Due Giorni di Algebra Lineare Numerica
www.dima.unige.it/∼dibenede/2gg/home.html
Genova, 16–17 Febbraio 2012

Bernoulli, Ramanujan, Toeplitz e le matrici triangolari

Carmine Di Fiore, Francesco Tudisco, Paolo Zellini

Speaker: Carmine Di Fiore

By using one of the definitions of the Bernoulli numbers, we observe that they solve particular *odd* and *even* lower triangular Toeplitz (l.t.T.) systems.

In a paper Ramanujan writes down a sparse lower triangular system solved by Bernoulli numbers; we observe that such system is equivalent to a sparse l.t.T. system.

The attempt to obtain the sparse l.t.T. Ramanujan system from the l.t.T. odd and even systems, leads us to study efficient methods for solving generic l.t.T. systems.

Bernoulli numbers are the rational numbers satisfying the following identity

$$\frac{t}{e^t-1}=\sum_{n=0}^{+\infty}\frac{B_n(0)}{n!}t^n=-\frac{1}{2}t+\sum_{k=0}^{+\infty}\frac{B_{2k}(0)}{(2k)!}t^{2k}.$$

So, they satisfy the following linear equations

$$-\frac{1}{2}j+\sum_{k=0}^{[\frac{j-1}{2}]}\binom{j}{2k}B_{2k}(0)=0,\ j=2,3,4,\dots,$$

$$j\ \text{ even }:\ \begin{bmatrix}\binom{2}{0} &&&&\\ \binom{4}{0} & \binom{4}{2} &&&\\ \binom{6}{0} & \binom{6}{2} & \binom{6}{4} &&\\ \binom{8}{0} & \binom{8}{2} & \binom{8}{4} & \binom{8}{6} &\\ \cdot & \cdot & \cdot & \cdot & \cdot\end{bmatrix}\begin{bmatrix}B_0(0)\\ B_2(0)\\ B_4(0)\\ B_6(0)\\ \cdot\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\\ 4\\ \cdot\end{bmatrix},$$

$$j\ \text{ odd }:\ \begin{bmatrix}\binom{1}{0} &&&&\\ \binom{3}{0} & \binom{3}{2} &&&\\ \binom{5}{0} & \binom{5}{2} & \binom{5}{4} &&\\ \binom{7}{0} & \binom{7}{2} & \binom{7}{4} & \binom{7}{6} &\\ \cdot & \cdot & \cdot & \cdot & \cdot\end{bmatrix}\begin{bmatrix}B_0(0)\\ B_2(0)\\ B_4(0)\\ B_6(0)\\ \cdot\end{bmatrix}=\begin{bmatrix}1\\ 3/2\\ 5/2\\ 7/2\\ \cdot\end{bmatrix}.$$

In other words, the Bernoulli numbers can be obtained by solving (by forward substitution) a lower triangular linear system (one of the above two). For example, by forward solving the first system, I have obtained the first Bernoulli numbers:

$$B_0(0)=1,\ B_2(0)=\frac{1}{6},\ B_4(0)=-\frac{1}{30},\ B_6(0)=\frac{1}{42},\ B_8(0)=-\frac{1}{30},\ B_{10}(0)=\frac{5}{66},$$
$$B_{12}(0)=-\frac{691}{2730},\ B_{14}(0)=\frac{7}{6}\approx 1.16,\ B_{16}(0)=-\frac{47021}{6630}\approx -7.09,\ \dots$$

Bernoulli numbers appear in the Euler-Maclaurin summation formula, and, in particular, in the expression of the error of the trapezoidal quadrature rule as sum of even powers of the integration step h (the expression that justifies the efficiency of the Romberg-Trapezoidal quadrature method).
Bernoulli numbers are also often involved when studying the Riemann-Zeta function. For example, well known is the following Euler formula:

$$\zeta(s)=\sum_{k=1}^{+\infty}\frac{1}{k^s},\quad \zeta(2n)=\frac{4^n|B_{2n}(0)|\pi^{2n}}{2(2n)!},\ n\in\ 1,2,3,\dots$$

(see also [Riemann's Zeta Function, H. M. Edwards, 1974]).
The Ramanujan's paper we refer in the following is entitled "Some properties of Bernoulli's numbers" (1911).

The coefficient matrices of the previous two lower triangular linear systems are submatrices of the matrix X displayed here below:

$$X = \begin{bmatrix} \binom{0}{0} & & & & & & & \\ \binom{1}{0} & \binom{1}{1} & & & & & & \\ \binom{\mathbf{2}}{\mathbf{0}} & \binom{2}{1} & \binom{2}{2} & & & & & \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & & & & \\ \binom{\mathbf{4}}{\mathbf{0}} & \binom{4}{1} & \binom{\mathbf{4}}{\mathbf{2}} & \binom{4}{3} & \binom{4}{4} & & & \\ \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} & & \\ \binom{\mathbf{6}}{\mathbf{0}} & \binom{6}{1} & \binom{\mathbf{6}}{\mathbf{2}} & \binom{6}{3} & \binom{\mathbf{6}}{\mathbf{4}} & \binom{6}{5} & \binom{6}{6} & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

One can easily observe that X can be rewritten as a power series:

$$X = \sum_{k=0}^{+\infty} \frac{1}{k!} Y^k, \quad Y = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 2 & 0 & & & \\ & & 3 & 0 & & \\ & & & 4 & 0 & \\ & & & & \cdot & \cdot \end{bmatrix}$$

$$\text{Proof:} \quad [X]_{ij} = \frac{1}{(i-j)!}[Y^{i-j}]_{ij} = \frac{1}{(i-j)!} j \cdots (i-2)(i-1) = \binom{i-1}{j-1}, \quad 1 \le j \le i \le n.$$

This remark is the starting point in order to show that

$$\begin{bmatrix} 2 & & & & \\ & 12 & & & \\ & & 30 & & \\ & & & 56 & \\ & & & & \cdot \end{bmatrix} \sum_{k=0}^{+\infty} \frac{1}{(2k+2)!} \phi^k = \begin{bmatrix} \binom{2}{0} & & & & \\ \binom{4}{0} & \binom{4}{2} & & & \\ \binom{6}{0} & \binom{6}{2} & \binom{6}{4} & & \\ \binom{8}{0} & \binom{8}{2} & \binom{8}{4} & \binom{8}{6} & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$\begin{bmatrix} 1 & & & & \\ & 3 & & & \\ & & 5 & & \\ & & & 7 & \\ & & & & \cdot \end{bmatrix} \sum_{k=0}^{+\infty} \frac{1}{(2k+1)!} \phi^k = \begin{bmatrix} \binom{1}{0} & & & & \\ \binom{3}{0} & \binom{3}{2} & & & \\ \binom{5}{0} & \binom{5}{2} & \binom{5}{4} & & \\ \binom{7}{0} & \binom{7}{2} & \binom{7}{4} & \binom{7}{6} & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

where $\phi = \begin{bmatrix} 0 & & & & & \\ 2 & 0 & & & & \\ & 12 & 0 & & & \\ & & 30 & 0 & & \\ & & & 56 & 0 & \\ & & & & \cdot & \cdot \end{bmatrix}$, $2 = 1 \cdot 2$, $12 = 3 \cdot 4$, $30 = 5 \cdot 6$, $\ldots$.

It follows that the linear systems solved by the Bernoulli numbers, can be rewritten as follows, in terms of the matrix ϕ:

$$\sum_{k=0}^{+\infty} 2\frac{1}{(2k+2)!}\phi^k \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ B_6(0) \\ \cdot \end{bmatrix} = 2 \begin{bmatrix} 1/2 \\ 2/12 \\ 3/30 \\ 4/56 \\ 5/90 \\ \cdot \end{bmatrix} = 2 \begin{bmatrix} 1/2 \\ 1/6 \\ 1/10 \\ 1/14 \\ 1/18 \\ \cdot \end{bmatrix} =: \mathbf{q}^e, \qquad \text{(almosteven)}$$

$$\sum_{k=0}^{+\infty} \frac{1}{(2k+1)!}\phi^k \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ B_6(0) \\ \cdot \end{bmatrix} = \begin{bmatrix} 1/1 \\ (3/2)/3 \\ (5/2)/5 \\ (7/2)/7 \\ (9/2)/9 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ \cdot \end{bmatrix} =: \mathbf{q}^o. \qquad \text{(almostodd)}$$

Now we transform ϕ into a Toeplitz matrix. We have that

$$D\phi D^{-1} = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & d_4 & \\ & & & & \cdot \end{bmatrix} \begin{bmatrix} 0 & & & & & \\ 2 & 0 & & & & \\ & 12 & 0 & & & \\ & & 30 & 0 & & \\ & & & 56 & 0 & \\ & & & & \cdot & \cdot \end{bmatrix} \begin{bmatrix} d_1^{-1} & & & & \\ & d_2^{-1} & & & \\ & & d_3^{-1} & & \\ & & & d_4^{-1} & \\ & & & & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} 0 & & & & & \\ 2\frac{d_2}{d_1} & 0 & & & & \\ & 12\frac{d_3}{d_2} & 0 & & & \\ & & 30\frac{d_4}{d_3} & 0 & & \\ & & & 56\frac{d_5}{d_4} & 0 & \\ & & & & \cdot & \cdot \end{bmatrix} = xZ, \quad Z = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & 0 & & \\ & & & 1 & 0 & \\ & & & & \cdot & \cdot \end{bmatrix},$$

iff $d_k = \dfrac{x^{k-1}d_1}{(2k-2)!}$, $k = 1, 2, 3, \ldots$, iff

$$D = d_1 D_x, \quad D_x = \begin{bmatrix} 1 & & & & & \\ & \frac{x}{2!} & & & & \\ & & \frac{x^2}{4!} & & & \\ & & & \cdot & & \\ & & & & \frac{x^{n-1}}{(2n-2)!} & \\ & & & & & \cdot \end{bmatrix}.$$

We are ready to introduce the two *even* and *odd* lower triangular Toeplitz (l.t.T.) systems solved by the Bernoulli numbers. Set

$$\mathbf{b} = \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ \cdot \end{bmatrix}$$

where the $B_{2i}(0)$, $i = 0, 1, 2, \ldots$, are the Bernoulli numbers.

Then the (almosteven) system $\sum_{k=0}^{+\infty} 2\frac{1}{(2k+2)!}\phi^k \mathbf{b} = \mathbf{q}^e$ is equivalent to the system $\sum_{k=0}^{+\infty} 2\frac{1}{(2k+2)!}(D_x\phi D_x^{-1})^k(D_x\mathbf{b}) = D_x\mathbf{q}^e$, i.e. to the following l.t.T. even system:

$$\sum_{k=0}^{+\infty} 2\frac{x^k}{(2k+2)!} Z^k\,(D_x\mathbf{b}) = D_x\mathbf{q}^e. \qquad \text{(even)}$$

Idem, the (almostodd) system $\sum_{k=0}^{+\infty} \frac{1}{(2k+1)!}\phi^k \mathbf{b} = \mathbf{q}^o$ is equivalent to the system $\sum_{k=0}^{+\infty} \frac{1}{(2k+1)!}(D_x\phi D_x^{-1})^k(D_x\mathbf{b}) = D_x\mathbf{q}^o$, i.e. to the following l.t.T. odd system:

$$\sum_{k=0}^{+\infty} \frac{x^k}{(2k+1)!} Z^k\,(D_x\mathbf{b}) = D_x\mathbf{q}^o. \qquad \text{(odd)}$$

So, *Bernoulli numbers can be computed by using a l.t.T. linear system solver.* Such solver yields the following vector $\mathbf{z}$:

$$\mathbf{z} = D_x\mathbf{b} = \begin{bmatrix} 1\cdot B_0(0) \\ \frac{x}{2!}B_2(0) \\ \frac{x^2}{4!}B_4(0) \\ \cdot \\ \frac{x^s}{(2s)!}B_{2s}(0) \\ \cdot \\ \frac{x^{n-1}}{(2n-2)!}B_{2n-2}(0) \\ \cdot \end{bmatrix},$$

from which one obtains the vector of the first n Bernoulli numbers:

$$\{\mathbf{b}\}_n = \{D_x^{-1}\mathbf{z}\}_n.$$

Why x positive different from 1 may be useful?
A suitable choice of x can make possible and more stable the computation via a l.t.T. solver of the entries z_i of $\mathbf{z}$ for very large i. In fact, since

$$\frac{x^i}{(2i)!}B_{2i}(0) \approx (-1)^{i+1}p_i, \quad p_i = \frac{x^i}{(2i)!}4\sqrt{\pi i}\frac{i^{2i}}{(\pi e)^{2i}}, \quad \frac{p_{i+1}}{p_i} \rightarrow \frac{x}{4\pi^2},$$

we have that $|\frac{x^i}{(2i)!}B_{2i}(0)| \rightarrow 0\ (+\infty)$ if $x < 4\pi^2$ $(x > 4\pi^2)$, both bad situations. Instead, for $x \approx 4\pi^2 = 39.47..$ the sequence $|\frac{x^i}{(2i)!}B_{2i}(0)|$, $i = 0, 1, 2, \ldots$, should be lower and upper bounded. ...

$|\frac{x^2}{(4)!}B_4(0)| \leq 1$ iff $|x| \leq 26.84$
$|\frac{x^4}{(8)!}B_8(0)| \leq 1$ iff $|x| \leq 33.2$
$|\frac{x^8}{(16)!}B_{16}(0)| \leq 1$ iff $|x| \leq 36.2$
$|\frac{x^{16}}{(32)!}B_{32}(0)| \leq 1$ about iff $|x|^{16} \leq \frac{1}{1293}\frac{(8.54)^{32}}{4\cdot 7.09}$ iff $|x| \leq 37.82$
$|\frac{x^s}{(2s)!}B_{2s}(0)| \leq 1$ about iff $|\frac{x^s}{(2s)!}4\sqrt{\pi s}\frac{s^{2s}}{(\pi e)^{2s}}| \leq 1$ iff $|x|^s \leq \frac{(2s)!}{s^{2s}}\frac{(\pi e)^{2s}}{4\sqrt{\pi s}}$ ...

More generally, the parameter x should be used to make more stable the l.t.T. solver.

Ramanujan in a paper states that the Bernoulli numbers $B_2(0),\ B_4(0),\ B_6(0),\ldots$, satisfy the following lower triangular *sparse* system of linear equations:

$$\begin{bmatrix} 1 &&&&&&&&&&& \\ 0 & 1 &&&&&&&&&& \\ 0 & 0 & 1 &&&&&&&&& \\ \frac{1}{3} & 0 & 0 & 1 &&&&&&&& \\ 0 & \frac{5}{2} & 0 & 0 & 1 &&&&&&& \\ 0 & 0 & 11 & 0 & 0 & 1 &&&&&& \\ \frac{1}{5} & 0 & 0 & \frac{143}{4} & 0 & 0 & 1 &&&&& \\ 0 & 4 & 0 & 0 & \frac{286}{3} & 0 & 0 & 1 &&&& \\ 0 & 0 & \frac{204}{5} & 0 & 0 & 221 & 0 & 0 & 1 &&& \\ \frac{1}{7} & 0 & 0 & \frac{1938}{7} & 0 & 0 & \frac{3230}{7} & 0 & 0 & 1 && \\ 0 & \frac{11}{2} & 0 & 0 & \frac{7106}{5} & 0 & 0 & \frac{3553}{4} & 0 & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} B_2(0) \\ B_4(0) \\ B_6(0) \\ B_8(0) \\ B_{10}(0) \\ B_{12}(0) \\ B_{14}(0) \\ B_{16}(0) \\ B_{18}(0) \\ B_{20}(0) \\ B_{22}(0) \\ \cdot \end{bmatrix} = \begin{bmatrix} 1/6 \\ -1/30 \\ 1/42 \\ 1/45 \\ -1/132 \\ 4/455 \\ 1/120 \\ -1/306 \\ 3/665 \\ 1/231 \\ -1/552 \\ \cdot \end{bmatrix}$$

(actually, since Ramanujan-Bernoulli numbers are the moduli of ours, his equations, obtained by an analytical proof, are a bit different).

So, for example, from the above equations I have easily computed the Bernoulli numbers $B_{18}(0), B_{20}(0), B_{22}(0)$ (from the ones already computed):

$$B_{18}(0) = \frac{43867}{798} \approx 54.97, \quad B_{20}(0) = -\frac{174611}{330} \approx -529.12, \quad B_{22}(0) = \frac{854513}{138} \approx 6192.12.$$

Problem.
Is it possible to obtain such Ramanujan lower triangular sparse system of equations from our odd and even l.t.T. linear systems ? Is it possible to obtain other sparse equations (hopefully more sparse than Ramanujan ones) defining the Bernoulli numbers ?

Note that the Ramanujan matrix, say R, has nonzero entries exactly in the places where the following Toeplitz matrix

$$\sum_{k=0}^{+\infty} \gamma_k \, (Z^3)^k, \quad Z = \begin{bmatrix} 0 &&& \\ 1 & 0 && \\ & 1 & 0 & \\ && \cdot & \cdot \end{bmatrix}$$

has nonzero entries. But R it is not a Toeplitz matrix !

... break for some months ...

Note that the vector of the indeterminates in the Ramanujan system is Z^T times our vector $\mathbf{b}$:

$$\begin{bmatrix} B_2(0) \\ B_4(0) \\ B_6(0) \\ \cdot \end{bmatrix} = \begin{bmatrix} 0 & 1 && \\ & 0 & 1 & \\ && 0 & \cdot \\ &&& \cdot \end{bmatrix} \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ \cdot \end{bmatrix} = Z^T \mathbf{b}.$$

So, the Ramanujan system can be rewritten as follows

$$R(Z^T\mathbf{b}) = \mathbf{f}, \quad \mathbf{f} = [\, f_1\, f_2\, f_3\, \cdot \,]^T.$$

Remark

The Ramanujan matrix R satisfies the following identity involving a sparse lower triangular *Toeplitz* matrix $\tilde{R}$:

$$R \begin{bmatrix} \frac{2!}{x} & & & \\ & \frac{4!}{x^2} & & \\ & & \frac{6!}{x^3} & \\ & & & \cdot \end{bmatrix} = \begin{bmatrix} \frac{2!}{x} & & & \\ & \frac{4!}{x^2} & & \\ & & \frac{6!}{x^3} & \\ & & & \cdot \end{bmatrix} \tilde{R}, \quad \tilde{R} = \sum_{s=0}^{+\infty} \frac{2\,x^{3s}}{(6s+2)!(2s+1)} Z^{3s}.$$

$\Rightarrow$

$$RZ^T\mathbf{b} = \mathbf{f}, \quad \mathbf{f} = [f_1\, f_2\, f_3 \cdot]^T \quad \text{iff}$$

$$R \begin{bmatrix} \frac{2!}{x} & & & \\ & \frac{4!}{x^2} & & \\ & & \frac{6!}{x^3} & \\ & & & \cdot \end{bmatrix} \begin{bmatrix} \frac{x}{2!} & & & \\ & \frac{x^2}{4!} & & \\ & & \frac{x^3}{6!} & \\ & & & \cdot \end{bmatrix} Z^T\mathbf{b} = \mathbf{f} \quad \text{iff}$$

$$\begin{bmatrix} \frac{2!}{x} & & & \\ & \frac{4!}{x^2} & & \\ & & \frac{6!}{x^3} & \\ & & & \cdot \end{bmatrix} \tilde{R} \begin{bmatrix} \frac{x}{2!} & & & \\ & \frac{x^2}{4!} & & \\ & & \frac{x^3}{6!} & \\ & & & \cdot \end{bmatrix} Z^T\mathbf{b} = \mathbf{f} \quad \text{iff}$$

$$\tilde{R} \begin{bmatrix} \frac{x}{2!} & & & \\ & \frac{x^2}{4!} & & \\ & & \frac{x^3}{6!} & \\ & & & \cdot \end{bmatrix} \begin{bmatrix} B_2(0) \\ B_4(0) \\ B_6(0) \\ \cdot \end{bmatrix} = \begin{bmatrix} \frac{x}{2!} & & & \\ & \frac{x^2}{4!} & & \\ & & \frac{x^3}{6!} & \\ & & & \cdot \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \cdot \end{bmatrix}.$$

So, *the Ramanujan system is is equivalent to the following sparse l.t.T. system*:

$$\tilde{R}\,(Z^T D_x \mathbf{b}) = \begin{bmatrix} \frac{x}{2!} & & & \\ & \frac{x^2}{4!} & & \\ & & \frac{x^3}{6!} & \\ & & & \cdot \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \cdot \end{bmatrix}, \quad \text{where}$$

$$\tilde{R} = L(\mathbf{a}^R) = \begin{bmatrix} 1 & & & & & & & & & & & \\ 0 & 1 & & & & & & & & & & \\ 0 & 0 & 1 & & & & & & & & & \\ \frac{2}{8!3}x^3 & 0 & 0 & 1 & & & & & & & & \\ 0 & \frac{2}{8!3}x^3 & 0 & 0 & 1 & & & & & & & \\ 0 & 0 & \frac{2}{8!3}x^3 & 0 & 0 & 1 & & & & & & \\ \frac{2}{14!5}x^6 & 0 & 0 & \frac{2}{8!3}x^3 & 0 & 0 & 1 & & & & & \\ 0 & \frac{2}{14!5}x^6 & 0 & 0 & \frac{2}{8!3}x^3 & 0 & 0 & 1 & & & & \\ 0 & 0 & \frac{2}{14!5}x^6 & 0 & 0 & \frac{2}{8!3}x^3 & 0 & 0 & 1 & & & \\ \frac{2}{20!7}x^9 & 0 & 0 & \frac{2}{14!5}x^6 & 0 & 0 & \frac{2}{8!3}x^3 & 0 & 0 & 1 & & \\ 0 & \frac{2}{20!7}x^9 & 0 & 0 & \frac{2}{14!5}x^6 & 0 & 0 & \frac{2}{8!3}x^3 & 0 & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \; \mathbf{a}^R = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{2}{8!3}x^3 \\ 0 \\ 0 \\ \frac{2}{14!5}x^6 \\ 0 \\ 0 \\ \frac{2}{20!7}x^9 \\ 0 \\ \cdot \end{bmatrix}.$$

$$\text{Note the new notation : } \mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \end{bmatrix} \quad \Rightarrow \quad L(\mathbf{a}) = \begin{bmatrix} a_0 & & & \\ a_1 & a_1 & & \\ a_2 & a_1 & a_2 & \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

THEOREM

Notations: Z is the lower shift matrix

$$Z = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \cdot & \cdot \end{bmatrix},$$

$L(\mathbf{a})$ is the lower triangular Toeplitz matrix with first column $\mathbf{a}$, i.e.

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \end{bmatrix}, \quad L(\mathbf{a}) = \sum_{i=0}^{+\infty} a_i\, Z^i = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$d(\mathbf{z})$ is the diagonal matrix with z_i as diagonal entries.

Set

$$\mathbf{b} = \begin{bmatrix} B_0(0) \\ B_2(0) \\ B_4(0) \\ \cdot \end{bmatrix}, \quad D_x = \operatorname{diag}\left(\frac{x^i}{(2i)!}, \ i = 0, 1, 2, \ldots\right), \ x \in \mathbb{R},$$

where $B_{2i}(0)$, $i = 0, 1, 2, \ldots$, are the Bernoulli numbers.

Then *the vectors* $D_x\mathbf{b}$ *and* $Z^T D_x\mathbf{b}$ *solve the following l.t.T. linear systems*

$$L(\mathbf{a})\,(D_x\mathbf{b}) = D_x\mathbf{q}, \quad L(\mathbf{a})\,(Z^T D_x\mathbf{b}) = d(\mathbf{z})Z^T D_x\mathbf{q},$$

where the vectors $\mathbf{a} = (a_i)_{i=0}^{+\infty}$, $\mathbf{q} = (q_i)_{i=0}^{+\infty}$, and $\mathbf{z} = (z_i)_{i=1}^{+\infty}$, can assume respectively the values:

$$a_i^R = \delta_{i=0 \bmod 3} \frac{2x^i}{(2i+2)!(\frac{2}{3}i+1)}, \quad q_i^R = \frac{1}{(2i+1)(i+1)}(1 - \delta_{i=2 \bmod 3}\frac{3}{2}), \ i = 0, 1, 2, 3, \ldots$$
$$z_i^R = 1 - \delta_{i=0 \bmod 3} \frac{1}{\frac{2}{3}i+1}, \quad i = 1, 2, 3, \ldots,$$

$$a_i^e = \frac{2x^i}{(2i+2)!}, \quad q_i^e = \frac{1}{2i+1}, \quad i = 0, 1, 2, 3, \ldots$$
$$z_i^e = \frac{i}{i+1}, \quad i = 1, 2, 3, \ldots,$$

$$a_i^o = \frac{x^i}{(2i+1)!}, \quad i = 0, 1, 2, 3, \ldots, \quad q_0^o = 1, \ q_i^o = \frac{1}{2}, \quad i = 1, 2, 3, \ldots$$
$$z_i^o = \frac{2i-1}{2i+1}, \quad i = 1, 2, 3, \ldots\ .$$

Problem (regarding the computation of the Bernoulli numbers).
Can the Ramanujan l.t.T. sparse system

$$L(\mathbf{a}^R)D_x\mathbf{b} = D_x\mathbf{q}^R,$$

be obtained as a consequence of the even and odd l.t.T. system

$$L(\mathbf{a}^e)D_x\mathbf{b} = D_x\mathbf{q}^e, \quad L(\mathbf{a}^o)D_x\mathbf{b} = D_x\mathbf{q}^o \quad ?$$

$\rightarrow$ find $\hat{\mathbf{a}}^e$, $\hat{\mathbf{a}}^o$ such that $L(\hat{\mathbf{a}}^e)L(\mathbf{a}^e) = L(\mathbf{a}^{(1)}) = L(\hat{\mathbf{a}}^o)L(\mathbf{a}^o)$,
i.e. such that $L(\mathbf{a}^e)\hat{\mathbf{a}}^e = \mathbf{a}^{(1)} = L(\mathbf{a}^o)\hat{\mathbf{a}}^o$,

with $\mathbf{a}^{(1)} = \mathbf{a}^R = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \bullet \\ \cdot \end{bmatrix}$ or $\mathbf{a}^{(1)}$ more sparse than $\mathbf{a}^R$.

Important: the computation of such vectors $\hat{\mathbf{a}}^e$, $\hat{\mathbf{a}}^o$ and $\mathbf{a}^{(1)}$ should be cheaper than solving the original even and odd (dense) systems.

A more general problem: is it possible to transform efficiently a generic l.t.T. matrix into a more sparse l.t.T. matrix ?

$\rightarrow$ Question: given a_i, $i = 1, 2, 3, \ldots$, is it possible to obtain "cheaply" $\hat{a}_i$ and $a_i^{(1)}$ such that

$$\begin{bmatrix} 1 & & & & \\ a_1 & 1 & & & \\ a_2 & a_1 & 1 & & \\ a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ a_1^{(1)} \\ 0 \\ \cdot \end{bmatrix};$$

$$\begin{bmatrix} 1 & & & & & \\ a_1 & 1 & & & & \\ a_2 & a_1 & 1 & & & \\ a_3 & a_2 & a_1 & 1 & & \\ a_4 & a_3 & a_2 & a_1 & 1 & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a_1^{(1)} \\ 0 \\ 0 \\ \cdot \end{bmatrix};$$

$$\begin{bmatrix} 1 & & & & & \\ a_1 & 1 & & & & \\ a_2 & a_1 & 1 & & & \\ a_3 & a_2 & a_1 & 1 & & \\ a_4 & a_3 & a_2 & a_1 & 1 & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{0} \\ a_1^{(1)} \\ \mathbf{0} \\ a_2^{(1)} \\ \mathbf{0} \\ \cdot \end{bmatrix} = \gamma, \ \mathbf{0} \in \mathbb{R}^{b-1} \quad ?$$

If the answer is yes, then a dense l.t.T system can be transformed efficiently into a sparse l.t.T. system:

$$L(\mathbf{a})\mathbf{z} = \mathbf{c} \quad \text{iff} \quad L(\hat{\mathbf{a}})L(\mathbf{a})\mathbf{z} = L(\hat{\mathbf{a}})\mathbf{c} \quad \text{iff} \quad L(\gamma)\mathbf{z} = L(\hat{\mathbf{a}})\mathbf{c}.$$

DEFINITIONS:

$$\mathbf{a}=\begin{bmatrix}1\\a_1\\a_2\\ \cdot\end{bmatrix}\in\mathbb{C}^{+\infty},\quad L(\mathbf{a})=\begin{bmatrix}1&&&\\a_1&1&&\\a_2&a_1&1&\\ \cdot&\cdot&\cdot&\cdot\end{bmatrix},$$

$$E=\begin{bmatrix}1&0&0&\cdot&\cdot\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\cdot&\cdot\\0&1&0&\cdot&\cdot\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\cdot&\cdot\\0&0&1&0&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\end{bmatrix},\ \mathbf{0}=\mathbf{0}_{b-1},\quad E^s=\begin{bmatrix}1&0&0&\cdot&\cdot\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\cdot&\cdot\\0&1&0&\cdot&\cdot\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\cdot&\cdot\\0&0&1&0&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\end{bmatrix},\ \mathbf{0}=\mathbf{0}_{b^s-1},$$

$$\mathbf{u}=\begin{bmatrix}1\\u_1\\u_2\\ \cdot\end{bmatrix},\quad E\mathbf{u}=\begin{bmatrix}1\\ \mathbf{0}\\u_1\\ \mathbf{0}\\u_2\\ \cdot\end{bmatrix},\ \mathbf{0}=\mathbf{0}_{b-1},\quad L(E\mathbf{u})=\begin{bmatrix}1&&&&&\\ \mathbf{0}&I&&&&\\u_1&\mathbf{0}^T&1&&&\\ \mathbf{0}&u_1I&\mathbf{0}&I&&\\u_2&\mathbf{0}^T&u_1&\mathbf{0}^T&1&\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{bmatrix},\ \mathbf{0}=\mathbf{0}_{b-1}.$$

LEMMA: If $\mathbf{u}=\begin{bmatrix}1\\u_1\\u_2\\ \cdot\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}1\\v_1\\v_2\\ \cdot\end{bmatrix}$, then

$$L(E\mathbf{u})E\mathbf{v}=EL(\mathbf{u})L(\mathbf{v}),\quad L(E^s\mathbf{u})E^s\mathbf{v}=E^sL(\mathbf{u})\mathbf{v},\quad \forall\, s\in\mathbb{N}\,.$$

PROBLEM: Given $\mathbf{a}=\begin{bmatrix}1\\a_1\\a_2\\ \cdot\end{bmatrix}$, find $\hat{\mathbf{a}}=\begin{bmatrix}1\\ \hat{a}_1\\ \hat{a}_2\\ \cdot\end{bmatrix}$ and $\mathbf{a}^{(1)}=\begin{bmatrix}1\\a_1^{(1)}\\a_2^{(2)}\\ \cdot\end{bmatrix}$ such that

$$L(\mathbf{a})\hat{\mathbf{a}}=E\mathbf{a}^{(1)}=\begin{bmatrix}1\\ \mathbf{0}\\a_1^{(1)}\\ \mathbf{0}\\ \cdot\end{bmatrix},\ \mathbf{0}=\mathbf{0}_{b-1},\quad L(\mathbf{a})L(\hat{\mathbf{a}})=L(E\mathbf{a}^{(1)})\,.$$

Questions:

Is it possible to obtain "cheaply" $\hat{a}_i$ and $a_i^{(1)}$?

There exist explicit formulas for the $\hat{a}_i$ and $a_i^{(1)}$?

At the moment, let us see in detail, with two examples, how
the solutions of the above Problem can lead to efficient methods for solving generic l.t.T. linear systems.

EXAMPLE: $n = 8 \quad (n = b^k,\ b = 2,\ k = 3)$

$$\mathbf{a} = \mathbf{a}^{(0)} = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ \cdot \end{bmatrix}, \quad L(\mathbf{a}) = \begin{bmatrix} 1 & & & & & & & & \\ a_1 & 1 & & & & & & & \\ a_2 & a_1 & 1 & & & & & & \\ a_3 & a_2 & a_1 & 1 & & & & & \\ a_4 & a_3 & a_2 & a_1 & 1 & & & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad E = \begin{bmatrix} 1 & & & & \\ 0 & & & & \\ & 1 & & & \\ & 0 & & & \\ & & 1 & & \\ & & 0 & & \\ & & & 1 & \\ & & & 0 & \\ & & & & \cdot \end{bmatrix}.$$

step 1: From $\mathbf{a} = \mathbf{a}^{(0)}$ find $\hat{\mathbf{a}} = \hat{\mathbf{a}}^{(0)}$ such that

$$L(\mathbf{a})\hat{\mathbf{a}} = \begin{bmatrix} 1 & & & & & & & & \\ a_1 & 1 & & & & & & & \\ a_2 & a_1 & 1 & & & & & & \\ a_3 & a_2 & a_1 & 1 & & & & & \\ a_4 & a_3 & a_2 & a_1 & 1 & & & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \\ \hat{a}_6 \\ \hat{a}_7 \\ \cdot \end{bmatrix} =: E\mathbf{a}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ a_1^{(1)} \\ 0 \\ a_2^{(1)} \\ 0 \\ a_3^{(1)} \\ 0 \\ \cdot \end{bmatrix},$$

$$L(\mathbf{a})L(\hat{\mathbf{a}}) = L(E\mathbf{a}^{(1)});$$

step 2: From $\mathbf{a}^{(1)}$ find $\hat{\mathbf{a}}^{(1)}$ such that

$$L(E\mathbf{a}^{(1)})E\hat{\mathbf{a}}^{(1)} = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ a_1^{(1)} & & 1 & & & & & & \\ & a_1^{(1)} & & 1 & & & & & \\ a_2^{(1)} & & a_1^{(1)} & & 1 & & & & \\ & a_2^{(1)} & & a_1^{(1)} & & 1 & & & \\ a_3^{(1)} & & a_2^{(1)} & & a_1^{(1)} & & 1 & & \\ & a_3^{(1)} & & a_2^{(1)} & & a_1^{(1)} & & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \hat{a}_1^{(1)} \\ 0 \\ \hat{a}_2^{(1)} \\ 0 \\ \hat{a}_3^{(1)} \\ 0 \\ \cdot \end{bmatrix} =: E^2\mathbf{a}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ a_1^{(2)} \\ 0 \\ 0 \\ 0 \\ \cdot \end{bmatrix},$$

$$\underline{L(E\mathbf{a}^{(1)})}L(E\hat{\mathbf{a}}^{(1)}) = L(E^2\mathbf{a}^{(2)});$$

step $3 = \log_2 8$: From $\mathbf{a}^{(2)}$ find $\hat{\mathbf{a}}^{(2)}$ such that

$$L(E^2\mathbf{a}^{(2)})E^2\hat{\mathbf{a}}^{(2)} = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ a_1^{(2)} & & & & 1 & & & & \\ & a_1^{(2)} & & & & 1 & & & \\ & & a_1^{(2)} & & & & 1 & & \\ & & & a_1^{(2)} & & & & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \hat{a}_1^{(2)} \\ 0 \\ 0 \\ 0 \\ \cdot \end{bmatrix} =: E^3\mathbf{a}^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \end{bmatrix},$$

$$\underline{L(E^2\mathbf{a}^{(2)})}L(E^2\hat{\mathbf{a}}^{(2)}) = L(E^3\mathbf{a}^{(3)}).$$

$$\Rightarrow \qquad L(E^2\hat{\mathbf{a}}^{(2)})L(E\hat{\mathbf{a}}^{(1)})L(\hat{\mathbf{a}})\big[\, L(\mathbf{a}) \,\big] = L(E^3\mathbf{a}^{(3)}) = \begin{bmatrix} I_8 & O \\ \cdot & \cdot \end{bmatrix},$$

so, one realizes that we have performed a kind of Gaussian elimination.

How many arithmetic operations (a.o.) ?

Actually, given $a_i = a_i^{(0)}$, $i = 1, \dots, 7$, we have to compute

$$\hat{a}_i = \hat{a}_i^{(0)},\ a_i^{(1)} \ \Big| \ \begin{bmatrix} 1 &&&&&&&& \\ a_1 & 1 &&&&&&& \\ a_2 & a_1 & 1 &&&&&& \\ a_3 & a_2 & a_1 & 1 &&&&& \\ a_4 & a_3 & a_2 & a_1 & 1 &&&& \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 &&& \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 && \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \\ \hat{a}_6 \\ \hat{a}_7 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ a_1^{(1)} \\ 0 \\ a_2^{(1)} \\ 0 \\ a_3^{(1)} \\ 0 \\ \cdot \end{bmatrix}, \ \varphi_8 \text{ a.o.,}$$

$$\hat{a}_i^{(1)},\ a_1^{(2)} \ \Big| \ \begin{bmatrix} 1 &&&& \\ a_1^{(1)} & 1 &&& \\ a_2^{(1)} & a_1^{(1)} & 1 && \\ a_3^{(1)} & a_2^{(1)} & a_1^{(1)} & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_1^{(1)} \\ \hat{a}_2^{(1)} \\ \hat{a}_3^{(1)} \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ a_1^{(2)} \\ 0 \\ \cdot \end{bmatrix}, \ \varphi_4 \text{ a.o.,}$$

$$\hat{a}_1^{(2)} \ \Big| \ \begin{bmatrix} 1 & 0 & \\ a_1^{(2)} & 1 & \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_1^{(2)} \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cdot \end{bmatrix}, \ \varphi_2 \text{ a.o..}$$

The general case: $n = 2^k$

Given $a_i = a_i^{(0)}$, $i = 1, \dots, n-1 = 2^k - 1$, we have to compute

$$\hat{a}_i^{(j)},\ a_i^{(j+1)} \ \Big| \ \underbrace{\begin{bmatrix} 1 &&&&&& \\ a_1^{(j)} & 1 &&&&& \\ a_2^{(j)} & a_1^{(j)} & 1 &&&& \\ \cdot & \cdot & \cdot & 1 &&& \\ \cdot & \cdot & \cdot & \cdot & \cdot && \\ a_{\frac{n}{2^j}-1}^{(j)} & \cdot & \cdot & a_2^{(j)} & a_1^{(j)} & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{\frac{n}{2^j} \times \frac{n}{2^j}} \begin{bmatrix} 1 \\ \hat{a}_1^{(j)} \\ \hat{a}_2^{(j)} \\ \cdot \\ \cdot \\ \hat{a}_{\frac{n}{2^j}-1}^{(j)} \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ a_1^{(j+1)} \\ 0 \\ \cdot \\ a_{\frac{n}{2^{j+1}}-1}^{(j+1)} \\ 0 \\ \cdot \end{bmatrix}, \ \varphi_{\frac{n}{2^j}} \text{ a.o.,}$$

$$j = 0, 1, \dots, k-2, k-1 \ \ (j = k-1 : \text{ only } \hat{a}_i^{(j)})$$

Total cost: $\sum_{j=0}^{k-1} \varphi_{\frac{n}{2^j}} \leq ?$

Remark. Note that, at step j, the $a_i^{(j+1)}$ are the $\frac{n}{2^{j+1}}$ nonzero entries of a matrix $\frac{n}{2^j} \times \frac{n}{2^j}$ $(2^{k-j} \times 2^{k-j})$ l.t.T. by vector product. So, if we assume such matrix by vector product computable in at most $c\,2^{k-j}(k-j)$ a.o., for some constant c, then

$$\begin{aligned} \sum_{j=0}^{k-1} \varphi_{\frac{n}{2^j}} &\leq c \sum_{j=0}^{k-2} 2^{k-j}(k-j) + \sum_{j=0}^{k-1} \text{CostCompOf}\,(\,\hat{a}_i^{(j)}\,) \\ &\leq O(2^k k) + \sum_{j=0}^{k-1} \text{CostCompOf}\Big(\,\hat{a}_i^{(j)},\ i = 1, \dots, \frac{n}{2^j} - 1\Big) = ? \end{aligned}$$

Computing the first column of $L(\mathbf{a})^{-1}$ $(n = 2^3 = 8)$

Let $\mathbf{v} = [v_0\, v_1\, v_2 \cdot]^T$ be any vector. From the identity

$$L(E^2\hat{\mathbf{a}}^{(2)})L(E\hat{\mathbf{a}}^{(1)})L(\hat{\mathbf{a}})\big[\,L(\mathbf{a})\,\big] = L(E^3\mathbf{a}^{(3)}) = \begin{bmatrix} I_8 & O \\ \cdot & \cdot \end{bmatrix}$$

and from the Lemma, it follows that

$$\begin{aligned} L(\mathbf{a})\mathbf{z} = E^2\mathbf{v} \ \text{ iff} & \\ L(E^3a^{(3)})\mathbf{z} &= L(\hat{\mathbf{a}})L(E\hat{\mathbf{a}}^{(1)})L(E^2\hat{\mathbf{a}}^{(2)})E^2\mathbf{v} \\ &= L(\hat{\mathbf{a}})L(E\hat{\mathbf{a}}^{(1)})E^2L(\hat{\mathbf{a}}^{(2)})\mathbf{v} \\ &= L(\hat{\mathbf{a}})EL(\hat{\mathbf{a}}^{(1)})EL(\hat{\mathbf{a}}^{(2)})\mathbf{v}. \end{aligned}$$

So, the system $L(\mathbf{a})\mathbf{z} = E^2\mathbf{v}$ is equivalent to the system

$$\begin{bmatrix} I_8 & O \\ \cdot & \cdot \end{bmatrix}\begin{bmatrix} \{\mathbf{z}\}_8 \\ \cdot \end{bmatrix} = L(E^3a^{(3)})\mathbf{z} = L(\hat{\mathbf{a}})EL(\hat{\mathbf{a}}^{(1)})EL(\hat{\mathbf{a}}^{(2)})\mathbf{v}$$

$$\begin{aligned} \Rightarrow\quad \{\mathbf{z}\}_8 &= \{L(\hat{\mathbf{a}})\}_8\{E\}_8\{L(\hat{\mathbf{a}}^{(1)})\}_8\{E\}_8\{L(\hat{\mathbf{a}}^{(2)})\}_8\{\mathbf{v}\}_8 \\ &= \{L(\hat{\mathbf{a}})\}_8\{E\}_{8,4}\{L(\hat{\mathbf{a}}^{(1)})\}_4\{E\}_{4,2}\{L(\hat{\mathbf{a}}^{(2)})\}_2\{\mathbf{v}\}_2. \end{aligned}$$

Thus the vector

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{bmatrix} = \begin{bmatrix} 1 &&&&&&& \\ \hat{a}_1 & 1 &&&&&& \\ \hat{a}_2 & \hat{a}_1 & 1 &&&&& \\ \hat{a}_3 & \hat{a}_2 & \hat{a}_1 & 1 &&&& \\ \hat{a}_4 & \hat{a}_3 & \hat{a}_2 & \hat{a}_1 & 1 &&& \\ \hat{a}_5 & \hat{a}_4 & \hat{a}_3 & \hat{a}_2 & \hat{a}_1 & 1 && \\ \hat{a}_6 & \hat{a}_5 & \hat{a}_4 & \hat{a}_3 & \hat{a}_2 & \hat{a}_1 & 1 & \\ \hat{a}_7 & \hat{a}_6 & \hat{a}_5 & \hat{a}_4 & \hat{a}_3 & \hat{a}_2 & \hat{a}_1 & 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ 0 &&& \\ & 1 && \\ & 0 && \\ && 1 & \\ && 0 & \\ &&& 1 \\ &&& 0 \end{bmatrix}\begin{bmatrix} 1 &&& \\ \hat{a}_1^{(1)} & 1 && \\ \hat{a}_2^{(1)} & \hat{a}_1^{(1)} & 1 & \\ \hat{a}_3^{(1)} & \hat{a}_2^{(1)} & \hat{a}_1^{(1)} & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ 0 & \\ & 1 \\ & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \hat{a}_1^{(2)} & 1 \end{bmatrix}\begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$$

is such that

$$\begin{bmatrix} 1 &&&&&&& \\ a_1 & 1 &&&&&& \\ a_2 & a_1 & 1 &&&&& \\ a_3 & a_2 & a_1 & 1 &&&& \\ a_4 & a_3 & a_2 & a_1 & 1 &&& \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 && \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{bmatrix} = \begin{bmatrix} v_0 \\ 0 \\ 0 \\ 0 \\ v_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

How many arithmetic operations (a.o.) ?

Case $n = 2^k$

It is clear that the above procedure requires the computation of matrix $2^j \times 2^j$ l.t.T. by vector products, with $j = 1, \ldots, k$ (the vectors are sparse for $j = 2, \ldots, k$). So, if we assume such matrix by vector product computable in at most $c\,2^j j$ a.o., for some constant c, then the above procedure requires at most

$$c\sum_{j=1}^{k} 2^j j \le O(2^k k)$$

arithmetic operations.

EXAMPLE: $n = 9$ $(n = b^k,\ b = 3,\ k = 2)$

$$\mathbf{a} = \mathbf{a}^{(0)} = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ \cdot \end{bmatrix},\ L(\mathbf{a}) = \begin{bmatrix} 1 &&&&&&&&& \\ a_1 & 1 &&&&&&&& \\ a_2 & a_1 & 1 &&&&&&& \\ a_3 & a_2 & a_1 & 1 &&&&&& \\ a_4 & a_3 & a_2 & a_1 & 1 &&&&& \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 &&&& \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 &&& \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 && \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},\ E = \begin{bmatrix} 1 &&& \\ 0 &&& \\ 0 &&& \\ & 1 && \\ & 0 && \\ & 0 && \\ && 1 & \\ && 0 & \\ && 0 & \\ &&& \cdot \end{bmatrix}.$$

step 1: From $\mathbf{a} = \mathbf{a}^{(0)}$ find $\hat{\mathbf{a}} = \hat{\mathbf{a}}^{(0)}$ such that

$$L(\mathbf{a})\hat{\mathbf{a}} = \begin{bmatrix} 1 &&&&&&&&& \\ a_1 & 1 &&&&&&&& \\ a_2 & a_1 & 1 &&&&&&& \\ a_3 & a_2 & a_1 & 1 &&&&&& \\ a_4 & a_3 & a_2 & a_1 & 1 &&&&& \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 &&&& \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 &&& \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 && \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \\ \hat{a}_6 \\ \hat{a}_7 \\ \hat{a}_8 \\ \cdot \end{bmatrix} =: E\mathbf{a}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a_1^{(1)} \\ 0 \\ 0 \\ a_2^{(1)} \\ 0 \\ 0 \\ \cdot \end{bmatrix},$$

$$L(\mathbf{a})L(\hat{\mathbf{a}}) = L(E\mathbf{a}^{(1)});$$

step $2 = \log_3 9$: From $\mathbf{a}^{(1)}$ find $\hat{\mathbf{a}}^{(1)}$ such that

$$L(E\mathbf{a}^{(1)})E\hat{\mathbf{a}}^{(1)} = \begin{bmatrix} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && 1 &&&&&& \\ a_1^{(1)} &&& 1 &&&&& \\ & a_1^{(1)} &&& 1 &&&& \\ && a_1^{(1)} &&& 1 &&& \\ a_2^{(1)} &&& a_1^{(1)} &&& 1 && \\ & a_2^{(1)} &&& a_1^{(1)} &&& 1 & \\ && a_2^{(1)} &&& a_1^{(1)} &&& 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \hat{a}_1^{(1)} \\ 0 \\ 0 \\ \hat{a}_2^{(1)} \\ 0 \\ 0 \\ \cdot \end{bmatrix} =: E^2\mathbf{a}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \end{bmatrix},$$

$$\underline{L(E\mathbf{a}^{(1)})}L(E\hat{\mathbf{a}}^{(1)}) = L(E^2\mathbf{a}^{(2)});$$

$$\Rightarrow \qquad L(E\hat{\mathbf{a}}^{(1)})L(\hat{\mathbf{a}})\big[\,L(\mathbf{a})\,\big] = L(E^2\mathbf{a}^{(2)}) = \begin{bmatrix} I_9 & O \\ \cdot & \cdot \end{bmatrix},$$

so, one realizes that we have performed a kind of Gaussian elimination.

How many arithmetic operations (a.o.) ?

Actually, given $a_i = a_i^{(0)}$, $i = 1, \ldots, 8$, we have to compute

$$\hat{a}_i = \hat{a}_i^{(0)},\ a_i^{(1)}\ \Big| \quad \begin{bmatrix} 1 & & & & & & & & & \\ a_1 & 1 & & & & & & & & \\ a_2 & a_1 & 1 & & & & & & & \\ a_3 & a_2 & a_1 & 1 & & & & & & \\ a_4 & a_3 & a_2 & a_1 & 1 & & & & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \\ \hat{a}_6 \\ \hat{a}_7 \\ \hat{a}_7 \\ \hat{a}_8 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a_1^{(1)} \\ 0 \\ 0 \\ a_2^{(1)} \\ 0 \\ 0 \\ \cdot \end{bmatrix}, \quad \varphi_9 \text{ a.o.,}$$

$$\hat{a}_i^{(1)}\ \Big| \quad \begin{bmatrix} 1 & & & \\ a_1^{(1)} & 1 & & \\ a_2^{(1)} & a_1^{(1)} & 1 & \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_1^{(1)} \\ \hat{a}_2^{(1)} \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \end{bmatrix}, \quad \varphi_3 \text{ a.o..}$$

The general case: $n = 3^k$

Given $a_i = a_i^{(0)}$, $i = 1, \ldots, n-1 = 3^k - 1$, we have to compute

$$\hat{a}_i^{(j)},\ a_i^{(j+1)}\ \Big| \quad \underbrace{\begin{bmatrix} 1 & & & & & & \\ a_1^{(j)} & 1 & & & & & \\ a_2^{(j)} & a_1^{(j)} & 1 & & & & \\ \cdot & \cdot & \cdot & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ a_{\frac{n}{3^j}-1}^{(j)} & \cdot & \cdot & a_2^{(j)} & a_1^{(j)} & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{\frac{n}{3^j} \times \frac{n}{3^j}} \begin{bmatrix} 1 \\ \hat{a}_1^{(j)} \\ \hat{a}_2^{(j)} \\ \cdot \\ \cdot \\ \hat{a}_{\frac{n}{3^j}-1}^{(j)} \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a_1^{(j+1)} \\ 0 \\ 0 \\ \cdot \\ a_{\frac{n}{3^{j+1}}-1}^{(j+1)} \\ 0 \\ 0 \\ \cdot \end{bmatrix}, \quad \varphi_{\frac{n}{3^j}} \text{ a.o.,}$$

$$j = 0, 1, \ldots, k-2, k-1 \quad (j = k-1: \text{ only } \hat{a}_i^{(j)})$$

Total cost: $\sum_{j=0}^{k-1} \varphi_{\frac{n}{3^j}} \leq$?

Remark. Note that, at step j, the $a_i^{(j+1)}$ are the $\frac{n}{3^{j+1}}$ nonzero entries of a matrix $\frac{n}{3^j} \times \frac{n}{3^j}$ $(3^{k-j} \times 3^{k-j})$ l.t.T. by vector product. So, if we assume such matrix by vector product computable in at most $c\,3^{k-j}(k-j)$ a.o., for some constant c, then

$$\begin{aligned} \sum_{j=0}^{k-1} \varphi_{\frac{n}{3^j}} &\leq c \sum_{j=0}^{k-2} 3^{k-j}(k-j) + \sum_{j=0}^{k-1} \text{CostCompOf}\,(\,\hat{a}_i^{(j)}\,) \\ &\leq O(3^k k) + \sum_{j=0}^{k-1} \text{CostCompOf}\Big(\,\hat{a}_i^{(j)},\ i = 1, \ldots, \frac{n}{3^j} - 1\Big) = \ ? \end{aligned}$$

Computing the first column of $L(\mathbf{a})^{-1}$ (case $n = 3^2 = 9$):

Let $\mathbf{v} = [v_0\, v_1\, v_2 \cdot]^T$ be any vector. From the identity

$$L(E\hat{\mathbf{a}}^{(1)})L(\hat{\mathbf{a}})\big[\, L(\mathbf{a})\,\big] = L(E^2\mathbf{a}^{(2)}) = \begin{bmatrix} I_9 & O \\ \cdot & \cdot \end{bmatrix}$$

and from the Lemma, it follows that

$$\begin{aligned} L(\mathbf{a})\mathbf{z} = E\mathbf{v} \ \text{ iff} & \\ L(E^2 a^{(2)})\mathbf{z} &= L(\hat{\mathbf{a}})L(E\hat{\mathbf{a}}^{(1)})E\mathbf{v} \\ &= L(\hat{\mathbf{a}})EL(\hat{\mathbf{a}}^{(1)})\mathbf{v}. \end{aligned}$$

So, the system $L(\mathbf{a})\mathbf{z} = E\mathbf{v}$ is equivalent to the system

$$\begin{bmatrix} I_9 & O \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \{\mathbf{z}\}_9 \\ \cdot \end{bmatrix} = L(E^2 a^{(2)})\mathbf{z} = L(\hat{\mathbf{a}})EL(\hat{\mathbf{a}}^{(1)})\mathbf{v}$$

$$\begin{aligned} \Rightarrow \quad \{\mathbf{z}\}_9 &= \{L(\hat{\mathbf{a}})\}_9\{E\}_9\{L(\hat{\mathbf{a}}^{(1)})\}_9\{\mathbf{v}\}_9 \\ &= \{L(\hat{\mathbf{a}})\}_9\{E\}_{9,3}\{L(\hat{\mathbf{a}}^{(1)})\}_3\{\mathbf{v}\}_3. \end{aligned}$$

Thus the vector

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{bmatrix} = \begin{bmatrix} 1 &&&&&&&& \\ \hat{a}_1 & 1 &&&&&&& \\ \hat{a}_2 & \hat{a}_1 & 1 &&&&&& \\ \hat{a}_3 & \hat{a}_2 & \hat{a}_1 & 1 &&&&& \\ \hat{a}_4 & \hat{a}_3 & \hat{a}_2 & \hat{a}_1 & 1 &&&& \\ \hat{a}_5 & \hat{a}_4 & \hat{a}_3 & \hat{a}_2 & \hat{a}_1 & 1 &&& \\ \hat{a}_6 & \hat{a}_5 & \hat{a}_4 & \hat{a}_3 & \hat{a}_2 & \hat{a}_1 & 1 && \\ \hat{a}_7 & \hat{a}_6 & \hat{a}_5 & \hat{a}_4 & \hat{a}_3 & \hat{a}_2 & \hat{a}_1 & 1 & \\ \hat{a}_8 & \hat{a}_7 & \hat{a}_6 & \hat{a}_5 & \hat{a}_4 & \hat{a}_3 & \hat{a}_2 & \hat{a}_1 & 1 \end{bmatrix} \begin{bmatrix} 1 && \\ 0 && \\ 0 && \\ & 1 & \\ & 0 & \\ & 0 & \\ && 1 \\ && 0 \\ && 0 \end{bmatrix} \begin{bmatrix} 1 && \\ \hat{a}_1^{(1)} & 1 & \\ \hat{a}_2^{(1)} & \hat{a}_1^{(1)} & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$$

is such that

$$\begin{bmatrix} 1 &&&&&&&& \\ a_1 & 1 &&&&&&& \\ a_2 & a_1 & 1 &&&&&& \\ a_3 & a_2 & a_1 & 1 &&&&& \\ a_4 & a_3 & a_2 & a_1 & 1 &&&& \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 &&& \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 && \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{bmatrix} = \begin{bmatrix} v_0 \\ 0 \\ 0 \\ v_1 \\ 0 \\ 0 \\ v_2 \\ 0 \\ 0 \end{bmatrix}.$$

How many arithmetic operations (a.o.) ?

Case $n = 3^k$

It is clear that the above procedure requires the computation of matrix $3^j \times 3^j$ l.t.T. by vector products, with $j = 1, \ldots, k$ (the vectors are sparse for $j = 2, \ldots, k$). So, if we assume such matrix by vector product computable in at most $c\,3^j j$ a.o., for some constant c, then the above procedure requires at most

$$c\sum_{j=1}^{k} 3^j j \le O(3^k k)$$

arithmetic operations.

For the general case $n = b^k$ see the Appendix.

→ PROBLEM

Answer to the quotation marks in the following equality:

$$L(\mathbf{a})\hat{\mathbf{a}} = \begin{bmatrix} 1 & & & & & & \\ a_1 & 1 & & & & & \\ a_2 & a_1 & 1 & & & & \\ a_3 & a_2 & a_1 & 1 & & & \\ a_4 & a_3 & a_2 & a_1 & 1 & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ ? \\ ? \\ ? \\ ? \\ ? \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ ? \\ 0 \\ ? \\ 0 \\ ? \\ \cdot \end{bmatrix} = E\mathbf{a}^{(1)}, \quad E = \begin{bmatrix} 1 & & & & \\ 0 & & & & \\ & 1 & & & \\ & 0 & & & \\ & & 1 & & \\ & & 0 & & \\ & & & 1 & \\ & & & 0 & \\ & & & & \cdot \end{bmatrix}.$$

- There is not a unique answer to the ?.
- There exists an answer that allows to obtain from the even and odd systems, a system solved by Bernoulli numbers where in the coefficient matrix null diagonals alternate with the non null ones. Find it ...
- *If there exists an answer such that the first 2^j entries of $\hat{\mathbf{a}}$ can be computed in at most $O(2^j j)$ arithmetic operations, for all $j \leq k$, then we have an algorithm of complexity $O(2^k k)$ for solving generic $2^k \times 2^k$ lower triangular Toeplitz systems.*

Here below is an answer such that *the first 2^j entries of $\hat{\mathbf{a}}$ can be computed with zero arithmetic operations*:

$$L(\mathbf{a})\hat{\mathbf{a}} = \begin{bmatrix} 1 & & & & & & \\ a_1 & 1 & & & & & \\ a_2 & a_1 & 1 & & & & \\ a_3 & a_2 & a_1 & 1 & & & \\ a_4 & a_3 & a_2 & a_1 & 1 & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ -a_1 \\ a_2 \\ -a_3 \\ a_4 \\ -a_5 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2a_2 - a_1^2 \\ 0 \\ 2a_4 - 2a_1a_3 + a_2^2 \\ 0 \\ 2a_6 - 2a_1a_5 + 2a_2a_4 - a_3^2 \\ 0 \\ 2a_8 - 2a_1a_7 + 2a_2a_6 - 2a_3a_5 + a_4^2 \\ 0 \\ \cdot \end{bmatrix} = E\mathbf{a}^{(1)},$$

$$L(\mathbf{a})\big(\mathbf{e}_1 + \sum_{i=1}^{+\infty}(-1)^i a_i \mathbf{e}_{i+1}\big) = \mathbf{e}_1 + \sum_{i=1}^{+\infty} \delta_{i=0 \bmod 2}\big(2a_i + \sum_{j=1}^{i-1}(-1)^j a_j a_{i-j}\big)\mathbf{e}_{i+1}.$$

→ PROBLEM

Answer to the quotation marks in the following equality:

$$
L(\mathbf{a})\hat{\mathbf{a}} = \begin{bmatrix} 1 & & & & & & \\ a_1 & 1 & & & & & \\ a_2 & a_1 & 1 & & & & \\ a_3 & a_2 & a_1 & 1 & & & \\ a_4 & a_3 & a_2 & a_1 & 1 & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ ? \\ ? \\ ? \\ ? \\ ? \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ ? \\ 0 \\ 0 \\ ? \\ 0 \\ 0 \\ ? \\ \cdot \end{bmatrix} = E\mathbf{a}^{(1)}, \quad E = \begin{bmatrix} 1 & & & \\ 0 & & & \\ 0 & & & \\ & 1 & & \\ & 0 & & \\ & 0 & & \\ & & 1 & \\ & & 0 & \\ & & 0 & \\ & & & \cdot \end{bmatrix}.
$$

- There is not a unique answer to the ?.
- There exists an answer that allows to obtain the Ramanujan system solved by Bernoulli numbers as a consequence of the odd (even) system. Find it ...
- *If there exists an answer such that the first 3^j entries of $\hat{\mathbf{a}}$ can be computed in at most $O(3^j j)$ arithmetic operations, for all $j \le k$, then we have an algorithm of complexity $O(3^k k)$ for solving generic $3^k \times 3^k$ lower triangular Toeplitz systems.*

Here below is an answer:

$$
L(\mathbf{a})\hat{\mathbf{a}} = \begin{bmatrix} 1 & & & & & & & & & & & & \\ a_1 & 1 & & & & & & & & & & & \\ a_2 & a_1 & 1 & & & & & & & & & & \\ a_3 & a_2 & a_1 & 1 & & & & & & & & & \\ a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & \\ a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & \\ a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & \\ a_{11} & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.
$$

$$
\begin{bmatrix} 1 \\ -a_1 \\ -a_2 + a_1^2 \\ 2a_3 - a_1a_2 \\ -a_4 - a_1a_3 + a_2^2 \\ -a_5 + 2a_1a_4 - a_2a_3 \\ 2a_6 - a_1a_5 - a_2a_4 + a_3^2 \\ -a_7 - a_1a_6 + 2a_2a_5 - a_3a_4 \\ -a_8 + 2a_1a_7 - a_2a_6 - a_3a_5 + a_4^2 \\ 2a_9 - a_1a_8 - a_2a_7 + 2a_3a_6 - a_4a_5 \\ -a_{10} - a_1a_9 + 2a_2a_8 - a_3a_7 - a_4a_6 + a_5^2 \\ -a_{11} + 2a_1a_{10} - a_2a_9 - a_3a_8 + 2a_4a_7 - a_5a_6 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a_1^{(1)} \\ 0 \\ 0 \\ a_2^{(1)} \\ 0 \\ 0 \\ a_3^{(1)} \\ 0 \\ 0 \\ \cdot \end{bmatrix} = E\mathbf{a}^{(1)},
$$

$a_1^{(1)} = 3a_3 - 3a_1a_2 + a_1^3$, $a_2^{(1)} = 3a_6 - 3a_1a_5 - 3a_2a_4 + 3a_3^2 - 3a_1a_2a_3 + 3a_1^2a_4 + a_2^3?$,
$a_3^{(1)} = 3a_9 - 3a_1a_8 - 3a_2a_7 + 6a_3a_6 - 3a_1a_2a_6 - 3a_1a_3a_5 - 3a_2a_3a_4 + 3a_1^2a_7 + 3a_1a_4^2 - 3a_4a_5 + 3a_5a_2^2 + a_3^3, \ \ldots$

$$
\hat{a}_i = -\sum_{r=0}^{\lfloor \frac{i-1}{2} \rfloor} a_r a_{i-r} + \delta_{i=0 \bmod 2} a_{\frac{i}{2}}^2 + 3 \begin{cases} \sum_{s \ge \frac{3-i}{6}}^{0} a_{\frac{i-3}{2}+3s} a_{\frac{i+3}{2}-3s} & i \text{ odd} \\ \sum_{s \ge \frac{6-i}{6}}^{0} a_{\frac{i-6}{2}+3s} a_{\frac{i+6}{2}-3s} & i \text{ even} \end{cases}, \quad i = 0,1,2,3,4,5,\ldots.
$$

Can such $\hat{a}_i$, $i = 0, 1, \ldots, 3^j - 1$, be computed in at most $O(3^j j)$ arithmetic operations ? →

$$
\left(
\left[\begin{array}{*{28}{c}}
1 & & & & & & & & & & & & & & & & & & & & & & & & & & & \\
 & 0 & & & & & & & & & & & & & & & & & & & & & & & & & & \\
 & a_1 & 0 & & & & & & & & & & & & & & & & & & & & & & & & & \\
 & & & 3 & & & & & & & & & & & & & & & & & & & & & & & & \\
 & & a_2 & & & 0 & & & & & & & & & & & & & & & & & & & & & & \\
 & & & & 3a_1 & 0 & & & & & & & & & & & & & & & & & & & & & & \\
 & & & a_3 & & & 3 & & & & & & & & & & & & & & & & & & & & & \\
 & & & & & 3a_2 & & 0 & & & & & & & & & & & & & & & & & & & & \\
 & & & & a_4 & & & 3a_1 & 0 & & & & & & & & & & & & & & & & & & & \\
 & & & & & & 3a_3 & & & 3 & & & & & & & & & & & & & & & & & & \\
 & & & & & a_5 & & & 3a_2 & & 0 & & & & & & & & & & & & & & & & & \\
 & & & & & & & 3a_4 & & & 3a_1 & 0 & & & & & & & & & & & & & & & & \\
 & & & & & & a_6 & & & 3a_3 & & & 3 & & & & & & & & & & & & & & & \\
 & & & & & & & & 3a_5 & & & 3a_2 & & 0 & & & & & & & & & & & & & & \\
 & & & & & & & a_7 & & & 3a_4 & & & 3a_1 & 0 & & & & & & & & & & & & & \\
 & & & & & & & & & 3a_6 & & & 3a_3 & & & 3 & & & & & & & & & & & & \\
 & & & & & & & & a_8 & & & 3a_5 & & & 3a_2 & & 0 & & & & & & & & & & & \\
 & & & & & & & & & & 3a_7 & & & 3a_4 & & & 3a_1 & 0 & & & & & & & & & & \\
 & & & & & & & & & a_9 & & & 3a_6 & & & 3a_3 & & & 3 & & & & & & & & & \\
 & & & & & & & & & & & 3a_8 & & & 3a_5 & & & 3a_2 & & 0 & & & & & & & & \\
 & & & & & & & & & & a_{10} & & & 3a_7 & & & 3a_4 & & & 3a_1 & 0 & & & & & & & \\
 & & & & & & & & & & & & 3a_9 & & & 3a_6 & & & 3a_3 & & & 3 & & & & & & \\
 & & & & & & & & & & & a_{11} & & & 3a_8 & & & 3a_5 & & & 3a_2 & & 0 & & & & & \\
 & & & & & & & & & & & & & 3a_{10} & & & 3a_7 & & & 3a_4 & & & 3a_1 & 0 & & & & \\
 & & & & & & & & & & & & a_{12} & & & 3a_9 & & & 3a_6 & & & 3a_3 & & & 3 & & & \\
 & & & & & & & & & & & & & & 3a_{11} & & & 3a_8 & & & 3a_5 & & & 3a_2 & & 0 & & \\
 & & & & & & & & & & & & & a_{13} & & & 3a_{10} & & & 3a_7 & & & 3a_4 & & & 3a_1 & 0 & \\
 & & & & & & & & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
\right.
$$

$$
\left.
-\left[\begin{array}{*{28}{c}}
 & & & & & & & & & & & & & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & & & & & & & & & & & & & \\
 & & a_1 & 1 & & & & & & & & & & & & & & & & & & & & & & & & \\
 & & & a_1 & 1 & & & & & & & & & & & & & & & & & & & & & & & \\
 & & & a_2 & a_1 & 1 & & & & & & & & & & & & & & & & & & & & & & \\
 & & & & a_2 & a_1 & 1 & & & & & & & & & & & & & & & & & & & & & \\
 & & & & a_3 & a_2 & a_1 & 1 & & & & & & & & & & & & & & & & & & & & \\
 & & & & & a_3 & a_2 & a_1 & 1 & & & & & & & & & & & & & & & & & & & \\
 & & & & & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & & & & & & & & & & & & \\
 & & & & & & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & & & & & & & & & & & \\
 & & & & & & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & & & & & & & & & & \\
 & & & & & & & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & & & & & & & & & \\
 & & & & & & & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & & & & & & & & \\
 & & & & & & & & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & & & & & & & \\
 & & & & & & & & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & & & & & & \\
 & & & & & & & & & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & & & & & \\
 & & & & & & & & & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & & & & \\
 & & & & & & & & & & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & & & \\
 & & & & & & & & & & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & & \\
 & & & & & & & & & & & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & & \\
 & & & & & & & & & & & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & & \\
 & & & & & & & & & & & & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & & \\
 & & & & & & & & & & & & a_{11} & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & & \\
 & & & & & & & & & & & & & a_{11} & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & & \\
 & & & & & & & & & & & & & a_{12} & a_{11} & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & & \\
 & & & & & & & & & & & & & & a_{12} & a_{11} & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \\
 & & & & & & & & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
\right)
\left[\begin{array}{c}
1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \\ a_{16} \\ a_{17} \\ a_{18} \\ a_{19} \\ a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{24} \\ a_{25} \\ a_{26} \\ \cdot
\end{array}\right]
$$

$\rightarrow$ Can the above $3^j \times 3^j$ (27×27) matrix by vector product be computed in at most $O(3^j j)$ arithmetic operations ?

If yes, then we would have a method which solves $3^k \times 3^k$ lower triangular Toeplitz linear systems in at most $O(3^k k)$ arithmetic operations.

If no, then look for another solution $\hat{\mathbf{a}}$ of the system $L(\mathbf{a})\hat{\mathbf{a}} = [1\,0\,0 \bullet 0\,0 \cdot]^T$ such that $\{\hat{\mathbf{a}}\}_{3^j}$ is computable from $\{\mathbf{a}\}_{3^j}$ in at most $O(3^j j)$ arithmetic operations ...

THE END

APPENDIX Introduce low complexity l.t.T. linear system solvers:

$$L(\mathbf{a}) = \begin{bmatrix} 1 & & & \\ a_1 & 1 & & \\ a_2 & a_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \mathbf{a}^{(0)} := \mathbf{a}$$

Find $\hat{\mathbf{a}}^{(0)}$, $\mathbf{a}^{(1)}$ such that

$$L(\mathbf{a}^{(0)})\hat{\mathbf{a}}^{(0)} = E\mathbf{a}^{(1)} = \begin{bmatrix} 1 \\ \mathbf{0} \\ \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b-1}, \ \text{ so that}$$

$$L(\mathbf{a}^{(0)})L(\hat{\mathbf{a}}^{(0)}) = L(E\mathbf{a}^{(1)}).$$

Find $\hat{\mathbf{a}}^{(1)}$, $\mathbf{a}^{(2)}$ such that

$$L(\mathbf{a}^{(1)})\hat{\mathbf{a}}^{(1)} = E\mathbf{a}^{(2)} = \begin{bmatrix} 1 \\ \mathbf{0} \\ \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b-1}, \quad \text{so that}$$

$$L(E\mathbf{a}^{(1)})E\hat{\mathbf{a}}^{(1)} = E^2\mathbf{a}^{(2)} = \begin{bmatrix} 1 \\ \mathbf{0} \\ \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b^2-1},$$

$$\underline{L(E\mathbf{a}^{(1)})}L(E\hat{\mathbf{a}}^{(1)}) = L(E^2\mathbf{a}^{(2)}).$$

(use Lemma). Find $\hat{\mathbf{a}}^{(2)}$, $\mathbf{a}^{(3)}$ such that

$$L(\mathbf{a}^{(2)})\hat{\mathbf{a}}^{(2)} = E\mathbf{a}^{(3)} = \begin{bmatrix} 1 \\ \mathbf{0} \\ \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b-1}, \quad \text{so that}$$

$$L(E^2\mathbf{a}^{(2)})E^2\hat{\mathbf{a}}^{(2)} = E^3\mathbf{a}^{(3)} = \begin{bmatrix} 1 \\ \mathbf{0} \\ \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b^3-1},$$

$$\underline{L(E^2\mathbf{a}^{(2)})}L(E^2\hat{\mathbf{a}}^{(2)}) = L(E^3\mathbf{a}^{(3)}).$$

... Find $\hat{\mathbf{a}}^{(k-2)}$, $\mathbf{a}^{(k-1)}$ such that

$$L(\mathbf{a}^{(k-2)})\hat{\mathbf{a}}^{(k-2)} = E\mathbf{a}^{(k-1)} = \begin{bmatrix} 1 \\ \mathbf{0} \\ \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b-1}, \quad \text{so that}$$

$$L(E^{k-2}\mathbf{a}^{(k-2)})E^{k-2}\hat{\mathbf{a}}^{(k-2)} = E^{k-1}\mathbf{a}^{(k-1)} = \begin{bmatrix} 1 \\ \mathbf{0} \\ \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b^{k-1}-1},$$

$$\underline{L(E^{k-2}\mathbf{a}^{(k-2)})}L(E^{k-2}\hat{\mathbf{a}}^{(k-2)}) = L(E^{k-1}\mathbf{a}^{(k-1)}).$$

Find $\hat{\mathbf{a}}^{(k-1)}$, $\mathbf{a}^{(k)}$ such that

$$L(\mathbf{a}^{(k-1)})\hat{\mathbf{a}}^{(k-1)} = E\mathbf{a}^{(k)} = \begin{bmatrix} 1 \\ \mathbf{0} \\ \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b-1}, \quad \text{so that}$$

$$L(E^{k-1}\mathbf{a}^{(k-1)})E^{k-1}\hat{\mathbf{a}}^{(k-1)} = E^{k}\mathbf{a}^{(k)} = \begin{bmatrix} 1 \\ \mathbf{0} \\ \cdot \end{bmatrix}, \ \mathbf{0} = \mathbf{0}_{b^{k}-1},$$

$$\underline{L(E^{k-1}\mathbf{a}^{(k-1)})}L(E^{k-1}\hat{\mathbf{a}}^{(k-1)}) = L(E^{k}\mathbf{a}^{(k)}).$$

Then

$$\begin{bmatrix} \overbrace{I}^{b^k} & O \cdot \\ \begin{bmatrix} a_1^{(k)} & \\ \cdot & \cdot \end{bmatrix} & \cdot \cdot \end{bmatrix} = L(E^k\mathbf{a}^{(k)}) = L(E^{k-1}\hat{\mathbf{a}}^{(k-1)})L(E^{k-2}\hat{\mathbf{a}}^{(k-2)}) \cdots L(E\hat{\mathbf{a}}^{(1)})L(\hat{\mathbf{a}}^{(0)})L(\mathbf{a}^{(0)}).$$

This implies that

$$L(\mathbf{a}^{(0)})\mathbf{z} = \mathbf{c} \;\text{ iff }\; L(E^k\mathbf{a}^{(k)})\mathbf{z} = L(\hat{\mathbf{a}}^{(0)})L(E\hat{\mathbf{a}}^{(1)}) \cdots L(E^{k-2}\hat{\mathbf{a}}^{(k-2)})L(E^{k-1}\hat{\mathbf{a}}^{(k-1)})\mathbf{c}.$$

Moreover, if

$$\mathbf{c} = E^{k-1}\mathbf{v} = \begin{bmatrix} v_0 \\ \mathbf{0} \\ v_1 \\ \mathbf{0} \\ v_2 \\ \mathbf{0} \\ \cdot \end{bmatrix}, \quad \mathbf{0} = \mathbf{0}_{b^{k-1}-1},$$

where $\mathbf{v} = (v_i)_{i=0}^{+\infty}$ is any vector (for example $\mathbf{v} = \mathbf{e}_1$), then by using the Lemma, we obtain the following result:

$$L(\mathbf{a}^{(0)})\mathbf{z} = \mathbf{c} \;\text{ iff}$$
$$\begin{bmatrix} I_{b^k} & O \cdot \\ \begin{bmatrix} a_1^{(k)} & \\ \cdot & \cdot \end{bmatrix} & \cdot \cdot \end{bmatrix} \mathbf{z} = L(E^k\mathbf{a}^{(k)})\mathbf{z} =$$
$$L(\hat{\mathbf{a}}^{(0)})EL(\hat{\mathbf{a}}^{(1)})E \cdots EL(\hat{\mathbf{a}}^{(k-2)})EL(\hat{\mathbf{a}}^{(k-1)})\mathbf{v}.$$

In other words, the vector $\{\mathbf{z}\}_n$, $n = b^k$, such that

$$\{L(\mathbf{a})\}_n \{\mathbf{z}\}_n = \begin{bmatrix} 1 & & & \\ a_1 & 1 & & \\ \cdot & \cdot & \cdot & \\ a_{b^k-1} & \cdot & a_1 & 1 \end{bmatrix} \{\mathbf{z}\}_n = \begin{bmatrix} v_0 \\ \mathbf{0} \\ v_1 \\ \mathbf{0} \\ \cdot \\ v_{b-1} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{0} = \mathbf{0}_{b^{k-1}-1}$$

(for example $\{L(\mathbf{a})\}_n^{-1}\{\mathbf{e}_1\}_n$, $v_0 = 1$, $v_i = 0$ $i \geq 1$), can be represented as follows

$$\begin{aligned} \{\mathbf{z}\}_n &= \{L(\hat{\mathbf{a}}^{(0)})\}_n \{E\}_n \{L(\hat{\mathbf{a}}^{(1)})\}_n \{E\}_n \cdots \{L(\hat{\mathbf{a}}^{(k-2)})\}_n \{E\}_n \{L(\hat{\mathbf{a}}^{(k-1)})\}_n \{\mathbf{v}\}_n \\ &= \{L(\hat{\mathbf{a}}^{(0)})\}_n \{E\}_{n,\frac{n}{b}} \{L(\hat{\mathbf{a}}^{(1)})\}_{\frac{n}{b}} \{E\}_{\frac{n}{b},\frac{n}{b^2}} \cdots \{L(\hat{\mathbf{a}}^{(k-2)})\}_{\frac{n}{b^{k-2}}} \{E\}_{\frac{n}{b^{k-2}},\frac{n}{b^{k-1}}} \{L(\hat{\mathbf{a}}^{(k-1)})\}_{\frac{n}{b^{k-1}}} \{\mathbf{v}\}_b \end{aligned}$$

FIRST: Compute the first n entries of $\hat{\mathbf{a}}^{(0)}$ and the first $\frac{n}{b}$ entries of $\mathbf{a}^{(1)}$ (cost φ_{b^k}); compute the first $\frac{n}{b}$ entries of $\hat{\mathbf{a}}^{(1)}$ and the first $\frac{n}{b^2}$ entries of $\mathbf{a}^{(2)}$ (cost $\varphi_{b^{k-1}}$); ... compute the first $\frac{n}{b^{k-2}}$ entries of $\hat{\mathbf{a}}^{(k-2)}$ and the first $\frac{n}{b^{k-1}}$ entries of $\mathbf{a}^{(k-1)}$ (cost φ_{b^2}); compute the first $\frac{n}{b^{k-1}}$ entries of $\hat{\mathbf{a}}^{(k-1)}$ (cost φ_b). Total cost of this FIRST operation: $\sum_{j=0}^{k-1} \varphi_{\frac{n}{b^j}}$.

SECOND: To such cost add $\sum_{j=1}^{k} \text{cost}\big((b^j \times b^j\, l.t.T) \cdot\; (b^j\, vector)\big)$ (the vector is sparse if $j = 2, \ldots, k$; the cost for $j = 1$ is zero if $\mathbf{v} = \mathbf{e}_1$). See also the next page.

Amount of operations.
In the following $n = b^k$ and $\mathbf{0} = \mathbf{0}_{b-1}$:

FIRST: For $j = 0, \ldots, k-1$ compute, by performing $\varphi_{\frac{n}{b^j}}$ arithmetic operations, the vectors $I^1_{\frac{n}{b^j}} \hat{\mathbf{a}}^{(j)}$ and $I^1_{\frac{n}{b^{j+1}}} \mathbf{a}^{(j+1)}$, i.e. scalars $\hat{a}_i^{(j)}$ and $a_i^{(j+1)}$ such that

$$\underbrace{\begin{bmatrix} 1 & & & & \\ a_1^{(j)} & 1 & & & \\ a_2^{(j)} & a_1^{(j)} & 1 & & \\ \cdot & & \cdot & \cdot & \\ a^{(j)}_{\frac{n}{b^j}-1} & \cdot & a_2^{(j)} & a_1^{(j)} & 1 \end{bmatrix}}_{\frac{n}{b^j} \times \frac{n}{b^j}} \begin{bmatrix} 1 \\ \hat{a}_1^{(j)} \\ \hat{a}_2^{(j)} \\ \cdot \\ \hat{a}^{(j)}_{\frac{n}{b^j}-1} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{0} \\ a_1^{(j+1)} \\ \mathbf{0} \\ \cdot \\ a^{(j+1)}_{\frac{n}{b^{j+1}}-1} \\ \mathbf{0} \end{bmatrix}, \quad j = 0, \ldots, k-1$$

(note that there is no $a_i^{(k)}$ to be computed).
Case $b = 2$. In this case, since $\hat{a}_i^{(j)} = (-1)^i a_i^{(j)}$, only $\frac{n}{b^j} \times \frac{n}{b^j}$ l.t.T. by vector products, $j = 0, \ldots, k-2$, need to be computed (the $a_i^{(j+1)}$ are the $\frac{n}{b^{j+1}}$ nonzero entries of the resulting vectors).

SECOND: Compute the $b \times b$ l.t.T. by vector product $\{L(\hat{\mathbf{a}}^{(k-1)})\}_{\frac{n}{b^{k-1}}} \begin{bmatrix} v_0 \\ \cdot \\ v_{b-1} \end{bmatrix}$, and $\frac{n}{b^j} \times \frac{n}{b^j}$ l.t.T. by vector products of type

$$\underbrace{\{L(\hat{\mathbf{a}}^{(j)})\}_{\frac{n}{b^j}}}_{\frac{n}{b^j} \times \frac{n}{b^j}} \begin{bmatrix} 1 \\ \mathbf{0} \\ \bullet \\ \mathbf{0} \\ \cdot \end{bmatrix}, \quad j = k-2, \ldots, 1, 0.$$

COMMENTS
So, in case $b = 2$, we have to perform $2^j \times 2^j$ l.t.T. by vector products, for $j = 1, \ldots, k$, twice. If we assume the cost of a $2^j \times 2^j$ l.t.T. by vector product bounded by $c2^j j$ (c constant), then the total cost of the above operations is smaller than $O(2^k k) = O(n \log_2 n)$. As a consequence we have obtained, in particular, a l.t.T. linear system solver of complexity $O(n \log_2 n)$

Analogously, for $b = 3$, if we assume both φ_{3^j} and the cost of a $3^j \times 3^j$ l.t.T. by vector product bounded by $c3^j j$, then the total cost of the above operations is smaller than $O(3^k k) = O(n \log_3 n)$. ...
But is φ_{3^j} bounded by $c3^j j$? ...

For me:
http://www.imsc.res.in/~rao/ramanujan/collectedindex.html
http://mathworld.wolfram.com/BernoulliNumber.html
http://numbers.computation.free.fr/Constants/Miscellaneous/bernoulli.html