Operator theory and its applications

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Chapter 1

Introduction

In this section we present some preliminary results and definitions about matrix theory and functional analysis. Some of them are left without the proof and have the scope to introduce the reader to the notation and to recall him some useful and maybe known properties.

Definition 1.1 *Let $\mathcal{K}$ be a field, we shall denote with $\mathbf{M}_n(\mathcal{K})$ the vectorial space of matrix defined over $\mathcal{K}$. Some important subspaces of $\mathbf{M}_n(\mathcal{K})$ are $\mathbf{U}_n(\mathcal{K})$, $\mathbf{PD}_n(\mathcal{K})$, $\mathbf{H}_n(\mathcal{K})$ with whom we shall indicate respectively the sets of unitary, positive defined and hermitian matrix.*

Definition 1.2 *A normed space $\mathcal{X}$ is said to be a Banach space if it is complete with respect the norm. A Banach space $\mathcal{H}$ is said to be a Hilbert space if on $\mathcal{H}$ is defined a scalar product that induce the norm.*

It is known that any Hilbert space $\mathcal{H}$ always has an orthonormal base $(e_i)_{i\in\mathcal{I}}$. If the set $\mathcal{I}$ is numerable then $\mathcal{H}$ is a separable Hilbert space. An important notion is the so called *weak topology.*

Definition 1.3 *Let $\mathcal{X}$ be a Banach space. The space of all the linear bounded functionals over $\mathcal{X}$ is called the* dual *of $\mathcal{X}$ and denoted with $\mathcal{X}^*$. The space $\mathcal{X}^{**} = (\mathcal{X}^*)^*$ is called* bidual *of $\mathcal{X}$ and $\mathcal{X}$ is said to be* reflexive *if $\mathcal{X}^{**} = \mathcal{X}$.*

Definition 1.4 *A sequence $(x_n)_n \subset \mathcal{X}$ is said to be* weakly convergent *to x if $f(x_n) \to f(x)$, $\forall f \in \mathcal{X}^*$. A sequence $(f_n)_n \subset \mathcal{X}^*$ is said to be* $*$-weakly convergent *if $f_n(x) \to f(x)$, $\forall x \in \mathcal{X}$.*

Let us introduce the space $\mathcal{B}(\mathcal{X},\mathcal{Y})$ of all the linear continuous maps (or operator) from $\mathcal{X}$ to $\mathcal{Y}$. It is known that if $\mathcal{Y}$ is complete then $\mathcal{B}(\mathcal{X},\mathcal{Y})$ is complete as well. Therefore for any normed space $\mathcal{X}$ the dual space $\mathcal{X}^*$ is a Banach space (since $\mathcal{B}(\mathcal{X},\mathbb{R}) = \mathcal{X}^*$). Moreover a reflexive space is necessarily a Banach space.

Let $A \in \mathcal{B}(\mathcal{X},\mathcal{Y})$, set

$$\|A\| = \sup_{\|x\|\leqslant 1} \|Ax\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}$$

it is not difficult to prove that the equalities above are true and that $\|A\|$ actually define a norm on $\mathcal{B}(\mathcal{X},\mathcal{Y})$. Moreover for any $A \in \mathcal{B}(\mathcal{X},\mathcal{Y})$, being A continuous, there exists $k_A \in \mathbb{R}$ such that $\|Ax\| \leqslant k_A\|x\|$ i.e. A is bounded. Precisely the two properties are equivalent, in fact it can be shown that a linear map $A : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if it is bounded. Therefore another definition for $\|A\|$ is $\|A\| = \inf\{k : \|Ax\| \leqslant k\|x\|, \forall x \in \mathcal{X}\}$.

Let $\mathcal{V} \subset \mathcal{H}$ be a closed subset of an Hilbert space. It is well known that $\forall x \in \mathcal{H}, \exists! x_1 \in \mathcal{V}, x_2 \in \mathcal{V}^\perp$ such that $x = x_1 + x_2$ and $\|x - x_1\| = \inf_{y \in \mathcal{V}} \|x - y\|$. It follows

Theorem 1.5 *Let $\mathcal{H}$ be a Hilbert space, then $\mathcal{H}^*$ is isometrically isomorphic to $\mathcal{H}$. In other words for any $f \in \mathcal{H}^*$ exists $x_f \in \mathcal{H}^*$ such that $f(y) = (x_f, y)$ and $\|f\| = \|x_f\|$.*

Proof. If such x_f exists then clearly $\|f\| = \|x_f\|$. In fact $|f(y)| \leqslant \|x_f\|\|y\|$ by Cauchy-Swartz and $|f(x_f)| = \|x_f\|^2$. For the existence just observe that $\ker(f) \subset \mathcal{H}$ is closed. Then fix $u \in \ker(f)^\perp$. For all $x \in \mathcal{H}$ set $v = f(x)u - f(u)x$. Clearly $v \in \ker(f)$ then $0 = (u, v) = f(x)\|u\|^2 - f(u)(u, x)$. Now by setting $x_f = \frac{f(u)u}{\|u\|^2}$ we have the thesis. ■

1.1 Banach Algebras

Definition 1.6 *A* Banach algebra *$\mathscr{A}$ is a Banach space such that $\|AB\| \leqslant \|A\|\|B\|$ for all the pair of elements $A, B \in \mathscr{A}$. A Banach algebra is said to be a* $*$-algebra *if it can be defined on $\mathscr{A}$ a map $* : \mathscr{A} \to \mathscr{A}$ such that, if $A \in \mathscr{A}$ then*

$$(A^*)^* = A, \quad (\lambda A)^* = \overline{\lambda} A^*, \quad A^{**} = A, \quad (AB)^* = B^* A^*,$$
$$(A + B)^* = A^* + B^*, \quad \|A\| = \|A^*\|$$

Lastly a $*$-algebra is said to be a C^*-algebra *if $\|A^*A\| = \|A\|^2$.*

Let $\mathcal{H}$ be an Hilbert space, then $\mathcal{B}(\mathcal{H})$ is a $*$-algebra. In particular if $\mathcal{H}$ is finite and $\dim(\mathcal{H}) = n$ then $\mathcal{H}$ is isomorphic to $\mathbb{C}^n$ and $\mathcal{B}(\mathcal{H}) = \mathbf{M}_n(\mathbb{C})$. Moreover, if $\|\cdot\|_2$ is the Spectral norm defined over $\mathbf{M}_n(\mathbb{C})$ then $\|A\|_2^2 = \|A^*A\|_2$, that is $\mathbf{M}_n(\mathbb{C})$ is a C^*-algebra. Therefore, from now on, if not explicitly specified, we shall denote the matrix norm $\|\cdot\|_2$ just with $\|\cdot\|$.

A very important set of functionals over a Banach algebra $\mathscr{A}$ is the so called *set of characters of $\mathscr{A}$*

Definition 1.7 *Let $\mathscr{A}$ be a Banach algebra. We denote with $\Omega(\mathscr{A})$ the set of all the linear (both for sum and product) non-zero functionals of $\mathscr{A}$. Precisely if $\phi \in \Omega(\mathscr{A})$ then*

$$\phi : \mathscr{A} \to \mathbb{C}, \quad \phi(A + B) = \phi(A) + \phi(B), \quad \phi(AB) = \phi(A)\phi(B)$$

$\Omega(\mathscr{A})$ is called the set of characters of the algebra $\mathscr{A}$.

It can be proved that the set $\Omega(\mathscr{A})$ is compact (i.e. any sequence in $\Omega(\mathscr{A})$ has a convergent subsequence), and Hausdorff (the limits are unique) under weak topology (recall that $\Omega(\mathscr{A})$ is a set of functionals).

A huge quantity of theorems is known for commutative C^* algebras. One of the most important is the following

Theorem 1.8 *Let $\mathscr{A}$ be a commutative C^* algebra . Then $\mathscr{A}$ is isomorphic with $\mathscr{C}(\Omega(\mathscr{A}))$, the space of the continuous functions from $\Omega(\mathscr{A})$ to $\mathbb{C}$. In other words any commutative C^* algebra is nothing else that the space of continuous functions over a compact Hausdorff set.*

As a consequence if $\mathbf{\Gamma}_n(\mathbb{C}) \subset \mathbf{M}_n(\mathbb{C})$ is a commutative subset of $\mathbf{M}_n$ (i.e. $AB = BA$, $\forall A, B \in \mathbf{\Gamma}_n$) then $\mathcal{L}(AB) = \mathcal{L}(A)\mathcal{L}(B)$, moreover there exists $\mathcal{X}$ compact and Hausdorff, $f : \mathcal{X} \to \mathbb{C}$ continuous and Ψ isomorphism such that $\Psi \circ A = f$, $\Psi^{-1} \circ f = A$.

Question 2. *Does we know any commutative class of matrices??*

Question 3. *Does we know any class of matrices that is commutative and C^*?*

An important class of operator in $\mathcal{B}(\mathcal{H})$, being $\mathcal{H}$ a Hilbert space, is the class of projection operators.

Definition 1.9 *A $\mathsf{P} \in \mathcal{B}(\mathcal{H})$ is said to be a* projector *if $\mathsf{P}^2 = \mathsf{P}$ and $\mathsf{P}^* = \mathsf{P}$.*

With the following we characterize such very special class of operators

Theorem 1.10 *Let $\mathsf{P} \in \mathcal{B}(\mathcal{H})$ be such that $\mathsf{P}^2 = \mathsf{P}$ and $\mathsf{P}^* = \mathsf{P}$. Then $\mathcal{H} = \text{range}(\mathsf{P}) + \text{range}(\mathsf{P})^\perp$. Viceversa, let $\mathcal{V}$ be a closed subset of $\mathcal{H}$. There exists $(v_i)_{i\in I}$ an orthonormal base for $\mathcal{V}$, and a map $\mathsf{P}_\mathcal{V} : \mathcal{H} \to \mathcal{V}$ defined by $x \mapsto \sum_{i\in I}(x, e_i)e_i$ such that $\text{range}(\mathsf{P}_\mathcal{V}) = \mathcal{V}$ and $\mathsf{P}_\mathcal{V}^2 = \mathsf{P}_\mathcal{V}$, $\mathsf{P}_\mathcal{V}^* = \mathsf{P}_\mathcal{V}$.*

Proof. First observe that

$$\ker(\mathsf{P}) = \text{range}(I - \mathsf{P})$$

In fact if $x \in \ker(\mathsf{P})$ then $(I - \mathsf{P})x = x$ i.e. $x \in \text{range}(\mathsf{P})$, viceversa if $x \in \text{range}(I - \mathsf{P})$ then $\exists y \in \mathcal{H}$ st $x = (I - \mathsf{P})y \implies \mathsf{P}x = \mathsf{P}(I - \mathsf{P})y = 0$, i.e. $x \in \ker(\mathsf{P})$. Secondly observe that

$$\ker(\mathsf{P}) = \text{range}(\mathsf{P})^\perp$$

In fact $x \in \ker(\mathsf{P}) \Rightarrow \forall y \in \text{range}(\mathsf{P})\ \exists z \in \mathcal{H}$ st $\mathsf{P}z = y$, but exists also $w \in \mathcal{H}$ st $(I - \mathsf{P})w = x$, therefore $(x, y) = ((I - \mathsf{P})w, \mathsf{P}z) = (\mathsf{P}^*(I - \mathsf{P})w, z) = (\mathsf{P}(I - \mathsf{P})w, z) = 0$ then $x \in \text{range}(\mathsf{P})^\perp$. Viceversa if $x \in \text{range}(\mathsf{P})^\perp$ then $(x, \mathsf{P}y) = 0$ $\forall y \in \mathcal{H} \Rightarrow 0 = (\mathsf{P}^*x, y) = (\mathsf{P}x, y) \Rightarrow \mathsf{P}x = 0$ i.e. $x \in \ker(\mathsf{P})$. Lastly since for all $x \in \mathcal{H}$ results $x = \mathsf{P}x + (I - \mathsf{P})x$ we have proved the first implication.

The proof of the inverse implication is analogous, but easier. In fact it is clear that $\mathsf{P}_\mathcal{V}$ maps x onto its unique projection in $\mathcal{V}$ such that $\|x - \mathsf{P}_V x\| = \inf_{y\in\mathcal{V}} \|x - y\|$. Therefore of course $\mathsf{P}_\mathcal{V} v = v$ if $v \in \mathcal{V}$ and thus $\mathsf{P}_\mathcal{V}^2 = \mathsf{P}_\mathcal{V}$. Moreover suppose $x \in \text{range}(\mathsf{P})^\perp$ then $0 = (x, \mathsf{P}y) = (\mathsf{P}^*x, y)$ i.e. $\mathsf{P}^* = \mathsf{P}$, while if $x \in \text{range}(\mathsf{P})$ ... ■

1.1.1 The spectrum

One of the key notions for a Banach algebra is the notion of spectrum. Given a Banach algebra $\mathscr{A}$ let us denote with $\mathscr{A}^{-1}$ the set of the invertible elements of

$\mathscr{A}$.

Definition 1.11 *Let $A \in \mathscr{A}$. The set $\rho(A) = \{z \in \mathbb{C} \mid (zI - A) \in \mathscr{A}^{-1}\}$ is called the* resolvent of A. *The set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called the* spectrum of A. *The set of all the eigenvalues of A is called the* punctual spectrum *of A and is denoted with $\mathcal{L}(A)$.*

It is important to observe that in general $\sigma(A) \supseteq \mathcal{L}(A)$. In fact $\lambda \in \mathcal{L}(A)$ if and only if $(\lambda I - A)$ is not injective (i.e. $\ker(\lambda I - A) \neq \{0\}$). While $\lambda \in \sigma(A)$ not only if $(\lambda I - A)$ is not injective but also if $(\lambda I - A)$ is bijective. Anyway it is known that in the case of matrices the two sets coincide.

Another important notion is the *spectral radius* of an $A \in \mathscr{A}$, defined by $\mathbf{r}(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$. It is a known, but not easy to prove, that $\mathbf{r}(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$.

Theorem 1.12 *Let $\mathscr{A}$ be a commutative Banach algebra (i.e. $AB = BA$, $\forall A, B \in \mathscr{A}$) then $\sigma(A) = \{\phi(A) \mid \phi \in \Omega(\mathscr{A})\}$.*

Observe that as a direct consequence we have that if two matrices commute then the eigenvalues of their product are given by the product of their eigenvalues, i.e. $\mathcal{L}(AB) = \mathcal{L}(A)\mathcal{L}(B)$. In fact if $\lambda \in \mathcal{L}(AB)$ then there exists $\phi \in \Omega(\mathbf{M}_n)$ such that $\lambda = \phi(AB) = \phi(A)\phi(B) = \nu\mu$ being $\nu \in \mathcal{L}(A)$, $\mu \in \mathcal{L}(B)$. *(Is it correct???)*

Example 1.13 Let $\mathscr{U} \subset \mathbf{M}_n(\mathbb{C})$ be an algebra of simultaneously diagonalizable matrices by unitary transform, i.e. fixed $U \in \mathbf{U}_n(\mathbb{C})$, $\mathscr{U} = \{U \operatorname{diag}(\lambda_1, \ldots, \lambda_n)U^* \mid \lambda_i \in \mathbb{C}\}$. Then $\mathscr{U}$ is a commutative sub-algebra of $\mathbf{M}_n(\mathbb{C})$ and hence $\mathcal{L}(AB) = \mathcal{L}(A)\mathcal{L}(B)$, $\forall\, A, B \in \mathscr{U}$.

The example above is not so restrictive, in fact

Proposition 1.14 *A sub-algebra $\mathscr{Y} \subset \mathbf{M}_n(\mathbb{C})$ is a commutative unital C^*-algebra of diagonalizable matrices if and only if there exists a non-singular $Y \in \mathbf{M}_n(\mathbb{C})$ such that $\mathscr{Y} = \{Y \operatorname{diag}(a)Y^{-1} \mid a \in \mathbb{C}^n\}$.*

Proof. One implication is trivial since two elements of $\mathscr{Y}$ obviously commute, are diagonalizable and invertible. Viceversa suppose that $A, B \in \mathbf{M}_n(\mathbb{C})$ commute and are diagonalizable. Let $\{\lambda_i\}_i, \{\mu_i\}_i$ be the eigenvalues of A and B, respectively, and $\{x_i\}_i, \{y_i\}_i$ their correspondent eigenvectors. Then $\forall x \in \mathbb{C}^n$ we have $Ax = \sum_{i=1}^n \lambda_i a_i x_i$ and $Bx = \sum_{i=1}^n \mu_i b_i y_i$. Therefore

$$BAx = \textstyle\sum_i \lambda_i a_i(x) B x_i = \sum_j \left[\sum_i \lambda_i a_i(x) b_j(x_i)\right] \mu_i y_j = \\ \textstyle\sum_j \left[\sum_i \mu_i b_i(x) a_j(y_i)\right] \lambda_j x_j = \sum_i \mu_i b_i(x) A y_i = ABx$$

or in other words there exists two diagonal matrices D_1, D_2 such that $AB = BA = X^{-1}D_1X = Y^{-1}D_2Y$. As a consequence both A and B are diagonalized by the same matrix. ∎

Definition 1.15 *Let $A \in \mathbf{M}_n(\mathbb{C})$, the set*

$$\mathcal{F}(A) = \left\{ \frac{(x, Ax)}{\|x\|^2} \mid x \in \mathbb{C}^n \right\}$$

is called the field of values *of A.*

It is known and not difficult to prove that $\mathcal{L}(A) \subset \mathcal{F}(A)$.

Proposition 1.16 *Let $A \in \mathbf{M}_n(\mathbb{C})$. Then $\mathcal{F}(A)$ is a compact subset of $\mathbb{C}$.*

Proof. Define $\varphi_A : \mathbb{C}^n \longrightarrow \mathbb{R}$ by $x \mapsto (x, Ax)$ and $\mathbb{T}^n = \{x \in \mathbb{C}^n \mid \|x\| = 1\}$. Clearly φ_A is a continuous function and $\mathcal{D}_1(0)$ a compact subset of $\mathbb{C}^n$. Moreover $\mathcal{F}(A) = \varphi_A(\mathcal{D}_1(0))$. It follows that $\mathcal{F}(A)$ is the image through a continuous function of a compact set, thus it is compact as well. ■

An analogous argument let us to prove that the set of non-singular matrices is an open set, thus the limit of a sequence of invertible matrices could be a singular matrix.

Proposition 1.17 *The set of invertible matrices is an open subset of $\mathbf{M}_n(\mathbb{C})$.*

Proof. It is known that $\det : \mathbf{M}_n(\mathbb{C}) \longrightarrow \mathbb{R}$ is continuous, then $\det^{-1}(0)$, the set of singular matrices, is a closed subset of $\mathbf{M}_n(\mathbb{C})$ and its complement is open. ■

It is known that the spectrum has particular characteristics for some special classes of matrices, but such spectral properties are preserved under a more general context

Theorem 1.18 *Let $\mathscr{A}$ be a C^*-algebra with unit and $A \in \mathscr{A}$. Then*

(i) $\sigma(A^*) = \overline{\sigma(A)}$

(ii) If A is normal (i.e. $AA^* = A^*A$) then $\mathbf{r}(A) = \|A\|$.

(iii) If P is a projector then $\sigma(\mathsf{P}) \subset \{0, 1\}$.

(iv) If U is unitary, (i.e. $UU^* = U^*U = I$) then $\sigma(U) \subset \mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

(v) If A is self-adjoint then $\sigma(A) \subset \mathbb{R}$.

Proof. ————- ■

Proposition 1.19 *Let $\mathcal{H}$ be an Hilbert space, $U \in \mathcal{B}(\mathcal{H})$ and $(e_i)_{i \in \mathcal{I}}$ an orthonormal basis for $\mathcal{H}$. The following conditions are equivalent*
(i) U è un isometria biettiva.
(ii) U è suriettivo e $(Ux, Uy) = (x, y)$ per ogni $x, y \in \mathcal{H}$.
(iii) $(Ue_i)_{i \in \mathcal{I}}$ is an orthonormal basis for $\mathcal{H}$.
(iv) $UU^* = U^*U = \mathbf{1}$.

Proof. *(ii)* $\Rightarrow$ *(iii)* Infatti $(Ue_i, Ue_j) = \delta_{ij}$[1], inoltre per l'ipotesi di suriettività la famiglia $(Ue_i)_{i \in \mathcal{I}}$ è una base. *(ii)* $\Rightarrow$ *(iv)* Osserviamo che

$$\|U^*Ux\|^2 = (Ux, UU^*Ux) = (x, U^*Ux) = (Ux, Ux) = \|x\|^2$$

ed analogamente $(x, U^*Ux) = \|x\|^2$ per ogni $x \in \mathcal{H}$. Dunque $\|(U^*U - \mathbf{1})x\|^2 = \|U^*Ux\|^2 - 2(x, U^*Ux) + \|x\|^2 = 0$, $\forall x \in \mathcal{H}$, ovvero $U^*U = \mathbf{1}$. Allo stesso modo, per la suriettività di U , $\forall x \in \mathcal{H}$, $\exists y \in \mathcal{H}$ tale che $x = Uy$, dunque $(x, UU^*x) = (Uy, UU^*x) = (y, U^*x) = (Uy, x) = \|x\|^2$. Pertanto gli stessi

[1] δ_{ij} indica la delta di Kronecker, che vale 1 se $i = j$ e zero altrimenti.

passaggi fatti per U^*U permettono di concludere $UU^* = \mathbf{1}$. $(iv) \Rightarrow (i)$ Per ipotesi esiste l'inverso bilatero di U, dunque U è biettiva ed è un isometria perchè $\|Ux\|^2 = (Ux, Ux) = (x, U^*Ux) = \|x\|^2$, per ogni $x \in \mathcal{H}$. $(i) \Rightarrow (ii)$ Per ipotesi U conserva la norma. Allora U conserva il prodotto scalare per l'identità di polarizzazione.[2] Inoltre U è suriettiva per ipotesi. Per concludere proviamo l'implicazione $(iii) \Rightarrow (ii)$; per ogni $x, y \in \mathcal{H}$ si ha $x = \sum_k a_k e_k$ e $y = \sum_h b_h e_h$. Allora $(x, y) = \sum_k \overline{a_k} b_k$ e

$$(Ux, Uy) = \textstyle\sum_{k,h} \overline{a_k} b_h (Ue_k, Ue_h) = \sum_k \overline{a_k} b_k = (x, y)$$

■

[2] Ovvero $(x, y) = \frac{1}{4} \sum_{\alpha:\alpha^4=1} \alpha^{-1} \|x + \alpha y\|^2$

Chapter 2

Infinite matrix operators

Let us consider a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$, and an orthonormal basis $(e_i)_{i\in\mathcal{I}}$, where $\mathcal{I}$ is any at most countable set. For any operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ one can naturally define a matrix $A = (a_{ij})_{i,j\in\mathcal{I}}$ given by $a_{ij} = (\mathcal{A}e_j, e_i)$. Observe that in the case $\mathcal{I}$ is finite $\mathcal{B}(\mathcal{H})$ is isomorphic to $\mathbf{M}_n(\mathbb{C})$ and the correspondence operator-matrix is one to one. In this chapter we are interested in the more general case $\mathcal{I}$ countable but not finite. In this case, obviously, the matrix A associated to $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ is an infinite matrix, and we can describe the action of $\mathcal{A}$ on $\mathcal{H}$ by the means of A

$$Ax = \left[\begin{array}{cc|cc} \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{-1,-1} & a_{-1,0} & \cdots \\ \hline \cdots & a_{0,-1} & a_{0,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}\right] \begin{bmatrix} \vdots \\ x_{-1} \\ x_0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ \vdots \end{bmatrix} = y$$

where $(x_i)_i$ and $(y_i)_i$ are the coefficients of $x, y \in \mathcal{H}$ with respect $(e_i)_i$. In fact A describes exactly the action of $\mathcal{A}$ onto every element of the basis $(e_i)_i$ of $\mathcal{H}$.[1] Every operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ can be represented by a matrix A as done above, but is the converse always true? Suppose that A is an infinite matrix with element in $\mathbb{C}$, A generates a linear bounded operator $\mathcal{A}$ on $\mathcal{H}$ only if there exists a finite constant $c > 0$ such that for every $x \in \mathcal{H}$ the following hold
(i) The series $\sum_{j\in\mathcal{I}} a_{ij}x_j$ converges $\forall i \in \mathcal{I}$.
(ii) If $y_i = \sum_j a_{ij}x_j$ then $y = \sum_{i\in\mathcal{I}} y_i e_i$ belongs to $\mathcal{H}$.
(iii) $\|y\| \leqslant c\|x\|$.
In this case, the matrix A and the operator $\mathcal{A}$ are substantially the same think, exactly like the finite-matrices case. Therefore we shall denote both with the same symbol, failing to distinguish from each other.

2.1 Laurent Matrices

In this section and throughout the rest of the chapter we shall assume $\mathcal{H} = \ell^2(\mathcal{I})$. Notice that a separable Hilbert space $\mathcal{H}$ can always be identified with $\ell^2(\mathcal{I})$ by the construction described above. In other words for any separable Hilbert

[1]For the sake of simplicity we denote an element x of $\mathcal{H}$ with the same symbol of its infinite vector of coefficients $x = (\dots, x_{-1}, x_0, \dots)$

space there exists an obvious bijection $\Psi : \mathcal{H} \to \ell^2(\mathcal{I})$ such that $x \mapsto \{(x, e_i)\}_{i\in\mathcal{I}}$. Therefore the hypothesis $\mathcal{H} = \ell^2$ is not restrictive. Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, we denote with μ the translation invariant Radon measure such that $\mu(\mathbb{T}) = 1$ and with $L^p(\mathbb{T}) = L^p(\mathbb{T}, \mu)$, for $1 \leqslant p \leqslant \infty$.

Let us consider a sequence $(a_n)_n \subset \mathbb{C}$ and associate to it the infinite matrix

$$\left[\begin{array}{cc|cccc} \cdots & & & & & \\ & a_0 & a_{-1} & a_{-2} & & \\ \hline & a_1 & a_0 & a_{-1} & a_{-2} & \\ & a_2 & a_1 & a_0 & a_{-1} & \\ & & a_2 & a_1 & a_0 & \\ & & \cdots & & & \cdots \end{array}\right], \tag{2.1}$$

the following is a well know result due to Laurent

Theorem 2.1 *The matrix* (2.1) *generates a linear bounded operator on* $\ell^2(\mathbb{Z})$ *if and only if there exists* $a \in L^\infty(\mathbb{T})$ *such that* $(a_n)_n$ *are the Fourier coefficients of* a, *and*

$$a(t) = \sum_{n\in\mathbb{Z}} a_n t^n, \quad a_n = \int_{\mathbb{T}} a(\zeta)\zeta^{-n} d\mu(\zeta), \quad t \in \mathbb{T}$$

Note that since a is bounded on $\mathbb{T}$ then $a \in \cap_{p\geqslant 1} L^p(\mathbb{T})$ and hence the Fourier series above converges to a at least in $L^2(\mathbb{T})$. Given $a \in L^\infty(\mathbb{T})$ we shall denote the matrix (2.1) as $L(a)$ and refer to a as the *symbol* of $L(a)$.

An important property of a Laurent matrix is that $\{L(a)\}_{a\in L^\infty(\mathbb{T})}$ is an algebra of simultaneously diagonalizable matrices (or operators).

Definition 2.2 *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* $M(\varphi)$ *be the multiplication operator on* $L^2(\mathcal{X}, \nu)$ *such that* $f \mapsto M(\varphi)f = \varphi f$, $(\mathcal{X}, \nu)$ *a suitable measure space. A is said to be a diagonalizable operator iff there exists an isomorphism* $\Phi : L^2(\mathcal{X}, \nu) \to \mathcal{H}$ *such that* $A = \Phi M(\varphi)\Phi^{-1}$. *We refer to* φ *as the diagonal function of* A.

Remark 2.3 Notice that two multiplication operators always commute and, as a consequence, the diagonal function of an operator A is not unique. In fact, for instance, if $A = \Phi M(\varphi)\Phi^{-1}$ one can choose any invertible $\psi \in L^2(\mathcal{X}, \nu)$, then $A = \Phi M(\varphi)\Phi^{-1} = (\Phi M(\psi))M(\varphi)(\Phi M(\psi))^{-1} = \Phi M(\psi \circ \varphi \circ \psi^{-1})\Phi^{-1}$, namely $\psi \circ \varphi \circ \psi^{-1}$ is another diagonal function of A. Finally let us recall that any normal operator $N \in \mathcal{B}(\mathcal{H})$ is diagonalized by an unitary transform U, moreover a diagonal function of A is the simplest $\xi : \sigma(N) \to \mathbb{C}$, $x \mapsto \xi(x) = x$, i.e. $N = UM(\xi)U^*$.

Given $a \in L^\infty(\mathbb{T})$ we consider the multiplication operator $M(a) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$, $f \mapsto af$ and the isometric isomorphism $\Phi : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$, $f \mapsto (f_n)_n$ which sends an $L^2(\mathbb{T})$ function to its sequence of Fourier coefficients. Then it is clear that

$$\|M(a)\| = \|a\|_\infty, \quad \|\Phi f\|_{\ell^2} = \|f\|_{L^2}, \quad L(a) = \Phi M(a)\Phi^{-1},$$

where the first identity is a trivial computation, the second one is a direct consequence of the Parseval identity, the last one is not difficult to prove and is a consequence of the definitions of M, Φ and L. As a consequence we have the following two properties for a Laurent matrix $L(a)$:

$$L(ab) = L(a)L(b), \quad \forall a, b \in L^\infty(\mathbb{T}) \tag{2.2}$$

$$\|L(a)\| = \|a\|_\infty \quad \forall a \in L^\infty(\mathbb{T}) \tag{2.3}$$

It is known that that $L^\infty(\mathbb{T})$ is a C^*-algebra therefore one can consider the spectrum of a as an element of $L^\infty(\mathbb{T})$, such compact and non-empty set is referred to as the *essential range* of a

$$\sigma_{L^\infty(\mathbb{T})}(a) = \mathcal{R}(a) = \{\lambda \in \mathbb{C} : \mu\big(|a(t) - \lambda| = 0\big) > 0\}$$

Observe that in general the essential range $\mathcal{R}(a)$ is a proper subset of the range $a(\mathbb{T})$, obtained removing from $a(\mathbb{T})$ the null-measure sets, and it is closed. In fact given $(\lambda_n)_n \subset \mathcal{R}(a)$ such that $\lambda_n \to \lambda$, modulo reordering and considering a sub-sequence, we can assume that $\mu(|a(t) - \lambda_n| \leqslant \frac{1}{n}) > \frac{1}{n}$, $\forall n \geqslant 1$, therefore by monotony it follows

$$\bigcap_{n\geqslant 1} \{\mu\left(|a(t) - \lambda_n| \leqslant \tfrac{1}{n}\right) > \tfrac{1}{n}\} = \{\mu(|a(t) - \lambda| = 0) > 0\},$$

and thus $\lambda \in \mathcal{R}(a)$. Moreover notice that if $\lambda \in \mathcal{R}(a)$ then $\lambda \in \sigma(L(a))$ then $\mathcal{R}(a) \subset \sigma(L(a))$, anyway the converse inclusion holds too

Theorem 2.4 *Let $a \in L^\infty(\mathbb{T})$, then $\sigma(L(a)) = \sigma(M(a)) = \mathcal{R}(a)$. Moreover if $0 \notin \mathcal{R}(a)$ then $L(a)$ is invertible and $L(a)^{-1} = L(a^{-1})$.*

Proof. It is obvious that $\sigma(M(a)) = \mathcal{R}(a)$ therefore by the above observations $\sigma(L(a)) = \sigma(\Phi M(a)\Phi^{-1}) = \sigma(M(a)) = \mathcal{R}(a)$. Moreover if $0 \notin \mathcal{R}(a)$ then a is invertible as an element of $L^\infty(\mathbb{T})$ and thus if $\mathrm{Id} \in L^\infty(\mathbb{T})$ is the identity $x \mapsto \mathrm{Id}(x) = x$, by (2.2) we have $I = L(\mathrm{Id}) = L(aa^{-1}) = L(a^{-1}a) = L(a)L(a^{-1}) = L(a^{-1})L(a)$. ■

2.2 Toeplitz and Hankel matrices

A *Toeplitz matrix* defined by a sequence $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ is a matrix of the form

$$T = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_{-1} & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{2.4}$$

Observe that such a matrix is the bottom-right part of the Laurent matrix (2.1). As a consequence, if we consider the projection $P : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$ and a function $a \in L^\infty(\mathbb{T})$, there follows $T = PL(a)P$. This fact, together with Theorem 2.1, implies that T define a bounded (linear) operator on $\ell^2(\mathbb{N})$ if $(a_n)_{n\in\mathbb{Z}}$ are the Fourier coefficients of a function $a \in L^\infty(\mathbb{T})$, in fact

$$\|T\| = \|PL(a)P\| \leqslant \|P\|\|L(a)\|\|P\| \leqslant \|L(a)\| = \|a\|_{L^\infty(\mathbb{T})}.$$

It was proved by Toeplitz that also the converse implication is true, so that we have

Theorem 2.5 *A Toeplitz matrix (2.4) generates an operator of $\mathcal{B}(\ell^2(\mathbb{N}))$ if and only if there exists $a \in L^\infty(\mathbb{T})$ such that $(a_n)_n$ are the Fourier coefficients of a. In this case we shall denote the matrix (2.4) as $T(a)$ and refer to $a \in L^\infty(\mathbb{T})$ as the symbol of $T(a)$.*

The observations done above also yield to the following

Theorem 2.6 *Let a be the symbol a Toeplitz matrix $T(a)$. Then $\|T(a)\| = \|a\|_{L^\infty(\mathbb{T})}$.*

Proof. We have just observed that $\|T(a)\| \leqslant \|a\|_{L^\infty(\mathbb{T})}$. Let us prove the reverse implication. Let Q_n be the $\ell^2(\mathbb{Z})$ projection defined by

$$(Q_n x)_k = \begin{cases} 0 & k < -n \\ x_k & k \geqslant -n \end{cases}, \quad \forall x \in \ell^2(\mathbb{Z})$$

It is obvious that $\|(Q_n - I)x\| \to 0$, $\forall x \in \ell^2(\mathbb{Z})$ then $Q_n \to I$ strongly. Moreover observe that for any $x \in \ell^2(\mathbb{Z})$

$$(L(a)Q_n x)_k = \textstyle\sum_{i\in\mathbb{Z}} a_{i+k}(Q_n x)_i = \sum_{i\geqslant -n} a_{i+k} x_i$$

therefore $Q_n L(a) Q_n x = T(a)x$ that obviously imply $\|Q_n L(a) Q_n\| = \|T(a)\|$. Now by the Banach-Steinhaus theorem we have

$$\|a\|_{L^\infty(\mathbb{T})} = \|L(a)\| \leqslant \liminf_{n\to\infty} \|Q_n L(a) Q_n\| = \|T(a)\|$$

and thus we conclude the proof. ∎

To any sequence $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ one can associate two *Hankel* matrices of the form

$$H_+ = \begin{bmatrix} a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & a_4 & \dots \\ a_3 & a_4 & a_5 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}, \quad H_- = \begin{bmatrix} a_{-1} & a_{-2} & a_{-3} & \dots \\ a_{-2} & a_{-3} & a_{-4} & \dots \\ a_{-3} & a_{-4} & a_{-5} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \tag{2.5}$$

An condition analogue to the Laurent and Toeplitz cases can be given on the sequence $(a_n)_n$ ensuring that H_+ and H_- define two linear bounded operators on $\ell^2(\mathbb{Z})$:

Theorem 2.7 *The matrices H_+ and H_- generate a bounded linear operator on $\ell^2(\mathbb{Z})$ if and only if there exists an $f \in L^\infty(\mathbb{T})$ with Fourier coefficient $(f_n)_{n\in\mathbb{Z}}$ such that $a_n = f_n$ and $a_{-n} = f_{-n}$, for all $n \geqslant 1$, respectively.*

Let $a \in L^\infty(\mathbb{T})$ we shall denote with $H(a)$ the matrix H_+ we shall refer to a as the symbol of $H(a)$. Observe that if i is the "inverse" function $x \mapsto i(x) = x^{-1}$ then $a \circ i$ is the symbol defining H_-, in fact, recalling that if $\zeta \in \mathbb{T}$ then $i(\zeta)$ is a translation and that μ is a Radon translation invariant measure, we have $a \circ i \in L^\infty(\mathbb{T})$ and

$$a_{-k} = \int_{\mathbb{T}} a(\zeta)\zeta^k d\mu(\zeta) = \int_{\mathbb{T}} (a \circ i)(\zeta)\zeta^{-k} d\mu(\zeta) = (a \circ i)_k.$$

Observe that the sufficient condition of Theorem 2.7 can be easily obtained the same way we do for the Toeplitz case. Let us show how. Consider the two orthogonal projections $P : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$, $Q : \ell^2(\mathbb{Z}) \to \ell^2(-\mathbb{N})$ and the so called *reverse identity* J defined by $(Jx)_k = x_{-k-1}$. Notice that P and Q are complementary projections, i.e. $P + Q = I$, and that $J^2 = I$. It is an intuitive fact to suppose that H_+ and H_- are related to the bottom-left and up-right parts of a Laurent matrix, respectively, and this is actually the case. In fact, it is easy to observe that

$$H(a) = PL(a)QJ, \quad H(a \circ i) = JQL(a)P.$$

As a consequence we have the desired norm bound: $\|H(f)\| \leqslant \|f\|_{L^\infty(\mathbb{T})}$.

Of course the equality can not be obtained as in the Toeplitz case since there are infinitely many functions in $f \in L^\infty(\mathbb{T})$ such that $H(a) = H(f)$. Anyway one can show that

$$\|H(a)\| = \inf\{\|f\|_\infty : H(f) = H(a)\}.$$

Suppose that we are given $a, b \in L^\infty(\mathbb{T})$, what can be said about the Toeplitz matrix $T(ab)$? Unluckily it does not hold the same simple formula obtained for the Laurent $L(ab)$, but it is not difficult to prove the following

Proposition 2.8 *Let $a, b \in L^\infty(\mathbb{T})$, then $T(ab) = T(a)T(b) + H(a)H(b \circ i)$.*

Proof. Let P, Q and J be as defined above. There follows $T(ab) = PL(a)L(b)P = PL(a)PL(b)P + PL(a)QL(b)P = (PL(a)P)(PL(b)P) + (PL(a)QJ)(JQL(b)P) = T(a)T(b) + H(a)H(b \circ i)$. ∎

2.3 Approximation methods

We have observed that one can always think to a linear bounded operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ as its infinite matrix A. Therefore the problem of solving an operator equation $\mathcal{A}x = b$ is equivalent to the problem of solving an infinite linear system $Ax = b$. Of course one expect that the solution of such infinite system could be approximated by considering a truncated problem of dimension n. Let us formalize such concept.

Observe that any matrix $A \in \mathbf{M}_n(\mathbb{C})$ can be thought as immersed into the whole $\ell^2(\mathbb{N})$, i.e. we can think of A as an infinite matrix whose j-th columns and rows are null for all $j \geqslant n + 1$. Let P_n be the projection defined by

$$(P_n x)_k = \begin{cases} x_k & 1 \leqslant k \leqslant n \\ 0 & k > n \end{cases}, \quad \forall x \in \ell^2(\mathbb{N}),$$

denote with $\ell^2_n = \text{range}(P_n) \subset \ell^2(\mathbb{N})$ and with $\ell^2 = \ell^2(\mathbb{N})$, for brevity.

Definition 2.9 *Let $(A_n)_n$ be a sequence of $n \times n$ matrices and think them immersed into ℓ^2. We say that $(A_n)_n$ is an approximating sequence for some operator $A \in \mathcal{B}(\ell^2)$ if A_n converges strongly to A, that is*

$$\|A_n x - Ax\| \overset{n\to\infty}{\longrightarrow} 0, \quad \forall x \in \ell^2.$$

Let us recall that for a sequence $(W_n)_n \subset \mathcal{B}(\mathcal{H})$ of operators one can consider at least three different type of convergence: W_n converges *uniformly* to W iff $\|W_n - W\| \to 0$; W_n converges *strongly* to W iff $\|W_n x - Wx\| \to 0$, $\forall x \in \mathcal{H}$; W_n converges *weakly* to W iff $|(y, W_n x) - (y, Wx)| \to 0$, $\forall x, y \in \mathcal{H}$. Notice that the uniform convergence implies the strong convergence that in turn implies the weak convergence, exactly as one expect.

Definition 2.10 *Given $A \in \mathcal{B}(\ell^2)$ and $(A_n)_n$ an approximating sequence for A we say that $(A_n)_n$ is an applicable approximation method for A if*

(i) the matrices A_n are invertible for all sufficiently large n

(ii) the unique solution $x^{(n)} \in \ell^2_n$ of $A_n x^{(n)} = P_n b$ converges in ℓ^2 to the unique solution of $Ax = b$.

In this case we write $A \in \mathcal{M}(A_n)_n$

Observe that given $A \in \mathcal{B}(\ell^2)$ one can easily construct an approximating sequence for A by considering $(P_n A P_n)_{n\in\mathbb{N}}$, what is called the *finite sections method*. Let us consider an approximating sequence $(A_n)_n$ for a given $A \in \mathcal{B}(\ell^2)$ such that $A \in \mathcal{M}(A_n)_n$. Then A_n are invertible for all $n \geqslant \overline{n}$ sufficiently large and there follows from the property *(ii)* that A_n^{-1} converges strongly to A^{-1}. In fact, for every $b \in \ell^2$ we have $0 = \lim_n \|x^{(n)} - x\| = \lim_n \|A_n^{-1} P_n b - A^{-1} b\|$. It also holds the reverse implication, thus we have another characterization for a sequence as in Definition 2.10

Proposition 2.11 *Let $A \in \mathcal{B}(\ell^2)$ and $(A_n)_n$ an approximating sequence for A. Then $A \in \mathcal{M}(A_n)_n$ if and only if A_n are definitively invertible and A_n^{-1} converges strongly to A^{-1}.*

Proof. Observe that we only have to show the invertiblity of A..... ■

On putting $\|A_n^{-1}\| = \infty$ if A_n is not invertible, we have

$$\exists \overline{n} > 1 : \sup_{n \geqslant \overline{n}} \|A_n^{-1}\| < \infty \Longleftrightarrow \limsup_{n\to\infty} \|A_n^{-1}\| < \infty$$

and we refer to such property as the *stability* or *uniform stability* of the sequence $(A_n)_n$. Notice that with the above conditions we are requiring the uniform convergence of the inverses A_n^{-1} to A^{-1}, not only the strong convergence, thus for a stable approximating sequence $(A_n)_n$ always holds $A \in \mathcal{M}(A_n)_n$. However a stable sequence may be not an approximating sequence since the convergence of A_n^{-1} does not imply the convergence of A_n. Moreover, notice that the stability property is an adjoint invariant while the approximating property is not. In fact it is not difficult to observe that $(A_n)_n$ stable implies $(A_n^*)_n$ stable, while if $(A_n)_n$ is an approximating sequence the adjoint sequence may not converges at all. For instance, consider the shift-forward matrix $[Z_n]_{ij} = 1$ if $i = j \mod n+1$ and $[Z_n]_{ij} = 0$ otherwise. Then Z_n converges strongly to the shift-forward operator $(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$. On the other hand the adjoint Z_n^* defined by $[Z_n^*]_{ij} = 1$ if $j = i \mod n+1$ and 0 otherwise, does not converges, in fact for every $x \in \ell^2_n$, $(Z_n^* x)_n = x_1$, $\forall n \geqslant 1$ that is it can not converges to a bounded linear operator on ℓ^2.

The following theorem gives reason to our interest in the concept of stability

Theorem 2.12 *Let $A \in \mathcal{B}(\ell^2)$ and let $(A_n)_n$ be an approximating sequence for A. Then $A \in \mathcal{M}(A_n)_n \iff A$ is invertible and $(A_n)_n$ is stable.*

Proof. The "if" portion comes easily from what we have seen above. Let us prove the "only if" portion....... ∎

Asymptotic inverse

Suppose we are given a sequence $(A_n)_n$ of finite matrices and consider any sequence of projections $(P_n)_n$ defined above, such that $P_n \to I$, uniformly. The *asymptotic inverse* of $(A_n)_n$ is a sequence $(M_n)_n$ such that $\limsup_{n\to\infty} \|M_n\| < \infty$

$$\|M_n A_n - P_n\| \overset{n\to\infty}{\longrightarrow} 0, \quad \|A_n M_n - P_n\| \overset{n\to\infty}{\longrightarrow} 0.$$

We have the following

Proposition 2.13 *Let $(A_n)_n$ be an asymptotic invertible sequence. Then $(A_n)_n$ is stable and*

$$A_n^{-1} = M_n + R_n, \quad \|R_n\| \overset{n\to\infty}{\longrightarrow} 0$$

being $(M_n)_n$ the asymptotic inverse of $(A_n)_n$.

Proof. Let us set $B_n = M_n A_n - P_n$, then for any $\varepsilon > 0$ there exists $n_\varepsilon > 0$ such that $\|B_n\| < \varepsilon$, $\forall n \geqslant n_\varepsilon$. As a consequence the sequence $P_n + B_n$ is definitely invertible and $A_n^{-1} = (P_n + B_n)^{-1} M_n$. Fix an $\varepsilon > 0$, we have $\|(P_n + B_n)^{-1}\| \leqslant 2\varepsilon^{-1}$ for all $n \geqslant n_\varepsilon$ or rather

$$\limsup_{n\to\infty} \|A_n^{-1}\| \leqslant 2\varepsilon^{-1} \limsup_{n\to\infty} \|M_n\|$$

that is $(A_n)_n$ is invertible (recall we are assuming $\|A_n^{-1}\| = \infty$ whenever A_n is not invertible). The matrix R_n can easily be obtained by setting $R_n = -B_n A_n^{-1}$ since

$$A_n^{-1} = P_n A_n^{-1} = P_n (P_n + B_n)^{-1} M_n$$

∎

2.3.1 The Toeplitz case

Let us now consider the case of Toeplitz operators. We shall denote with $(T_n(a))_n$ the finite section method obtained by $P_n T(a) P_n = T_n(a)$. Stability and approximation properties of $T_n(a)$ strictly depend on the function a. We have observed in the previous section that $T_n(a)$ always is an approximating sequence, therefore by Theorem 2.12 we have the implication

$$(T_n(a))_n \text{ stable} \quad \Longrightarrow \quad T(a) \text{ invetible.}$$

This is interesting recalling Theorem 2.12, in fact if we are given a function $a \in L^\infty(\mathbb{T})$ such that $T_n(a)$ is stable, we easily obtain an applicable method, since $T(a) \in \mathcal{M}(A_n)_n$. Unluckily the reverse implication is not true in general,

Proposition 2.14 *There exists a function $a \in L^\infty(\mathbb{T})$ such that $T(a)$ is invertible but $T_n(a)$ is not stable.*

This result is fairly recent and this is because the construction of such symbol a is rather difficult and for a large class of symbols a the invertibility of $T(a)$ implies the stability of $T_n(a)$, so that $T(a) \in \mathcal{M}(T_n(a))_n$.

This is the case, for instance, of continuous symbols

Theorem 2.15 (Gohberg-Feldman) *Let* $a \in \mathscr{C}(\mathbb{T})$*. Then*

$$T(a) \text{ invertible} \quad \Longrightarrow \quad (T_n(a))_n \text{ stable}$$

and, as a consequence, $T(a) \in \mathcal{M}(T_n(a))_n$.

This is a classical result and they are known at least three way to prove it.

First proof of Gohberg-Feldman theorem. .... ∎

Chapter 3

Infinite graphs

In this chapter we would try to formalize the idea of describing the web-graph matrix as the restriction of an infinite matrix being the limit behaviour of a dynamic increasing set of pages and links. The idea has origin from the Google pagerank matrix describing the existence of hyperlinks between the world wide web pages.

The question can be pointed out as follows: consider a simple oriented finite graph $\mathscr{G}_n = (V\mathscr{G}_n, E\mathscr{G}_n)$ defined by a set of enumerated vertexes $V\mathscr{G}_n = \{1, \ldots, n\}$ and a set of oriented edges e_{ij}, from i to j, univocally determined by the ordered pairs (i, j), $i, j \in V\mathscr{G}_n$. The number of vertex n, is referred to as the dimension of the graph. Of course the edge e_{ij} does not exist for any two vertex i and j, moreover the existence of e_{ij} does not implies the existence of e_{ji}. To such a graph we associate the adjacency matrix $E_n \in \mathbf{M}_n(\mathbb{R})$ defined by

$$[E_n]_{ij} = \begin{cases} 1 & e_{ij} \in E\mathscr{G}_n \\ 0 & e_{ij} \notin E\mathscr{G}_n \end{cases}.$$

It is well known that such matrix is irreducible if and only if the graph $\mathscr{G}_n$ is strongly connected and that, in this case, Perron-Frobenious theory ensures the existence of an unique maximal right eigenvector v of E_n, corresponding to its maximal eigenvalue $\rho(E_n)$.

Suppose we are given a graph $\mathscr{G}_n$ describing a set of data which is not static, i.e. the number of objects in the set and the relations that may occur between them could increase or decrease in time. Let us introduce a equivalence relation $\sim$ between the graphs by setting $\mathscr{G}^1_k \sim \mathscr{G}^2_h$ if and only if $k = h$. By restricting the attention to the increasing situation and by considering the graphs modulo the equivalence $\sim$, we can identify the sequence of graphs defined by the dynamic set of data we are considering by $(\mathscr{G}_n)_{n\in\mathbb{N}}$. In other words we are assuming that the dimension of the graph we are considering always increase in time, and we shall refer to such situation with the symbol $\mathscr{G}_n^{\bar{\wedge}}$.

The pagerank problem concerns the situation of an increasing non-connected graphs sequence $\mathscr{G}_n^{\bar{\wedge}}$, to which is associated a stochastic matrix P_n obtained by the normalization of the **reducible** adjacency matrix E_n. Precisely, consider the diagonal matrix Δ_n defined by $(\Delta_n)_{ii} = E_n e_i$, $i = 1, \ldots, n$, we set $P_n = \Delta_n^{-1} E_n$, were the inverse of Δ_n is obtained by inverting only the non-null rows of Δ_n and taking no account of null ones.