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Householder-Type Matrix Algebras in Displacement Decompositions

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0. Introduction

The idea of informational content of a matrix is a wide and not-well defined concept, although Forsythe [14] suggested that a measure of it could be the amount of memory required to store the matrix. Indeed, for several classes of matrices, to identify a matrix it is sufficient a number of parameters k smaller than the number of elements. During the last decades it has become clear that a measure of the informational content of a matrix is related to the structure of the matrix, as remarked in [5],[6].

Displacement decompositios are a tool that, besides computational benefits, allows to point out different aspects of the informational content of some classes of matrices, for example the number of parameters required to identify uniquely a matrix in such a class.

In literature there are several displacement formulas that can be usefull to decompose a matrix as a combination of structured and low-complexity matrices; these formulas find also some interesting applications, for example to compute the solution of Toeplitz, Hankel or Toeplitz plus Hankel linear systems. Most of these decompositions can be formulated in terms of Hessenberg algebras, as it has been demonstrated in [1]; although, several of the algebras involved in displacement formulas are, not only Hessenberg algebras, but also SDU algebras (algebras of matrices simultaneously diagonalized by a unitary matrix U). For example we can cite the Gohberg-Olshevsky formulas that decompose a matrix as combinations of ε-circulant matrices (that are diagonalized by the ε-Fourier matrix [7]); other examples, found in [3], are formulas involving the algebras $\tau_{\varepsilon\varphi}$, which are related to the τ class and diagonalized by different types of sine and cosine transforms.

Our aim was to prove some displacement theorems for generic SDU algebras;

we have succeeded in it (see the contents of the first chapter) and we have observed that our theorems can be applied to some low complexity SDU algebras as the $\tau_{\varepsilon\varphi}$. Thanks to our formulas it is possible to decompose a generic matrix A as a sum of products of Hermitian matrices in two "near each other" SDU algebras, $\mathcal{U}$ and $\mathcal{V}$, plus one matrix in one of the two algebras; the number of the terms in the sum is proportional to the rank of the commutator of A with an Hermitian matrix in $\mathcal{U}$.
In the second chapter, we have studied some particular low complexity matrix algebras that are the Householder algebras; with the aim to apply our displacement theorems to the Householder algebras, we have investigated when they have some properties that can be interesting for an algebra of matrices: closure by conjugation, symmetry, persymmetry and non trivial intersection with algebras of the same class.
Working with the Householder matrices, we met the necessity to generalize them, so in the third chapter we present our generalization, the *Householder-type matrices*. We have noticed that this generalization had been already proposed in the 90' by Venkaiah, Krishna and Paulraj [11], although , in literature, we have found just a small number of basic results of those that we present in the third chapter.
We can say that the Householder-type matrices are a good generalization of the Householder matrices mainly for four reasons:

- If w and v are two real vectors of equal 2-norm, there always exists an Householder matrix U such that $Uw = v$. Nevertheless when w and v are two vectors as above, but with complex entries, the above property is true if and only if the scalar product $<w, v>$ is real. The Householder-type matrices have the capability to fill this gap of the Householder matrices in the complex case; indeed, given w and v two complex vectors of equal norm, there always exists an Householder-type matrix U_α such that $U_\alpha w = v$.

- The result described here, obtainable thanks to the introduction of the Householder-type matrices, can be seen as a consequence of the above one but, in our opinion, it is worth of consideration as much as the above.
  If U is a real unitary matrix, we know that U can be decomposed as a product of Householder matrices; but, again, in the complex case

this is not true anymore. Indeed a generic complex unitary matrix can be, at most, decomposed as a product of $n-1$ Householder matrices and a diagonal unitary matrix. Also in this case the Householder-type matrices fill this gap and we can demonstrate that every complex unitary matrix can be decomposed as a product of n Householder-type matrices.

- An Householder-type matrix has the shape $I-\alpha uu^h$, where u is a unitary vector and α is a coefficient that can move on the circumference (in the complex field) with center in 1 and radius 1. Fixed the vector u, the set of Householder-type matrices that we can define by such vector is a commutative group that contains the identity I and the Householder matrix $I-2uu^h$.

- Any unitary matrix that is a 1-rank variation of the identity matrix must be an Householder-type matrix. So the 1-rank matrix perturbing I must be Hermitian unless a complex factor α.

In the first part of the third chapter, after a presentation of the above results, we deepen some related questions. In particular we describe an *optimal* decomposition of a generic unitary matrix U in terms of Householder-type matrices, underlining its link with the spectral decomposition of U. Moreover, we find the best approximation, in both Frobenius and 2-norm, of a generic unitary matrix U by the product of a fixed number k of Householder-type matrices. This is obtained by simply removing the $n-k$ smallest singular values of $I-U$. We prove also that if two matrices, A and B, have the same Gram-matrix (i.e. $A^hA=B^hB$) then it is possible to transform A in B through a number of Householder-type matrices equal to the rank of A (this result is a generalization of an analogous one, for real matrices, found in [9] and usefull in conceving optimization and preconditioning procedures).
In the second part of the third chapter we investigate two possible numerical applications of the Householder-type matrices. However the study of both the applications needs still some work to be concluded.
The first one is a modification of the classical Householder QR decomposition of a matrix A(see [8],[15]) that, in order to define A_i at step i of the triangularization, uses the Householder-type matrix "closest" to the identity I to map in span$\{e_1\}$ the first column of the $(n+1-i)\times(n+1-i)$ tail submatrix of A_{i-1} $(A_0=A)$. This version of the QR algorithm could have

the capability to bound the expansion of the errors as much as possible, but this is still to prove. An interesting fact that comes from the study of this, in principle, optimal QR algorithm is that, before each step, it seems convenient to introduce a partial or a total pivoting, as it happens for the LU algorithm. In the second application, which has yet to be investigated in many details, Householder-type matrices are used to conceive an iterative procedure for the construction of the normal matrix closest to a generic matrix A (the problem is studied also in [12],[13]). The fact that every normal matrix is diagonalized by a unitary transform and the fact that every unitary matrix can be decomposed as a product of (at most n) Householder-type matrices, lead naturally to the proposed algorithm, which, starting from $A_0 = A$, at the generic step k, computes the Householder-type matrix U_α that minimizes the distance of a certain matrix A_k from its projection in the Householder-type SDU algebra defined by U_α. Such projections turn out to be normal matrices that approximate A better and better.
Apart the applications already described we expect that algebras SDU with U = Householder-type or U = products of Householder-type can be used to construct new efficient displacement formulas, and that the results of the first chapter (or some their improvements) can help to reach this aim, but this needs further work to be verified.

1. SDU Algebras

In the first part of this chapter we recall some properties and definitions of the algebras of matrices simultaneously diagonalized by a unitary matrix (SDU algebras) and then discuss the way an SDU algebra can be characterized by a combination of his rows or columns. In the second part of the chapter we present some new displacement theorems that allow to decompose a generic matrix into a sum of products of matrices in two different SDU algebras, which have the property to contain two matrices whose difference is a rank one Hermitian(symmetric) matrix with some good properties.

1.1 Definitions and basic properties

Definition. : If U is a unitary matrix we can define the space $\mathcal{U} = sdU$ as the algebra of the matrices simultaneously diagonalized by the matrix U. In particular $\forall\, z \in \mathbb{C}^n$ let's define $U[z] = UD(z)U^h$ the matrix $\in \mathcal{U}$ with eigenvalues $\lambda_i = z_i$. (So $D(z) = D(z_i) = diag(z_1, \ldots, z_n)$).

Remark 1.1.1. : The space sdV is equal to the space sdU iff $V = UPD$, where D is a unitary diagonal matrix and P a permutation matrix. Indeed

$$\mathcal{V}[z] = VD(z)V^h = UPDD(z)\overline{D}P^TU^h = UPD(z)D\overline{D}P^TU^h = UD(Pz)U^h = \mathcal{U}[Pz]\,.$$

On the other hand, if $sdV = sdU$ then $\forall\, z\ \exists y_z$ s.t. $\mathcal{U}[z] = \mathcal{V}[y_z]$. Noting that $(z)_i$ *and* $(y_z)_i$ are the eigenvalues of $\mathcal{U}[z]$, they have to be uniquely defined and so necessarily there exists a permutation matrix P^z s.t $y_z = P^z z$. So

$$UD(z)U^h = \mathcal{U}[z] = \mathcal{V}[P^z z] = VP^zD(z)(P^z)^TV^h.$$

Moreover, if we choose z s.t. $\forall i \neq j \quad z_i \neq z_j$, we can observe that the i-th columns of U and VP^z are both eigenvectors of the eigenvalue z_i and so they have to be unitary multiples one of the other; that is $Ue_i = e^{i\theta_i} VP^z e_i \quad \forall i = 1, ..., n$ or, equivalently, in matrix form : $U = VPD$ with $P = P^z$ (so P doesn't depend on z) and D unitary diagonal matrix.

Theorem 1.1.2. *Let $A \in \mathbb{C}^{n \times n}$. Then the best approximation in Frobenius norm of A in $\mathcal{U}$ is $\mathcal{U}_A := UD\big((U^h AU)_{ii}\big)U^h$.*

Proof. Consider the Hilbert space $\mathbb{C}^{n \times n}$ with the Frobenius inner product $(< M, N >_F = \sum_{i,j=1}^n \overline{M}_{ij} N_{ij})$. Then $\mathcal{U}$ is a closed subspace of $\mathbb{C}^{n \times n}$ and so, thanks to the Hilbert projection theorem, it is well defined the projection of A in $\mathcal{U}$, that is also its best approximation in the Frobenius norm
$(||M||_F^2 = \sum_{i,j=1}^n |M_{ij}|^2)$.
We want to minimize:

$$||A - UD(z)U^h||_F^2 = ||U^h AU - D(z)||_F^2 \tag{1.1}$$

So we have to take $z = Diag(U^h AU)$. □

Let's state a result that follows from the definition of $\mathcal{U}_A$ and the linearity of the projection:

Corollary 1.1.3. *$\mathcal{U}_A$ is real whenever A is real ($\overline{A} = A$) if and only if $\overline{\mathcal{U}} = \mathcal{U}$.*

Proof. ($\Leftarrow$) $||\mathcal{U}_A - A|| = ||\overline{\mathcal{U}_A} - \overline{A}|| = ||\overline{\mathcal{U}_A} - A||$
then since $\overline{\mathcal{U}_A} \in \mathcal{U}$ and the projection is unique: $\overline{\mathcal{U}_A} = \mathcal{U}_A$.
($\Rightarrow$) Consider a generic $\mathcal{U}[z]$, then it exists a matrix A such that
$\mathcal{U}[z] = \mathcal{U}_A \quad (Diag(U^h AU) = z)$. Now decompose:
$\mathcal{U}[z] = \mathcal{U}_{Re(A)} + i\,\mathcal{U}_{Im(A)} = \mathcal{U}[z'] + i\,\mathcal{U}[z'']$.
Since $Re(A)$ and $Im(A)$ are real matrices, $\mathcal{U}_{Re(A)}$ and $\mathcal{U}_{Im(A)}$ have to be real; this implies that:
$\Rightarrow \overline{\mathcal{U}[z]} = \overline{\mathcal{U}_A} = \mathcal{U}_{Re(A)} - i\,\mathcal{U}_{Im(A)} = UD(z')U^h - i\,UD(z'')U^h \in \mathcal{U}$

□

1.2 Characterization of a SDU Algebra

Let's now analyze when the algebra $\mathcal{U}$ is characterized uniquely by a combination of its rows or columns and let's introduce the notations $\mathcal{U}^{(v)}(z)$ and $\mathcal{U}_{(w)}(z)$. Similiar definitions can be found in [4] where is presented the concept of *s*pace of class $\mathbb{V}$, these are n-dimensional spaces such that each matrix of the space is uniquely identified thanks to its product by a fixed vector v.

Definition 1.2.1. Let v be a vector s.t. $(U^h v)_i \neq 0 \ \ \forall i\,; \quad \forall z \in \mathbb{C}^n$ let's define $\mathcal{U}^{(v)}(z) := UD(U^h z)D(U^h v)^{-1}U^h$.

It is the only matrix of $\mathcal{U}$ s.t. $\mathcal{U}^{(v)}(z)v = z$; indeed:

$$UD(U^h z)D(U^h v)^{-1}U^h v = UD(U^h v)^{-1}D(U^h z)U^h v = UD(U^h v)^{-1}D(U^h v)U^h z = z.$$

To demonstrate the uniqueness assume by contradiction that $\exists\, \mathcal{U}[y_1], \mathcal{U}[y_2]$ s.t. $\mathcal{U}[y_i]v = z, \ \ i = 1,2 \ \Rightarrow \ (\mathcal{U}[y_1] - \mathcal{U}[y_2])\, v = 0$
$\Rightarrow \ UD(y_1 - y_2)U^h v = 0 \Rightarrow \ D(y_1 - y_2)U^h v = 0 \Rightarrow \ (y_1 - y_2)_i = 0 \ \ \forall i.$

Note moreover that $\mathcal{U}^{(v)}(z)q = \mathcal{U}^{(v)}(q)z$ (see also [1]).

Similarly we can define $\mathcal{U}_{(w)}(z)$:

Definition 1.2.2. If w is a vector s.t. $(w^T U)_i \neq 0 \ \ \forall i$;
$\forall z \in \mathbb{C}^n$ let's define $\mathcal{U}_{(w)}(z) := UD(U^T z)D(U^T w)^{-1}U^h$.

It is the only matrix of $\mathcal{U}$ s.t. $w^T \mathcal{U}_{(w)}(z) = z^T$:

$$w^T UD(U^T z)D(U^T w)^{-1}U^h = z^T UD(U^T w)D(U^T w)^{-1}U^h = z^T\,.$$

The uniqueness can be easily demonstrated as above.
Note moreover that $q^t \mathcal{U}_{(w)}(z) = z^t \mathcal{U}_{(w)}(q)$.

Proposition 1.2.3. *If v characterizes $\mathcal{U}$ by columns, i.e.* $(U^h v)_i \neq 0 \ \ \forall i$, *then:*
$\{$ *w s.t. characterizes $\mathcal{U}$ by columns* $\} = \{\, \mathcal{U}[z]v \mid \mathcal{U}[z]$ *is nonsingular* $\}$

Proof. ($\subseteq$) If w characterizes $\mathcal{U}$ by columns $(U^h w)_i \neq 0\ \forall i$, then $\exists\, \mathcal{U}[z]$ nonsingular s.t. $\mathcal{U}[z]v = w$

$\Leftrightarrow\; D(z)U^hv = U^hw \quad \Leftrightarrow\; \forall i \quad z_i = \dfrac{(U^hw)_i}{(U^hv)_i}\,.$

The vector z is well defined and $\mathcal{U}[z]$ is nonsingular because product of nonsingular matrices.

($\supseteq$) If $\mathcal{U}[z]$ is nonsingular we want to show that $(U^h\mathcal{U}[z]v)_i \neq 0\ \ \forall i$. But

$$(U^h\mathcal{U}[z]v)_i = (D(z)U^hv)_i = z_i(U^hv)_i \neq 0\,,$$

indeed $\mathcal{U}[z]$ is nonsingular iff $z_i \neq 0\ \forall i$ and $(U^hv)_i$ are non zero because v characterizes $\mathcal{U}$.

□

A similar result can be easily formulated for a vector that characterizes $\mathcal{U}$ by rows.

Corollary 1.2.4. *Let v characterize $\mathcal{U}$ by columns and $w = \mathcal{U}[z]v$ with $\mathcal{U}[z]$ nonsingular. Then:*

$$\mathcal{U}^{(w)}(y)\mathcal{U}^{(v)}(w) = \mathcal{U}^{(w)}(y)\mathcal{U}[z] = \mathcal{U}^{(v)}(y).$$

Proof. It is easy to observe that the three matrices have all the property to be in $\mathcal{U}$ and to map v in y. So, because of the uniqueness, they have to be equal. □

Example 1.2.5. *If $U = I - 2uu^h$, $||u|| = 1$, is an Householder matrix and $u_i \neq 0\ \forall i$, then $\{w$ that characterize $\mathcal{U}\} = \{\mathcal{U}[z]u \mid \mathcal{U}[z]$ is nonsingular $\}$. Indeed, $U^hu = Uu = -u$ and so we can apply Proposition 1.2.3.*

Proposition 1.2.6. *Note that, if v characterizes $\mathcal{U}$ by columns, $\overline{v}$ characterizes $\mathcal{U}$ by rows indeed:*

$$(U^hv)_i \neq 0\ \ \forall\, i \iff (\overline{v}^tU)_i \neq 0\ \ \forall i\,.$$

In particular we can observe that $\mathcal{U}^{(v)}(z) = \mathcal{U}_{(\overline{v})}\left(\overline{U}D\Big(\dfrac{(\overline{U^hv})_i}{(U^hv)_i}\Big)U^hz\right)$*:*

$$\begin{aligned} v^h\mathcal{U}^{(v)}(z) &= v^hUD(U^hv)^{-1}D(U^hz)U^h \\ &= v^hUD(U^t\overline{v})^{-1}D(U^t\overline{v})D(U^hv)^{-1}D(U^hz)U^h = z^t\overline{U}D\left(\frac{(\overline{U^hv})_i}{(U^hv)_i}\right)U^h\,. \end{aligned}$$

1.3 Displacement Formulas

Now we will present two new diplacement theorems, the second one is a complex generalization of the first, but we have distinguished the two cases because, in order to demonstrate the complex version, one needs some extra work. The idea of the proofs is similar to the one adopted in [1], where some general displacement theorems, involving Hessenberg algebras, are proved; as a consequence of them, the authors deduce the displacement formulas of Gohberg-Semencul, Gader, Bini-Pan, Gohberg-Olshevsky and some other formulas, for example formulas involving matrices of the same algebra but of different dimensions. Our new theorems seem not to be a generalization of any of the above known results, nevertheless, their hypotheses are satisfiable for example by some low complexity SDU algebras as the ones presented in [2] and [3]. However, these are just an example that ensures that the theorems can be applied; it is an open problem to look for other low complexity unitary matrices such that the associated SDU algebras satisfy our hypotheses.

Lemma 1.3.1. *(See also [1], [2])*
Given $A \in \mathcal{C}^{n\times n}$*, if* $\mathfrak{C}_{\mathcal{U}[z]}(A) = A\mathcal{U}[z] - \mathcal{U}[z]A = \sum_{i=1}^{k} x_i y_i^t$*,*
then $\sum_{i=1}^{k} x_i^t \mathcal{U}^t[w] y_i = 0 \quad \forall w \in \mathbb{C}^n$*.*

Proof.

$$\sum_{i=1}^{k} x_i^t \mathcal{U}^t[w] y_i = \sum_{i=1}^{k} \sum_{h,j=1}^{n} x_{ih} \mathcal{U}^t[w]_{hj} y_{ij} = \sum_{h,j=1}^{n} \sum_{i=1}^{k} x_{ih} \mathcal{U}^t[w]_{hj} y_{ij} =$$
$$\sum_{h,j=1}^{n} \mathcal{U}[w]_{jh} \sum_{i=1}^{k} (x_i y_i^t)_{hj} = \sum_{h,j=1}^{n} \mathcal{U}[w]_{jh} (A\mathcal{U}[z] - \mathcal{U}[z]A)_{hj} =$$
$$Tr\big(\mathcal{U}[w](A\mathcal{U}[z] - \mathcal{U}[z]A)\big) = Tr\big(\mathcal{U}[w]A\mathcal{U}[z] - \mathcal{U}[z]\mathcal{U}[w]A)\big) = 0\,,$$

where the last equality follows since the matrices $\mathcal{U}[w]A\mathcal{U}[z]$ and $\mathcal{U}[z]\mathcal{U}[w]A$ have the same characteristic polynomial. □

Lemma 1.3.2. $Ker(\mathfrak{C}_{\mathcal{U}[z]}) = \{A \in \mathbb{C}^{n\times n}\ s.t.\ A\mathcal{U}[z] - \mathcal{U}[z]A = 0\} = \{UAU^h\ with\ A \in \mathbb{C}^{n\times n}\ s.t.\ AD(z) - D(z)A = 0\}\,.$

Proof. Trivial. □

Definition 1.3.3. A matrix A is non derogatory if and only if there is only one eigenvector associated with each distinct eigenvalue λ, if and only if its minimum polynomial is equal to its characteristic polynomial.

Note that if A is a diagonalizable nonderogatory matrix, then it has all distinct eiegnvalues.

Theorem 1.3.4. (I displacement theorem for SDU algebras)
Let $U, V \in \mathbb{R}^{n\times n}$ be two unitary real matrices and $w \in \mathbb{R}^n$ a vector s.t. characterizes $\mathcal{U}$ by colums and $\mathcal{V}$ by rows; assume that there exists $z \in \mathbb{R}^n$ such that $\mathcal{U}[z] + ww^t = \mathcal{V}[z'] \in \mathcal{V}$. Then we can say :

1. Given $A \in \mathbb{R}^{n\times n}$, if $\mathfrak{C}_{\mathcal{U}[z]}(A) = A\mathcal{U}[z] - \mathcal{U}[z]A = \sum_{i=1}^k x_i y_i^t$ [1], then $A = \sum_{i=1}^k \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i) + C$ where C is a matrix that commutes with $\mathcal{U}[z]$.
2. If $\mathcal{U}[z]$ is non-derogatory, that is $z_i \neq z_j \ \forall i \neq j$, then $C = \mathcal{U}^{(w)}\Big(Aw - \sum_{i=1}^k \big(\mathcal{U}^{(w)}(x_i)y_i\big)\Big)$.

Proof. Let's show that $\mathfrak{C}_{\mathcal{U}[z]}(\sum_{i=1}^k \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)) = \sum_{i=1}^k x_i y_i^t = \mathfrak{C}_{\mathcal{U}[z]}(A)$:

$$\sum_{i=1}^k \big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)\big)\mathcal{U}[z] - \mathcal{U}[z]\sum_{i=1}^k \big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)\big) =$$
$$\sum_{i=1}^k \big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)\big)(\mathcal{V}[z'] - ww^t) - \sum_{i=1}^k \big(\mathcal{U}^{(w)}(x_i)\mathcal{U}[z]\mathcal{V}_{(w)}(y_i)\big) =$$
$$\sum_{i=1}^k \big(\mathcal{U}^{(w)}(x_i)\mathcal{V}[z']\mathcal{V}_{(w)}(y_i)\big) - \sum_{i=1}^k \big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)\big)(ww^t) -$$
$$-\sum_{i=1}^k \big(\mathcal{U}^{(w)}(x_i)\mathcal{U}[z]\mathcal{V}_{(w)}(y_i)\big) =$$
$$\sum_{i=1}^k \Big(\mathcal{U}^{(w)}(x_i)\Big(\mathcal{V}[z'] - \mathcal{U}[z]\Big)\mathcal{V}_{(w)}(y_i)\Big) - \sum_{i=1}^k \big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)\big)(ww^t) =$$
$$\sum_{i=1}^k \Big(\mathcal{U}^{(w)}(x_i)\Big(ww^t\Big)\mathcal{V}_{(w)}(y_i)\Big) - \sum_{i=1}^k \big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)\big)(ww^t) =$$
$$\sum_{i=1}^k \big(x_i y_i^t\big) - \sum_{i=1}^k \big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)\big)(ww^t).$$

So, if we show that $\sum_{i=1}^k \big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)\big)(ww^t) = 0$, we have proved the first statement.

[1] We assume the x_i and y_i real vectors.

Let's analyze the j-th row of the above equation (remind that $\mathcal{V}_{(w)}(y_i)$, $\mathcal{U}^{(w)}(x_i)$ are symmetric and remind Lemma 1.3.1):

$$e_j^t \sum_{i=1}^{k} \big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)\big)(ww^t) = \sum_{i=1}^{k} \Big(e_j^t\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)w\Big)w^t =$$
$$\sum_{i=1}^{k} \Big(e_j^t\mathcal{U}^{(w)}(x_i)\big(w^t\mathcal{V}_{(w)}(y_i)\big)^t\Big)w^t = \sum_{i=1}^{k} \Big(\big(\mathcal{U}^{(w)}(x_i)e_j\big)^t\big(y_i^t\big)^t\Big)w^t =$$
$$\sum_{i=1}^{k} \Big(\big(\mathcal{U}^{(w)}(e_j)x_i\big)^t y_i\Big)w^t = \sum_{i=1}^{k} \Big(x_i^t\big(\mathcal{U}^{(w)}(e_j)\big)^t y_i\Big)w^t = 0\,.$$

To prove the second statement we can observe that, thanks to Lemma 1.3.3, if $\mathcal{U}[z]$ is non-derogatory, it generates the whole algebra $\mathcal{U}$ and the commutator of $\mathcal{U}[z]$ is the algebra $\mathcal{U}$. From this it follows that C is a matrix of the algebra $\mathcal{U}$; that is $\exists z'' \in \mathbb{C}^n$ s.t. $C = \mathcal{U}^{(w)}(z'')$. Since

$$C = A - \sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)\,,$$

it follows that:

$$z'' = \Big(A - \sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)\Big)w = Aw - \sum_{i=1}^{k} \big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)\big)w =$$
$$= Aw - \sum_{i=1}^{k} \big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)w\big) = Aw - \sum_{i=1}^{k} \big(\mathcal{U}^{(w)}(x_i)y_i\big).$$

□

Remark. :
If A is a complex matrix we can apply the preceding theorem, where $\mathcal{U}[z] \in \mathbb{R}^{n\times n}$, to the matrices $Re(A)$ and $Im(A)$.
If $\mathfrak{C}_{\mathcal{U}[z]}(A) = A\mathcal{U}[z] - \mathcal{U}[z]A = \sum_{i=1}^{k} x_i y_i^t$, decomposing each x_i and y_i as $x_i = Re(x_i) + iIm(x_i),\ y_i = Re(y_i) + iIm(y_i)$, it follows that:

$$\mathfrak{C}_{\mathcal{U}[z]}(A) = \sum_{i=1}^{k} Re(x_i)Re(y_i)^t - Im(x_i)Im(y_i)^t$$
$$+\, i\Big(\sum_{i=1}^{k} Re(x_i)Im(y_i)^t + Im(x_i)Re(y_i)^t\Big)\,,$$

where $\left(\sum_{i=1}^{k} Re(x_i)Re(y_i)^t - Im(x_i)Im(y_i)^t\right)$ is a (non optimal) skeleton decomposition of $\mathfrak{C}_{\mathcal{U}[z]}(Re(A))$ and $\left(\sum_{i=1}^{k} Re(x_i)Im(y_i)^t + Im(x_i)Re(y_i)^t\right)$ is a skeleton decomposition of $\mathfrak{C}_{\mathcal{U}[z]}(Im(A))$. So A can always be decomposed as:

$$A = \sum_{i=1}^{k} \mathcal{U}^{(w)}(Re(x_i))\mathcal{V}_{(w)}(Re(y_i)) - \mathcal{U}^{(w)}(Im(x_i))\mathcal{V}_{(w)}(Im(y_i))$$
$$+ i\left(\sum_{i=1}^{k} \mathcal{U}^{(w)}(Re(x_i))\mathcal{V}_{(w)}(Im(y_i)) + \mathcal{U}^{(w)}(Im(x_i))\mathcal{V}_{(w)}(Re(y_i))\right) + C_1 + iC_2\,,$$

where C_1 and C_2 are matrices that can be determined as in Theorem1.3.4.

Now we want to generalize Theorem 1.3.4 to the complex case; to do it we want two matrices $\mathcal{U}[z]$, $\mathcal{V}[z']$ such that $\mathcal{V}[z'] - \mathcal{U}[z] = ww^h$, where w characterizes $\mathcal{U}$ by columns and $\overline{w}$ characterizes $\mathcal{V}$ by rows.
Assume that $\mathcal{U}[z] \in \mathcal{U}$ and $w \in \mathbb{C}^n$ is such that characterizes $\mathcal{U}$ by columns; if $\mathcal{U}[z] + ww^h \in \mathcal{V}$ for some space SDV (i.e. $A + ww^h$ is a normal matrix), then z has to be such that $z - \overline{z} = k\underline{1}$.
Indeed if w characterizes $\mathcal{U}$ by columns, thanks to Def 1.2.1, it can be written as $w = Ux$ where $x_i \neq 0 \ \forall i$.
So we are looking for a matrix $\mathcal{U}[z]$ such that $\mathcal{U}[z] + Uxx^hU^h \in \mathcal{V}$;
let's impose that $\mathcal{U}[z] + Uxx^hU^h$ is a normal matrix:

$$U\Big(D(z) + xx^h\Big)U^hU\Big(D(\overline{z}) + xx^h\Big)U^h = U\Big(D(\overline{z}) + xx^h\Big)U^hU\Big(D(z) + xx^h\Big)U^h\,,$$
$$D(|z|^2) + ||x||^2xx^h + D(z)xx^h + xx^hD(\overline{z}) = D(|z|^2) + ||x||^2xx^h + D(\overline{z})xx^h + xx^hD(z)\,,$$
$$D(z)xx^h + xx^hD(\overline{z}) = D(\overline{z})xx^h + xx^hD(z)\,,$$
$$D(z-\overline{z})xx^h = xx^hD(z-\overline{z})\,.$$

Since $D(z-\overline{z})$ has to commute with xx^h, that is a full non-zero matrix, $D(z-\overline{z})$ must be equal to kI for some $k \in \mathbb{C}$.
Hence, as new hypotheses for the generalization of Thm 1.3.4, let's assume that z and z' are real vectors (i.e. $\mathcal{U}[z]$, $\mathcal{V}[z']$ are Hermitian matrices), in this way the above condition is easily satisfied.
Another important property that we have used in the proof of Theorem 1.3.4 is the symmetry of the matrices $\mathcal{U}^{(w)}(x_i)$ and $\mathcal{V}_{(w)}(y_i)$; so, to generalize it in

the complex case, we want to make them Hermitian matrices.
The matrix $\mathcal{U}^{(w)}(x)$ is an Hermitian matrix iff $D(U^h w)^{-1}U^h x$ is a real vector and the matrix $\mathcal{V}_{\overline{w}}(y)$ is an Hermitian matrix iff $D(V^t\overline{w})^{-1}V^t y$ is a real vector. So, analogously to before when we have applied the theorem to the real matrices, now we want to apply the theorem to matrices A such that

$$D(U^h w)^{-1}U^h \mathfrak{C}_{\mathcal{U}[z]}(A) V D(V^t\overline{w})^{-1}$$

is a real matrix.
The next lemma is usefull to understand which frame have such matrices A.

Lemma 1.3.5. *:*
Let $U, V \in \mathbb{C}^{n\times n}$ be two unitary matrices, $w \in \mathbb{C}^n$ a vector s.t. characterizes $\mathcal{U}$ by colums and $\overline{w}$ characterizes $\mathcal{V}$ by rows, and $z \in \mathbb{R}^n$ a real vector such that $\mathcal{U}[z] + ww^h = \mathcal{V}[z'] \in \mathcal{V}$.
If $D(U^h w)^{-1}U^h A V D(V^t\overline{w})^{-1}$ is a real matrix, then $D(U^h w)^{-1}U^h\Big(A\mathcal{U}[z] - \mathcal{U}[z]A\Big)VD(V^t\overline{w})^{-1}$ is still a real matrix.

Proof. Let's consider the real skeleton decomposition

$$D(U^h w)^{-1}U^h A V D(V^t\overline{w})^{-1} = \sum_{i=1}^{k} h_i k_i^t\,,$$

then we can consider the following skeleton decomposition of A:
$A = \sum_{i=1}^{k} U D(U^h w) h_i k_i^t D(V^t\overline{w}) V^h = \sum_{i=1}^{k} x_i y_i^t$,
where we have defined $x_i := U D(U^h w) h_i$ and $y_i := \overline{V} D(V^t\overline{w}) k_i$.
Then

$$D(U^h w)^{-1}U^h\Big(A\mathcal{U}[z] - \mathcal{U}[z]A\Big)VD(V^t\overline{w})^{-1} =$$

$$D(U^h w)^{-1}U^h\Big(\sum_{i=1}^{k} x_i y_i^t\,\mathcal{U}[z] - \mathcal{U}[z]\sum_{i=1}^{k} x_i y_i^t\Big)VD(V^t\overline{w})^{-1} =$$

$$\Big(\sum_{i=1}^{k} D(U^h w)^{-1}U^h x_i y_i^t\,\mathcal{U}[z] V D(V^t\overline{w})^{-1} - \sum_{i=1}^{k} D(U^h w)^{-1}U^h\mathcal{U}[z] x_i y_i^t V D(V^t\overline{w})^{-1}\Big)\,.$$

Here we already know that $D(U^h w)^{-1}U^h x_i = h_i$ and $y_i^t V D(V^t\overline{w})^{-1} = k_i^t$ are real vectors; so, to demonstrate the thesis, it is enough to observe that $y_i^t\,\mathcal{U}[z]VD(V^t\overline{w})^{-1}$ and $D(U^h w)^{-1}U^h\mathcal{U}[z]x_i$ are real vectors.

Let's analyze $D(U^h w)^{-1} U^h \mathcal{U}[z] x_i$:

$$D(U^h w)^{-1} U^h \mathcal{U}[z] x_i = D(U^h w)^{-1} U^h U D(z) U^h x_i =$$
$$D(z) D(U^h w)^{-1} U^h x_i = D(z) h_i$$

Since both z and h_i are real, $D(U^h w)^{-1} U^h \mathcal{U}[z] x_i$ is real.
Now analyze $y_i^t \mathcal{U}[z] V D(V^t \overline{w})^{-1}$:

$$y_i^t \mathcal{U}[z] V D(V^t \overline{w})^{-1} = y_i^t \Big(\mathcal{V}[z'] - w w^h\Big) V D(V^t \overline{w})^{-1} =$$
$$y_i^t V D(z') V^h V D(V^t \overline{w})^{-1} - y_i^t w w^h V D(V^t \overline{w})^{-1} =$$
$$y_i^t V D(V^t \overline{w})^{-1} D(z') - y_i^t w \Big(V^t \overline{w}\Big)^t D(V^t \overline{w})^{-1} =$$
$$k_i^t D(z') - \big(w^h \mathcal{V}_{(\overline{w})}(y_i) w\big)\, \underline{1}^t$$

Here, k_i is real by construction, z' is real since $\mathcal{V}[z']$ is an Hermitian matrix and[2] $w^h \mathcal{V}_{(\overline{w})}(y_i) w$ is real since, as we can easily observe, $\mathcal{V}_{(\overline{w})}(y_i)$ is an Hermitian matrix :

$$\mathcal{V}_{(\overline{w})}(y_i) = V D(V^t \overline{w})^{-1} D(V^t y_i) V^h = V D(V^t \overline{w})^{-1} D(V^t \overline{V} D(V^t \overline{w}) k_i) V^h =$$
$$V D(V^t \overline{w})^{-1} D(V^t \overline{w}) D(k_i) V^h = V D(k_i) V^h = \big(V D(k_i) V^h\big)^h = \big(\mathcal{V}_{(\overline{w})}(y_i)\big)^h.$$

So also $y_i^t \mathcal{U}[z] V D(V^t \overline{w})^{-1}$ is a real vector. □

Theorem 1.3.6. ***(II displacement theorem for SDU algebras)***
Let $U, V \in \mathbb{C}^{n \times n}$ be two unitary matrices, $w \in \mathbb{C}^n$ a vector s.t. characterizes $\mathcal{U}$ by colums and $\overline{w}$ characterizes $\mathcal{V}$ by rows. Assume that $\exists z \in \mathbb{R}^n$ such that $\mathcal{U}[z] + w w^h = \mathcal{V}[z'] \in \mathcal{V}$. Then:

1. *Given $A \in \mathbb{C}^{n \times n}$ such that $D(U^h w)^{-1} U^h A V D(V^t \overline{w})^{-1}$ is a real matrix, if $\mathfrak{C}_{\mathcal{U}[z]}(A) = A\mathcal{U}[z] - \mathcal{U}[z] A = \sum_{i=1}^k x_i y_i^t$, where $\Big(D(U^h w)^{-1} U^h x_i\Big)$ and $\Big(D(V^t \overline{w})^{-1} V^t y_i\Big)$ are real vectors,[3] then $A = \sum_{i=1}^k \mathcal{U}^{(w)}(x_i) \mathcal{V}_{(\overline{w})}(y_i) + C$ where C is a matrix that commutes with $\mathcal{U}[z]$.*

[2] $\underline{1}$:is the vector with each component equal to 1
[3] this is possible thanks to Lemma 1.3.5

2. *If $\mathcal{U}[z]$ is non-derogatory, that is $z_i \neq z_j \; \forall i \neq j$, then* $C = \mathcal{U}^{(w)}\Big(Aw - \sum_{i=1}^{k} \big(\mathcal{U}^{(w)}(x_i)\overline{y_i}\big)\Big)$.

Proof. C commutes with $\mathcal{U}[z]$ iff $\mathfrak{C}_{\mathcal{U}[z]}(A) = \mathfrak{C}_{\mathcal{U}[z]}\Big(\sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i)\Big)$. Hence, let's analyze the right side of the latter equality :

$$\sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i)\mathcal{U}[z] - \mathcal{U}[z]\sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i) =$$
$$\sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i)\Big(- ww^h + \mathcal{V}[z']\Big) - \sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{U}[z]\mathcal{V}_{(\overline{w})}(y_i) =$$
$$\sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\Big(\mathcal{V}[z'] - \mathcal{U}[z]\Big)\mathcal{V}_{(\overline{w})}(y_i) - \sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i)\Big(ww^h\Big) =$$
$$\sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\Big(ww^h\Big)\mathcal{V}_{(\overline{w})}(y_i) - \sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i)\Big(ww^h\Big) =$$
$$\sum_{i=1}^{k} x_i y_i^t - \sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i)\Big(ww^h\Big) = \mathfrak{C}_{\mathcal{U}[z]}(A) - \sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i)\Big(ww^h\Big).$$

Thus we have only to show that $\sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i)\Big(ww^h\Big) = 0$.
Look at its j-th row:

$$e_j^t \sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i)\Big(ww^h\Big) = \sum_{i=1}^{k} e_j^t \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i) w\Big(w^h\Big) =$$
$$\sum_{i=1}^{k} \Big(\big(\mathcal{U}^{(w)}(x_i)\big)^h e_j\Big)^h \Big(w^h \big(\mathcal{V}_{(\overline{w})}(y_i)\big)^h\Big)^h \Big(w^h\Big). \tag{1.2}$$

But, both $\Big(D(U^h w)^{-1} U^h x_i\Big) =: \xi_i$ and $\Big(D(V^t \overline{w})^{-1} V^t y_i\Big) =: \eta_i$ are real vectors, so it is easy to demonstrate that the matrices $\big(\mathcal{U}^{(w)}(x_i)\big)$ and $\big(\mathcal{V}_{(\overline{w})}(y_i)\big)$ are Hermitian matrices :

$$\big(\mathcal{U}^{(w)}(x_i)\big) = UD(U^h w)^{-1} D(U^h x_i) U^h =$$
$$UD(U^h w)^{-1} D(U^h U D(U^h w)\xi_i) U^h = UD(\xi_i)U^h ,$$

$$\big(\mathcal{V}_{(\overline{w})}(y_i)\big) = VD(V^t \overline{w})^{-1} D(V^t y_i) V^h =$$
$$VD(V^t \overline{w})^{-1} D(V^t \overline{V} D(V^t \overline{w})\eta_i) V^h = VD(\eta_i)V^h .$$

So, using Lemma 1.3.1, the (1.2) becomes:

$$\sum_{i=1}^{k}\Big(\big(\mathcal{U}^{(w)}(x_i)\big)^h e_j\Big)^h\Big(w^h\big(\mathcal{V}_{(\overline{w})}(y_i)\big)^h\Big)^h=\sum_{i=1}^{k}\Big(\big(\mathcal{U}^{(w)}(x_i)\big)e_j\Big)^h\Big(w^h\big(\mathcal{V}_{(\overline{w})}(y_i)\big)\Big)^h=$$
$$\sum_{i=1}^{k}\Big(\big(\mathcal{U}^{(w)}(e_j)\big)x_i\Big)^h\Big(y_i^t\Big)^h=\sum_{i=1}^{k}x_i^h\big(\mathcal{U}^{(w)}(e_j)\big)^h\overline{y_i}=$$
$$conj\Big(\sum_{i=1}^{k}x_i^t\big(\mathcal{U}^{(w)}(e_j)\big)^t y_i\Big)=0\,.$$

If the matrix $\mathcal{U}[z]$ is non derogatory, the commutator of $\mathcal{U}[z]$ is the algebra $\mathcal{U}$; thus C is in the algebra $\mathcal{U}$,

$$C=\mathcal{U}^{(w)}(\varphi)=A-\sum_{i=1}^{k}\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i)\,,$$

and it can be characterized by the combination of his columns given by the multiplication on the right by w, that is:

$$\begin{aligned}\varphi&=\Big(A-\sum_{i=1}^{k}\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i)\Big)w\\&=Aw-\sum_{i=1}^{k}\mathcal{U}^{(w)}(x_i)\Big(w^h\big(\mathcal{V}_{(\overline{w})}(y_i)\big)^h\Big)^h\\&=Aw-\sum_{i=1}^{k}\mathcal{U}^{(w)}(x_i)\overline{y_i}.\end{aligned}$$

This proves also the second statement. □

Remark. This theorem is a generalization of Theorem 1.3.4; indeed, in the real case, it is easy to observe that the term $D(U^hw)^{-1}U^hAVD(V^t\overline{w})^{-1}$ is real if and only if A is real.

Remark. If A is a generic matrix, we can always decompose it as a sum of two matrices $A=A_1+iA_2$ where A_1 and A_2 are defined as:

$$A_1=UD(U^hw)\Big(Re\Big(D(U^hw)^{-1}U^hAVD(V^t\overline{w})^{-1}\Big)\Big)D(V^t\overline{w})V^h\,,$$

$$A_2=UD(U^hw)\Big(Im\Big(D(U^hw)^{-1}U^hAVD(V^t\overline{w})^{-1}\Big)\Big)D(V^t\overline{w})V^h\,.$$

Both A_1 and A_2 satisfy the hypotheses of the theorem; so, to find a decomposition of the matrix A, we can apply the previous result separately to the

matrices A_1 and A_2.
In particular, if

$$\mathfrak{C}_{\mathcal{U}[z]}(A) = \sum_{i=1}^{k} \varphi_i \psi_i^t \,,$$

is a skeleton decomposition of $\mathfrak{C}_{\mathcal{U}[z]}(A)$, we can decompose each of φ_i and ψ_i as:

$$\varphi_i = UD(U^h w)\Big(Re\Big(D(U^h w)^{-1} U^h \varphi_i\Big)\Big) + iUD(U^h w)\Big(Im\Big(D(U^h w)^{-1} U^h \varphi_i\Big)\Big) \,,$$

$$\psi_i = \overline{V} D(V^t \overline{w})\Big(Re\Big(D(V^t \overline{w})^{-1} V^t \psi_i\Big)\Big) + i\overline{V} D(V^t \overline{w})\Big(Im\Big(D(V^t \overline{w})^{-1} V^t \psi_i\Big)\Big) \,;$$

for simplicity, if we rename

$$x_i := UD(U^h w)\Big(Re\Big(D(U^h w)^{-1} U^h \varphi_i\Big)\Big) \,,$$

$$y_i := UD(U^h w)\Big(Im\Big(D(U^h w)^{-1} U^h \varphi_i\Big)\Big) \,,$$

$$h_i := \overline{V} D(V^t \overline{w})\Big(Re\Big(D(V^t \overline{w})^{-1} V^t \psi_i\Big)\Big) \,,$$

$$k_i := \overline{V} D(V^t \overline{w})\Big(Im\Big(D(V^t \overline{w})^{-1} V^t \psi_i\Big)\Big) \,,$$

then we have $\mathfrak{C}_{\mathcal{U}[z]}(A) = \sum_{i=1}^{k} \Big(x_i h_i^t - y_i k_i^t\Big) + i \sum_{i=1}^{k} \Big(x_i k_i^t + y_i h_i^t\Big) \,,$
where
$\sum_{i=1}^{k} \Big(x_i h_i^t - y_i k_i^t\Big)$ is a not-optimal skeleton decomposition of the matrix A_1
and
$\sum_{i=1}^{k} \Big(x_i k_i^t + y_i h_i^t\Big)$ is a not-optimal skeleton decomposition of the matrix A_2.

Thus

$$\begin{aligned} A = & \sum_{i=1}^{k} \Big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(h_i) - \mathcal{U}^{(w)}(y_i)\mathcal{V}_{(\overline{w})}(k_i)\Big) + \\ & i \sum_{i=1}^{k} \Big(\mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(k_i) - \mathcal{U}^{(w)}(y_i)\mathcal{V}_{(\overline{w})}(h_i)\Big) + C_1 + iC_2, \end{aligned} \tag{1.3}$$

where, if $\mathcal{U}[z]$ is nonderogatory, C_1 and C_2 can be determined as in the theorem .

Remark. Note that if $\mathcal{U}[z] + Uxx^hU^h = \mathcal{V}[z']$, the unitary matrix $W = U^hV$ is such that $U^hVD(z')V^hU = D(z) + xx^h$; so the columns of W, $\{w_i\}$, are the eigenvectors of $D(z) + xx^h$:

$$\lambda_i w_i = (D(z) + xx^h)w_i \,,$$
$$w_i = (x^h w_i)(\lambda_i I - D(z))^{-1}x.$$

Thus:

$$W_{ij} = (x^h w_j)\frac{x_i}{(\lambda_j - z_i)} \,;$$

this means that W must be a unitary Cauchy-like matrix (see [18] section 12.1).

We can also define the real unitary Cauchy-like matrix

$$N := \left(diag\Big(\frac{|x_i|}{x_i}\Big)\right)W\left(diag\Big(\frac{|x^h w_i|}{x^h w_i}\Big)\right)$$

that is :

$$N_{ij} = |x^h w_j|\frac{|x_i|}{\lambda_i - z_j}.$$

So, if we want to look for SDU algebras that satisfy the hypotheses of Theorems 1.3.4 and 1.3.6, we could try to study when a Cauchy-like matrix (real or complex) is also unitary.

1.3.1 Applications

Let's start recalling some results; these will ensure that some SDU algebras satisfy all the hypotheses of Theorems 1.3.6 and 1.3.4.

- By proposition (2.1) of [1] we know that an Hessenberg matrix X is non-derogatory if and only if $b_i \neq 0 \;\forall\, i$.

$$X = \begin{pmatrix} r_{11} & b_1 & 0 & \dots & 0 \\ r_{21} & r_{22} & b_2 & & \vdots \\ \vdots & & \ddots & \ddots & . \\ . & . & . & . & b_{n-1} \\ r_{n1} & . & . & \dots & r_{nn} \end{pmatrix}$$

- By proposition (4.2) of [2] we know that the algebra generated by $T^{\beta,\beta}_{\varepsilon,\varphi}$ is a symmetric 1-algebra if the matrix $I_{\beta\varphi}$ is nonsingular.
  A 1-algebra is an algebra characterized by its first row; note that, since the algebra is symmetric, if e_1 characterizes the algebra by rows then it characterizes the algebra also by columns.

$$T^{\beta,\beta}_{\varepsilon,\varphi} = \begin{pmatrix} \varepsilon & 1 & 0 & . & . & & \beta \\ 1 & 0 & 1 & . & & & 0 \\ 0 & 1 & . & . & . & & . \\ . & . & . & . & . & . & . \\ . & & . & . & . & 1 & 0 \\ 0 & & & & 1 & 0 & 1 \\ \beta & 0 & . & . & 0 & 1 & \varphi \end{pmatrix}$$

$$I_{\beta\varphi} = \begin{pmatrix} \beta & & & & \\ & . & & & \\ & & . & & \\ & & & . & \\ & & & & \beta \end{pmatrix} + \begin{pmatrix} & & & & 1 \\ & & & 1 & -\varphi \\ & & . & . & \\ & 1 & . & & \\ 1 & -\varphi & & & \end{pmatrix}$$

- From [3] we know the matrices that diagonalize the algebra generated by $T_{\varepsilon,\varphi}$:

$$T_{\varepsilon,\varphi} = \begin{pmatrix} \varepsilon & 1 & 0 & . & & 0 \\ 1 & 0 & 1 & . & & 0 \\ 0 & 1 & . & . & . & \\ . & . & . & . & . & . \\ 0 & & & 1 & 0 & 1 \\ 0 & 0 & . & 0 & 1 & \varphi \end{pmatrix}$$

  1. $T_{0,0} = \tau$ is diagonalized by the matrix:

$$M_{00} = \sqrt{\frac{2}{n+1}}\Big(\sin\frac{ij\pi}{n+1}\Big), \quad i,j = 1,\dots,n,$$

$$M_{00}T_{00}M_{00} = 2Diag\Big(\cos\frac{j\pi}{n+1}\Big), \quad j = 1,\dots,n.$$

2. $T_{1,1}$ is diagonalized by the matrix:

$$M_{11} = \sqrt{\frac{2}{n}}\left(k_j \cos\frac{(2i+1)j\pi}{2n}\right), \quad i,j = 0,\dots,n-1,$$

$$M_{11}^t T_{11} M_{11} = 2Diag\left(\cos\frac{j\pi}{n}\right), \quad j = 0,\dots,n-1,$$

where $k_j = \frac{1}{\sqrt{2}}$ for $j = 0, n-1$ and $k_j = 1$ otherwise.

3. $T_{-1,-1}$ is diagonalized by the matrix:

$$M_{-1-1} = \sqrt{\frac{2}{n}}\left(k_j \sin\frac{(2i-1)j\pi}{2n}\right), \quad i,j = 1,\dots,n,$$

$$M_{-1-1}^t T_{-1-1} M_{-1-1} = 2Diag\left(\cos\frac{j\pi}{n}\right), \quad j = 1,\dots,n,$$

where $k_j = \frac{1}{\sqrt{2}}$ for $j = 1, n$ and $k_j = 1$ otherwise.

4. $T_{1,-1}$ is diagonalized by the matrix:

$$M_{1\,-1} = \sqrt{\frac{2}{n}}\left(\cos\frac{(2i+1)(2j+1)\pi}{4n}\right), \quad i,j = 0,\dots,n-1,$$

$$M_{1\,-1} T_{1\,-1} M_{1\,-1} = 2Diag\left(\cos\frac{(2j+1)\pi}{2n}\right), \quad j = 0,\dots,n-1.$$

5. $T_{-1,1}$ is diagonalized by the matrix:

$$M_{-1\,1} = \sqrt{\frac{2}{n}}\left(\sin\frac{(2i+1)(2j+1)\pi}{4n}\right), \quad i,j = 0,\dots,n-1,$$

$$M_{-1\,1} T_{-1\,1} M_{-1\,1} = 2Diag\left(\cos\frac{(2j+1)\pi}{2n}\right), \quad j = 0,\dots,n-1.$$

All of these cosine and sine transforms are fast transforms. Moreover, all the algebras generated by these $T_{\varepsilon,\varphi}$, thanks to the previous remark, are 1-algebras and the matrices $T_{\varepsilon,\varphi}$, from the first remark, are non-derogatory matrices.
Note also that, for example:

$$T_{-1,-1} + 2e_1 e_1^t = T_{1,-1} \qquad T_{-1,1} + 2e_1 e_1^t = T_{1,1},$$

So we can apply our diplacement theorems to this algebras.
Note that it is also possible to apply our results to the algebras generated by two matrices $T^{\beta,\beta}_{\varepsilon,\varphi}$ and $T^{\beta,\beta}_{\varepsilon',\varphi}$ such that $I_{\beta,\varphi}$ is non singular and $\varepsilon, \varepsilon', \varphi, \beta$ are real, indeed:

$$T^{\beta,\beta}_{\varepsilon,\varphi} + (\varepsilon' - \varepsilon)e_1 e_1^t = T^{\beta,\beta}_{\varepsilon',\varphi}$$

Nevertheless in this case we have the problem that we don't know the matrix that diagonalize a generic $T^{\beta,\beta}_{\varepsilon,\varphi}$.

It's right to indicate that the formulas in terms of the algebras $\tau_{\varepsilon,\varphi}$ that we can obtain as corollaries of Theorems 1.3.6, 1.3.4 are basically weaker than the ones that are presented in [3]; here "weaker" means that the number of terms of the sum, in our case is more or less 4-times the rank of $AT_{\varepsilon,\varphi} - T_{\varepsilon,\varphi}A$, whereas, in [3], it is equal to the rank of the above commutator. This is due to the fact that we use less hypotheses on the algebras than the ones used in [3]; indeed Bozzo and Di Fiore always use the symmetry or persymmetry of the algebra (or both of them).
So we can state that, even if our strategy is slightly more expensive, our theorems are much more adaptable.

2. Householder Algebras

In this chapter we study the particular case of the Householder-SDU algebras (the algebras of matrices simultaneously diagonalized by an Householder matrix). In particular we study some basic properties of these algebras: symmetry, closure by conjugation and intersections with algebras of the same class.

2.1 Householder Matrices

Definition. : Let $u \in \mathbb{C}^n$ be a vector such that $||u||_2 = 1$, then it is possible to define the Householder matrix: $U := I - 2uu^h$.

Remark 2.1.1. : We can observe that U is not uniquely defined by u, indeed, if we consider the vector $u' = e^{i\theta}u$, where $\theta \neq 0$, it is easy to note that $U' = U$.
Similarly, if $U' = U$, looking at the element (i, i), we can observe that $|u_i|^2 = |u'_i|^2$ so $\forall j \quad u'_j = u_j e^{i\theta_j}$ for some $\theta_j \in [0, 2\pi)$; and looking at the element (i, j), it follows that $u_i \overline{u}_j = u'_i \overline{u'}_j$, that is $u_i \overline{u}_j = u_i e^{i\theta_i} \overline{u}_j e^{-i\theta_j}$, so $e^{i(\theta_i - \theta_j)} = 1 \Rightarrow \theta_j = \theta_i := \theta \quad \Rightarrow \quad u' = ue^{i\theta}$

Properties 2.1.1. *:*

1. *U is a unitary matrix ;*
2. *U is an Hermitian matrix ;*
3. *U is the reflection matrix with respect to the hyperplane orthogonal to u.*

Proposition 2.1.2. *U is real iff* $u_i\overline{u_j} = u_j\overline{u_i} \quad \forall i,j = 1,...,n$ *iff* $u = e^{i\theta}v$ *with* $v \in \mathbb{R}^n$.

Proof. (1)⇒(2) If U is real , thanks to the property (2), U is symmetric and so $U_{ij} = U_{ji} \quad \forall i,j \quad \Rightarrow \quad u_i\overline{u}_j = u_j\overline{u}_i$

(2)⇒(3) Assume without loss of generality that $u_1 \neq 0$, then we have:

$$\begin{aligned}
& \forall i \quad \exists\, \theta_i \;\; s.t. \;\; u_i = |u_i|e^{i\theta_i} \\
\Rightarrow \quad & \forall i,j \;\; s.t. \;\; u_i, u_j \neq 0\,, \;\; e^{2i\theta_i} = \frac{u_i}{\overline{u}_i} = \frac{u_j}{\overline{u}_j} = e^{2i\theta_j} \\
\Rightarrow \quad & fixed\; i = 1 \;\; \forall j \;\; s.t. \;\; u_j \neq 0 \qquad \theta_j = \theta_1 \;\; or \;\; \theta_j = \theta_1 + \pi \\
\Rightarrow \quad & \forall i \;\; u_i = \pm|u_i|e^{i\theta_1}
\end{aligned}$$

this relation can obviously be extendend also to the $u_i = 0$

(3)⇒(1)trivial.

□

Theorem 2.1.3. *(See [9])*
Let be given $A, B \in \mathbb{R}^{n\times m}$ *non singular matrices with* $m \leq n$ *s.t.* $A^tA = B^tB$. *Then* $\exists\, U_m, ..., U_1$ *s.t.* $U_m...U_1A = B$.

Proof. Prove it by induction on m.
$m = 1$ it is easy true taking $U = I - 2\dfrac{(a_1 - b_1)(a_1 - b_1)^t}{||a_1 - b_1||^2}$.
Supposing it is verified for $m - 1$, study the case m.
Let's consider a generic unitary matrix Q;
and define $u_i := \dfrac{Qa_i - b_i}{||Qa_i - b_i||}$, and $\tilde{U}_i = I - 2u_iu_i^h \Rightarrow \tilde{U}_i(Qa_i) = b_i \;\; \forall i = 1, ..., m$.
If it exists a matrix Q product of $m-1$ Householder matrices, s.t. $u_1 = ... = u_m$, *then*
$\tilde{U_1} = ... = \tilde{U_m} =: U_i$ and it will be true the thesis. $\exists Q$ as above iff:

$$\begin{aligned}
& Qa_i - b_i = Qa_1 - b_1 \\
\Longleftrightarrow \quad & Q(a_i - a_1) = b_i - b_1
\end{aligned}$$

But the last one is true by induction since the matrices
$\Big(a_2 - a_1\,, \;\; a_3 - a_1\,, \;\; \ldots\,, \;\; a_m - a_1\Big)$ and $\Big(b_2 - b_1\,, \;\; b_3 - b_1\,, \;\; \ldots\,, \;\; b_m - b_1\Big)$

still satisfy the hypotheses of the theorem. So the theorem is proved.
In particular it follows that the vector u_i that defines the i-th Householder matrix, can be obtained by setting :

$$u_i := (-1)^{m-i}[U_{i-1} \dots U_1(A_{m-i+1} - A_{m-i}) - (B_{m-i+1} - B_{m-i})]$$
$$u_i := \frac{u_i}{||u_i||}$$

where we set $w_0 = v_0 = 0$. □

2.2 Householder SDU-Algebras and their properties

When U is an Householder matrix, exploiting the structure of $\mathcal{U}[z]$ we can observe that :

$$\begin{aligned}
&\mathcal{U}[z] = (I - 2uu^h)D(z)(I - 2uu^h) = \\
&D(z) - 2(uu^h)D(z) - 2D(z)(uu^h) + 4(u^h D(z)u)uu^h = \\
&D(z) + 2(uu^h)\big((u^h D(z)u)I - D(z)\big) + 2\big((u^h D(z)u)I - D(z)\big)uu^h = \\
&D(z) + 2uu^h M + 2Muu^h
\end{aligned} \tag{2.1}$$

where M is the diagonal matrix defined as:

$$M := \Big((u^h D(z)u)I - D(z)\Big). \tag{2.2}$$

From (2.1) it's easy to note that $\mathcal{U}[z]$ is always a 2-rank variation of a diagonal matrix.

Remark 2.2.1. : If $U = I - 2uu^h$ and $u_i = 0$ for some i, let's assume, without loss of generality, $u_i = 0 \quad \forall i = m+1, ..., n$; then $\mathcal{U}[z]$ has a block-structure and can be taken back to a direct sum of matrices of smaller dimension. Let $u' \in \mathbb{C}^m$ be the vector s.t. $u'_i = u_i \quad \forall i = 1, ..., m$, then $||u'|| = ||u|| = 1$ and taking $z' \in \mathbb{C}^m$ the vector s.t. $z'_i = z_i \quad \forall i = 1, ..., m$, we can note that:

$$\mathcal{U}[z] = \mathcal{U}\begin{bmatrix} z' \\ z'' \end{bmatrix} =$$

$$D(z) - 2D(z)uu^h - 2uu^hD(z) + 4(u^hD(z)u)uu^h =$$

$$\begin{pmatrix} D(z') & 0 \\ 0 & D(z'') \end{pmatrix} - 2\begin{pmatrix} D(z')u' \\ 0 \end{pmatrix}\begin{pmatrix} u'^h, & 0 \end{pmatrix} - 2\begin{pmatrix} u' \\ 0 \end{pmatrix}\begin{pmatrix} u'^hD(z'), & 0 \end{pmatrix}$$

$$+ 4(u'^hD(z')u')\begin{pmatrix} u' \\ 0 \end{pmatrix}\begin{pmatrix} u'^h & ,0 \end{pmatrix} =$$

$$\begin{pmatrix} D(z') - 2D(z')u'u'^h - 2u'u'^hD(z') + 4(u'^hD(z')u')u'u'^h & 0 \\ 0 & D(z'') \end{pmatrix} =$$

$$\begin{pmatrix} \mathcal{U}'[z'] & 0 \\ 0 & D(z'') \end{pmatrix} = \mathcal{U}'[z'] \bigoplus D(z'').$$

Theorem 2.2.2. ***(Householder algebras closed under conjugation)***
Let be given $u \in \mathbb{C}^n$, $n > 4$, $u_i \neq 0 \quad \forall i$:
The Householder algebra $\mathcal{U}$ is closed under conjugation $\Leftrightarrow$ U is real.

Remark. It's easy to observe that even though $u_i = 0$ for some i, thanks to Remark 2.2.1, it is possible to come back to the case $u_i \neq 0 \quad \forall i$. Indeed $\mathcal{U}$ is closed under conjugation iff $\mathcal{U}'$ is closed under conjugation, where u' is the subvector of u with all non-zero entries.

I Proof.
($\Leftarrow$):

$$\overline{\mathcal{U}[z]} = \overline{UD(z)U} = U\overline{D(z)}U = \mathcal{U}[\overline{z}]\,.$$

($\Rightarrow$) : $\mathcal{U}$ is closed under conjugation $\quad\Leftrightarrow\qquad \forall i\,,\quad \overline{\mathcal{U}[e_i]} \in \mathcal{U}$

$$\Leftrightarrow \quad \forall i \quad \exists\{\alpha^i_j\}_{j=1}^n \quad s.t. \quad \overline{\mathcal{U}[e_i]} = \sum_{j=1}^n \alpha^i_j \mathcal{U}[e_j]\,,$$

that is

$$(I - 2\overline{u}u^t)e_ie_i^t(I - 2\overline{u}u^t) = \sum_{j=1}^n \alpha^i_j (I - 2uu^h)e_je_j^t(I - 2uu^h)\,,$$

$$(e_i - 2u_i\overline{u})(e_i^t - 2\overline{u_i}u^t) = \sum_{j=1}^n \alpha^i_j (e_j - 2\overline{u}_ju)(e_j^t - 2u_ju^h)\,.$$

Selecting the element (k,l), with $l \neq k$, we obtain:

$$(\delta_{ik}-2u_k\overline{u}_i)(\delta_{il}-2u_i\overline{u}_l) = \sum_{j=1}^{n} \alpha_j^i(-2u_j\overline{u}_k)(\delta_{jl}-2\overline{u}_j u_l)+\alpha_k^i(\delta_{lk}-2\overline{u}_k u_l)\,. \quad (2.3)$$

From (2.3), if we take $k=i, l \neq i$,
and define $C_l := \sum_{j=1}^{n} \alpha_j^i\Big(-2\frac{u_j}{\overline{u}_l}\Big)(\delta_j l - 2\overline{u}_j u_l)$
it follows that:

$$\begin{aligned}
&(1-2|u_i|^2)(-2u_i\overline{u}_l) + 2\alpha_i^i\overline{u}_i u_l \\
&= \sum_{j=1}^{n} \alpha_j^i(-2u_j\overline{u}_i)(\delta_j l - 2\overline{u}_j u_l)\left(\frac{\overline{u}_l}{\overline{u}_l}\right) =: C_l\overline{u}_i\overline{u}_l \ , \\
\Rightarrow\ &(1-2|u_i|^2)\Big(-2\,\frac{u_i}{\overline{u}_i}\Big) + 2\alpha_i^i\left(\frac{u_l}{\overline{u}_l}\right) = C_l\,. \qquad (2.4)
\end{aligned}$$

From (2.3), taking $k,l \neq i$ with $k \neq l$ we obtain:

$$\begin{aligned}
&(4|u_i|^2 u_k\overline{u}_l) + 2\alpha_k^i\overline{u}_k u_l \\
&= \sum_{j=1}^{n} \alpha_j^i(-2u_j\overline{u}_k)(\delta_{jl} - 2\overline{u}_j u_l)\left(\frac{\overline{u}_l}{\overline{u}_l}\right) =: C_l\overline{u}_k\overline{u}_l \ , \\
\Rightarrow\ &(4|u_i|^2)\left(\frac{u_k}{\overline{u}_k}\right) + 2\alpha_k^i\left(\frac{u_l}{\overline{u}_l}\right) = C_l\,. \qquad (2.5)
\end{aligned}$$

Now let's equal the (2.4) with the (2.5), then:

$$\begin{aligned}
&(1-2|u_i|^2)\left(-2\frac{u_i}{\overline{u}_i}\right) + 2\alpha_i^i\,\frac{u_l}{\overline{u}_l} = 4|u_i|^2\frac{u_k}{\overline{u}_k} + 2\alpha_k^i\,\frac{u_l}{\overline{u}_l}\ , \\
&2\,\frac{u_l}{\overline{u}_l}(\alpha_k^i - \alpha_i^i) = 4|u_i|^2\left(\frac{u_i}{\overline{u}_i} - \frac{u_k}{\overline{u}_k}\right) - 2\,\frac{u_i}{\overline{u}_i}. \qquad (2.6)
\end{aligned}$$

From the latter equality we have to distinguish two cases :

1. $\alpha_k^i - \alpha_i^i = 0 \quad \forall k \neq i\,.$
   Taking out $\frac{u_k}{\overline{u}_k}$ we can observe that, changing k,
   $\frac{u_k}{\overline{u}_k}$ is constant for $k \neq i$, so:

$$\frac{u_k}{\overline{u}_k} = -\frac{1}{2}\,\frac{u_i}{\overline{u}_i}\,\frac{1}{|u_i|^2} + \frac{u_i}{\overline{u}_i}\,; \qquad (2.7)$$

2. $\exists\, k_i$ *s.t.* $\alpha^i_{k_i} - \alpha^i_i \neq 0$.
   In this case, taking out $\frac{u_l}{\overline{u}_l}$, we can observe that, varying l, $\frac{u_l}{\overline{u}_l}$ is constant for $l \neq i, k_i$, so :

$$\frac{u_l}{\overline{u}_l} = \frac{1}{2(\alpha^i_{k_i} - \alpha^i_i)} \left(4|u_i|^2 \left(\frac{u_i}{\overline{u}_i} - \frac{u_{k_i}}{\overline{u}_{k_i}} \right) - 2\, \frac{u_i}{\overline{u}_i} \right). \tag{2.8}$$

I claim Varying i $\exists$ not more than a unique k_0 s.t. $\frac{u_j}{\overline{u}_j}$ is constant for $j \neq k_0$, *and* $\frac{u_{k_0}}{\overline{u}_{k_0}} \neq \frac{u_j}{\overline{u}_j}$, $j \neq k_0$.
proof of claim I
If there exists some i such that (2.9) is true, there is nothing to prove.
Otherwise, if $\forall i$ it is true the (2.8) $\exists k_i$ and C_i constants s.t. $\frac{u_j}{\overline{u}_j} =: C_i\ \forall j \neq i, k_i$. Let's fix i_0 and the corresponding k_{i_0}; considering $i_1 \neq i_0, k_{i_0}$ it has to be true that one among k_{i_0} and i_0 is different from k_1. Without loss of generality let's assume that it is i_0, so

$$\frac{u_{i_0}}{\overline{u}_{i_0}} = C_1 = \frac{u_j}{\overline{u}_j} = C_0 = \frac{u_{i_1}}{\overline{u}_{i_1}}$$

where we have considered $j \neq i_0, i_1, k_0, k_1$.
So $\forall j \neq k_{i_0}$ $\frac{u_j}{\overline{u}_j} =: C$ constant.
This completes the proof of the first claim.
II claim If $\exists\,!k_0$ s.t. $\frac{u_j}{\overline{u}_j}$ is constant for $j \neq k_0$, and $\frac{u_{k_0}}{\overline{u}_{k_0}} \neq \frac{u_j}{\overline{u}_j}$, $j \neq k_0$ then $\forall i \neq k_0$, $\alpha^i_k = \alpha^i_i\ \forall k \neq k_0$.
proof of claim II
By contradiction suppose that $\exists k_1, \tilde{i} \neq k_0$ s.t. $\alpha^{\tilde{i}}_{k_1} \neq \alpha^{\tilde{i}}_{\tilde{i}}$; then, from (2.8) it follows that

$$\frac{u_{k_0}}{\overline{u}_{k_0}} = \frac{u_j}{\overline{u}_j} \qquad \forall j \neq \tilde{i}, k_1 .$$

This completes the proof of the II claim.
Note that if it is true the thesis of claim II, we would have that $\forall\, i \neq k_0$

$$\overline{\mathcal{U}[e_i]} = \alpha^i_i \sum_{j \neq k_0} \mathcal{U}[e_j] + \alpha^i_{k_0} \mathcal{U}[e_{k_0}].$$

From this it follows the absurd because $\left\{ \overline{\mathcal{U}[e_i]} \right\}_i$ are linearly independent and so it can't be true that $\overline{\mathcal{U}[e_i]} \in span\left\{ \mathcal{U}[v], \mathcal{U}[e_{k_0}] \right\}$, $\forall i \neq k_0$

with $v = \sum_{j \neq k_0} e_j$. Hence, $\forall j$, $\dfrac{u_j}{\overline{u}_j}$ is constant and therefore, thanks to Proposition 2.1.2, U is real. □

II proof. ($\Leftarrow$) Easy, as in the first proof.
($\Rightarrow$) $\overline{\mathcal{U}[z]} \in \mathcal{U}$ $\forall z$ implies that $\forall z \quad \exists\, v_z$ s.t. $\overline{\mathcal{U}[z]} = \mathcal{U}[v_z]$, that means that the algebra $\overline{\mathcal{U}}$ is a subalgebra of $\mathcal{U}$. On the other hand, since both of them are n-dimensional algebras, they have to be the same algebra, $sdU = sd\overline{U}$. Hence, thanks to Remark 1.1.1, $\exists$ a permutation matrix P and a diagonal unitary matrix D s.t.

$$U = \overline{U} P D\,.$$

So $\forall j \ \exists\, \theta_j, k_j$ s.t. multiplying on the right by e_j:

$$\begin{aligned} & e_j - 2\overline{u}_j u = e^{i\theta_j}\left(e_{k_j} - u_{k_j}\overline{u}\right) \\ \Rightarrow \quad & e_j - e^{i\theta_j} e_{k_j} = 2\overline{u}_j u - 2e^{i\theta_j} u_{k_j}\overline{u}. \end{aligned} \tag{2.9}$$

If, by contradiction $e_j - e^{i\theta_j} e_{k_j} \neq 0 \ \ \forall j$, consider the set $\{e_j - e^{i\theta_j} e_{k_j}\}_j$; this one would contain a set of more than $\lceil \frac{n}{2} \rceil$ linearly independent vectors and at the same time it would be contained in $span\{u, \overline{u}\}$, which is absurd for $n > 4$.
So $\exists j$ s.t. $k_j = j \ and \ \theta_j = 0$; replacing in (2.9) we obtain:

$$\begin{aligned} & 2\overline{u}_j u - 2u_j\overline{u} = 0 \\ \Rightarrow \quad & u_i\overline{u}_j = u_j\overline{u}_i \quad \forall i \\ \Rightarrow \quad & \frac{u_i}{\overline{u}_i} = Constant \quad \forall i \end{aligned}$$

and, thanks to Proposition 2.1.2, we can conclude that U is real. □

Corollary 2.2.3. *In the same hypotheses of Theorem 2.2.2, the algebra $\mathcal{U}$ is closed under transposition iff U is real.*

Proof. $\mathcal{U}$ is closed under transposition iff it is closed under conjugation:
$\mathcal{U}[z]^T = (D(z) + 2Muu^h + 2uu^hM)^T = D(z) + 2M\overline{u}u^t + 2\overline{u}u^tM = \overline{\mathcal{U}}[z]$. □

Proposition 2.2.4. *The algebra $\mathcal{U}$ is symmetric iff U is real.*

Proof. As we have observed in the previous corollary, we have $\mathcal{U}[z]^T = \overline{\mathcal{U}}[z]$. So if $\mathcal{U}[z] = \overline{\mathcal{U}}[z] \;\; \forall z \in \mathbb{C}^n$, it follows that the i-th column of U has to be a multiple of the i-th column of $\overline{U}$ (since both of them are the eigenvectors relative to the eigenvalue z_i). So let's impose $\forall i \quad e_i - 2\overline{u_i}u = e_i - 2u_i\overline{u}$, that is $u_i\overline{u_j} = u_j\overline{u_i}$, but this means that U is real because of Proposition 2.1.2. The inverse is trivial. □

Proposition 2.2.5. *The algebra $\mathcal{U}$ is never persymmetric if $n > 4$.*

Proof. $\mathcal{U}$ is persymmetric iff $\forall z \in \mathbb{C}^n \;\; \mathcal{U}[z]^t = J\mathcal{U}[z]J$, where J is the exchange matrix ($J_{ij} = \delta_{i,n+1-j}$). This means $\overline{U}D(z)\overline{U} = JUD(z)UJ$, where JU is still a unitary matrix. So it has to be true that the i-th column of JU is a unitary multiple of the i-th column of $\overline{U}$, that is :

$$\begin{aligned} &JUe_i = e^{i\theta_i}\overline{U}e_i \,, \\ &J(I - 2uu^h)e_i = e^{i\theta_i}(I - 2\overline{u}u^t)e_i \,, \\ &Je_i - 2\overline{u}_i\tilde{u} = e^{i\theta_i}(e_i - 2u_i\overline{u}) \,, \\ &e_{n+1-i} - 2\overline{u}_i\tilde{u} = e^{i\theta_i}(e_i - 2u_i\overline{u}) \,. \end{aligned}$$

Thus it should be true that $\{e^{i\theta_i}e_i - e_{n+1-i}\}_{i=1}^n \subset span\{\tilde{u}, \overline{u}\}$, but this is absurd if $n > 4$. □

Theorem 2.2.6. ***(Intersections of Householder algebras)***
Let's be given two distinct Householder algebras $\mathcal{U}, \mathcal{V}$ s.t. $u_i, v_i \neq 0 \; \forall i$, $n > 4$. Then we have:

$$dim(\mathcal{U} \cap \mathcal{V}) \leq 2 \,.$$

Assume $\mathcal{U}[z] = \mathcal{V}[z]$, where $z = \begin{pmatrix}\alpha, & \dots, & \alpha, & \beta, & \dots, & \beta\end{pmatrix}$; let u be a vector that defines the Householder matrix U, and consider the partition of $u = \begin{pmatrix}u_1, & u_2\end{pmatrix}$ where the dimension of (u_1) is equal to number of entries of z equal to α. Then it has to be true one of the following assertions:

1. *The vector $v = \left(\frac{||u_2||}{||u_1||}u_1, \quad -\frac{||u_1||}{||u_2||}u_2\right)$ defines the matrix V.*
2. $||u_1||^2 = ||u_2||^2 = 1/2$, *and if v is a vector that defines the Householder matrix V, then $\exists\, \psi,\, \varphi$ such that $v = \begin{pmatrix}e^{i\psi}u_1\,, & e^{i\varphi}u_2\end{pmatrix}$*

Proof. We want to show that if $A \in \mathcal{U} \cap \mathcal{V}$ then, $A = kI$ or A generates a subalgebra of dimension 2 that is the whole $\mathcal{U} \cap \mathcal{V}$.

If $A \in \mathcal{U} \cap \mathcal{V}$, then $\exists z, y \in \mathbb{C}^n$ s.t. $A = \mathcal{U}[z] = \mathcal{V}[y]$ and, since $\{z_i\}$ *and* $\{y_i\}$ are the eigenvalues of A, it has to exist a permutation matrix, P, s.t. $Pz = y$, hence:

$$\begin{aligned} & A = \mathcal{U}[z] \; = \; \mathcal{V}[Pz] \\ \Rightarrow \quad & UD(z)U = VD(Pz)V \Leftrightarrow VUD(z) = D(Pz)VU \\ & \quad \Leftrightarrow \quad \forall (i,j) \quad (VU)_{ij} z_j = (VU)_{ij} (Pz)_i \, . \end{aligned} \tag{2.10}$$

So we can assert that :

$$\forall i, j \;\; s.t. \quad (VU)_{ij} \neq 0 \quad \Rightarrow \quad z_i = (Pz)_j \, .$$

Define $\forall i$ the index h_i s.t. $Pe_{h_i} = e_i$ and let's consider the graph $\mathcal{G}$ s.t.

$$\exists \text{ edge } (i \longleftrightarrow j) \quad \Leftrightarrow \quad (VU)_{h_i j} \neq 0 \;\; and/or \;\; (VU)_{h_j i} \neq 0.$$

It is easy to observe that if $\exists (i \longleftrightarrow j)$ than $z_j = z_i$.
So if the graph is connected, then necessarily $z = k\underline{1}$, $k \in \mathbb{C}$, and $\mathcal{U} \cap \mathcal{V} = kI$.
In detail, we can say that the graph has as many connected components as many different elements has z.
If the graph is totally disconnected the matrix VU must have just one non zero entry in each row and column and therefore it has to be verified the equality

$$VU = QD'$$

where Q is a permutation matrix and D' is a diagonal unitary matrix.
Thus the equation (3.26) becomes:

$$\begin{aligned} & VUD(z) = D(Pz)VU \\ \Leftrightarrow \quad & QD'D(z) = D(Pz)QD' = PD(z)P^tQD' \\ \Leftrightarrow \quad & P^tQD'D(z) = D(z)P^tQD' \, . \end{aligned}$$

The latter equality is true $\forall z$ taking $P = Q$ but, then, it has to be true $\mathcal{U} = \mathcal{V}$. So we can restrict to the case of a disconnected graph but not totally disconnected.
Let be

$$I_1 = \{i_1, ..., i_h\}, \; I_2 = \{j_1, ..., j_k\} \; s.t. \; I_1 \cup I_2 = \{1, ..., n\}, \;\; I_1 \cap I_2 = \varnothing$$

and without loss of generality assume

$$z_i = \alpha \;\; \forall i \in I_1, \;\; z_i \neq \alpha \;\; \forall i \in I_2 \,.$$

Thanks to the above observation (the graph is not totally disconnected) it is admissible to assume $\sharp I_1 \geq 2$.
Now, by the definition of $\mathcal{G}$, it has to be true

$$\forall i \in I_1, \; j \in I_2 \;\; (VU)_{h_i j} = (VU)_{h_j i} = 0$$

and this relation, if we define

$$J_1 = \{h_i | i \in I_1\}, \;\; J_2 = \{h_j | j \in I_2\} \,,$$

can be written as

$$(VU)_{ij} = 0 \quad \forall (i,j) \in (J_1 \times I_2) \cup (J_2 \times I_1).$$

Let's define

$$v_{J_1} := v|_{J_1}, \quad v_{J_2} := v|_{J_2}, \quad v_{I_1} := v|_{I_1}, \quad v_{I_2} := v|_{I_2}$$

and $L_1, \; L_2, \; G_1, \; G_2$ the rectangular matrices s.t.

$$L_1 v = v_{J_1}, \quad L_2 v = v_{J_2}, \quad v^t G_1 = v_{I_1}, \quad v^t G_2 = v_{I_2}.$$

Then we can state the following equivalent statement:

$$(VU)_{ij} = 0 \quad \forall (i,j) \in J_1 \times I_2 \cup J_2 \times I_1 \qquad \Leftrightarrow \qquad L_1 UVG_2 = L_2 UVG_1 = 0 \,.$$

Thus it has to be true:

$$\begin{cases} L_1 G_2 - 2u_{J_1} u^h_{I_2} - 2v_{J_1} v^h_{I_2} + 4v^h u v_{J_1} u^h_{I_2} = 0 \\ L_2 G_1 - 2u_{J_2} u^h_{I_1} - 2v_{J_2} v^h_{I_1} + 4v^h u v_{J_2} u^h_{I_1} = 0 \end{cases} ,$$

$$\begin{cases} L_1 G_2 + 2((v^h u) v_{J_1} - u_{J_1}) u^h_{I_2} + 2v_{J_1}((v^h u) u^h_{I_2} - v^h_{I_2}) = 0 \\ L_2 G_1 + 2((v^h u) v_{J_2} - u_{J_2}) u^h_{I_1} + 2v_{J_2}((v^h u) u^h_{I_1} - v^h_{I_1}) = 0 \end{cases} . \tag{2.11}$$

Let's note that the matrices L_1G_2 and L_2G_1 are matrices with not more than one entry equal to 1 in each row and column and, since they have to be equal to matrices of rank 2, we can immediately say that it is not possible that they have more than two nonzero entries.
At the same time we can also say that the number of rows plus the number of columns of L_1G_2 and L_2G_1 has to be equal to n.

claim If $\sharp I_2 > 1$, then L_1G_2 and L_2G_1 are the null matrices.
proof of the claim
If we assume $n > 4$, we have an equation of the type $A + u_1v_1^h + u_2v_2^h = 0$, where A has not more than two non zero entries in two different rows and columns and both v_1 and u_2 have all entries different from zero. So we can analyze just the two following cases

$$1)\ A = \begin{bmatrix} 1 & 0 & 0 & ... \\ 0 & 1 & 0 & ... \end{bmatrix} \qquad 2)\ A = \begin{bmatrix} 1 & 0 & 0 & ... \\ 0 & 0 & 0 & ... \end{bmatrix}$$

In the first case, looking at the third column, u_1 and u_2 should be lin.dep. whereas, from the first two columns, they should be lin.indep. taking to a contradiction.
In the second case, looking at the second column, u_1 and u_2 should be lin.dep. and, looking at the first one, they should be both multiples of e_1 against the hypothesis that u_2 has all non zero entries. This completes the proof of the claim.

It remains to study the cases :
1)$\sharp I_2 = 1$
2)$L_1G_2 = 0$ and $L_2G_1 = 0$
In the first case z must have $n-1$ components equals to each other and so $\dim(\mathcal{U} \cap \mathcal{V}) \leq 2$.
In the second case, if

$$L_1G_2 = 0 \qquad and \qquad L_2G_1 = 0\,,$$

we are in the situation that $J_1 = I_1$ and $J_2 = I_2$.
Note that we can assume without loss of generality that $I_1 = \{1, \ldots, k\}$ and $I_2 = \{k+1, \ldots, n\}$.

The equalities in (2.11) became:

$$\begin{cases} ((v^h u)v_{J_1} - u_{J_1})u^h_{I_2} + v_{J_1}((v^h u)u^h_{I_2} - v^h_{I_2}) = 0 \\ ((v^h u)v_{J_2} - u_{J_2})u^h_{I_1} + v_{J_2}((v^h u)u^h_{I_1} - v^h_{I_1}) = 0 \end{cases}$$

and multiplying them on the right and on the left the equation by the permutation matrices that switch the indexes J_2 with I_2 and I_1 with J_1, we obtain:

$$\begin{cases} ((v^h u)v_{J_1} - u_{J_1})u^h_{I_2} + v_{J_1}((v^h u)u^h_{I_2} - v^h_{I_2}) = 0 \\ ((v^h u)v_{I_2} - u_{I_2})u^h_{J_1} + v_{I_2}((v^h u)u^h_{J_1} - v^h_{J_1}) = 0\,. \end{cases}$$

Let's redefine $u_{I_2} =: u_2$, $v_{I_2} =: v_2$, $v_{J_1} =: v_1$ and $u_{J_1} =: u_1$. Observing that the vectors u_1, u_2, v_1 *e* v_2 are a partition of the vectors u and v, the system of equations becomes:

$$\begin{cases} ((v^h u)v_1 - u_1)u^h_2 + v_1((v^h u)u^h_2 - v^h_2) = 0 \\ ((v^h u)v_2 - u_2)u^h_1 + v_2((v^h u)u^h_1 - v^h_1) = 0 \\ ||u_1||^2 + ||u_2||^2 = 1 \\ ||v_1||^2 + ||v_2||^2 = 1 \end{cases}. \tag{2.12}$$

Note that the above system of equations is equivalent to impose that the non-diagonal blocks of the matrix VU are equal to 0:

$$VU = \left(I - 2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\begin{bmatrix} v^h_1 & v^h_2 \end{bmatrix}\right)\left(I - 2\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} u^h_1 & u^h_2 \end{bmatrix}\right) \tag{2.13}$$

$$= \left(I - 2\begin{bmatrix} v_1 v^h_1 & v_1 v^h_2 \\ v_2 v^h_1 & v_2 v^h_2 \end{bmatrix} - 2\begin{bmatrix} u_1 u^h_1 & u_1 u^h_2 \\ u_2 u^h_1 & u_2 u^h_2 \end{bmatrix} + 4v^h u\begin{bmatrix} v_1 u^h_1 & v_1 u^h_2 \\ v_2 u^h_1 & v_2 u^h_2 \end{bmatrix}\right). \tag{2.14}$$

Using the equations of (2.12) and computing the difference between the second equation and the conjugate of the first one, we have:

$$-4(v^h u)v_1 u^h_2 + 4(u^h v)u_1 v^h_2 = 0\,, \tag{2.15}$$

from this it follows that must exist α and β such that

$$u_1 = \alpha v_1 \qquad \textit{and} \qquad u_2 = \beta v_2.$$

Now use the third equation to obtain :

$$|\alpha|^2||v_1||^2 + |\beta|^2||v_2||^2 = 1\,; \tag{2.16}$$

replace again in the (2.15) and remind that $v_1 v_2^h$ is a full matrix:

$$\begin{aligned}
& -(\alpha v_1^h v_1 + \beta v_2^h v_2)\overline{\beta} v_1 v_2^h + (\overline{\alpha} v_1^h v_1 + \overline{\beta} v_2^h v_2)\alpha v_1 v_2^h = 0 \\
\Rightarrow \quad & -(\alpha\overline{\beta}||v_1||^2 + |\beta|^2||v_2||^2)v_1 v_2^h + (|\alpha|^2||v_1||^2 + \overline{\beta}\alpha||v_2||^2)v_1 v_2^h = 0 \\
\Rightarrow \quad & (|\alpha|^2||v_1||^2 + \overline{\beta}\alpha||v_2||^2) - (\alpha\overline{\beta}||v_1||^2 + |\beta|^2||v_2||^2) = 0 \\
\Rightarrow \quad & ((|\alpha|^2 - \alpha\overline{\beta})||v_1||^2 + (\overline{\beta}\alpha - |\beta|^2)||v_2||^2 = 0.
\end{aligned} \tag{2.17}$$

Now, without loss of generality, we can divide by $\overline{\beta}(\overline{\alpha} - \overline{\beta})$:

$$\begin{aligned}
& \frac{((|\alpha|^2 - \alpha\overline{\beta})}{\overline{\beta}(\overline{\alpha} - \overline{\beta})}||v_1||^2 + \frac{(\overline{\beta}\alpha - |\beta|^2)}{\overline{\beta}(\overline{\alpha} - \overline{\beta})}||v_2||^2 = 0 \\
\Rightarrow \quad & \frac{\alpha}{\overline{\beta}}||v_1||^2 + \frac{(\alpha - \beta)}{\overline{\alpha} - \overline{\beta}}||v_2||^2 = 0\,.
\end{aligned} \tag{2.18}$$

Let's define

$$\frac{(\alpha - \beta)}{\overline{\alpha} - \overline{\beta}} =: e^{i\theta}\,,$$

replace it in (2.18) and obtain :

$$\begin{aligned}
& \frac{\alpha}{\overline{\beta}} = -\frac{||v_2||^2}{||v_1||^2}e^{i\theta} \\
\Rightarrow \quad & \alpha = -\frac{||v_2||^2}{||v_1||^2}e^{i\theta}\overline{\beta}.
\end{aligned} \tag{2.19}$$

Now consider again the identity (2.16) which yields:

$$\begin{aligned}
& \frac{||v_2||^4}{||v_1||^2}|\beta|^2 + |\beta|^2||v_2||^2 = 1 \\
\Rightarrow \quad & ||v_2||^2|\beta|^2\left(\frac{||v_2||^2}{||v_1||^2} + 1\right) = 1 \\
\Rightarrow \quad & \frac{||v_2||^2}{||v_1||^2}|\beta|^2 = 1 \\
\Rightarrow \quad & |\beta|^2 = \frac{||v_1||^2}{||v_2||^2}, \qquad |\alpha|^2 = \frac{||v_2||^2}{||v_1||^2}.
\end{aligned} \tag{2.20}$$

So we can assume
$$\beta = \frac{||v_1||}{||v_2||}e^{i\psi}, \qquad \alpha = \frac{||v_2||}{||v_1||}e^{i\varphi} \tag{2.21}$$
where φ must be different from ψ, otherwise $U = V$.

This is sufficient to prove that $dim(\mathcal{U} \cap \mathcal{V}) \leq 2$; indeed, replacing the last two formulas and the equations $u_1 = \alpha v_1$, $u_2 = \beta v_2$ in (2.14), we obtain:

$$\left(I-2\begin{bmatrix} v_1v_1^h & v_1v_2^h \\ v_2v_1^h & v_2v_2^h \end{bmatrix}-2\begin{bmatrix} |\alpha|^2v_1v_1^h & \alpha\overline{\beta}v_1v_2^h \\ \overline{\alpha}\beta v_2v_1^h & |\beta|^2v_2v_2^h \end{bmatrix}+4(\alpha||v_1||^2+\beta||v_2||^2)\begin{bmatrix} \overline{\alpha}v_1v_1^h & \overline{\beta}v_1v_2^h \\ \overline{\alpha}v_2v_1^h & \overline{\beta}v_2v_2^h \end{bmatrix}\right)$$

$$= \begin{bmatrix} I-2(1+|\alpha|^2-2(\alpha||v_1||^2+\beta||v_2||^2)\overline{\alpha})v_1v_1^h & -2(1+\alpha\overline{\beta}-2(\alpha||v_1||^2+\beta||v_2||^2)\overline{\beta})v_1v_2^h \\ -2(1+\overline{\alpha}\beta-2(\alpha||v_1||^2+\beta||v_2||^2)\overline{\alpha})v_2v_1^h & (I-2(1+|\beta|^2-2(\alpha||v_1||^2+\beta||v_2||^2)\overline{\beta})v_2v_2^h \end{bmatrix}$$

$$= \begin{bmatrix} I-2(1+|\alpha|^2-2(\alpha||v_1||^2+\beta||v_2||^2)\overline{\alpha})v_1v_1^h & 0 \\ 0 & (I-2(1+|\beta|^2-2(\alpha||v_1||^2+\beta||v_2||^2)\overline{\beta})v_2v_2^h \end{bmatrix}$$

$$= \begin{bmatrix} I-2\left(\dfrac{1-2||v_2||^2||v_1||^2(1+e^{i(\psi-\varphi)}}{||v_1||^2}\right)v_1v_1^h & 0 \\ 0 & (I-2\left(\dfrac{1-2||v_2||^2||v_1||^2(1+e^{i(\varphi-\psi)}}{||v_2||^2}\right)v_2v_2^h \end{bmatrix}.$$

So the matrix VU belongs to the commutator of every diagonal matrix with all equal elements in the first part and all equal elements in the second part of the diagonal, that means that $dim\Big(\mathcal{U} \cap \mathcal{V}\Big) = 2$. At the same time, since it is easy to observe that necessarily

$$\left(\frac{1-2||v_2||^2||v_1||^2(1+e^{i(\psi-\varphi)}}{||v_1||^2}\right) \neq 0\,,$$

$$\left(\frac{1-2||v_2||^2||v_1||^2(1+e^{i(\varphi-\psi)}}{||v_2||^2}\right) \neq 0$$

and

$$v_i \neq 0 \qquad \forall i,$$

the two blocks in the head and in the tail of the matrix VU are full matrices. Hence this matrix cannot belong to the commutator of none diagonal matrix that has more than two distinct elements in the diagonal. Indeed the commutator of a diagonal matrix that is direct sum of identity matrices multiplied by different scalars, is made of blocks-diagonal matrices with as many blocks on the diagonal as many different elements there are in the direct sum.

Now it is also easy to demonstrate the second part of the theorem, indeed we are already in its hypotheses; so let's replace in the first equation of (2.12), at first the (2.16) and then the equations for α and β in (2.21):

$$\begin{aligned} & 2\overline{\beta}(\alpha||v_1||^2 + \beta||v_2||^2)v_1 v_2^h - \alpha\overline{\beta}v_1 v_2^h - v_1 v_2^h = 0 \\ \Rightarrow \quad & 2\overline{\beta}(\alpha||v_1||^2 + \beta||v_2||^2) - \alpha\overline{\beta} - 1 = 0 \\ \Rightarrow \quad & 2\overline{\beta}\alpha||v_1||^2 + 2 - 2|\alpha|^2||v_1||^2 - \alpha\overline{\beta} - 1 = 0 \\ \Rightarrow \quad & 2(|\alpha|^2 - \overline{\beta}\alpha)||v_1||^2 - 1 + \alpha\overline{\beta} = 0\,, \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & 2\left(\frac{||v_2||^2}{||v_1||^2} - e^{i(\varphi-\psi)}\right)||v_1||^2 = 1 - e^{i(\varphi-\psi)} \\ \Rightarrow \quad & 2(||v_2||^2 - e^{i(\varphi-\psi)}||v_1||^2) = 1 - e^{i(\varphi-\psi)} \\ \Rightarrow \quad & 2(1 - ||v_1||^2 - e^{i(\varphi-\psi)}||v_1||^2) = 1 - e^{i(\varphi-\psi)} \\ \Rightarrow \quad & (1 - (1 + -e^{i(\varphi-\psi)})||v_1||^2) = \frac{1 - e^{i(\varphi-\psi)}}{2} \\ \Rightarrow \quad & (1 + -e^{i(\varphi-\psi)})||v_1||^2) = \frac{e^{i(\varphi-\psi)}}{2} + \frac{1}{2} \\ \Rightarrow \quad & ||v_2||^2 = \frac{1}{2} \quad or \quad (\varphi - \psi) = \pi. \end{aligned}$$

Now, if $||v_2||^2 = 1/2$, then also $||v_1||^2 = 1/2$.
Whereas in the case $(\varphi - \psi) = \pi$, we have that

$$\begin{aligned} u_1 &= \frac{||v_2||}{||v_1||} e^{i\varphi} v_1 \\ u_2 &= \frac{||v_1||}{||v_2||} e^{i\varphi - \pi} v_2 = -\frac{||v_1||}{||v_2||} e^{i\varphi} v_2 \end{aligned}$$

and, since u is defined unless unitary multiples (because of Remark 2.1.1), we can assume

$$u_1 = \frac{||v_2||}{||v_1||} v_1 \qquad u_2 = -\frac{||v_1||}{||v_2||} v_2\,,$$

that is the first possibility in the statement of the theorem (note that in this case $u \perp v$).

In the other case, when $||v_2||^2 = ||v_1||^2 = 1/2$, then $|\alpha| = |\beta| = 1$, hence $u_1 = e^{i\psi} v_1$, $u_2 = e^{i\varphi} v_2$, that is the second possility in the statement of the theorem.

□

As a consequence of this theorem we can formulate the following conjecture: If U is the product of k Householder matrices and V is the product of h Householder matrices it could be true that $dim\big(\mathcal{U} \cap \mathcal{V}\big) \leq k + h$.

The results in the next Lemma and Corollary could be usefull to find an extension of Theorems 1.3.4, 1.3.6 that involves two Householder SDU algebras.

Lemma 2.2.7. *Let be given $u \in \mathbb{C}^n$, $||u|| = 1$ e $z \in \mathbb{C}^n$. Then, $\forall\, u' \in \mathbb{C}^n$ such that $||u'|| = 1$, $u'D(z)u'^h = uD(z)u^h$ and $D(z)(u' - u) = \lambda(u' - u)$, we have that $\mathcal{U}[z] - \mathcal{U}'[z]$ is a two rank matrix.*

Proof. Defining $w = u' - u$ and using (2.1):

$$\begin{aligned}
\mathcal{U}'[z] &= D(z) - 2Mu'u'^h - 2u'u'^h M \\
&= D(z) - 2M(u + w)(u + w)^h - 2(u + w)(u + w)^h M \\
&= D(z) - 2M(u)(u)^h - 2(u)(u)^h M - 2M(w)(w)^h - \\
&\quad - 2(w)(w)^h M - 2M(uw^h + wu^h) - 2(uw^h + wu^h)M \\
&= \mathcal{U}[z] - Mw(w^h + 2u^h) - (Mw + 2Mu)w^h - w(w^h M + 2u^h M) - (w + 2u)w^h M.
\end{aligned}$$

Now since $D(z)w = \lambda w$ and $M := ((u^h D(z)u)I - D(z))$, it follows that $Mw = \nu w$ and so:

$$\begin{aligned}
\mathcal{U}'[z] &= \mathcal{U}[z] - \nu w(w^h + 2u^h) - (Mw + 2Mu)w^h - w(w^h M + 2u^h M) - (w + 2u)\nu w^h \\
&= \mathcal{U}[z] - w(w^h + 2u^h)(\nu I + M) - (\nu I + M)(w + 2u)w^h\,.
\end{aligned}$$

□

Corollary 2.2.8. *If $w = u' - u = ke_i$ where $k = u_i(e^{i\theta} - 1)$, then all the hypotheses of the above lemma are satisfied:*

$$u' = \sum_{j=1}^{n} u_j e_j + u_i(e^{i\theta} - 1)e_i = \sum_{j \neq i, j=1}^{n} u_j e_j + u_i(e^{i\theta})e_i\,.$$

So it follows :

$$||u'|| = ||u|| = 1, \quad u'D(z)u'^h = uD(z)u^h$$

and

$$Mke_i = k(uD(z)u^h - z_i)e_i\,.$$

3. Householder-Type

3.1 Householder-Type Matrices

It is well known that, if we have two vectors $u, v \in \mathbb{C}^n$ s.t. $||w|| = ||v||$, then there exists an Householder matrix U such that $Uw = v$ if and only if the scalar product $<w, v>$ is real; so if w and v are real vectors it's always possible. A consequence of this fact is also that each real unitary matrix is a product of Householder matrices, whereas to represent a generic complex unitary matrix, we need a product of Householder matrices and a unitary diagonal matrix. In this chapter we will study a new class of unitary matrices that generalize the Householder matrices and that work, in complex case, as well as the Householder matrices do in the real case; it is right to point out that these matrices and some of their basic properties have already been introduced in the past (see [10], [11]). To understand how to define these matrices let's understand why the Householder matrices fail in the complex case. A good explanation of this fact can come from Theorem 4.0.1 that says us that if we have two vectors as above, it exists an orthonormal basis u_i and $n+1$ coefficients :$\{\alpha_i\}_{i=1}^{n-1}$, β, $\gamma\,(\beta \neq \gamma)$ such that $w = \sum_{i=1}^{n-1} \alpha_i u_i + \beta u_n$ and $v = \sum_{i=1}^{n-1} \alpha_i u_i + \gamma u_n$ and, since $||w|| = ||v||$, it has to be true that $|\gamma| = |\beta|$. Now, if we want a matrix that maps w in v, it is sufficient a matrix that sends βu_n in γu_n and leaves invariate the other $\{u_i\}$. It is easy to observe that $<w, v>$ is real iff $\beta = -\gamma$, so, if $<w, v>$ is real, we just need a matrix that sends u_n in $-u_n$, that is exactly the Householder matrix generated by u_n. Nevertheless, in the general case, (since $\gamma = e^{i\theta}\beta$) what we need is a matrix that sends u_n in $e^{i\theta} u_n$ and leaves invariate the subspace orthogonal to u_n. To meet this requirement we have introduced the Householder-type matrices.

Definition. : Similarly as we did for the Householder matrices, we can define the matrix $U_\alpha = I - \alpha u u^h$ where $\alpha = 1 - e^{i\theta}$ and $||u|| = 1$. We name this class of matrices Householder-type.

Remark. Let α be defined as above, it's easy to note that:

- U_α is unitary and $U_\alpha^h = U_{\overline{\alpha}} = I - \overline{\alpha} u u^h$

$$
\begin{aligned}
U_\alpha^h U_\alpha = U_{\overline{\alpha}} U_\alpha = (I - \overline{\alpha} u u^h)(I - \alpha u u^h) = I - \overline{\alpha} u u^h - \alpha u u^h + |\alpha|^2 u u^h = \\
I + \big(|\alpha|^2 - 2Re(\alpha)\big) u u^h = I + \big(Re(\alpha)^2 + Im(\alpha)^2 - 2Re(\alpha)\big) u u^h = \\
I + \big((Re(\alpha) - 1)^2 + Im(\alpha)^2 - 1\big) u u^h = \\
I + \big(Re^2(e^{i\theta}) + Im^2(e^{i\theta}) - 1\big) u u^h = I
\end{aligned}
$$

- U_α is Hermitian iff $\alpha = 0, 2$

$$
U_\alpha^h = I - \overline{\alpha} u u^h = U_{\overline{\alpha}} = U_\alpha \iff \alpha = \overline{\alpha} \iff \alpha = 0, 2 \tag{3.1}
$$

- $U_\alpha u = e^{i\theta} u$ and $U_\alpha v = v \ \forall\, v \perp u$.

$$
\big(I - (1 - e^{i\theta}) u u^h\big) u = u - (1 - e^{i\theta}) u = e^{i\theta} u \tag{3.2}
$$

Lemma 3.1.1. *Let be given $w, v \in \mathbb{C}^n$ s.t. $||w|| = ||v||$. Then $\exists u \in \mathbb{C}^n, ||u|| = 1$ and α with $|\alpha - 1| = 1 \quad (|\alpha|^2 = 2Re\alpha)$ s.t. $U_\alpha v = w$:*
$u := \dfrac{v - w}{||v - w||}$ *and* $\alpha = 1 + \dfrac{< v, v - w >}{< v - w, v >} = 1 + \dfrac{< v, w - v >}{< w - v, v >}$

Proof. Without loss of generality assume $||w|| = ||v|| = 1$ and define $u := \dfrac{v - w}{||v - w||}$.
To get α let's impose :

$$
\begin{aligned}
&\Big(I - \frac{\alpha}{||v - w||^2}(v - w)\big((v - w)^h\Big) v = w \\
\Rightarrow \quad & v - \Big(\frac{\alpha}{||v - w||^2}(1 - < w, v >)\Big)(v - w) = w \,,
\end{aligned}
$$

from which it follows:

$$\begin{aligned}
&\frac{\alpha}{||v-w||^2}(1- < w, v >) = 1\,,\\
&\alpha(1- < w, v >) = ||v-w||^2\,,\\
&\alpha(1- < w, v >) =< v-w, v-w >= 2- < v, w > - < w, v >\,,\\
&(\alpha-1)(1- < w, v >) = 1- < v, w >\,,\\
&(\alpha-1) = \frac{1- < v, w >}{1- < w, v >} = \frac{< v, v-w >}{< v-w, v >}\,.
\end{aligned}$$

Note that : $|\alpha - 1| = 1$.

□

Theorem 3.1.2. *If $u \in \mathbb{C}^n$ is a unitary vector and we define the function $\alpha(\theta) := 1 - e^{i\theta}$, then the set of the Householder-type matrices $U_{\alpha(\theta)} := I - (1 - e^{i\theta})uu^h$, $\theta \in \mathbb{R}$, is a commutative subgroup of the unitary matrices.*

Proof. The thesis follows from the following equalities:

$$U_{\alpha(0)} = I\,,$$

$$U_{\alpha(-\theta)} = U^h_{\alpha(\theta)} = U^{-1}_{\alpha(\theta)}\,,$$

$$\begin{aligned}
U_{\alpha(\theta)}U_{\alpha(\varphi)} &= \big(I - (1-e^{i\theta})uu^h\big)\big(I - (1-e^{i\varphi})uu^h\big)\\
&= I - (1-e^{i\theta})uu^h - (1-e^{i\varphi})uu^h + (1-e^{i\theta})(1-e^{i\varphi})uu^h\\
&= I + \big(-(1-e^{i\theta}+1-e^{i\varphi}) + (1-e^{i\theta}-e^{i\varphi}+e^{i(\theta+\varphi)})\big)uu^h\\
&= I - (1-e^{i(\theta+\varphi)})uu^h = U_{\alpha(\theta+\varphi)} = U_{\alpha(\varphi)}U_{\alpha(\theta)}\,.
\end{aligned}$$

□

3.2 Characterization of Householder-Type Matrices

Theorem 3.2.1. *: In* $\mathbb{C}^{n\times n},\ n>2$

1. *If a matrix is a rank-1 variation of a diagonal matrix it has to be the product of an Householder-Type matrix by a unitary diagonal matrix*

2. *The Householder-type matrices are the only unitary matrices that are a 1-rank variation of the identity matrix.*

Proof. Let $U = D(z) - vw^h$ be a unitary matrix; we want to show that there exist a diagonal unitary matrix $D,\ u \in \mathbb{C}^n\ s.t.\ ||u|| = 1$ and $\alpha = 1 - e^{i\theta} \in \mathbb{C}$ such that $U = (I - \alpha uu^h)D$. Impose that U is unitary:

$$\big(D(z) - vw^h\big)\big(D(z) - vw^h\big)^h = \big(D(z) - vw^h\big)^h\big(D(z) - vw^h\big) = I\,.$$

This can be written as:

$$D(|z|^2) - D(z)wv^h - vw^hD(\overline{z}) + ||w||^2vv^h = I\,, \tag{3.3}$$

$$D(|z|^2) - D(\overline{z})vw^h - wv^hD(z) + ||v||^2ww^h = I\,, \tag{3.4}$$

That are equivalent to:

$$D(|z|^2) - I + \Big(\frac{||w||^2}{2}v - D(z)w\Big)v^h + v\Big(\frac{||w||^2}{2}v^h - w^hD(\overline{z})\Big) = 0\,, \tag{3.5}$$

$$D(|z|^2) - I + \Big(\frac{||v||^2}{2}w - D(\overline{z})v\Big)w^h + w\Big(-v^hD(z) + \frac{||v||^2}{2}w^h\Big) = 0\,. \tag{3.6}$$

From this it follows that $D(|z|^2) - I$ can have rank not bigger than 2, so we can assume that $\exists j$ s.t. $\big(D(|z|^2) - I\big)e_j = 0$ and this implies:

$$\Big(\big(\frac{||w||^2}{2}v - D(z)w\big)v^h + v\big(\frac{||w||^2}{2}v^h - w^hD(\overline{z})\big)\Big)e_j = 0\,.$$

From this it follows that $D(z)w /\!/ v\ \Rightarrow D(z)w = \alpha v$.

Now replacing it in (3.3), it follows that :

$$D(|z|^2) - I - \alpha vv^h - \overline{\alpha}vv^h + ||w||^2vv^h = 0 \tag{3.7}$$

$$\Rightarrow \big(D(|z|^2) - I\big) + \big(||w||^2 - 2Re(\alpha)\big)vv^h = 0\,. \tag{3.8}$$

From the last equality, noting that it is allowed to assume that $v \neq e_i \ \forall i$ (otherwise it should be $U = D(z')$), it follows that

$$D(|z|^2) - I = 0$$

and

$$||w||^2 - 2Re(\alpha) = 0\,.$$

So $\forall j \ \exists \theta_j$ s.t. $z_j = e^{i\theta_j}$; moreover, assuming without loss of generality [1] $||v|| = 1$, we have

$$||w|| = ||D(z)w|| = |\alpha|||v|| = |\alpha|\,.$$

Thus

$$|\alpha|^2 - 2Re(\alpha) = 0 \quad \Rightarrow \quad (Re(\alpha) - 1)^2 + Im(\alpha)^2 = 1 \quad \Rightarrow \quad \alpha = 1 + e^{i\theta}$$

$$\Rightarrow \quad U = D(e^{i\theta_j}) - v\Big(\alpha D(e^{i\theta_j})^{-1} v\Big)^h = D(e^{i\theta_j}) - \overline{\alpha} v v^h D(e^{-i\theta_j})^h$$
$$= D(e^{i\theta_j}) - \overline{\alpha} v v^h D(e^{i\theta_j}) = \Big(I - \overline{\alpha} v v^h\Big) D(e^{i\theta_j})\,.$$

If $D(z) = I$ we have $D(z)w = w = \alpha v$ and so U is an Householder-Type matrix. □

Corollary 3.2.2. *Let be given $A \in \mathbb{C}^{n\times n}$ normal and non-derogatory. If $\exists v \in \mathbb{C}^n$ and $\{(h_i, k_i)\}_{i=1}^n \in \mathbb{C}^2$ s.t. $\{h_i e_i - k_i v\}_{i=1}^n$ are eigenvectors of A, Then $\exists\, \alpha = 1 - e^{i\theta}$ and $u \in \mathbb{C}^n, ||u|| = 1$ s.t. $A \in \mathcal{U}_\alpha = SDU_\alpha$.*

Proof. Since A is normal $\exists\, U$ unitary matrix which diagonalizes A, that is:

$$A = UD(\lambda_i)U^H\,;$$

moreover, since it is non derogatory and diagonalizable, each eigenspace must have dimension equal to 1. So, unless permutations of columns, it has to be verified that

$$UP = D(\beta_i) - vw^h \quad where \quad (\beta_i, w_i) = c_i(h_i, k_i)\ \forall i\,,$$

and, since UP is unitary, thanks to Theorem 3.2.1, we obtain:

$$UP = U_\alpha D(e^{i\theta_j})\,.$$

[1] $U = D(z) - vw^h = D(z) - \frac{v}{||v||}||v||w^h = D(z) - v'w'^h\,.$

From this it follows that:

$$A = U_\alpha D(e^{i\theta_j})P^t D(\lambda_i) P D(e^{-i\theta_j}) U_\alpha^h$$

and thanks to Remark 1.1.1 one has $A \in \mathcal{U}_\alpha$. □

3.3 Decomposition of Unitary Matrices

Note that every unitary matrix, U, since is a normal matrix, can be diagonalized by another unitary matrix, V, and, since all the eigenvalues of U have absolute value equal to 1, we can write:

$$\begin{aligned} U &= VD(e^{i\theta_j})V^h = VV^h - V\Big(I - D(e^{i\theta_j})\Big)V^h \\ &= I - VD(1 - e^{i\theta_j})V^h = \prod_{j=1}^{n} \Big(I - (1 - e^{i\theta_j}) v_j v_j^h\Big) \end{aligned}$$

where the last equality holds since the columns of V are orthogonal.
So, every unitary matrix U can be decomposed as product of Householder-type matrices and more eigenvalues of U are equal to 1 less are the not-trivial Householder-Type matrices we need to represent U. We will show that the one above is an "optimal decomposition"; optimal means that the number of not-trivial Householder-type matrices involved is the minimum possible. We will also show that the QR algorithm adapted to work with the Householder-Type matrices can be usefull to find an optimal decomposition of a unitary matrix in terms of Householder-type matrices.

Theorem 3.3.1. *Let be given* $V, W \in \mathbb{C}^{n\times m}$ *s.t.* $V^H V = W^H W = I_{m\times m},\ m \leq n$. *Then* $\exists\, U_{\alpha_m} ... U_{\alpha_1}$ *s.t.* $W = U_{\alpha_m} ... U_{\alpha_1} V$.

Proof. Let us name $\{v_i\}_{i=1}^m, \{w_i\}_{i=1}^m$ the columns of V and W. Proceed by induction on m.
By lemma 3.1.1, $\exists\, U_{\alpha_1} = I - \alpha_1 u_1 u_1^h$ s.t. $U_{\alpha_1} v_1 = w_1$ where $u_1 = \frac{(v_1 - w_1)}{||v_1 - w_1||}$, so the thesis is true for $m = 1$.
$(k \Rightarrow k+1)$: Let $U_{\alpha_k}, ..., U_{\alpha_1}$ be s.t. $U_{\alpha_k} ... U_{\alpha_1} v_i = w_i\ \forall i = 1, ..., k$, then consider the vectors $\big(U_{\alpha_k} ... U_{\alpha_1} v_{k+1}\big)$, w_{k+1}. Thanks to Lemma 3.1.1, we can say that:

$$\exists\, U_{\alpha_{k+1}} = I - \alpha_{k+1} u_{k+1} u_{k+1}^h$$

where
$$u_{k+1} = \frac{(U_{\alpha_k}...U_{\alpha_1}v_{k+1} - w_{k+1})}{||U_{\alpha_k}...U_{\alpha_1}v_{k+1} - w_{k+1}||}$$
such that
$$U_{\alpha_{k+1}}\big(U_{\alpha_k}...U_{\alpha_1}v_{k+1}\big) = w_{k+1}\,.$$
Since the image of orthogonal vectors, through unitary tranformations, are still orthogonal vectors, we can observe that $\forall i = 1, ..., k$:
$\big(U_{\alpha_k}...U_{\alpha_1}v_i\big) = w_i$ is orthogonal to the vectors $\big(U_{\alpha_k}, ..., U_{\alpha_1}v_{k+1}\big)$ and w_{k+1}.
Hence
$$\big(U_{\alpha_k}...U_{\alpha_1}v_i\big) \perp \big(U_{\alpha_k}, ..., U_{\alpha_1}v_{k+1}\big) - w_{k+1}\,.$$
From this and (3.2) it follows that
$$U_{\alpha_{k+1}}\big(U_{\alpha_k}...U_{\alpha_1}v_i\big) = \big(U_{\alpha_k}...U_{\alpha_1}v_i\big) = w_i\,,\ \ \forall i = 1, ..., k\,,$$
and, thanks to the definition, $U_{\alpha_{k+1}}\big(U_{\alpha_k}...U_{\alpha_1}v_{k+1}\big) = w_{k+1}$. □

Corollary 3.3.2. *Each unitary matrix can be decomposed as a product of n Householder-type matrices.*

Remark 3.3.3. (This note will be usefull later).
If we want to decompose a unitary matrix Q as product of Householder-type matrices we can find $U_{\alpha_1}, \dots, U_{\alpha_n}$ such that
$$U_{\alpha_n}...U_{\alpha_1}Q = I\,.$$
We can note that the vector that defines the $(k+1)$-th Householder-type matrix in the proof of the preceeding theorem is
$$u_{k+1} := (U_{\alpha_k}...U_{\alpha_1}q_{k+1} - e_{k+1})\,,$$
where both $(U_{\alpha_k}...U_{\alpha_1}q_{k+1})$ and e_{k+1} are orthogonal to $\{e_i\}_{i=1}^k$.
So u_{k+1} must have the shape $\begin{pmatrix}0 & \dots & 0 & u'_{k+1}\end{pmatrix}$.
From this it is easy to note that the k-th Householder-type matrix has the shape
$$U_{\alpha_k} = \begin{pmatrix} I & 0 \\ 0 & U'_{\alpha_k}\end{pmatrix}$$
where U'_{α_k} is the Householder-type matrix of dimension $n+1-k$ generated by the vector u'_{k+1} and with the same α of U_{α_k}.

Hence, to find the decomposition of a unitary matrix Q as product of Householder-type matrices, it is enough to find at the k-th step the Householder-type matrix U'_{α_k} of dimension $n+1-k$ such that

$$U'_{\alpha_k} q_k^k = e_1$$

where q_k^k is the first column of the tail-submatrix Q^k of dimension $n+1-k$ of the matrix $\left(U_{\alpha_{k-1}} \dots U_{\alpha_1} Q\right)$:

$$\left(U_{\alpha_{k-1}} \dots U_{\alpha_1} Q\right) = \begin{pmatrix} I & 0 \\ 0 & U'_{k-1} \end{pmatrix} \dots \begin{pmatrix} 1 & 0 \\ 0 & U'_2 \end{pmatrix} U'_1 Q = \begin{pmatrix} I & 0 \\ 0 & Q^k \end{pmatrix} .$$

Note: this algorithm is equivalent to the QR algorithm.

Remark. With the Householder matrices it is possible to prove the same theorem if the matrices are real, whereas, in the complex case it is only possible to decompose a unitary matrix as product of $n-1$ Householder matrices and a unitary diagonal matrix.

Lemma 3.3.4. *:*

- *If $U = I - V\Omega V^h$ is a unitary matrix where V is a matrix in $\mathbb{C}^{n\times k}$ s.t. $V^hV = I_{k\times k}$, then $\exists W \in \mathbb{C}^{n\times k}$ s.t. $W^hW = I_{k\times k}$ and $D \in \mathbb{C}^{k\times k}$ a diagonal matrix with $D_{jj} = 1 - e^{i\theta_j}$ such that $U = I - WDW^h = \prod_{j=1}^{k} \left(I - D_{jj} w_j w_j^h\right) = \prod_{j=1}^{k} U_{\alpha_j}$.*

- *If $U = I - V\Omega V^h$ is a unitary matrix where V is matrix in $\mathbb{C}^{n\times m}$, then $\exists W \in \mathbb{C}^{n\times k}$ with $k = rk(V)$ s.t. $W^hW = I_{k\times k}$ and $D \in \mathbb{C}^{k\times k}$ a diagonal matrix with $D_{jj} = 1 - e^{i\theta_j}$ such that $U = I - WDW^h$.*

- *If $U = I - V\Omega V^h$ is a unitary matrix where $V \in \mathbb{C}^{n\times m}$ has full rank m, $\Omega \in \mathbb{C}^{m\times m}$ is non singular, then U can be decomposed as a product of m non-trivial Householder-type matrices s.t. the vectors that define the Householder-type matrices are orthogonal.*

Proof. Let's consider the Schur decomposition of Ω

$$\Omega = V'TV'^h .$$

Then

$$U = I - VV'TV'^hV^h$$

and thanks to the unitarity of V' we have $(VV')^h(VV') = V'^h V^h V V' = I$, so we can define

$$W := VV'\,.$$

Now let's impose that $I - WTW^h$ is unitary:

$$\begin{aligned}
&\big(I - WTW^h\big)\big(I - WT^hW^h\big) = \big(I - WT^hW^h\big)\big(I - WTW^h\big) = I \\
\Rightarrow\quad & I - W\big(T + T^h - TT^h\big)W^h = I - W\big(T + T^h - T^hT\big)W^h = I \\
\Rightarrow\quad & T + T^h - TT^h = T + T^h - T^hT = 0 \\
\Rightarrow\quad & T + T^h = TT^h = T^hT\,.
\end{aligned}$$

Since T is a triangular matrix that has to be also normal, T is a diagonal matrix.
From the last equation we have also that $|T_{ii}|^2 = 2Re(T_{ii})$, that is, each T_{ii} is equal to $1 - e^{i\theta_i}$ for some θ_i.
If

$$U = I - V\Omega V^h$$

and the $\{v_i\}$ are not linearly independent, V could be written as $V = V'X$ where $V' \in \mathbb{C}^{n\times k}$ is a full rank matrix and $X \in \mathbb{C}^{k\times m}$ is an upper triangular matrix, and so, defining

$$\Omega' := X\Omega X^h\,,$$

U can be written as

$$U = I - V\Omega V^h = U = I - V'X\Omega X^hV'^h = I - V'\Omega'V'^h\,.$$

Now with an easy ploy we can bring the above equation to the hypotheses of the first statement of the lemma; we use the fact that V'^hV' is positive definite and so we can consider it's square root. Thus

$$\begin{aligned}
U =& I - V'\Omega'V'^h \\
=& I - V'\big(V'^hV'\big)^{-\frac{1}{2}}\big(V'^hV'\big)^{\frac{1}{2}}\Omega'\big(V'^hV'\big)^{\frac{1}{2}}\big(V'^hV'\big)^{-\frac{1}{2}}V'^h
\end{aligned}$$

and, if we define the matrices

$$W' := V'\big(V'^hV'\big)^{-\frac{1}{2}}\,,$$

$$\Omega'' = \big(V'^hV'\big)^{\frac{1}{2}}\Omega'\big(V'^hV'\big)^{\frac{1}{2}}\,,$$

then W' is such that

$$W'^h W' = \left(V'^h V'\right)^{-\frac{1}{2}} V'^h V' \left(V'^h V'\right)^{-\frac{1}{2}} = I_{k\times k}\,.$$

Now we are in the hypothesss of the first statement of the lemma, and therefore $\exists W$and D s.t. $U = I - WDW^h$, where $W^h W = I_{k\times k}$ and $D_{jj} = 1 + e^{i\theta_j}$. The third statement follows from the two above: from the last one we observe that U can be decomposed as

$$U = I - WDW^h\,, \quad where \quad W^h W = I_{m\times m}\,,$$

$$D_{jj} = 1 - e^{i\theta_j}\,;$$

we can easily observe that D is nonsingular ($D_{jj} \neq 0\ \forall j$) since Ω is non singular and we have used only non singular transformations. Now it is simple to observe that if we decompose the last matrix equation in a product of Householder-type matrices (as we did in the first part of the theorem) all of them are non trivial. □

Theorem 3.3.5. *Let U be a unitary matrix whose eigenvalues are 1 with multiplicity k and $\{e^{i\theta_j}\}_{j=1}^{n-k}$, then:*

1. *$\exists\{\alpha_j\}_{j=1}^{n-k}$, $\{u_j\}_{j=1}^{n-k}$ s.t. $||u_j|| = 1\ \forall j$ and $u_j \perp u_i\ \forall\, i \neq j$ s.t. $U = \prod_{j=1}^{n-k} U_{\alpha_j} = \prod_{j=1}^{n-k}(I - \alpha_j u_j u_j^h)\,.$*
2. *If U is product of m non-trivial Householder-type matrices $U = \prod_{i=1}^{m} U_{\alpha_i}$ s.t. the vectors that define the matrices U_{α_i} are orthogonal, then $m = n - k\,.$*
3. *U can't be decomposed as a product of less than $n-k$ Householder-type matrices.*
4. *The algorithm that comes from Theorem 3.3.1 converges in exactly $n-k$ not-trivial steps.*

Proof. Let's consider the spectral decompostion of

$$U = VD(\lambda)V^h = VV^h - V\big(I - D(\lambda)\big)V^h = I - V\big(I - D(\lambda)\big)V^h\,;$$

we can observe that the matrix $I - D(\lambda)$ is a diagonal matrix with $n-k$ non-zeros entries each one of the type $1 - e^{i\theta_j} := \alpha_j$.
So $U = I - \sum_{h=1}^{n-k} \alpha_{j_h} v_{j_h} v_{j_h}^h = \prod_{h=1}^{n-k}\left(I - \alpha_{j_h} v_{j_h} v_{j_h}^h\right)$ where the latter equality

is true since the eigenvectors v_{j_h} are orthogonal.
To demonstrate the second statement it's enough to observe that if

$$U = \prod_{i=1}^{m} \left(I - \alpha_i u_i u_i^h\right) \quad with \quad u_i \perp u_j \; \forall i \neq j$$

then U has $n-m$ eigenvalues equal to 1 so $n-m=k$.
To demonstrate the third statement let's consider an optimal decomposition of U , that is a decompositin in terms of a number of nontrivial Householder-type matrices as minimum as possible

$$U = \prod_{j=1}^{m} U_{\alpha_j} = \prod_{j=1}^{m} \left(I - \alpha_j u_j u_j^h\right).$$

If we define V the $n \times m$ matrix that has as columns the $\{u_i\}$, it is easy to see that the above equality can be written in matrix form as

$$U = I - V \Omega V^h \, ,$$

where Ω is an upper triangular not-singular matrix with the α_j on the diagonal.
Because of Lemma 3.3.4 it's necessary that V has maximum rank otherwise the decomposition wouldn't be optimal. So, again thanks to Lemma 3.3.4, there exist W and D s.t.

$$U = I - W D W^h$$

where

$$W^h W = I_{m \times m}$$

and

$$D_{jj} = 1 + e^{i\theta_j} \, .$$

But this means that U has only m eigenvalues different from 1 where $m \leq k$, so $m = k$.
The algorithm that comes from Theorem 3.3.1 is such that the vectors $\{u_j\}$, which define the Householder-type matrices involved in the decomposition of U, are linearly independent, as observed in Remark 3.3.3; so if

$$U = \prod_{j=1}^{m} (I - \alpha_j u_j u_j^h)$$

is the decomposition that comes from the algorithm, then it can be written as

$$U = I - V\Omega V^h$$

where both V and Ω (V is the $n \times m$ matrix having $\{u_j\}$ as columns and Ω is an upper triangular matrix with the $\alpha_j \neq 0$ on the diagonal) are nonsingular matrices. Thanks to Lemma 3.3.4, this has to be an optimal decomposition, that is $m = n - k$.

□

Now we will show that, from the Householder-type decomposition of a real unitary matrix, we can derive its decomposition in terms of real Householder matrices. First of all let's remember some basic properties of a real unitary matrix. Consider U, a real unitary matrix, and λ one of its eigenvalues; then if λ is not real ($\lambda \neq \pm 1$), also $\overline{\lambda}$ is an eigenvalue. In the same way if v is an eigenvector related to an eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $\overline{v}$ is an eigenvector associated to $\overline{\lambda}$. Thus, if we look at the Householder-type decomposition of U that is obtained from its spectral decomposition, if $U_\alpha = I - \alpha u u^h$ is one the factors then, necessarily, in the decomposition there is also $\overline{U_\alpha} = I - \overline{\alpha}\overline{u}u^t$. So, to obtain a decomposition of U in terms of real Householder matrices, it's enough to find the decomposition, in terms of Householder matrices, of a matrix like $(\overline{U}_\alpha U_\alpha)$, where u and $\overline{u}$ are orthogonal.
Hence we want to show that $\forall \theta \in [0, 2\pi),\ u \in \mathbb{C}^n\ s.t.\ ||u|| = 1,\ u^t u = 0$ there exist $v, w \in \mathbb{R}^n$, $||v|| = ||w|| = 1$, such that:

$$\Big(I - (1 - e^{i\theta})uu^h\Big)\Big(I - (1 - e^{-i\theta})\overline{u}u^t\Big) = \Big(I - 2vv^t\Big)\Big(I - 2ww^t\Big)\,. \tag{3.9}$$

Note that, since $u^t u = 0$, it has to be true:

$$(Re\,u)^t(Re\,u) = (Im\,u)^t(Im\,u) = \frac{1}{2} \quad and \quad (Re\,u)^t(Im\,u) = 0\,.$$

If we develop the first member of the equality (3.9) we obtain:

$$I - (1 - e^{i\theta})uu^h - (1 - e^{-i\theta})\overline{u}u^t =$$

$$I - 2\Big((1 - cos\,\theta)\big((Re\,u)(Re\,u)^t + (Im\,u)(Im\,u)^t\big) + (sen\,\theta)\big((Im\,u)(Re\,u)^t - (Re\,u)(Im\,u)^t\big)\Big)\,.$$

Similarly the second member of (3.9) becomes:

$$I - 2\Big(vv^t + ww^t - 2(v^t w)vw^t\Big)\,. \tag{3.10}$$

Now, if we rename $Re\, u = x$ and $Im\, u = y$, we are looking for v and w such that:

$$(1 - cos\,\theta)\big(xx^t + yy^t\big) + (sen\,\theta)\big(yx^t - xy^t\big) = vv^t + ww^t - 2(v^t w)vw^t\,. \quad (3.11)$$

Let's impose $v = \sqrt{2}\,\sin\varphi\, x + \sqrt{2}\,\cos\varphi\, y$
and $w = \sqrt{2}\,\sin\psi\, x + \sqrt{2}\,\cos\psi\, y\,.$
Then the second term of the above equality becomes:

$$\begin{aligned}
&vv^t + ww^t - 2(v^t w)vw^t = \\
&2\Big((\sin^2\varphi + \sin^2\psi)xx^t + (\cos^2\varphi + \cos^2\psi)yy^t + (\sin\varphi\,\cos\varphi + \sin\psi\,\cos\psi)(xy^t + yx^t)\Big) \\
&-4(\sin\varphi\,\sin\psi + \cos\varphi\,\cos\psi)\Big((\sin\varphi\sin\psi)xx^t + (\cos\varphi\,\cos\psi)yy^t\Big) \\
&-4(\sin\varphi\,\sin\psi + \cos\varphi\,\cos\psi)\Big((\sin\varphi\,\cos\psi)xy^t + (\cos\varphi\,\sin\psi)yx^t)\Big)\,.
\end{aligned}$$

So ψ and φ have to satisfy the following equalities:

$$\begin{aligned}
1 - \cos\theta =& 2(\sin^2\varphi + \sin^2\psi) - 4(\sin\varphi\,\sin\psi + \cos\varphi\,\cos\psi)(\sin\varphi\sin\psi) \\
=& 2(\cos^2\varphi + \cos^2\psi) - 4(\sin\varphi\,\sin\psi + \cos\varphi\,\cos\psi)(\cos\varphi\,\cos\psi)\,, \\
\sin\theta =& 2(\sin\varphi\,\cos\varphi + \sin\psi\,\cos\psi) - 4(\sin\varphi\,\sin\psi + \cos\varphi\,\cos\psi)(\sin\varphi\,\cos\psi) \\
=& -2(\sin\varphi\,\cos\varphi + \sin\psi\,\cos\psi) + 4(\sin\varphi\,\sin\psi + \cos\varphi\,\cos\psi)(\cos\varphi\,\sin\psi)\,,
\end{aligned}$$

$$\begin{aligned}
\cos\theta =& (1 - 2\sin^2\varphi)(1 - 2\sin^2\psi) + 4(\cos\varphi\,\cos\psi\sin\varphi\sin\psi) \\
=& (1 - 2\cos^2\varphi)(1 - 2\cos^2\psi) + 4(\sin\varphi\,\sin\psi\cos\varphi\,\cos\psi)\,, \\
\sin\theta =& 2(\sin\varphi\,\cos\varphi)(1 - 2\cos^2\psi) + 2(\sin\psi\,\cos\psi)(1 - 2\sin^2\varphi) \\
=& -2(\sin\varphi\,\cos\varphi)(1 - 2\sin^2\psi) - 2(\sin\psi\,\cos\psi)(1 - 2\cos^2\varphi)\,,
\end{aligned}$$

$$\begin{aligned}
\cos\theta =& (\cos 2\varphi)(\cos 2\psi) + (\sin 2\varphi)(\sin 2\psi)\,, \\
\sin\theta =& (\sin(-2\varphi))(\cos 2\psi) + (\sin 2\psi)(\cos 2\varphi)\,,
\end{aligned}$$

$$\begin{aligned}
\cos\theta =& \big(\cos(2\psi - 2\varphi)\big)\,, \\
\sin\theta =& \big(\sin(2\psi - 2\varphi)\big)\,.
\end{aligned}$$

Hence, to satisfy (3.9), we can take

$$v = \sqrt{2}\,\sin\varphi\, x + \sqrt{2}\,\cos\varphi\, y \quad \textit{and} \quad w = \sqrt{2}\,\sin\psi\, x + \sqrt{2}\,\cos\psi\, y\,,$$

where φ and ψ are such that $\theta = 2(\psi - \varphi)$.

Remark 3.3.6. We can easily observe from the preceding theorems that the algorithm that comes from the demonstration of Theorem 3.3.1 can also be used to reduce the problem of computing the eigenvalues and eigenvectors of a $n \times n$ unitary matrix, U, to the problem of computing eigenvalues and eigenvectors of a $m \times m$ unitary matrix where m is equal to number of eigenvalues of U different from 1.
Indeed, given a unitary matrix U, we can do the following steps:

1. define $U^1 := U$ and u_1^1 it's first column.

2. $for\ i = 1, ..., n$

   (a) find the Householder-type matrix $U_{\alpha_i} = I - \alpha_i u_i u_i^h$ of dimension $n + 1 - i$, such that $U_{\alpha_i} u_i^i = e_1$

   (b) define U^{i+1} the tail submatrix of dimension $n - i$ of the matrix $\left(U_{\alpha_i} U^i\right)$ and u_{i+1}^{i+1} its first column.

   end for

3. after at most n steps we can say that:

$$U = U_{\overline{\alpha_1}} \begin{pmatrix} 1 & 0 \\ 0 & U_{\overline{\alpha_2}} \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 & \dots \\ 0 & \ddots & 0 \\ \vdots & 0 & U_{\overline{\alpha_n}} \end{pmatrix} \tag{3.12}$$

$$= \tilde{U}_{\overline{\alpha_1}} \dots \tilde{U}_{\overline{\alpha_n}} \tag{3.13}$$

$$= \prod_{i=1}^{n} \left(I - \overline{\alpha_i} \tilde{u} \tilde{u}^h\right) \tag{3.14}$$

   where $\tilde{u_i}$ is defined as the vector $\begin{pmatrix} 0 \\ u_i \end{pmatrix}$. In particular, thanks to Theorem 3.3.5, we can say that only $n - k$ of the terms in the above equation are non-trivial, where k is the multiplicity of 1 as eigenvalue of the matrix U.

4. Write in matrix form the above equation

$$U = I - V \Omega V^h$$

   where V is defined as the $n \times (n - k)$ matrix having as columns the $\tilde{u}_i$ relative to $\alpha_i \neq 0$.

5. Observe that

$$U = I - V\left(V^hV\right)^{-\frac{1}{2}}\left(V^hV\right)^{\frac{1}{2}}\Omega\left(V^hV\right)^{\frac{1}{2}}\left(V^hV\right)^{-\frac{1}{2}}V^h$$

   and define $V' := V\left(V^hV\right)^{-\frac{1}{2}}$ and $\Omega' := \left(V^hV\right)^{\frac{1}{2}}\Omega\left(V^hV\right)^{\frac{1}{2}}$.

6. Compute the spectral decomposition of the $(n-k)\times(n-k)$ unitary matrix $I-\Omega'$:

$$I - \Omega' = WD(e^{i\theta_i})W^h\,.$$

7. The columns of the matrix $V'W$ are the eigenvectors of the matrix U relative to the eigenvalues different from 1. The eigenvalues of $I-\Omega'$ are all the eigenvalues different from 1 of the matrix U.

Lemma 3.3.7. *Let be given $A, B \in \mathbb{C}^{n\times m}$, $m \le n$, s.t. $A^hA = B^hB$. Then*

- $\forall\{i_j\}_{j=1}^k \subseteq \{1,...,m\}$ *we have* $rk(A_{i_1,...,i_k}) = rk(B_{i_1,...,i_k})$, *where* $(A_{i_1,...,i_k})$ *is the submatrix of A obtained selecting the columns $i_1,...,i_k$.*

- *If $A_{i_1,...,i_k}$ is a submatrix of A with maximum rank, then* $\exists T \in \mathbb{C}^{k\times m}$ *s.t.* $A = A_{i_1,...,i_k}T$ *and* $B = B_{i_1,...,i_k}T$.

Proof. Let be given $\{i_j\}_{j=1}^k \subseteq \{1,...,m\}$; at first we can observe that, unless of permutations of columns, it is allowed to assume that $\{i_j\}_{j=1}^k = \{1,...,k\}$. Now it is easy to note that the columns $A_1,...,A_k$ are independent iff $det\Big(\left(A^hA\right)_{k\times k}\Big) \neq 0$ where $\left(A^hA\right)_{k\times k}$ is the head submatrix of A^hA of dimension $k\times k$.
Indeed, if the k-th column is combination of the first $k-1$, it should follow $A_k = \sum_{i=1}^{k-1}\alpha_iA_i$ and, from this, the k-th row of $\left(A^hA\right)_{k\times k}$ would be of the type:

$$\Big(\sum_{i=1}^{k-1}\overline{\alpha_i} <A_i, A_1>, ..., \sum_{i=1}^{k-1}\overline{\alpha_i} <A_i, A_{k-1}>, \sum_{i=1}^{k-1}\overline{\alpha_i} <A_i, \sum_{j=1}^{k-1}\alpha_jA_j> \Big)\,.$$

But the latter is a combination of the first $k-1$ rows of $\left(A^hA\right)_{k\times k}$ and so the determinat should be equal to zero.

On the other hand it is true also the contrary;
indeed if $det\big(A^hA\big)_{k\times k} = 0$, then $\exists\, v \in \mathbb{C}^k$ s.t. $\big(A^hA\big)_{k\times k}v = 0$. So:

$$\sum_{j=1}^{k} v_j <A_i, A_j>= 0 \quad \forall i \qquad \Rightarrow \sum_{j=1}^{k} <A_i, v_jA_j>= 0$$
$$\Rightarrow <\sum_{i=1}^{k} v_iA_i, \sum_{j=1}^{k} v_jA_j>= 0 \qquad \Rightarrow \Big|\Big|\sum_{i=1}^{k} v_iA_i\Big|\Big|^2 = 0\,,$$

that means that the $\{A_i\}$ are linearly dependent.
From the above equivalence we can observe that, if $A^hA = B^hB$, the k-th column of A is a combination of the first $k-1$ iff the same is true about B, in particular it is possible to conclude that A_k and B_k are combinations of the first $k-1$ columns iff $\exists v \in \mathbb{C}^k$ t.c

$$v^h\big(A^hA\big)_{k\times k}v = v^h\big(B^hB\big)_{k\times k}v = 0$$

iff

$$\sum_{i=1}^{k} v_iA_i = \sum_{i=1}^{k} v_iB_i = 0$$

that means

$$A_k = \sum_{i=1}^{k-1} \alpha_iA_i \quad \textit{iff} \quad B_k = \sum_{i=1}^{k-1} \alpha_iB_i\,.$$

This in matrix form can be expressed as

$$A = A_{i_1,\dots,i_k}T \quad and \quad B = B_{i_1,\dots,i_k}T\,,$$

where T is the same matrix for both A and B and $A_{i_1,\dots,i_k}$, $B_{i_1,\dots,i_k}$ are submatrices of maximum rank. □

Theorem 3.3.8. *Let be given $A, B \in \mathbb{C}^{n\times m}$ with $m \le n$ s.t. $A^hA = B^hB$. Then $\exists\, U_{\alpha_{rk(A)}}, \dots, U_{\alpha_1}$ s.t. $U_{\alpha_{rk(A)}}\dots U_{\alpha_1}A = B$.*

Proof. Recalling Lemma 3.3.7 it follows that exist $T \in \mathbb{C}^{k\times m}$ and two submatrices of maximum rank of A and B, $\tilde{A}, \tilde{B}$ in $\mathbb{C}^{n\times k}$, that are obtained selecting some columns of A, B, s.t.

$$A = \tilde{A}T, \qquad B = \tilde{B}T\,.$$

Let's consider $\{e_i\}$, $\{f_i\}$, the orthonormalization of the columns of $\tilde{A}$, $\tilde{B}$ obtained with the Gram Schmidt algorithm:

$$e_1 = \frac{\tilde{A}_1}{||\tilde{A}_1||}, \dots, e_i = \frac{\tilde{A}_i - \sum_{j=1}^{i-1} < e_j, \tilde{A}_i > e_j}{||\tilde{A}_i - \sum_{j=1}^{i-1} < e_j, \tilde{A}_i > e_j||}, \tag{3.15}$$

$$f_1 = \frac{\tilde{B}_1}{||\tilde{B}_1||}, \dots, f_i = \frac{\tilde{B}_i - \sum_{j=1}^{i-1} < f_j, \tilde{B}_i > f_j}{||\tilde{B}_i - \sum_{j=1}^{i-1} < f_j, \tilde{B}_i > f_j||}. \tag{3.16}$$

Note that

$$B^h B = A^h A \Rightarrow <\tilde{B}_i, \tilde{B}_j>=<\tilde{A}_i, \tilde{A}_j> \ \forall i, j ,$$

Then, from the above equations, it follows that the coefficients of e_i, f_i , each one with respect to the basis $\{\tilde{A}_j\}$, $\{\tilde{B}_j\}$ are the same. Thus (3.15),(3.16) can be similary formulated saying that exists only one matrix C, non singular, s.t.

$$E = \tilde{A}C$$

$$F = \tilde{B}C ,$$

where E and F are the matrices having $\{e_i\}$ and $\{f_i\}$ as columns.
So:

$$EC^{-1}T = A ,$$

$$FC^{-1}T = B .$$

Now it is easy to conclude because, thanks to Theorem 3.3.1,

$$\exists\, U_{\alpha_1}, \dots, U_{\alpha_{rk(A)}} \quad s.t. \quad F = U_{\alpha_1} \dots U_{\alpha_{rk(A)}} E \quad \Rightarrow$$

$$B \ = \ FC^{-1}T \ = \ U_{\alpha_1} \dots U_{\alpha_{rkA}} EC^{-1}T \ = \ U_{\alpha_1} \dots U_{\alpha_{rk(A)}} A .$$

□

Theorem 3.3.9. ***: the best approximation of a unitary matrix with k-Householder-type matrices***
Let U be a unitary matrix and $\prod_{i=1}^m U_{\alpha_i}$ its decomposition in Householder-type matrices defined by orthogonal vectors as in Theorem 3.3.5. Then the best approximation of U (in Frobenius and 2-norm) with k Householder-type

matrices is $U^* = \prod_{h=1}^{k} U_{\alpha_{i_h}}$ *where the* $\{\alpha_{i_h}\}_{h=1}^{k}$ *are the* k *ones with bigger absolute values among the* α_i*. And*

$$||U - U^*||_F^2 = \sum_{j=k+1}^{n} |\alpha_j|^2 = \sum_{j=k+1}^{n} |1 - e^{i\theta_j}|^2 ,$$

$$||U - U^*||_2^2 = |\alpha_{k+1}|^2 = |1 - e^{i\theta_{k+1}}|^2 ,$$

where $\{\alpha_j\}_{j=k+1}^{n}$ *are the ones with smaller absoulte value among the* α_j *and* α_{k+1} *is the one with bigger absolute value among the* $\{\alpha_j\}_{j=k+1}^{n}$*.*

Proof. Let

$$U = VD(\lambda_i)V^h$$

be the spectral decomposition of U; as in Theorem 3.3.5 it can be written as

$$U = I - VDV^h$$

where D is the diagonal matrix $\in \mathbb{C}^{n\times n}$ s.t. $D_{ii} = \alpha_i = 1 - \lambda_i$.
Still from the previous results, we can say that to approximate U with a product of k Householder-type matrices is the same thing that to approximate U with a matrix of the type $I - W\Omega W^h$, where $W \in \mathbb{C}^{n\times k}$, $\Omega \in \mathbb{C}^{k\times k}$ upper triangular.
So our problem is equivalent to approximate the matrix VDV^h with a matrix of rank $\leq k$ of the form $W\Omega W^h$.
It is well known that the best approximation (in both the 2-norm and the Frobenius norm, see the appendix) of a matrix with a rank k matrix is the truncation of its SVD that leaves the k bigger singular values.
It is also well known that the SVD of VDV^h is[2]

$$VD(|\alpha_i|)D\left(\frac{\alpha_i}{|\alpha_i|}\right)V^h ,$$

where the $|\alpha_i|$ are the singular values and the matrices $V, D\left(\frac{\alpha_i}{|\alpha_i|}\right)V^h$ are the left and right singular matrices.
So we have that the best approximation of VDV^h with a matrix of rank $\leq k$ is :

- VDV^h if $k \geq rank(D)$ where $rank(D)$ is the number of eigenvalues different from 1;

[2] we adopt the convention $\frac{\alpha_i}{|\alpha_i|} = 1$ if $\alpha_i = 0$

- $VD'V^h$ if $k \leq rank(D)$ where D' is the diagonal matrix with zeros in place of the $(rank(D) - k)$ smaller singular values, that correspond to the $rank(D) - k$ smaller α_i, that are also the eigenvalues of the matrix U closer to 1.

Hence we have that

$$U^* = I - VD'V^h$$

and

$$U - U^* = V \begin{pmatrix} 0 & . & . & . & . \\ . & \ddots & . & . & . \\ . & . & \alpha_{k+1} & . & . \\ . & . & . & \ddots & . \\ . & . & . & . & \alpha_n \end{pmatrix} V^h \,,$$

where $\{\alpha_j\}_{j=k+1}^n$ are the ones with smaller absoulte value among the α_j. In particular we can conclude observing that

$$||U - U^*||_F^2 = \sum_{j=k+1}^{n} |\alpha_j|^2 = \sum_{j=k+1}^{n} |1 - e^{i\theta_j}|^2 \,,$$

and

$$||U - U^*||_2^2 = |\alpha_{k+1}|^2 = |1 - e^{i\theta_{k+1}}|^2 \,,$$

where α_{k+1} is the one with bigger absolute value among the $\{\alpha_j\}_{j=k+1}^n$. $\square$

3.4 Improving stability of QR Decomposition

Usually to obtain the QR decomposition of a matrix A, we apply a $(n-1)$-steps triangularization procedure to A in which at the i-th step an Householder matrix $\tilde{U}_i$ is introduced such that $\tilde{U}_i a_i^i = e^{i\theta_i} e_1$ where a_i^i is the normalized first column of the tail $(n+1-i) \times (n+1-i)$-submatrix of $\Big(U_{i-1} \dots U_1 A\Big)$ [3], and $e^{i\theta_i}$ is chosen in order to have $< a_i^i, e^{i\theta_i} e_1 > \in \mathbb{R}$ (see [8], [15]).
Now, with the Householder-type matrices, at each step we don't need any more to fix θ_i in order to have $< a_i^i, e^{i\theta_i} e_1 >$ real; in fact $\forall \theta$, $\exists \tilde{U}_{i,\alpha(\theta)}$ such that $\tilde{U}_{i,\alpha(\theta)} a_i^i = e^{i\theta} e_1$. So we can choose θ_i such that $\tilde{U}_{i,\alpha(\theta_i)}$ is as "good" as possible.

[3] U_i is the $n \times n$ Householder matrix direct sum of the identity matrix I_{i-1} and of $\tilde{U}_i$

We say that the matrix $U_\alpha = I - \alpha uu^h$ is better than the matrix $W_\beta = I - \beta ww^h$ if it is closer to the identity matrix, that is, since $||uu^h|| = ||ww^h|| = 1$, if $|\alpha|$ is smaller than $|\beta|$. Indeed, given a unitary vector v, our aim is to have the equality $(I - \alpha uu^h)v = e^{i\theta}e_1$ and to bound, as much as possible, the error that affects the vector $(I - \alpha uu^h)z$ because of an error ε generated by the computation of $(u^h z)$. Looking at the equation:

$$(I - \alpha uu^h)z \approx z - \alpha(\widetilde{u^h z})u = z - \alpha(u^h z + \varepsilon)u = (I - \alpha uu^h)z - \varepsilon\alpha u$$

we can observe that, smaller is $|\alpha|$, smaller is the perturbation of $(I - \alpha uu^h)z$ caused by ε.

If we want the matrix $U_{\alpha(\theta)}$ s.t. $U_{\alpha(\theta)}v = e^{i\theta}e_1$, thanks to Lemma 3.1.1, we have to choose $U_{\alpha(\theta)} = I - \alpha(\theta)uu^h$ where

$u = \dfrac{(v - e^{i\theta}e_1)}{||(v - e^{i\theta}e_1)||}$ and $\alpha(\theta) - 1 = \dfrac{<v, e^{i\theta}e_1> -1}{<e^{i\theta}e_1, v> -1} = \dfrac{\overline{v_1}e^{i\theta} - 1}{v_1 e^{-i\theta} - 1}$

Set $\alpha(\theta) = \alpha_\theta$. Then:

$$\alpha_\theta = \frac{2Re(v_1 e^{-i\theta}) - 2}{v_1 e^{-i\theta} - 1}\,, \tag{3.17}$$

$$\begin{aligned}
|\alpha_\theta|^2 &= \frac{4(Re(v_1 e^{-i\theta}) - 1)^2}{(v_1 e^{-i\theta} - 1)(\overline{v_1}e^{i\theta} - 1)} = \frac{4(Re(v_1 e^{-i\theta}) - 1)^2}{|v_1 e^{-i\theta}|^2 + 1 - \overline{v_1}e^{i\theta} - v_1 e^{-i\theta}} \\
&= \frac{4\big(Re(v_1 e^{-i\theta} - 1)\big)^2}{Re^2(v_1 e^{-i\theta}) + Im^2(v_1 e^{-i\theta}) - 2Re(v_1 e^{-i\theta}) + 1} \\
&= \frac{4\big(Re(v_1 e^{-i\theta} - 1)\big)^2}{(Re(v_1 e^{-i\theta}) - 1)^2 + Im^2(v_1 e^{-i\theta})} \\
&= 4\left(\frac{1}{1 + \frac{Im^2(v_1 e^{-i\theta})}{(Re(v_1 e^{-i\theta} - 1))^2}}\right).
\end{aligned} \tag{3.18}$$

Now, since we want to minimize $|\alpha_\theta|$, let's maximize

$$\left(\frac{Im^2(v_1 e^{-i\theta})}{(Re(v_1 e^{-i\theta} - 1))^2}\right). \tag{3.19}$$

Note that the derivative of $Im(v_1 e^{-i\theta})$ is $-Re(v_1 e^{-i\theta})$ and the derivative of $Re(v_1 e^{-i\theta})$ is $Im(v_1 e^{-i\theta})$, in fact:

$$\frac{d}{d\theta}Im(v_1 e^{-i\theta}) = \frac{d}{d\theta}\frac{v_1 e^{-i\theta} - \overline{v_1}e^{i\theta}}{2i} = \frac{-iv_1 e^{-i\theta} - i\overline{v_1}e^{i\theta}}{2i} = -Re(v_1 e^{-i\theta})\,,$$
$$\frac{d}{d\theta}Re(v_1 e^{-i\theta}) = \frac{d}{d\theta}\frac{v_1 e^{-i\theta} + \overline{v_1}e^{i\theta}}{2} = \frac{-iv_1 e^{-i\theta} + i\overline{v_1}e^{i\theta}}{2} = \frac{v_1 e^{-i\theta} - \overline{v_1}e^{i\theta}}{2i} = Im(v_1 e^{-i\theta})\,.$$

So we can calculate the derivative of (3.19):

$$\frac{d}{d\theta}\left(\frac{Im^2(v_1e^{-i\theta})}{(Re(v_1e^{-i\theta}-1))^2}\right) =$$
$$\frac{-2Im(v_1e^{-i\theta})Re(v_1e^{-i\theta})\big(Re(v_1e^{-i\theta})-1\big)^2-2Im(v_1e^{-i\theta})Im^2(v_1e^{-i\theta})\big(Re(v_1e^{-i\theta})-1\big)}{\big(Re(v_1e^{-i\theta})-1\big)^4} =$$
$$\frac{-2\big(Im(v_1e^{-i\theta})\big)\Big(Im^2(v_1e^{-i\theta})+\big(Re(v_1e^{-i\theta})-1\big)\big(Re(v_1e^{-i\theta})\big)\Big)}{\big(Re(v_1e^{-i\theta})-1\big)^3}.$$

Thus (3.19) has minimum for θ s.t. $Im(v_1e^{i\theta}) = 0$ (this follows since the function (3.19) is non negative)
and maximum for θ s.t. $\Big(Im^2(v_1e^{-i\theta})+\big(Re(v_1e^{-i\theta})-1\big)\big(Re(v_1e^{-i\theta})\big)\Big) = 0$, that is $|v_1|^2 = Re(v_1e^{-i\theta})$.
So if we rename $e^{i\varphi} = \frac{v_1}{|v_1|}$, then the θ for which $|\alpha_\theta|$ is minimum can be obtained as:

$$Re\big(e^{i\varphi}e^{-i\theta}\big) = |v_1| \quad \Rightarrow \quad Re\big(e^{i(\varphi-\theta)}\big) = |v_1|$$
$$\Rightarrow \quad cos(\varphi-\theta) = |v_1| \quad \Rightarrow \quad \theta = \varphi \pm arcos(|v_1|).$$

From this it follows that, defining $v' = \Big(0, \quad v_2, \ldots, v_n\Big)$,

$$u = \frac{\big(v-e^{i\theta}e_1\big)}{||\big(v-e^{i\theta}e_1\big)||} = \frac{\big(v-e^{i(\varphi\pm arcos(|v_1|))}e_1\big)}{||\big(v-e^{i(\varphi\pm arcos(|v_1|))}e_1\big)||}$$
$$= \frac{\big(v-\frac{v_1}{|v_1|}e^{\pm i(arcos(|v_1|))}e_1\big)}{||\big(v-\frac{v_1}{|v_1|}e^{\pm i(arcos(|v_1|))}e_1\big)||} = \frac{(v_1-\frac{v_1}{|v_1|}\big(|v_1|\pm isen(arcos(|v_1|))\big))e_1+v'}{||(v_1-\frac{v_1}{|v_1|}\big(|v_1|\pm isen(arcos(|v_1|))\big))e_1+v'||}$$
$$= \frac{\mp i\,\frac{v_1}{|v_1|}sen\big(arcos(|v_1|)\big)e_1+v'}{||\mp i\,\frac{v_1}{|v_1|}sen\big(arcos(|v_1|)\big)e_1+v'||}.$$

In the end let's evaluate $|\alpha_\theta|^2$ in the point of minimum; when $|v_1|^2 = Re(v_1e^{-i\theta})$

we have $Im^2(v_1 e^{-i\theta}) = |v_1|^2 - |v_1|^4$ and thus:

$$
\begin{aligned}
|\alpha_\theta|^2 &= 4\frac{(Re(v_1 e^{-i\theta} - 1))^2}{(Re(v_1 e^{-i\theta}) - 1)^2 + Im^2(v_1 e^{-i\theta})} \\
&= \frac{(|v_1|^2 - 1)^2}{(|v_1|^2 - 1)^2 + |v_1|^2 - |v_1|^4} = 4\frac{(|v_1|^2 - 1)^2}{(1 - |v_1|^2)} \\
&= 4\big(1 - |v_1|^2\big). \qquad (3.20)
\end{aligned}
$$

If we think α_θ as $1 + e^{i\tau}$, since $|\alpha_\theta|^2 = 2Re(\alpha_\theta)$, still from (3.20) we have:

$$
\begin{aligned}
&2(1 + cos(\tau)) = 4\big(1 - |v_1|^2\big) \\
&\Rightarrow cos(\tau) = 1 - 2|v_1|^2 \\
&\Rightarrow \tau = \pm arcos(1 - 2|v_1|^2).
\end{aligned}
$$

Hence we can state the following theorem:

Theorem 3.4.1. *Given a unitary vector v, the Householder-type matrix $U_{\alpha *}$ closest to the identity matrix I such that $U_{\alpha *} v \in span\{e_1\}$ is :*

$$U_{\alpha *} = I - (\alpha^*)uu^h$$

where

$$\alpha^* = \Big(1 + e^{\pm i\, arcos(1 - 2|v_1|^2)}\Big), \quad u = \frac{\Big(\pm i\, \frac{v_1}{|v_1|} sen(arcos(|v_1|))e_1 + v'\Big)}{\Big|\Big|\pm i\, \frac{v_1}{|v_1|} sen(arcos(|v_1|))e_1 + v'\Big|\Big|},$$

$$v' = \Big(0\,,\ v_2, \ldots, v_n\Big)\,.$$

And it is true:

$$|\alpha^*|^2 = ||U_{\alpha^*} - I||^2 = 4\big(1 - |v_1|^2\big).$$

(if we choose "+" in the formula of $\alpha *$ we have to choose "+" also in the formula for u)

Proof. It comes from the previous 2 pages.

□

Observe that as big it is $|v_1|$ as small we can get $|\alpha *|$; hence it comes quite spontaneous to apply before each step of the triangularization procedure a pivoting in such a way to have $|v_1|$ as big as possible.

In detail, after the i-th step we have that

$$U_{\alpha_i} \dots U_{\alpha_1} A = \begin{pmatrix} e^{i\theta_1} & \dots & \dots & \dots \\ 0 & \ddots & \dots & \dots \\ 0 & 0 & e^{i\theta_i} & \dots \\ 0 & 0 & 0 & A_i \end{pmatrix} \tag{3.21}$$

So if we consider the permutation matrix P_i s.t. $|\left(P_i A_i\right)_{11}|$ is the biggest possible we have:

$$\begin{pmatrix} I & 0 \\ 0 & P_i \end{pmatrix} U_{\alpha_i} \dots U_{\alpha_1} A = \begin{pmatrix} e^{i\theta_1} & \dots & \dots & \dots \\ 0 & \ddots & \dots & \dots \\ 0 & 0 & e^{i\theta_i} & \dots \\ 0 & 0 & 0 & A_i' \end{pmatrix} \tag{3.22}$$

where $|\left(A_i'\right)_{11}|$ is the biggest among the absolute values of the elements of the first column of A_i' and the matrix $\begin{pmatrix} I & 0 \\ 0 & P_i \end{pmatrix} U_{\alpha_i} \dots U_{\alpha_1}$ is still a unitary matrix. So after n-steps of this method we will still have a QR decomposition of the matrix A but the matrix Q will be a product of alternating Householder-type matrices and permutation matrices.

Instead of considering the pivoting only on the column, at the i-th step it could also be possible to consider a total-pivoting on the submatrix A_i, that is to find the two permutation matrices P_i^1 and P_i^2 such that $|(P_i^1 A_i P_i^2)_{11}|$ is the biggest possible. In this case we would have

$$\begin{pmatrix} I & 0 \\ 0 & P_i^1 \end{pmatrix} U_{\alpha_i} \dots U_{\alpha_1} A \begin{pmatrix} I & 0 \\ 0 & P_i^2 \end{pmatrix} = \begin{pmatrix} e^{i\theta_1} & \dots & \dots & \dots \\ 0 & \ddots & \dots & \dots \\ 0 & 0 & e^{i\theta_i} & \dots \\ 0 & 0 & 0 & A_i' \end{pmatrix} \tag{3.23}$$

where $|(A_i')_{11}|$ is the biggest among the absolute values of the elements of the matrix A_i'. But, differently from before, after n-steps of this method we will have a QRP decomposition of the matrix A where Q is a unitary matrix product of alternating Householder-type matrices and permutation matrices, R is an upper triangular matrix and P is a permutation matrix.
Note that in both the cases the alternating product of permuation matrices and Householder-type matrices can be expressed as a product of k

Householder-type matrices by a permuation matrix. Indeed, since it is true

$$(I - \alpha vv^h)P = P(I - \alpha P^t vv^h P)\,,$$

we can easily show that

$$P_1 U_{\alpha_1} P_2 U_{\alpha_2} \cdots P_k U_{\alpha_k} = P_1 P_2 \cdots P_k \tilde{U}_{\alpha_1} \cdots \tilde{U}_{\alpha_k}$$

Let's make a final remark. From (3.20) one observes that it is required an Householder matrix (to realize the minumum distance from I) if and only if v_1 is zero, that is the same thing of saying $v \perp e_1$. This was predictable, indeed, since α in $I - \alpha uu^h$ can be written also as $1 - e^{i\theta}$ for some θ, $|\alpha|$ is always smaller than 2, thus the Householder case is the "worst" possible. Moreover, from Theorem 4.0.1 and from the comments in the introduction to this chapter, it follows that one needs an Householder matrix to send v in e_1 if $<v, e_1>$ is real; but as we said before, the Householder case is the worst possible, hence we don't want that the above scalar product is real; any time $<v, e_1> \neq 0$ we can always get it complex multiplying e_1 for some $e^{i\theta}$ so it is a Householder-type with $|\alpha| < 2$ that realizes the minimum.

3.5 Best normal approximation via projection on Householder-Type algebras

In this section we will present just an idea of a possible way to find one best normal approximation, in the Frobenius norm, of a generic matrix A. Some possible algorithms have been already found in [12],[13], however they seem to be not so cheap.
Our idea is based on the following results:

- Every normal matrix can be diagonalized by a unitary matrix.
- We know the shape of the best approximation of a matrix in a fixed SDU space (Theorem 1.1.2).
- We can decompose every unitary matrix as a product of Householder-type matrices (Theorem 3.3.1).

So the problem is the following: find

$$N_{opt} = \text{argmin}_{(N \in \mathcal{N})} ||A - N|| = U_{opt} D\Big((U_{opt}^h A U_{opt})_{ii} \Big) U_{opt}^h$$

where $\mathcal{N}$ denotes the set of $n \times n$ normal matrices. This problem can be equivalently formulated in terms of U_{opt} and then, in terms of its decomposition as a product of Householder-type matrices:

$$U_{opt} = \text{argmin}_{(U \ unitary)} ||A - UD\big((U^h AU)_{ii}\big)U^h|| = \tag{3.24}$$

$$\text{argmin}_{(U_{\alpha_i}\,,\, i=1,\cdots,n)} \left|\left| A - \Big(U_{\alpha_1} \cdots U_{\alpha_n}\Big) D\Big(\Big((U_{\alpha_n}^h \cdots U_{\alpha_1}^h) A \big(U_{\alpha_1} \cdots U_{\alpha_n}\big)\Big)_{ii}\Big) \Big(U_{\alpha_n}^h \cdots U_{\alpha_1}^h\Big) \right|\right| =$$

$$\text{argmin}_{(U_{\alpha_i}\ i=1,\ldots,n)} \left|\left| \Big(U_{\alpha_n}^h \cdots U_{\alpha_1}^h\Big) A \Big(U_{\alpha_1} \cdots U_{\alpha_n}\Big) - D\Big(\Big((U_{\alpha_n}^h \cdots U_{\alpha_1}^h) A \big(U_{\alpha_1} \cdots U_{\alpha_n}\big)\Big)_{ii}\Big) \right|\right|$$

$$= (U_{\alpha_1})_{opt} \cdots (U_{\alpha_n})_{opt}\,.$$

Note that if we assume that $U = (U_{\alpha_1})_{opt} \cdots (U_{\alpha_{n-1}})_{opt}$ is already known, then the last equation can be formulated as:

$$(U_{\alpha_n})_{opt} = \text{argmin}_{(U_{\alpha_n})} \left|\left| \Big(U_{\alpha_n}^h\Big) U^h AU \Big(U_{\alpha_n}\Big) - D\Big((U_{\alpha_n}^h (U^h AU) U_{\alpha_n})_{ii} \Big) \right|\right|.$$

So it comes quite spontaneous to imagine an iterative algorithm that, at each step k, computes the Householder-type matrix $(U_\alpha)_k$ that minimizes an equation like the following:

$$(U_\alpha)_k = \text{argmin}_{U_\alpha} \left|\left| \Big(U_\alpha^h\Big) A_k \Big(U_\alpha\Big) - D\Big((U_\alpha^h A_k U_\alpha)_{ii} \Big) \right|\right| \tag{3.25}$$

where:

$$A_k = (U_\alpha)_{k-1}^h \ldots (U_\alpha)_1^h A (U_\alpha)_1 \ldots (U_\alpha)_{k-1}.$$

We still have not solved the above equation in both α and u s.t. $U_\alpha = I - \alpha u u^h$, but we can suggest a possible iterative solver for such equation. Our idea is, fixed a vector u, to look for the value of α that minimizes the equation (3.25); i.e. to look for

$$\hat{\alpha} = \text{argmin}_\alpha ||\big(I - \alpha u u^h\big)^h A \big(I - \alpha u u^h\big) - D\Big(\big((I - \alpha u u^h)^h A (I - \alpha u u^h)\big)_{ii} \Big)|| \tag{3.26}$$

where for the sake of simplicity we have set $A := A_k$.
Then look for

$$\hat{u} = \text{argmin}_u ||\big(I - \hat{\alpha} u u^h\big)^h A \big(I - \hat{\alpha} u u^h\big) - D\Big(\big((I - \hat{\alpha} u u^h)^h A (I - \hat{\alpha} u u^h)\big)_{ii} \Big)||, \tag{3.27}$$

set $u = \hat{u}$ and repeat until

$$I - \hat{\alpha}\hat{u}\hat{u}^h = (U_\alpha)_k.$$

Let us see some preliminary results in solving (3.26). If we work with the Frobenius norm, the minimization problem (3.26) is equivalent to maximize $||\cdot||^2 = \left|\left|D\Big(\big(U_\alpha^h A U_\alpha\big)_{ii}\Big)\right|\right|^2$ in α.
Expanding the function $||\cdot||^2$ and computing its derivative, we have found two different equations that the optimal α has to satisfy:

- Using the hypothesis $2Re\alpha = |\alpha|^2$, one obtains the following expression for $||\cdot||^2$ in terms of $x := Re(\alpha)$

$$||\cdot||^2 = \begin{cases} 8x\sqrt{x(2-x)}C_1 + 4\sqrt{x(2-x)}C_2 + 4x^2C_3 + 4xC_4 + C_5 \\ (if\ Im\alpha \geq 0) \\ -8x\sqrt{x(2-x)}C_1 - 4\sqrt{x(2-x)}C_2 + 4x^2C_3 + 4xC_4 + C_5 \\ (if\ Im\alpha \leq 0) \end{cases}$$

where:

$$C_1 = \sum_{i=1}^{n}\Big(\xi_i\tilde{\xi}_i + \eta_i\tilde{\eta}_i\Big), \qquad C_2 = \sum_{i=1}^{n}\Big(\xi_i\big(Rea_{ii}\big) + \eta_i\big(Ima_{ii}\big)\Big),$$

$$C_3 = \sum_{i=1}^{n}\Big(\tilde{\xi}_i^2 + \tilde{\eta}_i^2 - \xi_i^2 - \eta_i^2\Big), \qquad C_4 = \sum_{i=1}^{n}\Big(\tilde{\xi}_i\big(Rea_{ii}\big) + \tilde{\eta}_i\big(Ima_{ii}\big) + 2\big(\xi_i^2 + \eta_i^2\big)\Big),$$

$$C_5 = \sum_{i=1}^{n}|a_{ii}|^2$$

and

$$\xi_i = Im\Big(\overline{u}_i\, e_i^t \frac{A + A^h}{2}\, u\Big), \qquad \tilde{\xi}_i = |u_i|^2\, u^h \frac{A + A^h}{2}\, u - Re\Big(\overline{u}_i\, e_i^t \frac{A + A^h}{2}\, u\Big),$$

$$\eta_i = Im\Big(\overline{u}_i\, e_i^t \frac{A - A^h}{2i}\, u\Big), \qquad \tilde{\eta}_i = |u_i|^2\, u^h \frac{A - A^h}{2i}\, u - Re\Big(\overline{u}_i\, e_i^t \frac{A - A^h}{2i}\, u\Big).$$

Computing the derivative of $||\cdot||^2$ with respect to $x = Re\alpha$, we have

$$\frac{d||\cdot||^2}{dx} = \begin{cases} 4p_1(x) - \frac{4p_2(x)}{\sqrt{x(2-x)}} & (if\ \ Im\alpha > 0) \\ 4p_1(x) + \frac{4p_2(x)}{\sqrt{x(2-x)}} & (if\ \ Im\alpha < 0) \end{cases},$$

where

$$p_1(x) = 2xC_3 + C_4 \qquad and \qquad p_2(x) = 4x^2C_1 - (6C_1 - C_2)x - C_2.$$

Thus the optimale value of $x = Re\alpha$ must be $0, 2$ or must be a real zero of the following fourth degree algebraic equation

$$p_1(x)^2 x(2-x) = p_2(x)^2. \tag{3.28}$$

We can observe that the coefficients $\{\xi_i\}$, $\{\eta_i\}$, $\{\tilde{\xi}_i\}$, $\{\tilde{\eta}_i\}$ have the following good property (we use $||u|| = 1$ and the fact that $\frac{A+A^h}{2}$, $\frac{A-A^h}{2i}$ are Hermitian matrices):

$$\sum_{i=1}^{n} \xi_i = Im\Big(u^h \frac{A+A^h}{2} u\Big) = 0,$$
$$\sum_{i=1}^{n} \tilde{\xi}_i = ||u||^2 \Big(u^h \frac{A+A^h}{2} u\Big) - Re\Big(u^h \frac{A+A^h}{2} u\Big) = 0,$$
$$\sum_{i=1}^{n} \eta_i = Im\Big(u^h \frac{A-A^h}{2i} u\Big) = 0,$$
$$\sum_{i=1}^{n} \tilde{\eta}_i = ||u||^2 \Big(u^h \frac{A-A^h}{2i} u\Big) - Re\Big(u^h \frac{A-A^h}{2i} u\Big) = 0.$$

Hence it follows that all the equations become easier if $A_{ii} = A_{jj}\ \forall i \neq j$ [4], indeed in this case :

C_2 is zero, $\qquad C_4 = \sum_{i=1}^{n} 2\big(\xi_i^2 + \eta_i^2\big) \geq 0\,,$

$2C_3 + C_4 = \sum_{i=1}^{n} 2\big(\tilde{\xi}_i^2 + \tilde{\eta}_i^2\big) \geq 0\,, \qquad || \cdot ||^2 \,|_{x=0} \leq || \cdot ||^2 \,|_{x=2},$

and (3.28) can be reduced to a third degree algebraic equation.

Remark: If $C_4 = 0$, then $\xi_i = \eta_i = 0\ \forall i$, $C_1 = 0$, $C_3 = \sum_{i=1}^{n}(\tilde{\xi}_i^2 + \tilde{\eta}_i^2) \geq 0\,,\ || \cdot ||^2 = 4x^2C_3 + C_5$ and thus $\hat{\alpha} = 2$. Otherwise, if $C_4 > 0$, the value $\alpha = 2$ can be optimal only if $C_1 = 0$

[4] It should be in general possible to equal all the diagonal entries by applying a finite number of Givens similarity transforms to A (to be verified).

and $4C_3 + C_4 \geq 0$ (this follows from a simple study of the behaviour of $||\cdot||^2$ in a neighborhood of $\alpha = 2$); in all other cases the optimal $\hat{\alpha}$ is s.t. $0 < |\hat{\alpha}| < 2$, i.e. $I - \hat{\alpha}uu^h$ is not Householder.

- Writing $\alpha = 1+e^{i\theta}$, we have that α is optimal if $e^{i\theta}$ satisfies the following equation:

$$Im\Big(e^{2i\theta}(C) + e^{i\theta}(D+B+C)\Big) = 0 \tag{3.29}$$

where D, B and C are defined as follows

$$D = \sum_{i=1}^{n} \overline{u}_i\big(k_iA_{ii} + h_i\overline{A}_{ii}\big)\,,$$
$$B = \sum_{i=1}^{n} |u_i|^2\Big(|k_i|^2 + |h_i|^2\Big)\,, \qquad C = 2\sum_{i=1}^{n} \overline{u}_i^2\Big(k_ih_i\Big)$$

and

$$k_i = (u^hA^hu)u_i - \sum_{j=1}^{n} u_j\overline{A}_{ji}\,, \qquad h_i = (u^hAu)u_i - \sum_{j=1}^{n} u_jA_{ij}.$$

Also in this case the $\{k_i\}$ and $\{h_i\}$ satisfy some good property:

$$\sum_{i=1}^{n} \overline{u}_ik_i = ||u||^2(u^hA^hu) - \sum_{i,j=1}^{n} \overline{u}_i(A^h)_{ij}u_j = 0\,,$$
$$\sum_{i=1}^{n} \overline{u}_ih_i = ||u||^2(u^hAu) - \sum_{i,j=1}^{n} \overline{u}_iA_{ij}u_j = 0\,.$$

So if we equalize the diagonal elements of A, the coefficient D becomes equal to zero.

Regarding the minimization (3.27), we note here only that it is equivalent to a minimization problem in $\mathbb{R}^{2n-1}$, just set $u = D(e^{i\theta})z$, $\theta \in \mathbb{R}^n$, $z \geq 0$, $z^tz = 1$. An open question is to understand if the solution $(U_\alpha)_k$ of (3.25) generates the best possible approximation of type product of k Householder-type of the U_{opt} in (3.24) (see Theorem 3.3.9). If this is true, the iterative algorithm, generating $(U_\alpha)_1(U_\alpha)_2\cdots(U_\alpha)_k$ from $(U_\alpha)_1(U_\alpha)_2\cdots(U_\alpha)_{k-1}$, should converge to U_{opt} in no more than n-steps. Otherwise, an infinite number of steps, could be performed improving more and more the normal approximation to A , but it's not sure that the algorithm converges to U_{opt}.

4. Appendix

Theorem 4.0.1. *Given two vectors $w, v \in \mathbb{C}^n$, there exists an hyperplane τ of dimension n-1 s.t. w and v have the same projection on τ.*

Proof. Observe that the thesis is equivalent to say that exists an orthonormal basis $\{u_i\}_{i=1}^n$ s.t. $\alpha_i =< v, u_i >=< w, u_i >$,$\forall i = 1, \dots, n-1$, and $w = \sum_{i=1}^{n-1} \alpha_i u_i + \gamma u_n, \quad v = \sum_{i=1}^{n-1} \alpha_i u_i + \beta u_n$. But the latter assertion is trivial because it is enough to find a set of $n-1$ orthogonal vectors $\{u_i\}_{i=1}^{n-1}$ s.t. $u_i \perp (v - w)\ \forall i$.

□

Theorem 4.0.2. *(see [18], [20]) Let $A \in \mathbb{C}^{n \times n}$ and $A = U\Sigma V^h = \sum_{i=1}^{r} \sigma_i u_i v_i^h$, $\sigma_i > 0$, $r \leq n$, it's SVD decomposition. The best rank-k approximation of A in the 2-norm is $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^h$, (as usual we are assuming $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$).*

Proof. Observe that $U^h(A - A_k)V = diag(0, \dots, 0, \sigma_{k+1}, \dots, \sigma_r)$, so:

$$||A - A_k||_2 = \sigma_{k+1}.$$

If $B \in \mathbb{C}^{n \times n}$ is a rank-k matrix, let's consider $z \in ker(B) \cap span\{v_1, \dots, v_{k+1}\}$ s.t. $||z||_2 = 1$. Then:

$$||A - B||^2 \geq ||(A - B)z||^2 = ||Az||^2 = \sum_{i=1}^{k+1} \sigma_i^2 |(v_i^h z)|^2 \geq \sigma_{k+1}^2$$

and this completes the proof.

□

Theorem 4.0.3. *Let $A \in \mathbb{C}^{n\times n}$ and $A = U\Sigma V^h = \sum_{i=1}^{r} \sigma_i u_i v_i^h$, $\sigma_i > 0$, $r \leq n$, it's SVD decomposition. The best rank-k approximation of A in the Frobenius-norm is $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^h$, (as usual we are assuming $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$).*

Proof. Observe that $U^h(A - A_k)V = diag(0, \dots, 0, \sigma_{k+1}, \dots, \sigma_r)$, so:

$$||A - A_k||_F^2 = \sum_{i=k+1}^{r} \sigma_i^2 \,.$$

Recall the Weyl's inequality for singular values (see [19]) which says that

$$\sigma_{i+j-1}(A + B) \leq \sigma_i(A) + \sigma_j(B).$$

Then for any rank-k matrix B, since $\sigma_{k+1}(B) = 0$, taking $j = k+1$, we have that:

$$\sigma_{i+k}(A) \leq \sigma_i(A - B) \quad \forall i = 1, \dots, n-k \,.$$

So:

$$||A - B||_F^2 = \sum_{i=1}^{n} \sigma_i^2(A - B) \geq \sum_{i=1}^{n-k} \sigma_i^2(A - B) \geq \sum_{i=k+1}^{n} \sigma_i^2(A) = \sum_{i=k+1}^{r} \sigma_i^2(A) \tag{4.1}$$

□

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