A_R \mathbf{z}_R + \mathbf{z}_I^T A_R \mathbf{z}_I - \mathbf{z}_R^T (A_I - A_I^T)\mathbf{z}_I \\ &\quad + \mathbf{i}[\mathbf{z}_R^T (A_R - A_R^T)\mathbf{z}_I + \mathbf{z}_R^T A_I \mathbf{z}_R + \mathbf{z}_I^T A_I \mathbf{z}_I] \end{aligned}$$

where $\mathbf{z} = \mathbf{z}_R + \mathbf{i}\mathbf{z}_I$, $A = A_R + \mathbf{i}A_I$. Note that $\mathbf{z}^* A_h \mathbf{z}$ is real and $\mathbf{z}^* A_{ah}\mathbf{z}$ is purely immaginary.

Moreover

$$\begin{aligned} & \mathbf{z}_R^T A\mathbf{z}_R + \mathbf{z}_I^T A\mathbf{z}_I \text{ (if } A \text{ is real) } = \\ (\mathbf{z}^* A\mathbf{z})_R &= \mathbf{z}^* A_h \mathbf{z} = (\mathbf{z}_R - \mathbf{i}\mathbf{z}_I)^T [(A_R)_S + \mathbf{i}(A_I)_{AS}](\mathbf{z}_R + \mathbf{i}\mathbf{z}_I) \\ &= \mathbf{z}_R^T (A_R)_S \mathbf{z}_R + \mathbf{z}_I^T (A_R)_S \mathbf{z}_I + 2\mathbf{z}_I^T (A_I)_{AS}\mathbf{z}_R \\ &\quad + \mathbf{i}[\mathbf{z}_R^T (A_I)_{AS}\mathbf{z}_R + \mathbf{z}_I^T (A_I)_{AS}\mathbf{z}_I] \\ &= \text{ (if } A \text{ is real) } \mathbf{z}_R^T A_s \mathbf{z}_R + \mathbf{z}_I^T A_S \mathbf{z}_I, \\ (\mathbf{z}^* A\mathbf{z})_R &= \mathbf{z}_R^T A_R \mathbf{z}_R + \mathbf{z}_I^T A_R \mathbf{z}_I - \mathbf{z}_R^T (A_I - A_I^T)\mathbf{z}_I \\ &= \text{ (if } A \text{ is real) } \mathbf{z}_R^T A\mathbf{z}_R + \mathbf{z}_I^T A\mathbf{z}_I. \end{aligned}$$

Consequences:

1. A_h is p.d. iff $(\mathbf{z}^* A\mathbf{z})_R > 0$, $\forall\, \mathbf{z} \in \mathbb{C}^n$, $\mathbf{z} \neq \mathbf{0}$
2. For any eigenvalue $\lambda(A)$ there exists $\mathbf{z}$, $\|\mathbf{z}\|_2 = 1$, such that $(\lambda(A))_R = \mathbf{z}^* A_h \mathbf{z} \geq \min \lambda(A_h)$ [it is the vector $\mathbf{z}$ in $A\mathbf{z} = \lambda(A)\mathbf{z}$]
3. For any eigenvalue $\lambda(A_h)$ there exists $\mathbf{y}$, $\|\mathbf{y}\|_2 = 1$, such that $\lambda(A_h) = (\mathbf{y}^* A\mathbf{y})_R$ [it is the vector $\mathbf{y}$ in $A_h\mathbf{y} = \lambda(A_h)\mathbf{y}$]
4. Assume A real. Then the following assertions are equivalent

- $A_h = A_S$ is p.d. ($\mathbf{z}^* A_h \mathbf{z} > 0$, $\forall\, \mathbf{z} \in \mathbb{C}^n$, $\mathbf{z} \neq \mathbf{0}$)
- $\xi^T A \xi > 0$, $\forall\, \xi \in \mathbb{R}^n$, $\xi \neq \mathbf{0}$
- $\xi^T A_S \xi > 0$, $\forall\, \xi \in \mathbb{R}^n$, $\xi \neq \mathbf{0}$ ($\xi^T A \xi = \xi^T A_S \xi$ if $\xi \in \mathbb{R}^n$ and A is real)

Further results:
A_h p.d. ($\lambda(A_h) > 0$) $\Rightarrow (\lambda(A))_R > 0$ (consequence of 2.)
$(\lambda(A))_R > 0$ & A normal $\Rightarrow A_h$ p.d.
$(\lambda(A))_R > 0$ does not imply A_h p.d. (see Example with $a^2 \geq 4$)
There exist non normal matrices A with $(\lambda(A))_R > 0$ for which A_h is p.d. (see Example with $0 < a^2 < 4$)
Perhaps $(\lambda(A))_R$ "much" positive would imply $\lambda(A_h) > 0$ (A_h p.d.)
EXAMPLE.

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \quad a \in \mathbb{R},$$

$$[x\ y] \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [x\ xa + y] \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + axy + y^2,$$

$$x^2 + axy + y^2 > 0, \quad \forall\, x,\, y,\ (x, y) \neq (0, 0) \ \text{ iff } \ a^2 < 4$$

i.e. the hermitian part of A is p.d. iff $a^2 < 4$. Also observe that A is normal iff $a = 0$. So, $a \in \mathbb{R}$, $0 < a^2 < 4 \Rightarrow A$ satisfies the coditions: A real, $A_h = A_S$ p.d., A is not normal, $(\lambda(A))_R = \lambda(A) = 1 > 0$.

We know that A_h p.d. implies $\Re(\lambda(A)) > 0$ ... $\Re(\lambda(A)) > 0 \Rightarrow A_h$ p.d. ? If A is normal, yes; otherwise a stronger hypothesis of kind $\Re(\lambda(A)) > q \geq 0$ is sufficient to obtain the p.d. of A_h. The aim is to find a q as small as possible. A question is "q can be zero for a class of not normal matrices ?"

Let A be a generic $n \times n$ matrix. It is known that $AX = XT$, with $X = [\mathbf{x}_1\ \mathbf{x}_2\ \ldots\ \mathbf{x}_n]$ unitary and T upper triangular

$$T = \begin{bmatrix} \lambda_1 & t_{12} & \cdots & t_{1n} \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{n-1n} \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

with the eigenvalues of A as diagonal entries (Schur theorem). Equivalently, we have

$$A\mathbf{x}_j = \lambda_j \mathbf{x}_j + t_{1j}\mathbf{x}_1 + \ldots + t_{j-1j}\mathbf{x}_{j-1}, \quad j = 1, \ldots, n.$$

Now let $\lambda(A_h)$ be a generic eigenvalue of $A_h = \frac{1}{2}(A + A^*)$, the hermitian part of A. Then there exists $\mathbf{y} \neq \mathbf{0}$ such that

$$\lambda(A_h) = \frac{\mathbf{y}^* A_h \mathbf{y}}{\mathbf{y}^* \mathbf{y}} = \frac{\mathbf{y}^* A \mathbf{y}}{\mathbf{y}^* \mathbf{y}} - \frac{\mathbf{y}^* A_{ah} \mathbf{y}}{\mathbf{y}^* \mathbf{y}}$$

and, since $\lambda(A_h)$ is real and $\mathbf{y}^* A_{ah} \mathbf{y}$ purely immaginary, we have the formula:

$$\lambda(A_h) = \frac{\Re(\mathbf{y}^* A \mathbf{y})}{\mathbf{y}^* \mathbf{y}}. \tag{1}$$

Let us obtain, using (1), an expression of $\lambda(A_h)$ in terms of the eigenvalues λ_i of A. There exist $\alpha_i \in \mathbb{C}$ for which $\mathbf{y} = \sum_i \alpha_i \mathbf{x}_i$ (recall that $\mathbf{y}$ is an eigenvector of the $\lambda(A_h)$ we are considering), thus $\mathbf{y}^*\mathbf{y} = \sum_i |\alpha_i|^2$ and

$$\begin{aligned}
\mathbf{y}^* A\mathbf{y} &= \sum_i \overline{\alpha_i}\mathbf{x}_i^* \sum_j \alpha_j A\mathbf{x}_j \\
&= \sum_i \overline{\alpha_i}\mathbf{x}_i^* \sum_j \alpha_j(\lambda_j \mathbf{x}_j + \sum_{k=1}^{j-1} t_{kj}\mathbf{x}_k) \\
&= \sum_i |\alpha_i|^2 \lambda_i + \sum_i \overline{\alpha_i}\mathbf{x}_i^* \sum_{j=2}^{n} \alpha_j (\sum_{k=1}^{j-1} t_{kj}\mathbf{x}_k) \\
&= \sum_i |\alpha_i|^2 \lambda_i + \sum_i \overline{\alpha_i}\mathbf{x}_i^* \sum_{k=1}^{n-1} (\sum_{j=k+1}^{n} \alpha_j t_{kj})\mathbf{x}_k \\
&= \sum_i |\alpha_i|^2 \lambda_i + f(\{\alpha_i\}_{i=1}^n, \{t_{ij}\}_{i<j})
\end{aligned}$$

where

$$f(\{\alpha_i\}_{i=1}^n, \{t_{ij}\}_{i<j}) = \sum_{i=1}^n \overline{\alpha_i} \sum_{j=i+1}^n \alpha_j t_{ij} = \sum_{j=1}^n \alpha_j \sum_{i=1}^{j-1} \overline{\alpha_i} t_{ij}.$$

It follows that

$$\lambda(A_h) = \frac{\sum_i |\alpha_i|^2 \Re(\lambda_i)}{\sum_i |\alpha_i|^2} + \frac{\Re(f(\{\alpha_i\}_{i=1}^n, \{t_{ij}\}_{i<j}))}{\sum_i |\alpha_i|^2}. \tag{2}$$

Remark. Since $AX = XT$ implies $A_h X = XT_h$, $T_h == \frac{1}{2}(T + T^*)$, we have:

$$\begin{aligned}
&\min_{\alpha^{(k)},\, X\alpha^{(k)}\ indip.\ eigenvectors\ of\ A_h} \frac{\Re(f(\{\alpha_i\}_{i=1}^n, \{t_{ij}\}_{i<j}))}{\sum_i |\alpha_i|^2} \\
&= \min_{\alpha^{(k)},\, \alpha^{(k)}\ indip.\ eigenvectors\ of\ T_h} \frac{\Re(f(\{\alpha_i\}_{i=1}^n, \{t_{ij}\}_{i<j}))}{\sum_i |\alpha_i|^2}.
\end{aligned}$$

Let us see some consequences of (2):

1 If $t_{ij} = 0\ \forall\, i < j$ (i.e. if A is normal), then

$$\min \Re(\lambda_i) \le \lambda(A_h) = \frac{\sum_i |\alpha_i|^2 \Re(\lambda_i)}{\sum_i |\alpha_i|^2} \le \max \Re(\lambda_i).$$

So, if A is normal and $\Re(\lambda(A)) > 0$, then A_h is p.d. (all eigenvalues of A_h are positive)

2 Since $|\Re(f)| \le |f|$, in order to obtain bounds for $\Re(f)/\sum_i |\alpha_i|^2$ in (2) we look for bounds for $|f|$:

$$\begin{aligned}
|f| &= |\sum_{i=1}^n \overline{\alpha_i} \sum_{j=i+1}^n \alpha_j t_{ij}| \le \sum_{i=1}^n |\alpha_i| \sum_{j=i+1}^n |\alpha_j||t_{ij}| \\
&\le \sqrt{\sum_{i=1}^n |\alpha_i|^2} \sqrt{\sum_{i=1}^n (\sum_{j=i+1}^n |\alpha_j||t_{ij}|)^2} \\
&\le \sqrt{\sum_{i=1}^n |\alpha_i|^2} \sqrt{\sum_{i=1}^n (\sum_{j=i+1}^n |\alpha_j|^2)(\sum_{j=i+1}^n |t_{ij}|^2} \\
&\le \sqrt{\sum_{i=1}^n |\alpha_i|^2} \sqrt{\sum_{i=1}^n |\alpha_i|^2} \sqrt{\sum_{i=1}^n \sum_{j=i+1}^n |t_{ij}|^2} \\
&= \sum_{i=1}^n |\alpha_i|^2 \sqrt{\sum_{i=1}^n \sum_{j=i+1}^n |t_{ij}|^2},
\end{aligned}$$

$$|f| \le \max_{i<j} |t_{ij}| \sum_{i=1}^n |\alpha_i| \sum_{j=i+1}^n |\alpha_j| \le \max_{i<j} |t_{ij}| x \sum_{i=1}^n |\alpha_i|^2, \quad x \le \frac{n-1}{2}.$$

So we can say that

$$\frac{|\Re(f)|}{\sum_i |\alpha_i|^2} \le \min\{\sqrt{\sum_{i=1}^n \sum_{j=i+1}^n |t_{ij}|^2}, \frac{n-1}{2} \max_{i<j} |t_{ij}|\}. \tag{3}$$

Note how the min changes when passing from a matrix T with $|t_{ij}|$ constant (for $i < j$) to a T with $t_{ij} = 0$ for all but one (i, j), $i < j$.

The case $n = 2$:

$$\frac{|\Re(\overline{\alpha_1}\alpha_2 t_{12})|}{|\alpha_1|^2 + |\alpha_2|^2} \leq \frac{|\alpha_1||\alpha_2||t_{12}|}{|\alpha_1|^2 + |\alpha_2|^2} \leq \frac{1}{2}|t_{12}|$$

Theorem. We have

$$\min \Re(\lambda_i) - g(\{t_{ij}\}_{i<j}) \leq \lambda(A_h) \leq \max \Re(\lambda_i) + g(\{t_{ij}\}_{i<j})$$

whenever $\frac{|\Re(f)|}{\sum_i |\alpha_i|^2} \leq g(\{t_{ij}\}_{i<j})$*. So, if* $\Re(\lambda(A)) > g(\{t_{ij}\}_{i<j})$*, then* A_h *is p.d.*

Examples of functions $g(\{t_{ij}\}_{i<j})$ are given in (3). Notice, however, that the functions $g(\{t_{ij}\}_{i<j})$ in Theorem should be easily computable from the entries of A; in fact they should depend directly on the entries a_{ij}:

$$\sum_{i<j} |t_{ij}|^2 = \|T\|_F^2 - \sum_i |\lambda_i|^2 = \|A\|_F^2 - \sum_i |\lambda_i|^2 \leq \|A\|_F^2 - n \min \lambda(A^*A) \ \ldots$$

$(A\mathbf{v}_i = \lambda_i \mathbf{v}_i, \Rightarrow \mathbf{v}_i^* A^* A \mathbf{v}_i = |\lambda_i|^2 \mathbf{v}_i^* \mathbf{v}_i \ \ldots)$

Eigenstructure of normal matrices

If A is normal and $A\mathbf{x} = \lambda\mathbf{x}$, then also $A^*\mathbf{x}$ (besides $\mathbf{x}$) is eigenvector of A corresponding to λ.

If A is normal, $A\mathbf{x} = \lambda\mathbf{x}$ and λ is a simple eigenvalue, then there exists μ such that $A^*\mathbf{x} = \mu\mathbf{x}$. Note that $\overline{\mu}$ must be an eigenvalue of A. (The eigenvalues of A^* are the complex conjugates of the eigenvalues of A).

If A is normal, $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $i = 1, \ldots, n$, and all the λ_i are simple eigenvalues (so A has n distinct eigenvalues) , then there exist μ_i such that $A^*\mathbf{x}_i = \mu_i\mathbf{x}_i$, $i = 1, \ldots, n$. Note that $\overline{\mu}_i$ must be equal to an eigenvalue λ_j of A

If A is normal then $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $\mathbf{x}_i^*\mathbf{x}_j = \delta_{ij}$, $1 \leq i, j \leq n$, $AA^*\mathbf{x}_i = A^*A\mathbf{x}_i = \lambda_i A^*\mathbf{x}_i \Rightarrow \{A^*\mathbf{x}_i\}$ are eigenvectors of A (as $\{\mathbf{x}_i\}$). Moreover

$$\begin{aligned}(A^*\mathbf{x}_i)^*(A^*\mathbf{x}_j) &= \mathbf{x}_i^* A A^* \mathbf{x}_j = \mathbf{x}_i^* A^* A \mathbf{x}_j = (A\mathbf{x}_i)^*(A\mathbf{x}_j) \\ &= (\lambda_i\mathbf{x}_i)^*(\lambda_j\mathbf{x}_j) = \overline{\lambda_i}\lambda_j \mathbf{x}_i^*\mathbf{x}_j = |\lambda_i|^2\delta_{ij}.\end{aligned}$$

So, if A is also non singular, then $\{\frac{1}{\lambda_i}A^*\mathbf{x}_i\}$ are orthonormal eigenvectors of A (as $\{\mathbf{x}_i\}$).

Circulant-type matrix algebras

Let

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix}.$$

We have:

- $A^3 = I$ iff $abc = 1$

$$A^2 = \begin{bmatrix} 0 & 0 & ab \\ bc & 0 & 0 \\ 0 & ca & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} abc & 0 & 0 \\ 0 & bca & 0 \\ 0 & 0 & cab \end{bmatrix}.$$

- the characteristic polynomial of A is $\lambda^3 - abc$, so, if $abc = 1$ then the eigenvalues of A are: $1,\ \omega_3,\ \omega_3^2$, where $\omega_3 = e^{-\mathbf{i}2\pi/3}$.
- A is normal iff $|a| = |b| = |c|$.
- A is unitary iff $|a| = |b| = |c| = 1$.

By imposing the identity

$$\begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \omega^i \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for $i = 0$, $i = 1$, $i = 2$, and therefore by requiring, respectively, the conditions $abc = 1$, $abc = 1$ & $\omega^3 = 1$, $abc = 1$ & $\omega^6 = 1$, one obtains the equalities:

$$\begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ bc \\ c \end{bmatrix} = 1 \begin{bmatrix} 1 \\ bc \\ c \end{bmatrix}, \quad \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ bc\omega \\ c\omega^2 \end{bmatrix} = \omega \begin{bmatrix} 1 \\ bc\omega \\ c\omega^2 \end{bmatrix},$$
$$\begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ bc\omega^2 \\ c\omega^4 \end{bmatrix} = \omega^2 \begin{bmatrix} 1 \\ bc\omega^2 \\ c\omega^4 \end{bmatrix}.$$

So, if $abc = 1$ and $Q = \operatorname{diag}(1,\ bc,\ c)F$, where F is the 3×3 Fourier matrix, then $AQ = Q\operatorname{diag}(1,\ \omega_3,\ \omega_3^2)$.

Note that Q is unitary iff $|a| = |b| = |c| = 1$ (iff A is unitary).

Exercise. Consider the $n \times n$ case.

Proof of $A\mathbf{x}_i = \lambda_i \mathcal{A}\mathbf{x}_i$, $A = [t^{|i-j|}]$, $\mathcal{A}$ *GStrang*

Let A be the real symmetric Toeplitz matrix $[t^{|i-j|}]_{i,j=1}^n$ and $\mathcal{A}$ be the GStrang circulant matrix associated with A. Assume n even, set $m = n/2$ and consider the $m \times m$ matrices

$$S = \begin{bmatrix} 1 & t & \cdots & t^{m-1} \\ t & & & \\ \vdots & & & \\ t^{m-1} & & & \end{bmatrix}, \quad R = \begin{bmatrix} t^m & t^{m+1} & \cdots & t^{n-1} \\ t^{m-1} & & & \\ \vdots & & & \\ t & & & \end{bmatrix},$$
$$Q = \begin{bmatrix} t^m & t^{m-1} & \cdots & t \\ t^{m-1} & & & \\ \vdots & & & \\ t & & & \end{bmatrix}, \quad J = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & 1 & 0 \\ 0 & & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

(S, R, Q are Toeplitz). Observe that

$$A = \begin{bmatrix} S & R \\ R^T & S \end{bmatrix}, \quad SJ = JS, \quad RJ = JR^T, \quad \mathcal{A} = \begin{bmatrix} S & Q \\ Q & S \end{bmatrix}, \quad QJ = JQ.$$

Obviously we have the identities $A\mathbf{e}_m = \mathcal{A}\mathbf{e}_m$ and $A\mathbf{e}_{m+1} = \mathcal{A}\mathbf{e}_{m+1}$.

Moreover, if $\mathbf{x}$ is the $m \times 1$ vector $[t\ 0\ \cdots\ 0\ -t^m]^T$, then

$$S\mathbf{x} = \begin{bmatrix} t - t^{n-1} \\ t^2 - t^{n-2} \\ \vdots \\ t^m - t^m \end{bmatrix}, \quad QJ\mathbf{x} = \begin{bmatrix} -t^n + t^2 \\ -t^{n-1} + t^3 \\ \vdots \\ -t^{m+1} + t^{m+1} \end{bmatrix}, \quad RJ\mathbf{x} = \begin{bmatrix} -t^n + t^n \\ -t^{n-1} + t^{n-1} \\ \vdots \\ -t^{m+1} + t^{m+1} \end{bmatrix} = \mathbf{0}.$$

$$\Rightarrow \quad S\mathbf{x} \pm RJ\mathbf{x} = \begin{bmatrix} t - t^{n-1} \\ t^2 - t^{n-2} \\ \vdots \\ t^m - t^m \end{bmatrix}, \quad S\mathbf{x} \pm QJ\mathbf{x} = \begin{bmatrix} (t - t^{n-1})(1 \pm t) \\ (t^2 - t^{n-2})(1 \pm t) \\ \vdots \\ (t^m - t^m)(1 \pm t) \end{bmatrix}$$

$$\Rightarrow \quad \frac{1}{1 \pm t}(S\mathbf{x} \pm QJ\mathbf{x}) = S\mathbf{x} \pm RJ\mathbf{x}. \tag{1}$$

$$\Rightarrow \quad SJJ\mathbf{x} \pm RJ\mathbf{x} = \tfrac{1}{1 \pm t}(SJJ\mathbf{x} \pm QJ\mathbf{x})$$
$$\Rightarrow \quad JSJ\mathbf{x} \pm JR^T\mathbf{x} = \tfrac{1}{1 \pm t}(JSJ\mathbf{x} \pm JQ\mathbf{x})$$
$$\Rightarrow \quad SJ\mathbf{x} \pm R^T\mathbf{x} = \tfrac{1}{1 \pm t}(SJ\mathbf{x} \pm Q\mathbf{x})$$

$$\Rightarrow \quad \pm SJ\mathbf{x} + R^T\mathbf{x} = \frac{1}{1 \pm t}(\pm SJ\mathbf{x} + Q\mathbf{x}). \tag{2}$$

Equalities (1) and (2) imply

$$\begin{bmatrix} S & R \\ R^T & S \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \pm J\mathbf{x} \end{bmatrix} = \frac{1}{1 \pm t} \begin{bmatrix} S & Q \\ Q & S \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \pm J\mathbf{x} \end{bmatrix}, \quad \text{i.e.}$$

$$A \begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \\ -t^m \\ \mp t^m \\ 0 \\ \vdots \\ 0 \\ \pm t \end{bmatrix} = \frac{1}{1 \pm t} \mathcal{A} \begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \\ -t^m \\ \mp t^m \\ 0 \\ \vdots \\ 0 \\ \pm t \end{bmatrix}.$$

Finally, let $\mathbf{y}$ be any $m \times 1$ vector $[y_0 \; y_1 \; \cdots \; y_{m-1}]^T$ satisfying the following two linear equations:

(a) $y_0 + y_1 t + \ldots + y_j t^j + \ldots + y_{m-1} t^{m-1} = 0$,

(b) $y_{m-1} + y_{m-2} t + \ldots + y_{j-1} t^{m-j} + \ldots + y_0 t^{m-1} = 0$.

Multiplying (a) by $t^m, t^{m-1}, \ldots, t$ and (b) by $t, t^2, \ldots, t^m$, one obtains the identities:

$$R\mathbf{y} = \begin{bmatrix} t^m & t^{m+1} & \cdots & t^{n-1} \\ t^{m-1} & & & \\ \vdots & & & \\ t & & & \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{bmatrix} = \mathbf{0}, \;\; R^T\mathbf{y} = \begin{bmatrix} t^m & t^{m-1} & \cdots & t \\ t^{m+1} & & & \\ \vdots & & & \\ t^{n-1} & & & \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \quad A \begin{bmatrix} \mathbf{y} \\ \pm\mathbf{y} \end{bmatrix} = \begin{bmatrix} S & R \\ R^T & S \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \pm\mathbf{y} \end{bmatrix} = \begin{bmatrix} S\mathbf{y} \\ \pm S\mathbf{y} \end{bmatrix}. \tag{1}$$

On the other side we also have:

$$t^m S\mathbf{y} = t^m \begin{bmatrix} 1 & t & \cdots & t^{m-1} \\ t & & & \\ \vdots & & & \\ t^{m-1} & & & \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{bmatrix} = - \begin{bmatrix} t^m & t^{m-1} & \cdots & t \\ t^{m-1} & & & \\ \vdots & & & \\ t & & & \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{bmatrix} = -Q\mathbf{y}.$$

In fact, for $j = 0, 1, \ldots, m-1$ the $(j+1)$-row in the left is equal to (use (b) and (a), respectively)

$$
\begin{aligned}
&[t^{m+j} \cdots t^{m+1}\ t^m\ t^{m+1} \cdots t^{2m-1-j}] \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{bmatrix} \\
&= (t^{m+j} y_0 + t^{m+j-1} y_1 + \ldots + t^{m+1} y_{j-1}) + (t^m y_j + t^{m+1} y_{j+1} + \ldots + t^{2m-1-j} y_{m-1}) \\
&= (-y_{m-1} - y_{m-2} t \ldots - y_j t^{m-j-1}) t^{j+1} + (-y_0 - y_1 t \ldots - y_{j-1} t^{j-1}) t^{m-j} \\
&= -[t^{m-j} \cdots t^{m-1}\ t^m\ t^{m-1} \cdots t^{j+1}] \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{bmatrix}
\end{aligned}
$$

which is the $(j+1)$-row in the right. Thus

$$
\mathcal{A} \begin{bmatrix} \mathbf{y} \\ \pm\mathbf{y} \end{bmatrix} = \begin{bmatrix} S\mathbf{y} \pm Q\mathbf{y} \\ Q\mathbf{y} \pm S\mathbf{y} \end{bmatrix} = \begin{bmatrix} S\mathbf{y} \mp t^m S\mathbf{y} \\ -t^m S\mathbf{y} \pm S\mathbf{y} \end{bmatrix} = (1 \mp t^m) \begin{bmatrix} S\mathbf{y} \\ \pm S\mathbf{y} \end{bmatrix}. \tag{2}
$$

From (1) and (2) it follows that

$$
A \begin{bmatrix} \mathbf{y} \\ \pm\mathbf{y} \end{bmatrix} = \frac{1}{1 \mp t^m} \mathcal{A} \begin{bmatrix} \mathbf{y} \\ \pm\mathbf{y} \end{bmatrix}, \ \forall \mathbf{y} \mid \begin{bmatrix} 1 & t & \cdots & & t^{m-1} \\ t^{m-1} & \cdots & & t & 1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

So we have:

- $m-2$ eigenvectors of type $\begin{bmatrix} \mathbf{y} \\ \mathbf{y} \end{bmatrix}$ corresponding to the eigenvalue $\frac{1}{1-t^m}$
- $m-2$ eigenvectors of type $\begin{bmatrix} \mathbf{y} \\ -\mathbf{y} \end{bmatrix}$ corresponding to the eigenvalue $\frac{1}{1+t^m}$
- two eigenvectors $\mathbf{e}_m$ and $\mathbf{e}_{m+1}$ corresponding to the eigenvalue 1
- one eigenvector $\begin{bmatrix} \mathbf{x} \\ J\mathbf{x} \end{bmatrix}$ corresponding to the eigenvalue $\frac{1}{1+t}$
- one eigenvector $\begin{bmatrix} \mathbf{x} \\ -J\mathbf{x} \end{bmatrix}$ corresponding to the eigenvalue $\frac{1}{1-t}$

where $\mathbf{x} = [t\ 0 \cdots 0\ \ -t^m]^T$ and the vectors $\mathbf{y}$ are $m-2$ linearly independent solutions of the system:

$$
\begin{bmatrix} 1 & t & \cdots & & t^{m-1} \\ t^{m-1} & \cdots & & t & 1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

We have proved the equality $A\mathbf{x}_i = \lambda_i \mathcal{A} \mathbf{x}_i$ for n (eigenvalues,eigenvectors) $(\lambda_i, \mathbf{x}_i)$. Why the $\mathbf{x}_i$ are linearly independent ?

Let A, B be $n \times n$ (non null) matrices with complex entries. Assume that $A\mathbf{x} = \lambda B\mathbf{x}$, $A\mathbf{y} = \mu B\mathbf{y}$ for non null vectors $\mathbf{x}$ and $\mathbf{y}$ where $\lambda, \mu \in \mathbb{C}$, $\lambda \neq \mu$. Then $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

If B is non singular, then we have the equations $B^{-1}A\mathbf{x} = \lambda\mathbf{x}$ and $B^{-1}A\mathbf{y} = \mu\mathbf{y}$, and the thesis follows as in the classic eigenvalue problem case (but $B^{-1}A$ takes the role of A).

If A is non singular and both λ and μ are non zero (the case of GStrang) then we have the equations $A^{-1}B\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$ and $A^{-1}B\mathbf{y} = \frac{1}{\mu}\mathbf{y}$, and the proof is very similar to the classic eigenvalue problem case (but $A^{-1}B$ takes the role of A).

If B is singular, A is non singular and $\lambda = 0$ (or $\mu = 0$), then we have the equation $A\mathbf{x} = \mathbf{0}$ ($A\mathbf{y} = \mathbf{0}$) which implies $\mathbf{x} = \mathbf{0}$ ($\mathbf{y} = \mathbf{0}$), which is against our hypothesis.

If B and A are singular ... is the thesis true ?

Proof of eigenvalue minmax representation for a hermitian matrix A (and of the interlace theorem)

(1) $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $\mathbf{x}_i^*\mathbf{x}_j = \delta_{ij}$, $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ (recall that normal matrices can be diagonalized via unitary transforms). Let $V_j \subset \mathbb{C}^n$ be a generic space of dimension j. Then for any $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x} \in V_j \cap \text{Span}\{\mathbf{x}_j, \ldots, \mathbf{x}_n\}$, we have $\mathbf{x} = \sum_{i=j}^n \alpha_i\mathbf{x}_i$ with α_i not all zeroes, and

$$\begin{aligned}\frac{\mathbf{x}^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} &= \frac{(\sum_i \alpha_i\mathbf{x}_i)^*A(\sum_k \alpha_k\mathbf{x}_k)}{(\sum_i \alpha_i\mathbf{x}_i)^*(\sum_k \alpha_k\mathbf{x}_k)} = \frac{(\sum \overline{\alpha_i}\mathbf{x}_i^*)(\sum \alpha_k\lambda_k\mathbf{x}_k)}{(\sum \overline{\alpha_i}\mathbf{x}_i^*)(\sum \alpha_k\mathbf{x}_k)} \\ &= \frac{\sum_{i=j}^n |\alpha_i|^2\lambda_i}{\sum_{i=j}^n |\alpha_i|^2} \geq \lambda_j.\end{aligned}$$

Thus $\lambda_j \leq \max_{\mathbf{x}\in V_j}(\mathbf{x}^*A\mathbf{x}/\mathbf{x}^*\mathbf{x})$.

Moreover, we have $(\mathbf{x}_j^*A\mathbf{x}_j/\mathbf{x}_j^*\mathbf{x}_j) = \lambda_j$, and for any $\mathbf{x} = \sum_{i=1}^j \beta_i\mathbf{x}_i$, with β_i not all zeroes,

$$\frac{\mathbf{x}^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \frac{\sum_{i=1}^j |\beta_i|^2\lambda_i}{\sum_{i=1}^j |\beta_i|^2} \leq \lambda_j.$$

It follows that for $V_j = \text{Span}\{\mathbf{x}_1 \ldots \mathbf{x}_j\}$ it holds $\max_{\mathbf{x}\in V_j} \frac{\mathbf{x}^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \lambda_j$.

(2) A, B, C hermitian, $\alpha_i, \beta_i, \gamma_i$ their eigenvalues in non-decreasing order, $C = A + B$: proof of the interlace theorem

$$\begin{aligned}\gamma_j &= \min_{V_j}\max_{\mathbf{x}\in V_j} \frac{\mathbf{x}^*C\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \min_{V_j}\max_{\mathbf{x}\in V_j}\left(\frac{\mathbf{x}^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} + \frac{\mathbf{x}^*B\mathbf{x}}{\mathbf{x}^*\mathbf{x}}\right) \\ &\leq \min_{V_j}\max_{\mathbf{x}\in V_j}\left(\frac{\mathbf{x}^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} + \beta_n\right) = \min_{V_j}\max_{\mathbf{x}\in V_j} \frac{\mathbf{x}^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} + \beta_n = \alpha_j + \beta_n,\end{aligned}$$

$$\begin{aligned}\alpha_j &= \min_{V_j}\max_{\mathbf{x}\in V_j} \frac{\mathbf{x}^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \min_{V_j}\max_{\mathbf{x}\in V_j}\left(\frac{\mathbf{x}^*C\mathbf{x}}{\mathbf{x}^*\mathbf{x}} - \frac{\mathbf{x}^*B\mathbf{x}}{\mathbf{x}^*\mathbf{x}}\right) \\ &\leq \min_{V_j}\max_{\mathbf{x}\in V_j}\left(\frac{\mathbf{x}^*C\mathbf{x}}{\mathbf{x}^*\mathbf{x}} - \beta_1\right) = \min_{V_j}\max_{\mathbf{x}\in V_j} \frac{\mathbf{x}^*C\mathbf{x}}{\mathbf{x}^*\mathbf{x}} - \beta_1 = \gamma_j - \beta_1.\end{aligned}$$

Deflation

Le A be a $n \times n$ matrix. Denote by λ_i, $i = 1, \ldots, n$, the eigenvalues of A and by $\mathbf{y}_i$ the corresponding eigenvectors. So, we have $A\mathbf{y}_i = \lambda_i\mathbf{y}_i$, $i = 1, \ldots, n$.

Assume that $\lambda_1, \mathbf{y}_1$ are given and that $\lambda_1 \neq 0$. Choose $\mathbf{w} \in \mathbb{C}^n$ such that $\mathbf{w}^*\mathbf{y}_1 \neq 0$ (given $\mathbf{y}_1$ choose $\mathbf{w}$ not orthogonal to $\mathbf{y}_1$) and set

$$W = A - \frac{\lambda_1}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1\mathbf{w}^*.$$

It is known that the eigenvalues of W are

$$0,\ \lambda_2,\ \ldots,\ \lambda_j,\ \ldots,\ \lambda_n$$

i.e. they are the same of A except λ_1 which is replaced with 0. Let $\mathbf{w}_1$, $\mathbf{w}_2$, $\ldots$, $\mathbf{w}_j$, $\ldots$, $\mathbf{w}_n$ be the corresponding eigenvectors ($W\mathbf{w}_1 = \mathbf{0}$, $W\mathbf{w}_j = \lambda_j \mathbf{w}_j$ $j = 2, \ldots, n$). Is it possible to obtain the $\mathbf{w}_j$ from the $\mathbf{y}_j$?

First observe that

$$A\mathbf{y}_1 = \lambda_1 \mathbf{y}_1 \ \Rightarrow\ W\mathbf{y}_1 = \mathbf{0}:\ \mathbf{w}_1 = \mathbf{y}_1. \qquad (a)$$

Then, for $j = 2, \ldots, n$,

$$W\mathbf{y}_j = A\mathbf{y}_j - \frac{\lambda_1}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1\mathbf{w}^*\mathbf{y}_j = \lambda_j\mathbf{y}_j - \lambda_1\frac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1. \qquad (1)$$

If we impose $\mathbf{y}_j = \mathbf{w}_j + c\mathbf{y}_1$, $j = 2, \ldots, n$, then (1) becomes,

$$\begin{aligned} W\mathbf{w}_j + cW\mathbf{y}_1 &= \lambda_j\mathbf{w}_j + c\lambda_j\mathbf{y}_1 - \lambda_1\tfrac{\mathbf{w}^*\mathbf{w}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1 - c\lambda_1\mathbf{y}_1 \\ &= \lambda_j\mathbf{w}_j + \mathbf{y}_1[c\lambda_j - \lambda_1\tfrac{\mathbf{w}^*\mathbf{w}_j}{\mathbf{w}^*\mathbf{y}_1} - \lambda_1 c] \end{aligned}$$

So, if $\lambda_j \neq \lambda_1$ and

$$\mathbf{w}_j = \mathbf{y}_j - \frac{\lambda_1}{\lambda_j - \lambda_1}\frac{\mathbf{w}^*\mathbf{w}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1, \qquad (2)$$

then $W\mathbf{w}_j = \lambda_j\mathbf{w}_j$. If, moreover, $\lambda_j \neq 0$, then $\mathbf{w}^*\mathbf{y}_j = \mathbf{w}^*\mathbf{w}_j + \frac{\lambda_1}{\lambda_j - \lambda_1}\mathbf{w}^*\mathbf{w}_j \Rightarrow \mathbf{w}^*\mathbf{y}_j = \mathbf{w}^*\mathbf{w}_j\frac{\lambda_j}{\lambda_j - \lambda_1} \Rightarrow \mathbf{w}^*\mathbf{w}_j = \frac{\lambda_j - \lambda_1}{\lambda_j}\mathbf{w}^*\mathbf{y}_j$. So, by (2),

$$\begin{aligned} &\text{for all } j \in \{2 \ldots n\} \mid \lambda_j \neq \lambda_1, 0: \\ &A\mathbf{y}_j = \lambda_j\mathbf{y}_j \ \Rightarrow \\ &W(\mathbf{y}_j - \tfrac{\lambda_1}{\lambda_j}\tfrac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1) = \lambda_j(\mathbf{y}_j - \tfrac{\lambda_1}{\lambda_j}\tfrac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1):\ \mathbf{w}_j = \mathbf{y}_j - \tfrac{\lambda_1}{\lambda_j}\tfrac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1. \end{aligned} \qquad (b)$$

Note that a formula for $\mathbf{y}_j$ in terms of $\mathbf{w}_j$ holds: see (2).

As regards the case $\lambda_j = \lambda_1$, it is simple to show that

$$\begin{aligned} &\text{for all } j \in \{2 \ldots n\} \mid \lambda_j = \lambda_1: \\ &A\mathbf{y}_j = \lambda_j\mathbf{y}_j \ \Rightarrow \\ &W(\mathbf{y}_j - \tfrac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1) = \lambda_j(\mathbf{y}_j - \tfrac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1):\ \mathbf{w}_j = \mathbf{y}_j - \tfrac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1. \end{aligned} \qquad (c)$$

Note that the vectors $\mathbf{y}_j - \frac{\mathbf{w}^*\mathbf{y}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1$ are orthogonal to $\mathbf{w}$. Is it possible to find from (c) an expression of $\mathbf{y}_j$ in terms of $\mathbf{w}_j$?

It remains the case $\lambda_j = 0$: find ? in

$$\begin{aligned} &\text{for all } j \in \{2 \ldots n\} \mid \lambda_j = 0: \\ &A\mathbf{y}_j = \lambda_j\mathbf{y}_j = \mathbf{0} \ \Rightarrow\ W(?) = \lambda_j(?) = \mathbf{0}:\ \mathbf{w}_j = ? \end{aligned} \qquad (d?)$$

$(\mathbf{y}_j = \mathbf{w}_j - \frac{\mathbf{w}^*\mathbf{w}_j}{\mathbf{w}^*\mathbf{y}_1}\mathbf{y}_1 \Rightarrow \mathbf{w}^*\mathbf{y}_j = 0)$ ...

Choices of $\mathbf{w}$. Since $\mathbf{y}_1^*\mathbf{y}_1 \neq 0$ one can set $\mathbf{w} = \mathbf{y}_1$. In this way, if A is hermitian also W is hermitian. .... If i is such that $(\mathbf{y}_1)_i \neq 0$ then $\mathbf{e}_i^T A\mathbf{y}_1 = \lambda_1(\mathbf{y}_1)_i \neq 0$. So one can set $\mathbf{w}^* = \mathbf{e}_i^T A = \text{row } i \text{ of } A$. In this way the row i of W is null and therefore we can introduce a matrix of order $n-1$ whose eigenvalues are λ_2, ..., λ_n (the unknown eigenvalues of A).