Università degli studi di Roma "Tor Vergata"

Tesi di Laurea Magistrale in
Matematica Pura ed Applicata

Householder-Type Matrix Algebras in Displacement Decompositions

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Introduction

Displacement decompositions are usefull tools that allow to decompose classes of matrices in terms of structured, low-complexity matrices. As a conseguence of this, for example Toeplitz, Hankel or Toeplitz plus Hankel linear systems can be efficiently solved.

Let's see an example; for any $|\varepsilon| = 1$ define

$$P_\varepsilon = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ \varepsilon & 0 & \dots & \dots & 0 \end{pmatrix};$$

this is a matrix that generates the algebra C_ε of ε-circulant matrices, matrices diagonalized by a unitary matrix that is the product of a diagonal matrix by the Fourier matrix.

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this is a matrix that generates the algebra C_ε of ε-circulant matrices, matrices diagonalized by a unitary matrix that is the product of a diagonal matrix by the Fourier matrix.

The Gohberg-Olshevksy displacement formula

The Gohberg-Olshevksy displacement formula says that if

$$AP_\varepsilon - P_\varepsilon A = \sum_{m=1}^{r} x_m y_m^t$$

then A can be decomposed as follows:

$$(\varepsilon - \beta)A = \sum_{m=1}^{r} \Big(C_\beta(\hat{x}_m) C_\varepsilon(y_m) \Big) + (\varepsilon - \beta) C_\varepsilon(A^t e_1)\,,$$

where:

$C_\beta(\hat{x_m})$ is the only β-circulant matrix with first row equal to $\hat{x}_m^t$ with $(\hat{x})_i := x_{n+1-i}$,

$C_\varepsilon(y_m)$ is the only ε-circulant matrix with first row equal to y_m^t .

The Gohberg-Olshevksy displacement formula

The above formula can be used for example to solve Toeplitz linear systems.
Indeed for any Toeplitz matrix T

$$T = \begin{pmatrix} t_0 & t_{-1} & \dots & t_{-n} \\ t_1 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \\ t_n & . & \dots & t_0 \end{pmatrix} = (t_{i-j})_{i,j=1}^{n}$$

it is easy to show that

$$rk(T^{-1}P_\varepsilon - P_\varepsilon T^{-1}) = 2\,.$$

So, thanks to the Gohberg-Olshevksy displacement formula, we can express T^{-1} in terms of 2 β-circulant matrices and 3 ε-circulant matrices.
($O(nlogn)$ arithmetic operations to compute $T^{-1}b$)

Aim of the thesis

In [1][1] the authors prove some displacement theorems for generic Hessenberg algebras and observe that these theorems imply both well known and new displacement formulas.

In all of these theorems and formulas the structure of the algebra is very important, indeed in the hypotheses of every theorem there is the symmetry or persymmetry of the algebra.

The algebras involved in some well known displacement formulas are not only Hessenberg algebras, but also *SDU algebras*.

Our aim has been to look for new displacement formulas involving new SDU low-complexity algebras as the *Householder* ones.

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SDU algebras

Given a unitary matrix U we can define the SDU algebra $\mathcal{U}$ as the set of all the matrices simultaneously diagonalized by U.

$$\mathcal{U} := \{UD(z)U^h \mid z \in \mathbb{C}^n\}, \qquad \mathcal{U}[z] := UD(z)U^h$$

$$D(z) = \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix}.$$

A SDU algebra can be uniquely characterized, by columns or by rows, via multiplication on the right or on the left by a vector. For any v s.t. $(U^h v)_i \neq 0 \quad \forall i = 1, \dots, n$, we can define $\mathcal{U}^{(v)}(z)$ as the only matrix of $\mathcal{U}$ such that

$$\mathcal{U}^{(v)}(z)v = z.$$

For any w s.t. $(U^t w)_i \neq 0 \quad \forall i = 1, \dots, n$, we can define $\mathcal{U}_{(w)}(z)$ as the only matrix of $\mathcal{U}$ such that

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Some results about Householder SDU Algebras

Given a vector $u \in \mathbb{C}^n$, $||u|| = 1$, define the Householder matrix U:

$$U = I - 2uu^h \,.$$

U is an Hermitian unitary matrix.
Consider the associated Householder SDU algebra $\mathcal{U}$.
Note that if $\forall i \;\; u_i \neq 0 \;\; \Rightarrow \;\; \mathcal{U}^{(u)}(z)u = z$ and $u^h \mathcal{U}_{(\overline{u})}(z) = z$

Proposition

Suppose $u_i \neq 0 \;\; \forall i$ and $n > 4$; then
$\mathcal{U}$ is closed under conjugation if and only if U is real.

Proposition

The Householder algebra $\mathcal{U}$ is symmetric if and only if U is real.

Proposition

Suppose $u_i \neq 0 \;\; \forall i$.
The Householder algebra $\mathcal{U}$ is never persymmetric if $n > 4$.

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Some results about Householder Algebras

Theorem

Let's be given two distinct Householder algebras $\mathcal{U}, \mathcal{V}$ *s.t.* $u_i, v_i \neq 0 \;\; \forall i$, $(n > 4)$. *Then*

$$dim(\mathcal{U} \cap \mathcal{V}) \leq 2 \, .$$

Lemma

Let be given $u \in \mathbb{C}^n$, $||u|| = 1$ *and* $z \in \mathbb{C}^n$; *then* $\forall \; u' \in \mathbb{C}^n$ *s.t.*

- $||u'|| = 1 \, ,$
- $u' D(z) u'^h = u D(z) u^h \, ,$
- $D(z)(u' - u) = \lambda (u' - u) \, ,$

we have that $\mathcal{U}[z] - \mathcal{U}'[z]$ *is a matrix of rank 2.*

Corollary

If $u' - u = k e_i$, *where* $k = u_i(e^{i\theta} - 1)$, *then* $\forall z \in \mathbb{C}^n$ $\mathcal{U}[z] - \mathcal{U}'[z]$ *has rank 2.*

I displacement theorem for generic SDU algebras

Theorem

Let $U, V \in \mathbb{R}^{n\times n}$ *be two unitary real matrices and* $w \in \mathbb{R}^n$ *a vector s.t. characterizes* $\mathcal{U}$ *by colums and* $\mathcal{V}$ *by rows.*
Assume that there exists $z \in \mathbb{R}^n$ *s.t.* $\mathcal{U}[z] + ww^t = \mathcal{V}[z'] \in \mathcal{V}$ *and* $\mathcal{U}[z]$ *is non derogatory.*
Then $\forall\, A \in \mathbb{R}^{n\times n}$, *if* $A\mathcal{U}[z] - \mathcal{U}[z]A = \sum_{i=1}^{k} x_i y_i^t$, *we can say that*

$$A = \sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i) + C\,,$$

where $C = \mathcal{U}^{(w)}\left(Aw - \sum_{i=1}^{k} \left(\mathcal{U}^{(w)}(x_i)y_i\right)\right)$.

Proof: The commutator of $A - \sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(w)}(y_i)$ with respect to $\mathcal{U}[z]$ is zero.

II displacement theorem for SDU algebras

Theorem

Let $U, V \in \mathbb{C}^{n\times n}$ *be two unitary matrices,* $w \in \mathbb{C}^n$ *a vector s.t. characterizes* $\mathcal{U}$ *by colums and* $\overline{w}$ *characterizes* $\mathcal{V}$ *by rows.*
Assume that $\exists\, z \in \mathbb{R}^n$ *s.t.* $\mathcal{U}[z] + ww^h = \mathcal{V}[z'] \in \mathcal{V}$ *and* $\mathcal{U}[z]$ *is a non derogatory matrix.*
Then $\forall\, A \in \mathbb{C}^{n\times n}$ *s.t.* $D(U^h w)^{-1} U^h A V D(V^t \overline{w})^{-1} \in \mathbb{R}^{n\times n}$,

$$\text{if } A\mathcal{U}[z] - \mathcal{U}[z]A = \sum_{i=1}^{k} x_i y_i^t \,,$$

(where $\left(D(U^h w)^{-1} U^h x_i\right), \left(D(V^t \overline{w})^{-1} V^t y_i\right)$ *are real vectors), we can say that:*

$$A = \sum_{i=1}^{k} \mathcal{U}^{(w)}(x_i)\mathcal{V}_{(\overline{w})}(y_i) + C \,,$$

where $C = \mathcal{U}^{(w)}\left(Aw - \sum_{i=1}^{k} \left(\mathcal{U}^{(w)}(x_i)\overline{y_i}\right)\right)$.

Remark

With this theorem we have removed the hypotheses on the structure (symmetry/persymmetry) of the algebras but we have an hypothesis on the matrices on which we can apply the theorem.

However we have shown that any matrix A can be decomposed as

$$A = A_1 + i\, A_2\,,$$

where both A_1 and A_2 satisfy the hypothesis of the theorem.

As a conseguence of this, every matrix A, can be decomposed as

$$A = \sum_{i=1}^{k} \mathcal{U}^{(w)}(\xi_i)\mathcal{V}_{(\overline{w})}(\eta_i) + i \sum_{i=1}^{k} \mathcal{U}^{(w)}(\varphi_i)\mathcal{V}_{(\overline{w})}(\psi_i) + C_1 + iC_2$$

where,

$$k \leq 2\; rk(A\mathcal{U}[z] - \mathcal{U}[z]A)\,.$$

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Towards Householder-type matrices

Look for a generalization of Householder matrices.

Good properties we want to keep

- Householder matrices $U = I - 2uu^h$ are a 1-rank variation of the identity. This implies that they are cheap matrices;
- $Uu = -u, \quad Uv = v \;\; \forall v \perp u$;
- Given $v, w \in \mathbb{R}^n$ s.t. $||v|| = ||w|| \; \exists \; U$ real Householder s.t. $Uv = w$;
- If $Q \in \mathbb{R}^{n \times n}$ is a unitary matrix $\exists U_1, \ldots, U_n$ real Householder matrices s.t. $Q = U_1 \cdots U_n$.

Properties we would like to improve

- Given $v, w \in \mathbb{C}^n$ s.t. $||v|| = ||w||$ $\exists \; U$ Householder s.t. $Uv = w$ if and only if $< v, w >$ is real;
- If $Q \in \mathbb{C}^{n \times n}$ is a unitary matrix, then $\exists \; U_1, \ldots, U_{n-1}$ Householder matrices and a unitary diagonal matrix D s.t. $Q = U_1 \cdots U_{n-1} D$.

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We want some matrices that, in the complex case, behave as good as the Householder matrices do in the real case.

To do this, it is usefull the following Lemma:

Lemma

Given $v, w \in \mathbb{C}^n$ s.t. $||v|| = ||w||$, then $\exists \{u_i\}_{i=1}^n$ an othonormal basis and $\alpha_1, \dots, \alpha_{n-1}, \beta, \gamma \in \mathbb{C}$ s.t.

$$v = \sum_{i=1}^{n-1} \alpha_i u_i + \beta u_n \,, \qquad w = \sum_{i=1}^{n-1} \alpha_i u_i + \gamma u_n$$

and $|\beta| = |\gamma|$.

Hence, to map v in w, we need a matrix $\tilde{U}$ s.t.

$$\tilde{U} u_n = \frac{\gamma}{\beta} u_n = e^{i\theta} u_n \quad \text{and} \quad \tilde{U} u_i = u_i \, \forall i = 1, ..., n-1 \,.$$

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Householder-type

Definition

Given $u \in \mathbb{C}^n$, $||u|| = 1$ and $\theta \in \mathbb{R}$ define the **Householder-type matrix**

$$U_{\alpha(\theta)} := I - (1 - e^{i\theta})uu^h.$$

$U_{\alpha(\theta)}$ is a unitary matrix and $U^h_{\alpha(\theta)} = U_{\alpha(-\theta)} = U_{\overline{\alpha(\theta)}}$.

Note:

$$U_{\alpha(\theta)}u = e^{i\theta}u \quad \textit{and} \quad U_{\alpha(\theta)}v = v \ \ \forall v \perp u \ \ \Rightarrow$$

Lemma

$\forall v, w \in \mathbb{C}^n$, $||v|| = ||w||$, there exists an Householder-type matrix $U_{\alpha(\theta)}$, such that:

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Householder-type decomposition of unitary matrices I

If Q is a unitary matrix then we can consider its spectral decomposition: there exists $V = (v_1, \dots, v_n)$ unitary s.t.

$$
\begin{aligned}
Q = VD(e^{i\theta_j})V^h &= I - V\Big(I - D(e^{i\theta_j})\Big)V^h \\
&= I - \sum_{j=1}^{n} \Big((1 - e^{i\theta_j})v_j v_j^h\Big) \\
&= \prod_{j=1}^{n} \Big(I - (1 - e^{i\theta_j})v_j v_j^h\Big),
\end{aligned}
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where the number of trivial Householder-type matrices is equal to the multiplicity of the eigenvalue "1" .
Note: the number of classic Householder matrices involved in the decomposition is equal to the multiplicity of the eigenvalue "-1".

It is easy to prove that this is an optimal decomposition.

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Householder-type decomposition of unitary matrices II

An algorithm to find a decomposition of Q in terms of Householder-type matrices is the following:
(1-st step): let be U_{α_1} the Householder-type matrix such that

$$U_{\alpha_1} Q = \begin{pmatrix} 1 & 0 \\ 0 & Q_2^1 \end{pmatrix} ;$$

(k-th step): let be U_{α_k} the Householder-type matrix such that

$$U_{\alpha_k} \begin{pmatrix} 1 & 0 & \dots \\ 0 & \ddots & 0 \\ \vdots & 0 & Q_k^{k-1} \end{pmatrix} = U_{\alpha_k} \left(U_{\alpha_{k-1}} \cdots U_{\alpha_1} \right) Q = \begin{pmatrix} 1 & 0 & \dots & . \\ 0 & \ddots & . & . \\ \vdots & . & 1 & \underline{0}^t \\ . & . & \underline{0} & Q_{k+1}^k \end{pmatrix}$$

after **n-steps** we have

$$U_{\alpha_n} U_{\alpha_{n-1}} \dots U_{\alpha_1} Q = I \, .$$

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Some properties of the Householder-type matrices

Proposition

- *The Householder type matrices are the only 1-rank variation of the identity matrix that are unitary matrices.*
- *If U is a unitary matrix that is a 1-rank variation of a diagonal matrix, then there exist a diagonal unitary matrix D and an Householder-type matrix U_α such that*

$$U = U_\alpha D\,.$$

Two SDU algebras $\mathcal{U}$ and $\mathcal{V}$ are equal if and only if

$$U = VPD\,,$$

for some permutation matrix P and diagonal unitary matrix D.

So, the study of the SDU algebras diagonalized by a 1-rank variation of a diagonal matrix is reduced to the study of the algebras diagonalized by an Householder-type matrix.

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Given $u \in \mathbb{C}^n$ s.t. $||u|| = 1$, the set of the Householder-type matrices defined by u

$$\{U_{\alpha(\theta)} = I - (1 - e^{i\theta})uu^h\}$$

is a commutative subgroup of the group of unitary matrices, and

$$U_{\alpha(\theta)} \cdot U_{\alpha(\varphi)} = U_{\alpha(\theta+\varphi)} \, .$$

This Theorem gives us the possibility to dip each Householder matrix in a non trivial commutative group, where the Householder matrix generates the only subgroup of order 2.

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Best approximation of a Unitary matrix

Theorem

Let U be a unitary matrix and $U = VD(e^{i\theta_j})V^h$ its spectral decomposition. Then the best approximation of U (in Frobenius and 2-norm) with k Householder-type matrices is

$$\tilde{U} = \prod_{j=1}^{k} \left(I - (1 - e^{i\theta_j}) v_j v_j^h \right),$$

where the $\{e^{i\theta_j}\}_{j=1}^{k}$ are the k eigenvalues of U farthest from "1". Moreover the following equalities are true:

$$||U - \tilde{U}||_f^2 = \sum_{j=k+1}^{n} |1 - e^{i\theta_j}|^2, \quad ||U - \tilde{U}||_2^2 = |1 - e^{i\theta_{k+1}}|^2.$$

proof: it is enough to show that to approximate U with a product of k Householder-type matrices it is the same thing that to approximate $V\left(I - D(e^{i\theta_j})\right)V^h$ with a rank k matrix; then we can conclude thanks to the Sing.Val.Dec. theory.

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A possible way to improve the stability of QR algorithm

If A is a matrix, the QR algorithm is a n-step procedure, exploiting Householder (or Givens) transforms, to compute a factorization of the matrix A as product of a unitary matrix Q and an upper triangular matrix R.

We propose a modification that at each step uses an Householder-type tansform (it is still to prove if it is better).

The algorithm works as follows:

(1-st step): let be $U_{1,\alpha(\theta_1)}$ the Householder-type matrix such that

$$U_{1,\alpha(\theta_1)}A = \begin{pmatrix} ||a_1||e^{i\theta_1} & * \\ 0 & A^{(2)} \end{pmatrix}, \quad a_1 = a_1^1 = Ae_1\,;$$

(k-th step): let be $U_{k,\alpha(\theta_k)}$ the Householder-type matrix such that

$$U_{k,\alpha(\theta_k)} \begin{pmatrix} ||a_1||e^{i\theta_1} & * & \cdots \\ 0 & \ddots & * \\ \vdots & 0 & A^{(k)} \end{pmatrix} = \begin{pmatrix} ||a_1||e^{i\theta_1} & * & \cdots & . \\ 0 & \ddots & . & . \\ \vdots & . & ||a_1^k||e^{i\theta_k} & * \\ . & . & \underline{0} & A^{(k+1)} \end{pmatrix};$$

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$$U_{n,\alpha(\theta_n)} U_{n-1,\alpha(\theta_{n-1})} \dots U_{1,\alpha(\theta_1)} A = R \text{ (with } R \text{ upper triangular)}\,.$$

Using the Householder matrices we cannot choose θ_j ; instead, using the Householder-type matrices we can choose each θ_j as we prefer.

Doing the product of $U = I - \alpha u u^h$ by a vector z and considering only the error on the computation of $u^h z$, that is

$$(I - \alpha u u^h) z \approx z - \alpha (\widetilde{u^h z}) u = z - \alpha (u^h z + \varepsilon) u = (I - \alpha u u^h) z - \varepsilon \alpha u\,,$$

we can observe that, smaller is $|\alpha|$, smaller is the perturbation of $(I - \alpha u u^h) z$ caused by ε.

$\Rightarrow$ Choose at each step $e^{i\theta_j}$ in order to have $\alpha(\theta_j)$ as small as possible.

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Thus we have proved the following Theorem:

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Given a unitary vector v, the Householder-type matrix $U_{\alpha^*} = I - \alpha^* uu^h$ such that

$$U_{\alpha^*} v \in span\{e_1\} \quad \& \quad ||U_{\alpha^*} - I|| \text{ minimum}$$

is s.t. :

$$|\alpha^*|^2 = ||U_{\alpha^*} - I||^2 = 4\big(1 - |v_1|^2\big).$$

(There exists also an explicit formula for U_{α^*})

IMPORTANT NOTE: it can be convenient to introduce, at each step k, a partial or total pivoting on the submatrix A_k, in order to have $(A_k)_{11}$ as big as possible.
Using a partial pivoting we would still obtain a QR factorization; instead, using a total pivoting we would obtain a QRP factorization.

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Fine

GRAZIE PER L'ATTENZIONE!