Soluzioni esercizi foglio 5
(lezione del 14/01/2019)

Calcolare i seguenti integrali:

1) $$\int_1^e \frac{\operatorname{arctg}(\lg x)}{x}\,dx$$

$$y=\lg x \Rightarrow x=e^y \qquad dx=e^y\,dy$$

$x=1 \Rightarrow y=0$
$x=e \Rightarrow y=1$

$$\int_0^1 \frac{\operatorname{arctg} y}{\cancel{e^y}}\cancel{e^y}\,dy \underset{p.p.}{=} y\operatorname{arctg} y\Big|_0^1 - \int_0^1 \frac{2}{2}\, y\,\frac{1}{1+y^2}\,dy$$

$$=\operatorname{arctg} 1 - \frac{1}{2}\lg(1+y^2)\Big|_0^1$$

$$=\frac{\pi}{4}-\frac{1}{2}\lg 2$$

2) $$\int_e^{e^{\sqrt{3}}} \frac{\operatorname{arctg}(\lg x)}{x(\lg x)^2}\,dx$$

$y = \lg x \Rightarrow x = e^y \quad dx = e^y dy$

$x = e \Rightarrow y = 1$

$x = e^{\sqrt{3}} \Rightarrow y = \sqrt{3}$

$$= \int_1^{\sqrt{3}} \frac{\operatorname{arctg}(y)}{\cancel{e^y}\, y^2} \cancel{e^y}\, dy \underset{p.p.}{=}$$

$$-\frac{1}{y}\operatorname{arctg} y \Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{1}{y}\frac{1}{(1+y^2)}dy$$

$$= -\frac{1}{\sqrt{3}}\operatorname{arctg}\sqrt{3} + \operatorname{arctg} 1 + \int_1^{\sqrt{3}}\left(\frac{1}{y} - \frac{y}{y^2+1}\right)dy \quad (*)$$

$$\frac{1}{y(1+y^2)} = \frac{A}{y} + \frac{By+C}{1+y^2} = \frac{1}{y} - \frac{y}{1+y^2}$$

$\Rightarrow \quad A(1+y^2) + By^2 + Cy = 1$

$\Rightarrow A + B = 0 \quad \Rightarrow B = -1$

$C = 0$

$A = 1$

$$(*) = -\frac{1}{\sqrt{3}}\cdot\frac{\pi}{3} + \frac{\pi}{4} + \lg\sqrt{3} - \frac{1}{2}\lg(1+y^2)\Big|_1^{\sqrt{3}}$$

3) $$\int_{2\lg 2}^{2\lg 3} \frac{e^x}{1-\sqrt{e^x}}\,dx \quad (*)$$

$$\sqrt{e^x} = y \Rightarrow e^x = y^2 \Rightarrow x = \lg(y^2)$$

$$dx = \frac{2y\,dy}{y^2} = \frac{2}{y}\,dy$$

Se calcoliamo prima una primitiva

$$\int \frac{e^x}{1-(e^x)^{1/2}}\,dx \Rightarrow \int \frac{y^2}{1-y}\,\frac{2}{\cancel{y}}\,dy$$

$$= 2\int \frac{y}{1-y}\,dy = -2\int \frac{y-1+1}{y-1}\,dy$$

$$= -2y - 2\lg|y-1|$$

$$= -2\sqrt{e^x} - 2\lg|\sqrt{e^x}-1|$$

$$\underset{\smile}{y}$$

$$(*) = -2\sqrt{e^x} - 2\lg|\sqrt{e^x}-1| \Big|_{2\lg 2 = \lg 4}^{2\lg 3 = \lg 9}$$

$$= -2\left[3 + \lg 2 - 2 \qquad\right]$$

4) $\int_0^{\lg 2} e^{2x}\lg(1+e^x)\,dx$

$e^x = y \Rightarrow x = \lg y \qquad dx = \frac{1}{y}\,dy$

$x=0 \Rightarrow y = 1$

$x = \lg 2 \Rightarrow y = 2$

$$\int_1^2 y^2 \lg(1+y)\,\frac{dy}{y} = \int_1^2 y\lg(1+y)\,dy$$

$$\underset{pp.}{=} \frac{y^2}{2}\lg(1+y)\Big|_1^2 - \int_1^2 \frac{y^2}{2}\,\frac{1}{1+y}\,dy = (*)$$

$$\frac{y^2}{1+y} = \frac{y^2-1+1}{1+y} = \frac{(y-1)(y+1)+1}{1+y}$$

$$= y-1+\frac{1}{y+1}$$

$$\Rightarrow (*) = 2\lg 3 - \frac{1}{2}\lg 2 - \frac{1}{2}\int_1^2 (y-1)+\frac{1}{y+1}\,dy$$

$$= 2\lg 3 - \frac{1}{2}\lg 2 - \frac{1}{2}\left(\frac{y^2}{2} - y + \lg(1+y)\right)\Big|_1^2$$

$$= 2\lg 3 - \cancel{\frac{1}{2}\lg 2} - \frac{1}{2}\left(\lg 3 - \frac{1}{2} + 1 - \cancel{\lg(2)}\right) = \frac{3}{2}\lg 3 - \frac{1}{4}$$

5) $\int_0^{2/3} \operatorname{arctg}\sqrt{1-x}\, dx$

$y = \sqrt{1-x} \quad \Rightarrow 1-x = y^2 \Rightarrow x = 1-y^2$

$dx = -2y\, dy$

$x=0 \Rightarrow y=1 \quad ; \quad x=\frac{2}{3} \Rightarrow y = \frac{1}{\sqrt{3}}$

$$\int_1^{1/\sqrt{3}} -2y \operatorname{arctg} y\, dy = 2\int_{1/\sqrt{3}}^{1} y \operatorname{arctg} y\, dy$$

$$= \cancel{2}\,\frac{y^2}{\cancel{2}} \operatorname{arctg} y \Big|_{\frac{1}{\sqrt{3}}}^{1} - \int_{\frac{1}{\sqrt{3}}}^{1} \frac{y^2+1-1}{1+y^2}\, dy$$

$$= \frac{\pi}{4} - \frac{1}{3}\operatorname{arctg}\frac{1}{\sqrt{3}} - \left(1 - \frac{1}{\sqrt{3}} - \operatorname{arctg} y \Big|_{\frac{1}{\sqrt{3}}}^{1}\right)$$

$$\frac{\frac{\pi}{\sqrt{3}}}{3}$$

$$= \frac{\pi}{4} - \frac{1}{3}\frac{\pi}{6} - 1 + \frac{1}{\sqrt{3}} + \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{2} - \frac{\cancel{4}^{2}}{3}\frac{\pi}{\cancel{6}\,3} +$$

$$-1 + \frac{1}{\sqrt{3}}$$

6) $$\int \frac{5x^3-2}{x^2-x-2}\,dx \quad (*)$$

Va fatta prima la divisione tra i polinomi

$$\begin{array}{l|l}
5x^3 \qquad\qquad -2 & x^2-x-2 \\ \hline
-5x^3+5x^2+10x & 5x+5 \\
// \quad 5x^2+10x-2 & \\
\quad -5x^2+5x+10 & \\ \hline
// \quad 15x+8 &
\end{array}$$

$$\Rightarrow (*) = \int 5x+5\,dx + \int \frac{15x+8}{x^2-x-2}\,dx$$

$$= 5\frac{x^2}{2} + 5x + \frac{15}{2}\int \frac{2x-1+1+\frac{16}{15}}{x^2-x-2}\,dx$$

$$= \frac{5}{2}x^2 + 5x + \frac{15}{2}\lg|x^2-x-2| + \frac{\cancel{15}}{2}\cdot\frac{31}{\cancel{15}}\int\frac{dx}{x^2-x-2}$$

resta da calcolare

$$\int \frac{1}{x^2-x-2}\,dx = \int \frac{dx}{(x+1)(x-2)}$$

$x^2 - x - 2 = 0$

$$x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} \begin{cases} 2 \\ -1 \end{cases}$$

$$\frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} = \frac{1/3}{x-2} - \frac{1/3}{x+1}$$

$A + B = 0 \Rightarrow B = -A \qquad \Rightarrow B = 1/3$

$B - 2A = 1 \Rightarrow A = -1/3$

$$\int \frac{1}{x^2-x-2}\,dx = \frac{1}{3} \lg|x-2| - \frac{1}{3} \lg|x+1|$$

$$\Rightarrow (*) = \frac{5}{2}x^2 + 5x + \frac{15}{2} \lg|x^2-x-2|$$

$$+ \frac{31}{2} \cdot \frac{1}{3} \lg\frac{|x-2|}{|x+1|} + C$$

$$= \frac{5}{2}x^2 + 5x + \frac{15}{2} \lg|x+1| + \frac{15}{2} \lg|x-2|$$

$$+ \frac{31}{6} \lg|x-2| - \frac{31}{6} \lg|x+1| + C$$

$$= \frac{5}{2}x^2 + 5x + \left(\frac{15}{2} - \frac{31}{6}\right) \lg|x+1| + \left(\frac{15}{2} + \frac{31}{6}\right) \lg|x-2| + C$$

$$= \frac{\cancel{14}}{\cancel{6}\,3} \qquad \frac{45+31}{6} = \frac{\cancel{76}\,43}{\cancel{6}\,3}$$

7) $$\int_0^{1/3} \frac{\operatorname{arctg}\sqrt{x}}{\left(\frac{1}{9}+x\right)^2}\,dx$$

$\sqrt{x}=t \Rightarrow x=t^2 \quad dx=2t\,dt$

$x=0 \Rightarrow t=0$

$x=\frac{1}{3} \Rightarrow t=1/\sqrt{3}$

$$\int_0^{1/\sqrt{3}} \frac{\operatorname{arctg} t}{\left(\frac{1}{9}+t^2\right)^2}(2t)\,dt = (*)$$

Me $$\frac{2t}{\left(\frac{1}{9}+t^2\right)^2} = \frac{d}{dt}\left(-\frac{1}{\frac{1}{9}+t^2}\right)$$

$$(*) \underset{P.P.}{=} -\frac{1}{\frac{1}{9}+t^2}\operatorname{arctg} t\,\Big|_0^{1/\sqrt{3}} + \int_0^{1/\sqrt{3}} \frac{1}{1+t^2}\cdot\frac{1}{t^2+\frac{1}{9}}\,dt$$

$$= -\frac{1}{\frac{1}{9}+\frac{1}{3}}\operatorname{arctg}\frac{\sqrt{3}}{3} + \int_0^{1/\sqrt{3}} \frac{\cancel{At+B}=-9/8}{1+t^2} + \frac{\cancel{Ct+D}=9/8}{t^2+\frac{1}{9}}\,dt \qquad \frac{1}{9}\left(1+(3t)^2\right)$$

$$= -\frac{9}{4}\frac{\pi}{6} - \frac{9}{8}\operatorname{arctg}\frac{\sqrt{3}}{3} + \frac{9}{8}\cdot\frac{9}{3}\operatorname{arctg}(3t)\Big|_0^{1/\sqrt{3}}$$

dove A, B, C, D sono dati dalla scomposizione in fratti semplici

$$\frac{1}{(t^2+1)\left(t^2+\frac{1}{9}\right)} = \frac{At+B}{1+t^2} + \frac{Ct+D}{t^2+\frac{1}{9}}$$

$$\Rightarrow \begin{cases} A+C=0 \\ B+D=0 & B=-D \quad \Rightarrow D=9/8 \\ \frac{A}{9}+C=0 & \Rightarrow A=C=0 \\ \frac{B}{9}+D=1 & B\left(\frac{1}{9}-1\right)=1 \end{cases}$$

$$-\frac{8}{9}B=1 \quad \Rightarrow B=-\frac{9}{8}$$

8) $$\int \frac{1}{\underbrace{x^2+x+1}_{x^2+x+\frac{1}{4}+\frac{3}{4}}}\,dx$$

$$= \int \frac{1}{\underbrace{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}_{\frac{3}{4}\left[1+\left(\frac{x+1/2}{\sqrt{3}/2}\right)^2\right]}}\,dx = \frac{4}{3}\int \frac{dx}{1+\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2}$$

$$= \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + C$$

9) $$\int \frac{1}{(x-1)^2}\,dx = -\frac{1}{x-1} + C$$

(immediato)

10) $\int \frac{4x^2+x+1}{x^3-1}\,dx$

$x^3-1 = (x-1)(x^2+x+1)$

$$\boxed{a^3-b^3 = (a-b)(a^2+ab+b^2)}$$

$\Rightarrow \frac{4x^2+x+1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

$\begin{cases} A+B=4 \\ A-B+C=1 \\ A-C=1 \end{cases}$ $\Rightarrow 3A=6 \Rightarrow A=2$

$A-B+A-1=1 \Rightarrow B=2A-2=2$

$C=A-1=1$

$\Rightarrow \int \frac{2\,dx}{(x-1)} + \left(\frac{2x+1}{x^2+x+1}\right)dx$

$= 2\lg|x-1| + \lg(x^2+x+1)$

11) e 12) Cfr. lezione del

19/12/2018

13)
$$I_\alpha = \int_1^{+\infty} \underbrace{\frac{1}{(1+x)^2(x+2)^\alpha}}_{f(x)}\, dx \qquad \alpha \in \mathbb{R}$$

poiché i punti $x=-2$ e $x=-1$
non sono nell'intervallo $(1,+\infty)$

$\Rightarrow$ l'unica condizione da verificare
è il comportamento di $f(x)$ per
$x \to +\infty$

$$f(x) = O\left(\frac{1}{x^{2+\alpha}}\right) \qquad x \to +\infty$$

$\Rightarrow$ l'integrale I_α è finito $\Leftrightarrow$ $2+\alpha > 1$
$\alpha > -1$

per $\alpha=1$ perciò I_1 converge

$$I_1 = \int_1^{+\infty} \frac{1}{(1+x)^2(x+2)} \, dx$$

$$= \lim_{\omega \to +\infty} \int_1^{\omega} \frac{1}{(x+1)^2(x+2)} \, dx$$

$$\frac{1}{(x+2)(x+1)^2} = \frac{A=1}{x+2} + \frac{B=-1}{x+1} + \frac{C=+1}{(x+1)^2}$$

$$A = \lim_{x \to -2} \frac{1}{(x+1)^2} = 1$$

$$C = \lim_{x \to -1} \frac{1}{x+2} = +1$$

$$B = \lim_{x \to -1} \frac{d}{dx}\left(\frac{1}{x+2}\right) =$$

$$= \lim_{x \to -1} \left(-\frac{1}{(x+2)^2}\Big|_{-1} \right) = -1$$

$$I_1 = \lim_{\omega \to +\infty} \left[+ \lg(x+2) - \lg|x+1| - \frac{1}{x+1} \right]_1^{\omega}$$

$$= \lim_{\omega\to+\infty} \frac{\lg(\omega+2)}{\omega+1} - \frac{1}{\omega+1} - \lg\frac{3}{2}+\frac{1}{2}$$

$$= \frac{1}{2} - \lg\frac{3}{2}$$

14) Studiare la convergenza di

$$I_\alpha = \int_\alpha^{+\infty} \underbrace{\frac{e^{-x}}{(x-3)^2\sqrt{x-\frac{1}{2}}}}_{f(x)}\,dx \qquad \alpha\in\mathbb{R}$$

$f(x) \geq 0$

Per $x\to+\infty$ $\quad f(x) = o\left(\frac{1}{x^{2+1/2}}\right)$

$\Rightarrow$ l'integrale all'∞ converge

l'unico problema c'è per $x\to 3$

dove f ha un int. non conv.

$\Rightarrow$ è suff. che $\alpha>3$ perché $I_\alpha<+\infty$