Lower triangular Toeplitz matrices and the sequence of Bernoulli numbers

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Bernoulli numbers

$B(x+1) - B(x) = nx^{n-1}$, $\int_0^1 B(x)\,dx = 0$, $B(x)$ polynomial $\Rightarrow$ $B_n(x)$ *degree n Bernoulli polynomial*

Note: $B_n'(x) = nB_{n-1}(x)$, $B_n(1-x) = (-1)^n B_n(x)$, $B_{2j+1}(0) = 0$

Bernoulli numbers: $\underline{B_{2j} := B_{2j}(0) = B_{2j}(1)}$, $j = 0, 1, 2, \dots$

$1, \frac{1}{6}, -\frac{1}{30}, \frac{1}{42}, -\frac{1}{30}, \frac{5}{66}, -\frac{691}{2730}, \frac{7}{6}, -\frac{3617}{510}, \frac{43867}{798}, -\frac{174611}{330}, \frac{854513}{138}, \dots$

Euler formula $\sum_{k=1}^{+\infty} \frac{1}{k^{2j}} = \frac{(-1)^{j-1} B_{2j} (2\pi)^{2j}}{2(2j)!} \quad \Rightarrow \quad |B_{2j}| \approx \frac{2(2j)!}{(2\pi)^{2j}}$.

Moreover $\left.\frac{B_{2j}(x)}{B_{2j}(0)}\right|_{[0,1]} \to \cos(2\pi x)$, $\displaystyle\sum_{x=1}^{k} x^{n-1} = \frac{1}{n}[B_n(k+1) - B_n(1)]$.

B_{2j} appear in Euler-Maclaurin summation formula and error in the trapezoidal quadrature rule, in cyclotomic fields and irregular primes, ...

... in power series expansions, in particular

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{+\infty} \frac{B_n(x)}{n!} t^n \quad \Rightarrow$$

$$\frac{t}{e^t - 1} = -\frac{1}{2}t + \sum_{k=0}^{+\infty} \frac{B_{2k}}{(2k)!} t^{2k},$$

$$t = \Big(-\frac{1}{2}t + \sum_{k=0}^{+\infty} \frac{B_{2k}}{(2k)!} t^{2k} \Big) \Big(\sum_{r=0}^{+\infty} \frac{t^{r+1}}{(r+1)!} \Big), \quad \dots,$$

$$-\frac{1}{2}j + \sum_{k=0}^{[\frac{j-1}{2}]} \binom{j}{2k} B_{2k} = 0, \quad j = 2, 3, 4, 5, \dots$$

For j even and for j odd the latter equations can be rewritten respectively as follows:

$$\begin{bmatrix} \binom{2}{0} & & & & \\ \binom{4}{0} & \binom{4}{2} & & & \\ \binom{6}{0} & \binom{6}{2} & \binom{6}{4} & & \\ \binom{8}{0} & \binom{8}{2} & \binom{8}{4} & \binom{8}{6} & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} B_0 \\ B_2 \\ B_4 \\ B_6 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ \cdot \end{bmatrix}, \qquad (even)$$

$$\begin{bmatrix} \binom{1}{0} & & & & \\ \binom{3}{0} & \binom{3}{2} & & & \\ \binom{5}{0} & \binom{5}{2} & \binom{5}{4} & & \\ \binom{7}{0} & \binom{7}{2} & \binom{7}{4} & \binom{7}{6} & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} B_0 \\ B_2 \\ B_4 \\ B_6 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 5/2 \\ 7/2 \\ \cdot \end{bmatrix}. \qquad (odd)$$

Triangular Toeplitz systems satisfied by Bernoulli numbers

(even) *is equivalent to a lower triangular Toeplitz* (ltT) *system*:

$$Z_{a_2,a_3,a_4,\dots} = \begin{bmatrix} 0 & & & & \\ a_2 & 0 & & & \\ & a_3 & 0 & & \\ & & a_4 & \cdot & \\ & & & \cdot & \cdot \end{bmatrix}, \quad Z^2_{a_2,a_3,a_4,\dots} = \begin{bmatrix} 0 & & & & \\ 0 & 0 & & & \\ a_2a_3 & 0 & 0 & & \\ & a_3a_4 & 0 & \cdot & \\ & & a_4a_5 & \cdot & \cdot \\ & & & \cdot & \cdot \end{bmatrix},$$

$$Z^k_{a_2,a_3,a_4,\dots} = \begin{bmatrix} 0 & & & & \\ \cdot & 0 & & & \\ 0 & \cdot & 0 & & \\ a_2a_3\cdot a_{1+k} & 0 & \cdot & \cdot & \\ & a_3a_4\cdot a_{2+k} & 0 & \cdot & \cdot \\ & & a_4a_5\cdot a_{3+k} & \cdot & \cdot \\ & & & \cdot & \cdot \end{bmatrix}, \quad Z = Z_{a_2,a_3,a_4,\dots} \Rightarrow$$

$$[Z^k_{a_2,a_3,a_4,\dots}]_{ij} = \begin{cases} a_{j+1}a_{j+2}\cdot a_{j+k} & \textit{if } i=k+j,\ j=1,2,3,\dots \\ 0 & \textit{otherwise} \end{cases}, \quad \underline{[Z^{i-j}]_{ij} = a_{j+1}\cdots a_i}$$

$$Y = Z_{1,2,3,\ldots} \quad (\, a_k = k - 1 \,) \quad \Rightarrow$$

$$\sum_{k=0}^{+\infty} \frac{1}{k!} Y^k = \begin{bmatrix} \binom{0}{0} & & & & & & & \\ \binom{1}{0} & \binom{1}{1} & & & & & & \\ \binom{\mathbf{2}}{\mathbf{0}} & \binom{2}{1} & \binom{2}{2} & & & & & \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & & & & \\ \binom{\mathbf{4}}{\mathbf{0}} & \binom{4}{1} & \binom{\mathbf{4}}{\mathbf{2}} & \binom{4}{3} & \binom{4}{4} & & & \\ \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} & & \\ \binom{\mathbf{6}}{\mathbf{0}} & \binom{6}{1} & \binom{\mathbf{6}}{\mathbf{2}} & \binom{6}{3} & \binom{\mathbf{6}}{\mathbf{4}} & \binom{6}{5} & \binom{6}{6} & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$Proof: \quad \Big[\sum_{k=0}^{+\infty} \frac{1}{k!} Y^k\Big]_{ij} = \frac{1}{(i-j)!} \underline{[Y^{i-j}]_{ij}} = \frac{1}{(i-j)!} \underline{j \cdots (i-2)(i-1)} = \binom{i-1}{j-1}.$$

$$\Phi = Z_{2,12,30,56,\dots} \quad (\, a_k = (2k-3)(2k-2)\,) \;\Rightarrow$$

$$\mathrm{diag}\,(2,12,30,56,\dots)\sum_{k=0}^{+\infty}\frac{1}{(2k+2)!}\Phi^k = \begin{bmatrix} \binom{2}{0} & & & & \\ \binom{4}{0} & \binom{4}{2} & & & \\ \binom{6}{0} & \binom{6}{2} & \binom{6}{4} & & \\ \binom{8}{0} & \binom{8}{2} & \binom{8}{4} & \binom{8}{6} & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Proof: $[\,\mathrm{diag}\,(2,12,30,56,\dots)\sum_{k=0}^{+\infty}\frac{1}{(2k+2)!}\Phi^k]_{ij} = 2i(2i-1)\Big(\sum_{k=0}^{+\infty}\frac{1}{(2k+2)!}\phi^k\Big)_{ij}$

$= 2i(2i-1)\frac{1}{(2i-2j+2)!}\underline{[\phi^{i-j}]_{ij}} = 2i(2i-1)\frac{1}{(2i-2j+2)!}\underline{\frac{(2i-2)!}{(2j-2)!}} = \binom{2i}{2j-2}$. $\square$

Thus (even) is equivalent to

$$\sum_{k=0}^{+\infty}\frac{2}{(2k+2)!}\Phi^k\mathbf{b} = \mathbf{q}^e, \quad \mathbf{b} = [B_0\ B_2\ B_4\ B_6\ \cdots]^T,\ \mathbf{q}^e = [1\ \frac{1}{3}\ \frac{1}{5}\ \frac{1}{7}\ \cdots]^T.$$

$$D = \text{diag}\left(1, \tfrac{x}{2!}, \tfrac{x^2}{4!}, \cdots, \tfrac{x^{n-1}}{(2n-2)!}, \cdots\right), \; x \in \mathbb{R} \;\Rightarrow$$

$$D\Phi D^{-1} = xZ, \quad D\Phi^k D^{-1} = (D\Phi D^{-1})^k = x^k Z^k, \quad \underline{Z = Z_{1,1,1,\dots}}$$

Thus (even) is equivalent to the following ltT system:

$$\sum_{k=0}^{+\infty} \frac{2x^k}{(2k+2)!} Z^k D\mathbf{b} = D\mathbf{q}^e. \qquad (even\ ltT)$$

Analogously, (odd) is equivalent to the following ltT system:

$$\sum_{k=0}^{+\infty} \frac{x^k}{(2k+1)!} Z^k D\mathbf{b} = D\mathbf{q}^o, \qquad (odd\ ltT)$$

$$\mathbf{q}^o = [1 \;\; \tfrac{1}{2} \;\; \tfrac{1}{2} \;\; \tfrac{1}{2} \;\; \cdots]^T.$$

Ramanujan Toeplitz systems satisfied by Bernoulli numbers

$$\begin{bmatrix} 1 & & & & & & & & & & & \\ 0 & 1 & & & & & & & & & & \\ 0 & 0 & 1 & & & & & & & & & \\ \frac{1}{3} & 0 & 0 & 1 & & & & & & & & \\ 0 & \frac{5}{2} & 0 & 0 & 1 & & & & & & & \\ 0 & 0 & 11 & 0 & 0 & 1 & & & & & & \\ \frac{1}{5} & 0 & 0 & \frac{143}{4} & 0 & 0 & 1 & & & & & \\ 0 & 4 & 0 & 0 & \frac{286}{3} & 0 & 0 & 1 & & & & \\ 0 & 0 & \frac{204}{5} & 0 & 0 & 221 & 0 & 0 & 1 & & & \\ \frac{1}{7} & 0 & 0 & \frac{1938}{7} & 0 & 0 & \frac{3230}{7} & 0 & 0 & 1 & & \\ 0 & \frac{11}{2} & 0 & 0 & \frac{7106}{5} & 0 & 0 & \frac{3553}{4} & 0 & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} B_2 \\ B_4 \\ B_6 \\ B_8 \\ B_{10} \\ B_{12} \\ B_{14} \\ B_{16} \\ B_{18} \\ B_{20} \\ B_{22} \\ \cdot \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{30} \\ \frac{1}{42} \\ \frac{1}{45} \\ -\frac{1}{132} \\ \frac{4}{455} \\ \frac{1}{120} \\ -\frac{1}{306} \\ \frac{3}{665} \\ \frac{1}{231} \\ -\frac{1}{552} \\ \cdot \end{bmatrix}.$$

$$R(Z^T\mathbf{b}) = Z^T\mathbf{f}, \qquad \mathbf{f} = [\, f_0 \;\; 1/6 \;\; -1/30 \;\; 1/42 \;\; 1/45 \;\; \ldots]^T$$

[S. Ramanujan, Some properties of Bernoulli numbers, *J. Indian Math. Soc.*, 3 (1911), 219–234]

$$\underline{R\Lambda^{-1} = \Lambda^{-1}\tilde{R}}, \quad \Lambda = Z^T DZ = \operatorname{diag}\left(\frac{x^i}{(2i)!} : i = 1, 2, 3, \dots\right) \qquad (C)$$

$$\text{where} \quad \tilde{R} = \sum_{k=0}^{+\infty} \frac{2x^{3k}}{(6k+2)!(2k+1)} Z^{3k}$$

$$= \begin{bmatrix} 1 \\ 0 & 1 \\ 0 & 0 & 1 \\ \frac{2x^3}{8!3} & 0 & 0 & 1 \\ 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\ 0 & 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\ \frac{2x^6}{14!5} & 0 & 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\ 0 & \frac{2x^6}{14!5} & 0 & 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\ 0 & 0 & \frac{2x^6}{14!5} & 0 & 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\ \frac{2x^9}{20!7} & 0 & 0 & \frac{2x^6}{14!5} & 0 & 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\ 0 & \frac{2x^9}{20!7} & 0 & 0 & \frac{2x^6}{14!5} & 0 & 0 & \frac{2x^3}{8!3} & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

REMARK The 11×11 upper left submatrix of $R\Lambda^{-1}$ coincides with the 11×11 upper left submatrix of $\Lambda^{-1}\tilde{R}$.

Assuming that (C) is true, we have the equalities

$$R(Z^T\mathbf{b}) = R\Lambda^{-1}(\Lambda Z^T\mathbf{b}) = \Lambda^{-1}\tilde{R}(Z^T D\mathbf{b}),$$

and thus the Ramanujan *system* $R(Z^T\mathbf{b}) = Z^T\mathbf{f}$ *is equivalent to the following ltT system*

$$\tilde{R}(Z^T D\mathbf{b}) = Z^T D\mathbf{f} \qquad (\textit{Ramanujan ltT})$$

where, in the coefficient matrix

$$\tilde{R} = \sum_{k=0}^{+\infty} \frac{2x^{3k}}{(6k+2)!(2k+1)} Z^{3k},$$

two null diagonals alternate the nonnull ones.

THEOREM Set $\mathbf{b} = [B_0 \ B_2 \ B_4 \ \cdots \ ]^T$, and $D = \operatorname{diag}\left(\frac{x^i}{(2i)!} : i = 0, 1, 2, \ldots\right)$, $x \in \mathbb{R}$. Then

$$L(\mathbf{a})\,(D\mathbf{b}) = D\mathbf{q}, \quad L(\mathbf{a}) = \sum_{i=0}^{+\infty} a_i\, Z^i$$

where $\mathbf{a} = (a_i)_{i=0}^{+\infty}$ and $\mathbf{q} = (q_i)_{i=0}^{+\infty}$ can assume the values:

$$a_0^o = q_0^o = 1,\ a_i^o = \frac{x^i}{(2i+1)!},\ q_i^o = \frac{1}{2},\ i = 1, 2, 3, \ldots, \qquad (o)$$

$$a_i^e = \frac{2x^i}{(2i+2)!},\quad q_i^e = \frac{1}{2i+1},\ i = 0, 1, 2, 3, \ldots, \qquad (e)$$

$$a_i^R = \delta_{i\equiv 0} \frac{2x^i}{(2i+2)!(\frac{2}{3}i+1)},\quad q_i^R = \frac{1}{(2i+1)(i+1)}(1 - \delta_{i\equiv 2}\frac{3}{2}),\ i \geq 0.\ (R)$$

In (R) the symbols $i \equiv 2$ and $i \equiv 0$ mean $i \equiv 2 \bmod 3$ and $i \equiv 0 \bmod 3$, respectively.

COROLLARY

We have all coefficients of the (not Toeplitz) Ramanujan system $RZ^T\mathbf{b} = Z^T\mathbf{f}$, $\mathbf{f} = [f_0\ f_1\ f_2\ \cdots\ ]^T$:

$$R_{ij} = \begin{cases} \frac{(2i)!}{(2j)!}\frac{2}{(2i-2j+2)!(\frac{2}{3}(i-j)+1)} & i-j \equiv 0,\ \ i \geq j \\ 0 & \text{otherwise} \end{cases},\ \ i,j = 1,2,\ldots,$$

$$f_i = \frac{1 - \delta_{i\equiv 2}\,\frac{3}{2} - \delta_{i\equiv 0}\,\frac{1}{\frac{2}{3}i+1}}{(2i+1)(i+1)},\quad i = 1,2,3,\ldots\ .$$

Example: the 12^{th} Ramanujan equation:

$$\frac{506}{5}B_6 + \frac{29716}{5}B_{12} + \frac{4807}{3}B_{18} + B_{24} = \frac{8}{2925} \quad \Rightarrow$$

$$B_{24} = -\frac{236364091}{2730}.$$

Remarks on the parameter x:

For any choice of x, the coefficients a_i and $(D\mathbf{q})_i$ of the ItT systems stated in the Theorem converge to zero as $i \to +\infty$.

The rate of convergence of the a_i^R ($(D\mathbf{q}^R)_i$) is greater than the rate of convergence of the a_i^e ($(D\mathbf{q}^e)_i$), which in turn is greater than the rate of convergence of the a_i^o ($(D\mathbf{q}^o)_i$).

Instead, the sequence $|(D\mathbf{b})_i|$ tends to zero, to 2, or to $+\infty$, depending on the value of x.

For instance, the choice $x = (2\pi)^2$ would ensure the sequence $(D\mathbf{b})_i = x^i B_{2i}/(2i)!$ to be bounded; indeed in this case $|(D\mathbf{b})_i| \to 2$ if $i \to +\infty$, due to Euler formula.

Possible application in computing Bernoulli numbers

- *The coefficients* a_i, $(D\mathbf{q})_i$ *of the three ltT systems are easily computable*, in fact

$O(n)$ arithmetic operations for the first n diagonal entries of D:

$$D_{00} = 1, \quad D_{ii} = \frac{x^i}{(2i)!} = \frac{x}{2i(2i-1)} D_{i-1i-1}, \ i = 1, 2, 3, \dots ;$$

$O(n)$ arithmetic operations to compute the first n entries of $\mathbf{a}^o, \mathbf{a}^e, \mathbf{a}^R$, $D\mathbf{q}^o, D\mathbf{q}^e, D\mathbf{q}^R$:

$$a_i^o = \frac{1}{2i+1} D_{ii}, \qquad (D\mathbf{q}^o)_0 = 1, \ (D\mathbf{q}^o)_i = \frac{1}{2} D_{ii}, \ i > 0,$$

$$a_i^e = \frac{1}{(i+1)(2i+1)} D_{ii}, \qquad (D\mathbf{q}^e)_i = \frac{1}{2i+1} D_{ii},$$

$$a_i^R = a_i^e \frac{1}{\frac{2}{3}i + 1} \delta_{i \equiv 0}, \qquad (D\mathbf{q}^R)_i = a_i^o \frac{1}{i+1} (1 - \delta_{i \equiv 2} \frac{3}{2}).$$

- *A formula for the nth Bernoulli number*:

$$B_{2n-2} = \frac{(2n-2)!}{x^{n-1}} \{D\mathbf{q}\}_n^T J \underline{\{L(\mathbf{a})\}_n^{-1}\mathbf{e}_1}, \quad J = \begin{bmatrix} & & 1 \\ & \cdot & \\ 1 & & \end{bmatrix}$$

i.e. $O(n \log n)$ *arithmetic operations to approximate* B_{2n-2} *in* $\mathbb{R}$.

A formula for any vector of n consecutive Bernoulli numbers:

$$\begin{bmatrix} \{L(\mathbf{a})\}_{N-n} & O \\ \{Toe(\mathbf{a})\}_{n,N-n} & \{L(\mathbf{a})\}_n \end{bmatrix} \begin{bmatrix} \{D\mathbf{b}\}_{up} \\ \{D\mathbf{b}\}_{down} \end{bmatrix} := \{L(\mathbf{a})\}_N \{D\mathbf{b}\}_N$$

$$= \{D\mathbf{q}\}_N =: \begin{bmatrix} \{D\mathbf{q}\}_{up} \\ \{D\mathbf{q}\}_{down} \end{bmatrix},$$

$$\{D\mathbf{b}\}_{down} = \underline{\{L(\mathbf{a})\}_n^{-1}}\Big(\{D\mathbf{q}\}_{down} - \{Toe(\mathbf{a})\}_{n,N-n}\{D\mathbf{b}\}_{up}\Big)$$

i.e. $O(n \log n)$ *arithmetic operations to approximate* $B_{2N-2n}, \dots, B_{2N-2}$ *in* $\mathbb{R}$ (if the vector $\{Toe(\mathbf{a})\}_{n,N-n}\{D\mathbf{b}\}_{up}$ is known).

Proof: It is well known that if $A \in \mathbb{C}^{n\times n}$ is ltT and $\mathbf{v} \in \mathbb{C}^n$, then $O(n\log n)$ a.o. are sufficient to compute $\mathbf{z} := A\mathbf{v}$. *Thus*

$O(n\log n)$ a.o. are sufficient to compute $\mathbf{c} : A\mathbf{c} = \mathbf{e}_1$

(multiply on the left both members of $A\mathbf{c} = \mathbf{e}_1$ by $\log n$ suitable sparser and sparser ltT matrices so to nullify the sub-diagonals of the coefficient matrix until obtaining $\mathbf{c} = A^{-1}\mathbf{e}_1$. NOTE: Product of ltT matrices is ltT)

$O(n\log n)$ a.o. are sufficient to compute $\mathbf{z} : A\mathbf{z} = \mathbf{v}$

(compute $\mathbf{c} : A\mathbf{c} = \mathbf{e}_1$, call M the ltT matrix with $\mathbf{c}$ as first column, and compute $\mathbf{z} = M\mathbf{v}$. NOTE: Inverse of ltT is ltT) $\square$

• Once a sufficiently accurate approximation B^*_{2j} of B_{2j} is available, this approximation and the fact that the denominator of B_{2j} is known (it is the product of all primes p such that $p-1$ divide $2j$) allow to deduce the numerator of B_{2j}.

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