1) Studiare la funzione

$f(x) = e^{\frac{x}{1-|x|}}$ e disegnarne

un grafico qualitativo

$\text{dom} f = \{x \in \mathbb{R} : |x| \neq 1\} =$

$(-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$

$f(x) > 0 \quad \forall x \in \text{dom} f$

$$f(x) = \begin{cases} e^{\frac{x}{1-x}} & \text{se } x \geq 0 \ x \neq 1 \\ e^{\frac{x}{1+x}} & x < 0 \quad x \neq -1 \end{cases}$$

$\lim_{x \to 1^+} f(x) = 0 \qquad \lim_{x \to 1^-} f(x) = +\infty$

$x = 1$ as v.

$\lim_{x \to -1^-} f(x) = +\infty \qquad \lim_{x \to -1^+} f(x) = 0$

$x = -1$ as v.

$\lim_{x\to+\infty} f(x) = e^{-1}$ $\quad y = e^{-1}$ as. orr. a $+\infty$

$\lim_{x\to-\infty} f(x) = e$ $\quad y = e$ as. or. a $-\infty$

Se $x \neq 0$ $x \in$ dom f f è sicuramente derivabile e

$$f'(x) = \begin{cases} e^{\frac{x}{1-x}} \dfrac{1-x+x}{(1-x)^2} & x>0 \;\; x\neq 1 \\ e^{\frac{x}{1+x}} \dfrac{1+x-x}{(1+x)^2} & x<0 \;\; x\neq -1 \end{cases}$$

$$= \begin{cases} e^{\frac{x}{1-x}} \dfrac{1}{(1-x)^2} & x>0 \;\; x\neq 1 \\ e^{\frac{x}{1+x}} \dfrac{1}{(1+x)^2} & x<0 \;\; x\neq -1 \end{cases}$$

$f' > 0$ quindi f è ↗ in $(-\infty, -1)$

f ↗ in $(-1, 1)$

f ↗ in $(1, +\infty)$

Per $x \to 0^+$ $f' \to 1$
$x \to 0^-$ $f' \to 1$ $\Rightarrow$ f è derivabile anche in $x=0$

e $f'(0) = 1$

inoltre per $x \to (-1)^+$ $f'(x) \to 0$

infatti $\lim\limits_{x \to -1^+} \dfrac{e^{\frac{x}{1+x}}}{(1+x)^2} = \dfrac{0}{0}$

$t = \dfrac{1}{1+x}$ $\Rightarrow$ $x = \dfrac{1}{t} - 1$ per $x \to -1^+$ $t \to +\infty$

$\lim\limits_{t \to +\infty} t^2 e^{1-t} = 0$ $f'_+(-1) = 0$

Analogamente per $x \to 1^+$

$\lim\limits_{x \to 1^+} e^{\frac{1}{1-x}} \dfrac{1}{(1-x)^2}$ $t = \dfrac{1}{1-x}$ $t \to \infty$

$1-x = \dfrac{1}{t}$ $x = 1 - \dfrac{1}{t}$

$\lim\limits_{t \to -\infty} t^2 e^{t-1} = 0$ $f'_+(1) = 0$

$$f''(x) = \begin{cases} e^{\frac{x}{1-x}} \dfrac{3-2x}{(1-x)^4} & x>0 \ x \neq 1 \\ e^{\frac{x}{1+x}} \dfrac{(-1-2x)}{(1+x)^4} & x<0 \ x \neq -1 \end{cases}$$

Quindi

$f''>0$ se $x<-1/2 \vee x \in (0,1) \cup x \in (1, 3/2)$

$f''<0$ ~~$x\in$~~ $(-1/2, 0) \cup x > 3/2$

-1	$-1/2$	0	1	$3/2$

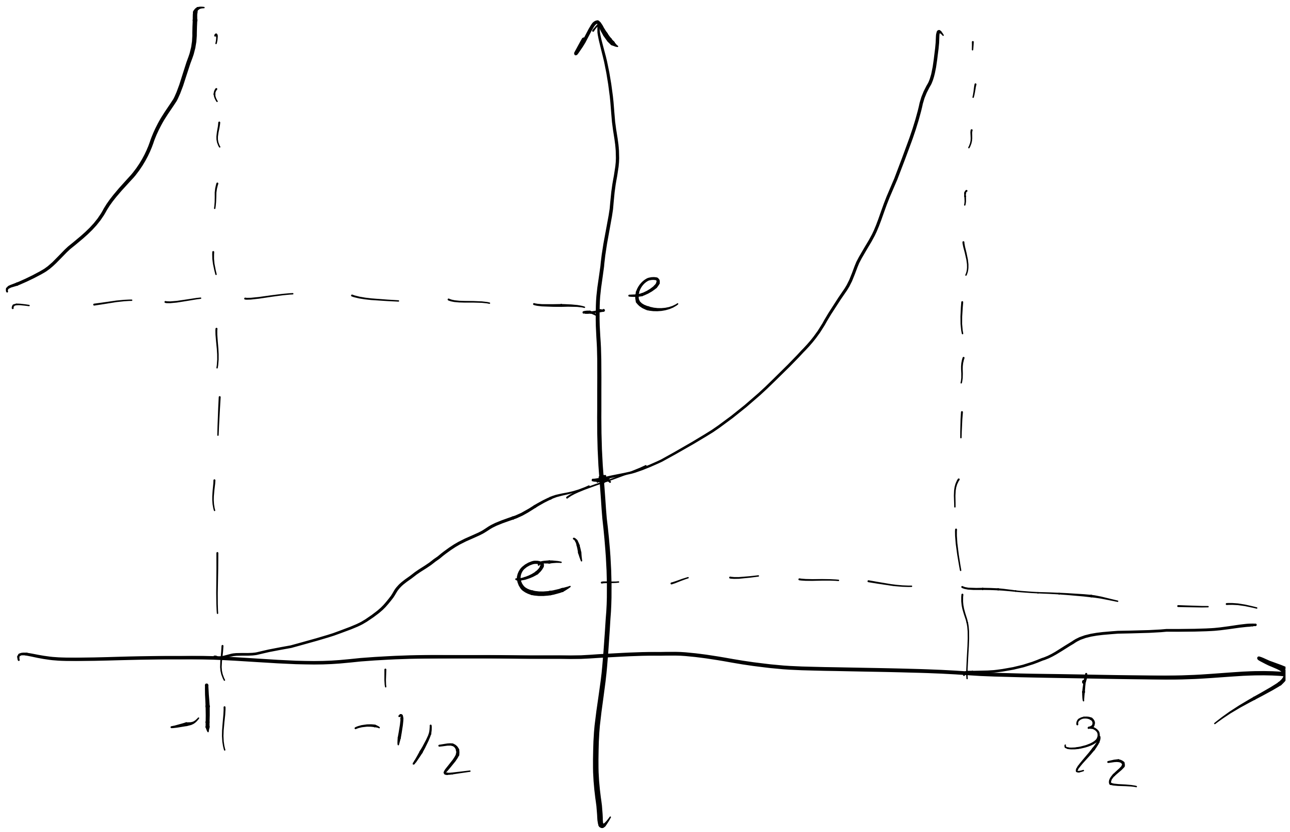
f conv. | conv | conc. | conv | conv | conc.

$x = -1/2$, $x = 3/2$ flesso

$x = 0$ ptodi

$\uparrow$

pto di derivabilità'



e
e'
-1
-1/2
3/2

2) Calcolare al variare di $\alpha > 0$

$$\lim_{x\to 0^+} \frac{4\sqrt{x}\,\lg(1+x^\alpha) + 1 - e^{x^2}}{\operatorname{tg}(x^\alpha)\operatorname{sen}\sqrt{x} + \cos(x^3) - 1}$$

NUM: $4\sqrt{x}\,x^\alpha(1+o(1)) + 1 - 1 - x^2 + o(x^3)$

$$= 4x^{\alpha+1/2} - x^2 + o(x^{\alpha+1/2}) + o(x^3)$$

DEN: $x^{\alpha+1/2} + o(x^{\alpha+1/2}) - \frac{x^6}{2} + o(x^7)$

$$\lim_{x\to 0^+} \frac{4x^{\alpha+1/2} - x^2 + o(x^{\alpha+1/2}) + o(x^3)}{x^{\alpha+1/2} - \frac{x^6}{2} + o(x^{\alpha+1/2}) + o(x^7)}$$

Quindi se $2 < \alpha + \frac{1}{2} < 6$ $\boxed{\frac{3}{2} < \alpha < \frac{11}{2}}$

$$\Rightarrow \frac{-x^2 + o(x^2)}{x^{\alpha+1/2}(1+o(1))} \longrightarrow \boxed{-\infty}$$

se $\boxed{\alpha < 3/2} \rightsquigarrow \frac{4x^{\alpha+1/2}}{x^{\alpha+1/2}}(1+o(1)) \to \boxed{4}$

se $\boxed{\alpha = 3/2} \rightsquigarrow \frac{3x^2}{x^2}(1+o(1)) \to \boxed{3}$

se $\boxed{\alpha = \frac{11}{2}} \rightsquigarrow \frac{x^2(1+o(1))}{\frac{1}{2}x^6} \to \boxed{-\infty}$

se $\boxed{\alpha > \frac{11}{2}} \rightsquigarrow \frac{-x^2(1+o(1))}{-\frac{1}{2}x^6} \to \boxed{+\infty}$

3) Studiare al variare di $\alpha \in \mathbb{R}$ la convergenza di

$$I_\alpha = \int_0^1 x^\alpha \left| \lg\left(\frac{x}{x^2+1}\right) \right| dx$$

e calcolarlo, se esiste finito, per $\alpha = 0$

$$f(x) = x^\alpha \left| \lg\left(\frac{x}{x^2+1}\right) \right|$$

$$= -x^\alpha \lg\left(\frac{x}{x^2+1}\right) \quad \text{se } x \in (0,1)$$

in quanto per $x \in (0,1)$ $\frac{x}{x^2+1} \in (0,1)$

$\Rightarrow \lg\left(\frac{x}{x^2+1}\right) < 0$

per $x \to 1^-$ $f(x) \to 0$

per $x \to 0^+$ $f(x) = x^\alpha |\lg x| (1 + o(1))$

Quindi se $\alpha > 0$ $f \to 0$

e non è un integrale improprio
(non essendoci altri problemi in $(0,1)$)

se $\alpha \leq 0$ $f \sim \frac{|\lg x|}{x^{-\alpha}}$

e I_α converge $\iff -\alpha < 1$

cioè $\alpha > -1$

Quindi I_α converge $\iff \alpha > -1$

se $\alpha = 0$

$$I_0 = -\int_0^1 \lg\left(\frac{x}{x^2+1}\right) dx \underset{p.p.}{=}$$

$$-x \lg\left(\frac{x}{x^2+1}\right)\Big|_0^1 + \int_0^1 \cancel{x} \cdot \frac{x^2+1}{\cancel{x}} \cdot \frac{1-x^2}{(x^2+1)^2}$$

$$= -\lg(1/2) + \int_0^1 \frac{1-x^2}{1+x^2} dx =$$

$$= \lg 2 - \int_0^1 \frac{x^2-1}{x^2+1} dx = \lg 2 - x\Big|_0^1 + 2\operatorname{arctg} x\Big|_0^1$$

$$= \lg 2 - 1 + 2\frac{\pi}{4} = \boxed{\lg 2 - 1 + \frac{\pi}{2}}$$

4a) Convergenza di $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n-\log n}$

(semplice e assoluta)

$a_n := \frac{1}{n-\log n} > 0$ def

$a_n \to 0$ e $a_n = \frac{1}{n}(1+o(1))$

$\Rightarrow \sum a_n = +\infty$ quindi la serie non converge assolut.

però

$\sum (-1)^n a_n$ converge perché

$a_n > 0$, $a_n \to 0$ e

$a_n \downarrow$ infatti

$$\frac{1}{n+1-\log(n+1)} < \frac{1}{n-\log n}$$

perché $\cancel{n} - \lg n < \cancel{n}+1 - \lg(n+1)$

$\Leftrightarrow \lg(n+1) - \lg n < 1$

$\Leftrightarrow \lg\left(\frac{n+1}{n}\right) < 1$

$\frac{n+1}{n} = 1 + \frac{1}{n} < e$

$\Leftrightarrow \frac{1}{n} < e-1$ definitivamente

vero perché $e>1$
e $\frac{1}{n} \to 0$

$\Rightarrow$ $a_n \downarrow$ definitivamente

$\Rightarrow$ per il criterio di Leibniz

$\sum (-1)^n a_n$ <u>converge</u>

4b) convergenza di $\sum \frac{n^n}{3^n n!}$

Serie a termini positivi

Criterio del rapporto

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{3^{n+1}(n+1)!} \quad \frac{3^n n!}{n^n}$$

$$= \frac{1}{3}\frac{\cancel{(n+1)}}{\cancel{(n+1)}} \left(\frac{n+1}{n}\right)^n$$

$$= \frac{1}{3}\left(1+\frac{1}{n}\right)^n \longrightarrow \frac{e}{3} < 1$$

$\Rightarrow$ la serie <u>converge</u>