Versione D

1) Calcolare al variare di $\alpha \in \mathbb{R}$ il

$$\lim_{x \to 0^+} \frac{x(\arctg x)\lg(1+x^\alpha)}{1-e^{x+x^2}+x+\frac{1}{2}x\,\mathrm{sen}\,x+x^2}$$

NUM: se $\alpha>0$ $x^\alpha \to 0$ per $x \to 0^+$

$\Rightarrow \lg(1+x^\alpha) = x^\alpha(1+o(1))$

Quindi $x \arctg x \lg(1+x^\alpha) = x \cdot x(1+o(1)) \cdot x^\alpha = x^{\alpha+2}(1+o(1))$

oss se $\alpha=0$ $\lg(1+x^\alpha) = \lg 2$

$\Rightarrow$ NUM $\sim x^2(\lg 2 + o(1))$

se $\alpha<0$ $x^\alpha \to +\infty$ per $x \to 0$

$\Rightarrow \lg(1+x^\alpha) = (\lg x^\alpha)(1+o(1)) = \alpha \lg x(1+o(1))$

$\Rightarrow$ NUM $= x^2 \lg(x)(\alpha+o(1))$

DENOMINATORE

$1-e^{x+x^2} = \cancel{1}-\cancel{1}-(x+x^2)-\frac{1}{2}(x+x^2)^2-\frac{1}{3!}(x+x^2)^3+o(x^3)$

$= -x-x^2-\frac{1}{2}x^2-x^3-\frac{1}{6}x^3+o(x^3)$

$\frac{1}{2}x\,\mathrm{sen}\,x = \frac{1}{2}x\left(x-\frac{1}{6}x^3+o(x^4)\right) = \frac{1}{2}x^2-\frac{1}{12}x^4+o(x^5)$

Quindi DEN $= \cancel{-x}\cancel{-x^2}\cancel{+x}\cancel{+x^2}\cancel{-\frac{1}{2}x^2}-\frac{1}{6}x^3\cancel{+\frac{1}{2}x^2}-x^3+o(x^3)$

$= -\frac{7}{6}x^3(1+o(1))$ $=$

Quindi, se $\alpha>0$ $\frac{x^{\alpha+2}(1+o(1))}{-\frac{7}{6}x^3} \to \begin{cases} 0 & \text{se } \alpha+2>3 \quad \alpha>1 \\ -6/7 & \text{se } \alpha+2=3 \quad \alpha=1 \\ -\infty & \text{se } \alpha\in(0,1) \end{cases}$

se $\alpha=0$ $\frac{x^2\lg 2}{-\frac{7}{6}x^3} \to -\infty$

se $\alpha<0$ $\frac{x^2 \lg x}{-7/6\, x^3} \sim \frac{\lg x}{-x} \to +\infty$

2) Discutere, al variare di $\alpha \in \mathbb{R}$ la convergenza del seguente integrale improprio e calcolarlo, se esiste finito, per $\alpha=0$

$$\int_2^4 \frac{\sqrt{x^2-4}}{3x(x-2)^\alpha}\,dx$$

In $[2,4]$ la funzione potrebbe non essere limitata per $x \to 2^+$ (dipende da $\alpha$).

Se $f_\alpha(x) = \frac{\sqrt{x^2-4}}{3x(x-2)^\alpha}$

per $x \to 2^+$ $\quad f_\alpha(x) = \frac{(x-2)^{1/2}(x+2)^{1/2}}{3x\,(x-2)^\alpha} = \frac{(x+2)^{1/2}}{3x\,(x-2)^{\alpha-1/2}}$

l'integrale converge se $\alpha - \frac{1}{2} < 1$ cioè $\alpha < \frac{3}{2}$

$\Rightarrow$ per $\alpha = 0$ l'integrale converge (anzi è un int. di Riemann)

$$\int_2^4 \frac{\sqrt{x^2-4}}{3x}\,dx$$

$= \quad t = \sqrt{x^2-4} \Rightarrow x^2-4 = t^2 \Rightarrow x^2 = t^2+4 \Rightarrow x = \sqrt{t^2+4}$

$dx = \frac{1}{\cancel{2}\sqrt{t^2+4}}\,\cancel{2}t\,dt$

$x=2 \Rightarrow t=0$

$x=4 \Rightarrow t=\sqrt{12} = 2\sqrt{3}$

$$\int_0^{2\sqrt{3}} \frac{t}{3\sqrt{t^2+4}}\,\frac{t}{\sqrt{t^2+4}}\,dt = \frac{1}{3}\int_0^{2\sqrt{3}} \frac{t^2+4-4}{t^2+4}\,dt$$

$$= \frac{1}{3}\,2\sqrt{3} - \frac{4}{3}\int_0^{2\sqrt{3}} \frac{1}{t^2+4}\,dt = \frac{2}{\sqrt{3}} - \frac{2}{3}\,\mathrm{arctg}\left(\frac{t}{2}\right)\Big|_0^{2\sqrt{3}}$$

$$\int \frac{1}{t^2+4}\,dt = \int \frac{1}{4\left(1+\left(\frac{t}{2}\right)^2\right)} \underset{y=\frac{t}{2}\quad dy=\frac{1}{2}dt}{=} \frac{1}{2}\,\mathrm{arctg}\left(\frac{t}{2}\right)$$

Quindi

$$I = \frac{2}{\sqrt{3}} - \frac{2}{3}\,\mathrm{arctg}(\sqrt{3}) = \frac{2}{\sqrt{3}} - \frac{2}{3}\,\frac{\pi}{3}$$

3) Dopo aver dato la def di ordine di infinitesimo per una successione $\{a_n\}$

a) det. se $a_n$ è infinitesima ed in caso, det. se esiste l'ordine di infinitesimo

$$a_n = \frac{\lg(1+n^{1/5}) - \frac{1}{5}\lg n}{\sqrt{n^3+3n^2}\,(\sqrt{n+7}-\sqrt{n})}$$

b) determinare, al variare di $\beta \in \mathbb{R}$ il carattere di $\sum_{n=1}^{\infty} n^\beta a_n$

Ordine di infinitesimo (def) vedere testi

a)

$$a_n = \frac{\lg\left(\frac{1+n^{1/5}}{n^{1/5}}\right)}{n^{3/2}(1+o(1))\,\sqrt{n}\left(\left(1+\frac{7}{n}\right)^{1/2}-1\right)}$$

$$= \frac{\lg\left(\frac{1}{n^{1/5}}+1\right)}{n^2\left(\cancel{1}+\frac{7}{2n}-\cancel{1}+o\left(\frac{1}{n}\right)\right)} = \frac{\frac{1}{n^{1/5}}(1+o(1))}{7n(1+o(1))}$$

$$= \frac{1}{n^{1+1/5}}(c+o(1))$$

$$= \frac{1}{n^{6/5}}(c+o(1))$$

$\Rightarrow a_n$ tende a zero per $n\to\infty$ con ordine $\frac{6}{5}$

b) $\sum n^\beta a_n$

$$b_n = n^\beta a_n = \frac{n^\beta}{n^{6/5}}(c+o(1)) = \frac{1}{n^{6/5-\beta}}(c+o(1))$$

$(a_n > 0)$

$\Rightarrow \sum$ converge $\Longleftrightarrow \frac{6}{5}-\beta > 1 \;\big|\; \beta < \frac{1}{5}$

4) Data la funzione

$$f(x) = -\frac{1}{4}\lg\left(\frac{x^2}{|x-3|}\right) + x + 4$$

studiarla e disegnarne un grafico qualitativo

dom f $\quad x \neq 3 \quad x \neq 0$

$\lim_{x \to 3} f(x) = -\infty \qquad x=3$ asintoto verticale dx e sn.

$\lim_{x \to 0} f(x) = +\infty \qquad x=0$ asintoto verticale

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x - \frac{1}{4}\lg(x)(1+o(1)) = +\infty$

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x - \lg(-x) = -\infty$ non ci sono asint. obliqui né a $+\infty$ né a $-\infty$

$$f(x) = \begin{cases} x+4 - \frac{1}{4}\lg\left(\frac{x^2}{x-3}\right) & x>3 \\ x+4 - \frac{1}{4}\lg\left(\frac{x^2}{3-x}\right) & x<3 \quad x\neq 0 \end{cases}$$

nel suo dominio f è derivabile e

$$f'(x) = \begin{cases} 1 - \frac{1}{4}\,\frac{\cancel{x-3}}{x^{\cancel{2}}}\,\frac{2x(x-3)-x^2}{(x-3)^{\cancel{2}}} \quad = x^2-6x = x(x-6) & x>3 \\ 1 - \frac{1}{4}\,\frac{\cancel{3-x}}{x^{\cancel{2}}}\,\frac{2x(3-x)+x^2}{(3-x)^{\cancel{2}}} = 6x-x^2 = x(6-x) & x<3 \end{cases}$$

$$= \begin{cases} 1 - \frac{1}{4}\,\frac{x-6}{x(x-3)} & x>3 \\ 1 - \frac{1}{4}\,\frac{6-x}{x(3-x)} & x<3 \quad x\neq 0 \end{cases}$$

$$= \begin{cases} \frac{4x(x-3) - x + 6}{4x(x-3)} & x>3 \\ \frac{4x(3-x) - 6 + x}{4x(3-x)} & x<3 \quad x \neq 0 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{4x^2-13x+6}{4x(x-3)} & x>3 \\ \dfrac{-4x^2+13x-6}{4x(3-x)} & x<3 \quad x \neq 0 \end{cases}$$

$$4x^2-13x+6=0$$

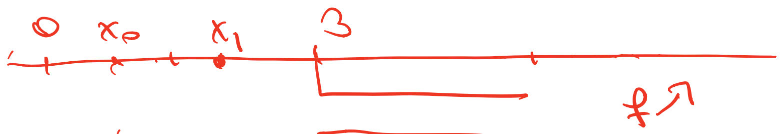
$$x = \frac{13 \pm \sqrt{169-96}}{8} = \frac{13 \pm \sqrt{73}}{8}$$

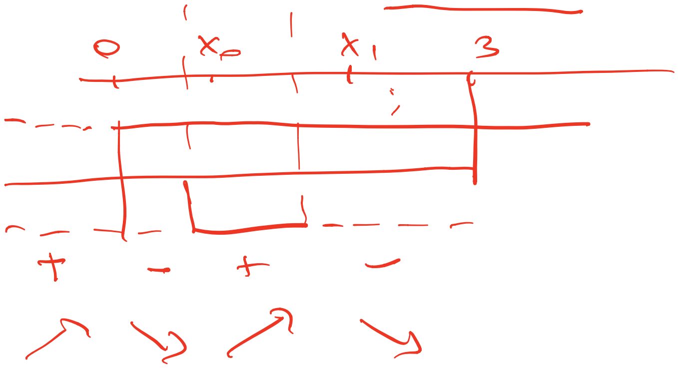
$$x_0 = \frac{13-\sqrt{73}}{8} \qquad x_1 = \frac{13+\sqrt{73}}{8} < 3$$

$x_0 > 0 \quad 13 > \sqrt{73}$ sì

$13+\sqrt{73} < 24$

$\sqrt{73} < 11$ sì




$x = x_0$ punto di minimo rel.

$x = x_1$ punto di max rel.


$$f'(x) = \begin{cases} \dfrac{4x^2-13x+6}{4x(x-3)} & x>3 \\ \dfrac{-4x^2+13x-6}{4x(3-x)} & x<3 \quad x \neq 0 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{4x^2-13x+6}{4x(x-3)} & x>3 \\[2ex] \dfrac{-4x^2+13x-6}{4x(3-x)} & x<3 \quad x\neq 0 \end{cases}$$

$$f' = \begin{cases} \dfrac{4x^2-13x+6}{4x(x-3)} & x>3 \\[2ex] + \dfrac{(4x^2-13x+6)}{4x(x-3)} & x<3 \end{cases}$$

quindi f' ha la stessa espressione per $x>3$ e $x<3$

$$4x^2-12x$$

$$f'' = \frac{(8x-13)(4x^2-12x)-(8x-12)(4x^2-13x+6)}{16x^2(x-3)^2}$$

segno $f''$ =

$$\text{segno}\left(\cancel{32x^3}-96x^2-52x^2+\cancel{156x}-\left(\cancel{32x^3}-104x^2+48x-48x^2+\cancel{156x}-72\right)\right)$$

$$(-96-52+104+48)x^2-48x+72$$

$$\text{segno}(4x^2-48x+72)=\text{segno}(x^2-12x+18)$$

ma $x^2-12x+18=0$

$$x_{2,3}=6\pm\sqrt{36-18}=6\pm\sqrt{18}$$

$x_3$ flesso $6+\sqrt{18} > 3$

$x_2 = 6-\sqrt{18} \in (x_0, x_1)$

f conv. se $x > x_3$

$x < x_2$

concava altr.

$$0 < \frac{13-\sqrt{73}}{8} < 6-\sqrt{18} < \frac{13+\sqrt{73}}{8} < 3$$

$$\left[\begin{array}{l} 13-\sqrt{73} < 48 - 8\sqrt{18} \\ 8\sqrt{18}-\sqrt{73} < 35 \end{array}\right]$$