MAXIMAL COMPLEXIFICATIONS OF CERTAIN RIEMANNIAN HOMOGENEOUS MANIFOLDS

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Abstract. Let \( M = G/K \) be a Riemannian homogeneous manifold with \( \dim_{\mathbb{C}} G^\mathbb{C} = \dim_{\mathbb{R}} G \), where \( G^\mathbb{C} \) denotes the universal complexification of \( G \). Under certain extensibility assumptions on the geodesic flow of \( M \), we give a characterization of the maximal domain of definition in \( TM \) for the adapted complex structure and show that it is unique. For instance, this can be done for generalized Heisenberg groups and naturally reductive Riemannian homogeneous spaces. As an application it is shown that the case of generalized Heisenberg groups yields examples of maximal domains of definitions for the adapted complex structure which are neither holomorphically separable, nor holomorphically convex.

1. Introduction

It is well known that complexifications of a real-analytic manifold \( M \) exist and are equivalent near \( M \), but differ usually very much in nature. If a complete real-analytic metric on \( M \) is given, one can construct canonical complexifications of \( M \) compatible with the metric by defining an adapted complex structure on a domain \( \Omega \) of the tangent bundle \( TM \) (see [GS] and [LS]). This structure can be characterized by the condition that the “complexification” \( (x + iy) \mapsto y \gamma'(x) \in \Omega \) of any geodesic \( x \mapsto \gamma(x) \) of \( M \) be a complex submanifold near the zero section. By the results of Guillemin-Stenzel and Lempert-Szöke cited above, the adapted complex structure exists and is unique on a sufficiently small neighborhood of \( M \). Here \( M \) is identified with its zero section in \( TM \).

In particular it is natural to ask for maximal domains around \( M \) with an adapted complex structure. By functoriality of the definition these may be regarded as invariants of the metric, i.e., isometric manifolds have biholomorphic maximal domains. For instance examples are known for symmetric spaces of non-compact type ([BHH]), compact normal Riemannian Homogeneous spaces ([Sz2]), compact symmetric spaces ([Sz1]) and spaces obtained by Kählerian reduction of these ([A]). Note that in the mentioned cases maximal domains turn out to be Stein.

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The aim of this work is to characterize maximal domains for the adapted complex structure for a class of Riemannian homogeneous spaces with “big” isometry group. Let $M = G/K$, with $G$ a Lie group of isometries and $K$ compact, and assume that $\dim_{\mathbb{C}} G^c = \dim_{\mathbb{R}} G$, where $G^c$ is the universal complexification of $G$. Then $K^c$ acts on $G^c$ and the left action on $M$ induces a natural $G$-action on $TM$. Our main result is that under certain extensibility assumptions on the geodesic flow of $M$ one obtains a real-analytic and $G$-equivariant map $P : TM \to G^c/K^c$ such that (see Theorem 3.2 for the precise statement)

The connected component of the non-singular locus of $DP$ containing $M$ is the unique maximal domain on which the adapted complex structure exists.

This applies to the case of naturally reductive Riemannian homogeneous spaces (corollary 3.3) and of generalized Heisenberg groups (see Sect. 4 and 5).

As an application it is shown that for all generalized Heisenberg groups such a maximal domain is neither holomorphically separable, nor holomorphically convex (Proposition 5.1). We are not aware of previous non-Stein examples. In the case of the 3-dimensional Heisenberg group we determine its envelope of holomorphy as well as a certain maximal Stein subdomain (Proposition 4.3).

2. Preliminaries

Here we introduce notations and briefly recall basic results we will need in the present paper. Let $M$ be a complete real-analytic Riemannian manifold which will be often identified with the zero section in its tangent bundle $TM$. Following [LS] (see also [GS] for an equivalent characterization) we say that a real-analytic complex structure defined on a domain $\Omega$ of $TM$ is adapted if all complex leaves of the Riemannian foliation are submanifolds with their natural complex structure, i.e., for any geodesic $\gamma : \mathbb{R} \to \mathbb{C}$ the induced map $f : \mathbb{C} \to TM$ defined by $(x + iy) \mapsto y\gamma'(x)$ is holomorphic on $f^{-1}(\Omega)$ with respect to the adapted complex structure. Here $y\gamma'(t) \in T_{\gamma(t)}M$ is the scalar multiplication in the vector space $T_{\gamma(t)}M$.

The adapted complex structure exists and is unique on a sufficiently small neighborhood of $M$. If $\Omega$ is a domain around $M$ in $TM$ on which it is defined, then we refer to it as an adapted complexification. Sometimes these are called Grauert tubes. For later use we need the following

**Lemma 2.1.** Let $F : TM \to \mathbb{C}$ be a real-analytic map which is holomorphic on any complex leaf of $TM$ in a neighborhood of $M$. Then $F$ is holomorphic on every adapted complexification.
Proof. Following the proof of [Sz1, Prop. 3.2 p. 416] one checks that the restriction of $F$ to $M$ extends to a holomorphic map $\hat{F}$ in a neighborhood $U$ of $M \subset TM$ where the adapted complex structure $J_0$ exists and, in order to have connected leaves, $U$ may be chosen to be starshaped. We can also assume that for any geodesic $\gamma : \mathbb{R} \to M$ the map $x + iy \mapsto F(y \gamma'(x))$ is holomorphic for all $x + iy$ such that $y \gamma'(x) \in U$. Now $F = \hat{F}$ on $\gamma(\mathbb{R}) \subset M \subset TM$, therefore $F = \hat{F}$ on every complex leaf, i.e., on $U$. In particular $DF \circ J_0 = i DF$ on $U$ and since all maps are real-analytic the statement follows from the identity principle.

A real Lie group $G$ acts on a complex manifold $X$, i.e., $X$ is a $G$-manifold, if there exists a real-analytic surjective map $G \times X \to X$ given by $(g, x) \mapsto g \cdot x$ such that for fixed $g \in G$ the map $x \mapsto g \cdot x$ is holomorphic and $(gh) \cdot x = g \cdot (h \cdot x)$ for all $h, g \in G$ and $x \in X$. Furthermore if $\dim_\mathbb{R} G = \dim_\mathbb{C} G^\mathbb{C}$, where $\iota : G \to G^\mathbb{C}$ is the universal complexification of $G$ (see, e.g., [Ho]), then $\text{Lie}(G^\mathbb{C}) = \mathfrak{g}^\mathbb{C}$ and one obtains an induced local holomorphic $G^\mathbb{C}$-action by integrating the holomorphic vector fields given by the $G$-action. Here $\mathfrak{g}$ denotes the Lie algebra of $G$.

Let $M = G/K$ be a Riemannian homogeneous manifold with $G$ a connected Lie group of isometries and $K$ compact, and consider the induced $G$-action on $TM$ defined by $g \cdot w := g_* w$ for all $g \in G$ and $w \in TM$. Then if $\Omega$ is a $G$-invariant adapted complexification, as an easy consequence of the definitions $g_*$ is a biholomorphic extension of the isometry $g$, i.e., $G \subset \text{Aut}(\Omega)$.

If one assumes that $\dim_\mathbb{R} G = \dim_\mathbb{C} G^\mathbb{C}$, then the natural map $\iota : G \to G^\mathbb{C}$ is an immersion and from the universality property of the universal complexification $K^\mathbb{C}$ of $K$ it follows that the restriction $\iota|_{K^\mathbb{C}}$ of $\iota$ to $K$ extends to an immersion $\iota^\mathbb{C} : K^\mathbb{C} \to G^\mathbb{C}$. Moreover the subgroup $\iota^\mathbb{C}(K^\mathbb{C})$ acts by right multiplication on $G^\mathbb{C}$ and one has a commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\iota} & G^\mathbb{C} \\
\downarrow & & \downarrow \\
G/K & \to & G^\mathbb{C}/\iota^\mathbb{C}(K^\mathbb{C})
\end{array}
$$

Also notice the $G$-action on $G^\mathbb{C}/\iota^\mathbb{C}(K^\mathbb{C})$ defined by $g \cdot h \iota^\mathbb{C}(K^\mathbb{C}) := \iota(g) h \iota^\mathbb{C}(K^\mathbb{C})$ for all $g \in G$ and $h \in G^\mathbb{C}$.

Lemma 2.2. Let $G$ be a connected Lie group, $K$ a compact subgroup and assume that $\dim_\mathbb{C} G^\mathbb{C} = \dim_\mathbb{R} G$. Then $G^\mathbb{C}/\iota^\mathbb{C}(K^\mathbb{C})$ is a complex $G$-manifold and $\dim_\mathbb{C} G^\mathbb{C}/\iota^\mathbb{C}(K^\mathbb{C}) = \dim_\mathbb{R} G/K$. 

Proof. One needs to show that $i^C(K^C)$ is closed in $G^C$. Note that $G^C/i^C(K^C)$ is the orbit space with respect to the $K^C$-action on $G^C$ defined by $k \cdot h := h i(k^{-1})$ for all $k \in K^C$ and $h \in G^C$. Since $G^C$ is Stein ([He]) and $K^C$ is reductive, it follows that every fiber of the categorical quotient $G^C \to G^C//K^C$ is equivariantly biholomorphic to an affine algebraic variety on which $K^C$ acts algebraically ([Sn]). In particular there exists at least one closed $K^C$-orbit and consequently $i^C(K^C)$ is closed in $G^C$. Thus $G^C/i^C(K^C)$ is a complex $G$-manifold and by construction its complex dimension is $\dim_R G/K$. □

3. A characterization of maximal adapted complexifications

If $M = G/K$ is a symmetric space of the non-compact type, then $G^C/K^C$ is a natural candidate for a complexification of $M$ and there exists a $G$-equivariant map $P : TM \to G^C/K^C$ embedding holomorphically a maximal adapted complexification $\Omega_M$ of $M$ (see [BHH], [Ha], [AG]). As a matter of fact one may show that $DP$ is singular on the boundary $\partial \Omega_M$ of $\Omega_M$.

Here we consider a Riemannian homogeneous manifold $M = G/K$ endowed with the additional data of a certain real-analytic $G$-equivariant map $P$ from $TM$ to a suitable complex $G$-manifold. Proposition 3.1 characterizes a maximal adapted complexification $\Omega_M$ as the connected component of $\{DP \text{ not singular}\}$ containing $M$. Unicity of $\Omega_M$ follows.

The existence of such data is proved when $\dim_C G^C = \dim_R G$ and the geodesic flow “extends” holomorphically on $G^C/i^C(K^C)$ (cf. Lemma 2.2). As a consequence the characterization applies to the case of naturally reductive Riemannian homogeneous spaces and of generalized Heisenberg groups.

**Proposition 3.1.** Let $M = G/K$ be an $n$-dimensional Riemannian homogeneous space and $X$ a $G$-complex manifold of complex dimension $n$ such that the induced local $G^C$-action is locally transitive. Assume there exists a real-analytic map $P : TM \to X$ which is

i) $G$-equivariant and

ii) holomorphic on every complex leaf of $TM$.

Then the connected component $\Omega_M$ of $\{p \in TM : DP_p \text{ not singular}\}$ containing $M$ is the unique maximal adapted complexification and $P|_{\Omega_M}$ is locally biholomorphic.

**Proof.** First we show that $\Omega_M$ is well defined, i.e., $DP$ has maximal rank along $M$. Since from Lemma 2.1 it follows that $P$ is holomorphic on $M$ with respect to the adapted complex structure, this is a consequence of the following
Claim: Assume that $P$ is holomorphic in $p \in TM$. Then $DP_p$ has maximal rank.

Proof of the claim: Since $G^C$ acts locally transitively on $X$, there exist elements $\xi_1, \ldots, \xi_n$ of $\mathfrak{g}$ such that the induced vector fields $\xi_{X,1}, \ldots, \xi_{X,n}$ on $X$ span a totally real and maximal dimensional subspace $V_{P(p)}$ of $T_{P(p)}X$, where

$$\xi_{X,j}(x) := \frac{d}{dt} \bigg|_0 \exp_{G^C(t\xi_j)} \cdot x$$

for $j = 1, \ldots, n$ and all $x \in X$. By equivariance it follows that $DP_p(V_p) = V_{P(p)}$, where $V_p$ is the subspace of $T_pTM$ spanned by $\xi_{TM,1}, \ldots, \xi_{TM,n}$, here $\xi_{TM,j}(q) := \frac{d}{dt} \bigg|_0 \exp_G(t\xi) \cdot q$ for all $q \in TM$. In particular $\dim \mathbb{R} V_p = n$ and since $P$ is holomorphic in $p$, $V_p$ is totally real and $DP_p$ has maximal rank, proving the claim.

Now we see that the pulled-back complex structure $J_o$ on $\Omega_M$ of the complex structure $J$ on $X$ is the adapted complex structure. For this consider a complex leaf $f : \mathbb{C} \to TM$ defined by $f(x + iy) := y \gamma'(x)$, where $\gamma$ is a geodesic of $M$, and note that by ii)

$$DP \circ Df(i\eta) = DP \circ J_o \circ Df(\eta)$$

for all $\eta$ tangent in $f^{-1}(\Omega_M)$. Since $DP$ has maximal rank on $\Omega_M$, one has

$$Df(i\eta) = J_o \circ Df(\eta)$$

showing that $J_o$ is the adapted complex structure. In particular $P|_{\Omega_M}$ is locally biholomorphic.

In order to prove maximality, assume that $J_o$ extends analytically in a neighborhood of a certain $p \in \partial \Omega_M \subset TM$. By construction $DP \circ J_o = J \circ DP$ on $\Omega_M$ and since all maps are real-analytic $P$ is holomorphic in $p$. Then the above claim shows that $DP_p$ has maximal rank, contradicting the definition of $\Omega_M$.

Finally we want to show that any adapted complexification $\Omega$ is contained in $\Omega_M$. If this is not the case, there exists a point $p$ in $\Omega \cap \partial \Omega_M$ and from Lemma 2.1 it follows that $P|_{\Omega}$ is holomorphic. In particular $P$ is holomorphic in $p$ and one obtains a contradiction arguing as above. Thus $\Omega_M$ is unique and this concludes the statement. 

Now we determine a class of Riemannian homogeneous spaces to which Proposition 3.1 may be applied in order to determine the maximal adapted complexification.

Theorem 3.2. Let $M = G/K$ be a Riemannian homogeneous space with $\dim \mathbb{R} G = \dim \mathbb{C} G^C$ and assume there exists a map $\varphi : \mathbb{C} \times T_KM \to g^C$ real-analytic and holomorphic on the first component such that $\varphi(\mathbb{R} \times T_KM) \subset g$ and
$t \mapsto \exp_G \circ \varphi(t, v) K$ is the unique geodesic tangent to $v$ at 0 for all $v \in T_K M$.

Then the map

$$P : TM \mapsto G^C / \iota^C(K^C)$$

defined by

$$P(g_*(v)) := \iota(g) \exp_{G^C}(\varphi(i, v)) \iota^C(K^C)$$

for all $g \in G$ and $v \in T_K M$ is as in Proposition 3.1. In particular the connected component $\Omega_M$ of $\{p \in TM : DP_p \text{ not singular}\}$ containing $M$ is the maximal adapted complexification and $P|_{\Omega_M}$ is locally biholomorphic.

**Proof.** In order to prove that $P$ is well defined we need to show that if $w = k_\ast v$ for some $k \in K$ and $v \in T_K M$ then $P(w) = P(k_\ast v)$, i.e.,

$$\exp_{G^C}(\varphi(i, w)) \iota^C(K^C) = \iota(k) \exp_{G^C}(\varphi(i, v)) \iota^C(K^C).$$

For this note that $t \mapsto k \exp_G \circ \varphi(t, v) K$ is the unique geodesic tangent to $v$ at 0 in $K$, thus

$$\exp_G \circ \varphi(t, w) K = k \exp_G \circ \varphi(t, v) K.$$

Then the commutativity of the diagram

$$\begin{array}{ccc}
g & \mapsto & g^C \\
\downarrow \exp_G & & \downarrow \exp_{G^C} \\
G & \overset{\iota}{\rightarrow} & G^C
\end{array}$$

implies that

$$\exp_{G^C} \circ \varphi(t, w) \iota^C(K^C) = \iota(k) \exp_{G^C}(\varphi(t, v)) \iota^C(K^C)$$

for all $t \in \mathbb{R}$ and equation (1) is a consequence of the identity principle for holomorphic maps.

In order to simplify notations we now assume that the canonical immersion $\iota : G \rightarrow G^C$ is injective so that once we identify $G$ with $\iota(G)$ the curve $\gamma(t) := \exp_{G^C}(\varphi(t, v)) K$ is the unique geodesic tangent to $v$ at 0 for all $v \in T_K M$. In what follows it is easy to check that all arguments apply to the case where $\iota$ is a non-injective immersion.

Fix $x \in \mathbb{R}$, let $g := \exp_{G^C}(\varphi(x, v))$ and note that

$$y \mapsto g \exp_{G^C}(\varphi(y, g^{-1}_\ast \gamma'(x))) K$$

is the unique geodesic tangent to $\gamma'(x)$ at 0. Therefore one has

$$g \exp_{G^C}(\varphi(y, g^{-1}_\ast \gamma'(x))) K = \exp_{G^C}(\varphi(x + y, v)) K$$

for all $y \in \mathbb{R}$ and from the identity principle it follows that

$$g \exp_{G^C}(\varphi(z, g^{-1}_\ast \gamma'(x))) \iota^C(K^C) = \exp_{G^C}(\varphi(x + z, v)) \iota^C(K^C)$$

(2)
for all $v \in T_KM$ and $z \in \mathbb{C}$. Now for $h \in G$, $v \in T_KM$ let $\gamma$, $g$ be as above and consider the unique geodesic $\tilde{\gamma} := h \cdot \gamma$ tangent to $h_*(v)$ at 0. One has

$$P(y\tilde{\gamma}'(x)) = P(h_*y\gamma'(x)) = h P(g_*g_*^{-1}y\gamma'(x)) =$$

$$h g \exp_{G^C}(\varphi(i, y g_*^{-1}\gamma'(x))) \iota^C(K^C) = h g \exp_{G^C}(\varphi(zy, v)) \iota^C(K^C),$$

where we used (2) and the fact that

$$\exp_{G^C}(\varphi(z, yv)) \iota^C(K^C) = \exp_{G^C}(\varphi(zy, v)) \iota^C(K^C)$$

for all $z \in \mathbb{C}$, since this holds for all $z \in \mathbb{R}$. As a consequence the map $(x+iy) \mapsto P(y\tilde{\gamma}'(x))$ is holomorphic for all geodesics $\tilde{\gamma}$ of $M$, i.e., $P$ is holomorphic on every complex leaf of $TM$.

Finally the map $P$ is $G$-equivariant by construction and the $G$-action on $G^C/\iota^C(K^C)$ induces a holomorphic $G^C$-action which may be obtained through left multiplication on $G^C$. Thus it is obviously transitive and this yields the statement.$\square$

Now let $M$ be a naturally reductive Riemannian homogeneous space and $M = G/K$ be a natural realization of $M$, i.e., there exists a reductive decomposition $\mathfrak{g} = \text{Lie}(K) \oplus \mathfrak{m}$ of the Lie algebra of $G$ such that every geodesic in $M$ is the orbit of a one parameter subgroup of $G$ generated by an element of $\mathfrak{m}$ (see, e.g., [BTV]). Consider the natural projection $\Pi : G \to M$ and note that $D\Pi_e(\mathfrak{m}) = T_KM$, where $e$ is the neutral element of $G$. Denote by $L : T_KM \to \mathfrak{m}$ the inverse of the restriction of $D\Pi_e$ to $\mathfrak{m}$. Since $L$ is linear it extends $\mathbb{C}$-linearly from $(T_KM)^C$ to $\mathfrak{m}^C$ and the map $\varphi : \mathbb{C} \times T_KM \to \mathfrak{g}^C$ defined by $\varphi(z, v) := zL(v)$ is as in the above Theorem. Therefore one has

**Corollary 3.3.** Let $M = G/K$ be a natural realization of a naturally reductive Riemannian homogeneous space and assume that $\dim_\mathbb{R} G = \dim_\mathbb{C} G^C$. Then the map

$$P : TM \to G^C/\iota(K^C)$$

defined by

$$P(g_*(v)) := \iota(g) \exp_{G^C}(iL(v)) \iota(K^C)$$

for all $g \in G$ and $v \in T_KM$ meets the conditions of Proposition 3.1. In particular the connected component $\Omega_M$ of $\{p \in TM : DP_p \text{ not singular} \}$ containing $M$ is the maximal adapted complexification and $P|_{\Omega_M}$ is locally biholomorphic.
Here we apply results of the previous section in order to give a concrete description of the unique maximal adapted complexification for the 3-dimensional Heisenberg group. It turns out that such domain is neither holomorphically separable, nor holomorphically convex. We also determine its envelope of holomorphy and a particular maximal Stein subdomain. We remark that in all previous examples we are aware of, maximal adapted complexifications are Stein.

Consider the 3-dimensional Heisenberg group defined as a subgroup of $GL_3(\mathbb{R})$ by

$$H := \left\{ \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\},$$

fix the inner product of the tangent space $T_eH$ in the neutral element $e$ for which the canonical basis determined by the global natural chart $(\alpha, \beta, \gamma)$ is orthonormal and let $(a, b, c)$ be coordinates of $T_eH$ with respect to this basis. Endow $H$ with the induced $H$-invariant metric

$$(da)^2 + (db)^2 + (d\gamma - \alpha \cdot d\beta)^2,$$

let $\mathfrak{h} = Lie(H)$ and define $\varphi : \mathbb{R} \times T_eH \to \mathfrak{h}$ by

$$\varphi(t, (a, b, c)) := \left( \begin{array}{c} \sin(tc) - b \cdot \frac{1 - \cos(tc)}{c} \\ \frac{\sin(tc)}{c} - a \cdot \frac{1 - \cos(tc)}{c} \\ \frac{1 - \cos(tc)}{c} \end{array} \right),$$

where the coordinates of $\mathfrak{h}$ are induced by those of $T_eH$ via the natural identification $\mathfrak{h} \cong T_eH$. Note that all singularities are removable and consequently $\varphi$ is real-analytic. Following [BTV, Th. p. 31] one checks that $t \mapsto \exp_{H} \circ \varphi(t, (a, b, c))$ is the unique geodesic tangent to $(a, b, c)$ at $0$. Furthermore by expanding the power series it is easy to verify that $\varphi(\cdot, (a, b, c))$ extends holomorphically on $\mathbb{C}$ to $(T_eH)^\mathbb{C}$ and by considering the polar decomposition $H \times \mathfrak{h} \to H^\mathbb{C}$ of $H^\mathbb{C}$ given by $(g, \xi) \mapsto g \exp_{H^\mathbb{C}}(i\xi)$, one obtains real-analytic functions $(a, b, c) \mapsto h_{(a,b,c)} \in H$ and $(a, b, c) \mapsto \xi_{(a,b,c)} \in \mathfrak{h}$ such that

$$\exp_{H^\mathbb{C}} \circ \varphi(i, (a, b, c)) = h_{(a,b,c)} \exp_{H^\mathbb{C}}(i\xi_{(a,b,c)}).$$

Define $P : TH \to H^\mathbb{C} \cong H \times \mathfrak{h}$ by

$$g_*(a, b, c) \mapsto g \exp_{H^\mathbb{C}} \circ \varphi(i, (a, b, c)) \cong \left( gh_{(a,b,c)}, \xi_{(a,b,c)} \right).$$

Then Theorem 3.2 implies that the connected component $\Omega_H$ containing $H$ of $\{DP \text{ not singular}\}$ is the maximal adapted complexification. Note that since $P$ is $H$-equivariant, $\Omega_H$ is $H$-invariant. Moreover $T_eH$ is a global slice for the $H$-action on $TH$, i.e., the map $H \times T_eH \to TH$ given by $(g, (a, b, c)) \mapsto g_*(a, b, c)$ is...
a $H$-equivariant real-analytic diffeomorphism, thus $\Omega_H$ is completely determined by its slice $\Omega_H \cap T_e H$.

Furthermore, since $H$ acts freely on the first component of $H \times \mathfrak{h}$, then the $H$-equivariance of $P$ implies that $DP_{g,(a,b,c)}$ has maximal rank if and only if $\tilde{P}_{(a,b,c)}$ has maximal rank, where $\tilde{P} := p_2 \circ P |_{T_e H} : T_e H \to \mathfrak{h}$ is given by

$$\tilde{P}(a,b,c) = \xi_{(a, b, c)} = \left( \frac{a \sinh(c)}{c}, \frac{b \sinh(c)}{c}, \left( 1 + \frac{a^2 + b^2}{2c^3} (c - \sinh(c) \cosh(c)) \right) c \right).$$

Here $p_2 : H \times \mathfrak{h} \to \mathfrak{h}$ is the canonical projection. It follows that $\Omega_H = H \cdot O_0$, where $O_0$ is the connected component of $\{ \det(D\tilde{P}) \neq 0 \}$ containing 0 in $T_e H$.

Now a straightforward computation shows that

$$\det(D\tilde{P}_{(a,b,c)}) = \sinh(c) \left( \frac{\sinh(c)}{c} + (a^2 + b^2) \left( \frac{\sinh(c) - c \cosh(c)}{c} \right) \right),$$

therefore

$$O_0 = \left\{ (a, b, c) \in T_e H : a^2 + b^2 < \frac{c^2 \sinh(c)}{c \cosh(c) - \sinh(c)} \right\}.$$

We want to discuss injectivity of $P |_{\Omega_H} : \Omega_H \to H \cong H \times \mathfrak{h}$ and again this is equivalent to injectivity of $\tilde{P} |_{O_0}$.

Note that $\tilde{P}$ is equivariant with respect to rotations around the $c$-axis as well as to the reflection $\sigma$ with respect to the plane $\{ c = 0 \}$. In particular for any

$$\tilde{P}(a, b, c) \in \left\{ 1 + \frac{a^2 + b^2}{2c^3} (c - \sinh(c) \cosh(c)) = 0 \right\}$$

one has $\tilde{P}(a, b, c) = \tilde{P}(a, b, -c) = \left( \frac{a \sinh(c)}{c}, \frac{b \sinh(c)}{c}, 0 \right)$. Therefore we are induced to investigate the domain

$$O_1 := \left\{ (a, b, c) \in T_e H : a^2 + b^2 < \frac{2c^3}{\sinh(c) \cosh(c) - c} \right\}.$$

**Lemma 4.1.** The domain $O_1$ is the maximal $\sigma$-invariant subdomain of $O_0$ containing 0 on which $\tilde{P}$ is injective. In particular $\tilde{P} |_{O_0}$ is not injective.

**Proof.** Let $f_j$ be the real function defining

$$O_j = \{ (a, b, c) \in T_e H : a^2 + b^2 < f_j(c) \}$$

for $j = 0, 1$. First we want to show that $O_1$ is a subdomain of $O_0$, i.e.,

$$f_1(c) \leq f_0(c)$$
for all $c \in \mathbb{R}$, which is equivalent to

$$2 \cosh(c) \leq \frac{\sinh(c)}{c} + \frac{\sinh^2(c)}{c^2} \cosh(c).$$

Expanding in power series one obtains

$$2 + c^2 + \frac{1}{12} c^4 + \cdots \leq \left(1 + \frac{1}{6} c^2 + \frac{1}{120} c^4 + \cdots \right) + \left(1 + \frac{5}{6} c^2 + \left(\frac{5}{6} + \frac{1}{24} + \cdots \right) c^4 + \cdots \right).$$

All coefficients are non-negative and one easily checks that for $k \geq 2$ the coefficient of $c^{2k}$ in the last series on the right side is strictly greater than that in the series on the left, hence $O_1 \subset O_0$. Moreover $\partial O_1 \cap \partial O_0 = \{(a, b, 0) \in T_eH : a^2 + b^2 = 3\}$, thus $O_1$ is a proper subdomain of $O_0$.

Furthermore by the previous remarks any $\sigma$-invariant domain containing 0 on which $\tilde{P}$ is injective is necessarily contained in $O_1$.

Assume that there exist $(a', b', c'), (a'', b'', c'') \in O_1$ such that $\tilde{P}(a', b', c') = \tilde{P}(a'', b'', c'') =: (A, B, C)$. If $C = 0$ then $c' = c'' = 0$ and consequently $a' = a'' = A$ and $b = b'' = B$. If $C \neq 0$ by eventually acting with $\sigma$ and a rotation around the $c$-axis we may assume that $a, A \geq 0$, $b = B = 0$ and $c > 0$. Now one has

$$a' \frac{\sinh(c')}{c'} = a'' \frac{\sinh(c'')}{c''} = A$$

therefore $(a', 0, c')$ and $(a'', 0, c'')$ lie on the same level curve $\rho_A : \mathbb{R} \to T_eH$ given by

$$\rho_A(t) := \left(A \frac{t}{\sinh(t)}, 0, t \right).$$

One has the following

**Claim:** Let $A \geq 0$ and $t_0 \in \mathbb{R}^\geq 0$ such that $\rho_A(t_0) \in \overline{O}_0$. Then $\rho_A(t) \in O_0$ for all $t > t_0$.

**Proof of the claim:** One needs to show that $A^2 \frac{t^2}{\sinh^2(t)} < f_0(t)$ for all $t > t_0$, that is

$$(3) \quad A^2 < \frac{\sinh^2(t)}{t^2} f_0(t).$$

By expanding in power series as above one has the estimate

$$2t \cosh^2(t) - 3 \cosh(t) \sinh(t) + t > 0,$$

for all $t > 0$, which by a straightforward computation implies that the derivative of the function at the right hand side of (3) is positive for all $t > 0$, proving the claim.

Now let $t_0 := \min(c', c'')$ and note that since $O_1 \subset O_0$, as a consequence of the above claim there exists $\epsilon > 0$ such that $\rho_A(t) \in O_0$ for $t > t_0 - \epsilon$. In
particular \((a',0,c')\) and \((a'',0,c'')\) lie in the same connected real one dimensional submanifold \(N := \rho_A(t_0 - \epsilon, \infty)\) of \(O_0\) and \(\tilde{P}|_N : N \to \{(A,0,\cdot) \in T_e H\} \cong \mathbb{R}\) is locally diffeomorphic. Then a classical argument implies that \(\tilde{P}|_N\) is injective, thus \((a',0,c') = (a'',0,c'')\) as wished. \(\square\)

We also want to determine the image of \(P|_{\Omega_H}\) in \(H^C\). Note that \(P(\Omega_H)\) is \(H\)-invariant and the polar decomposition implies that \(\exp_{H^C}(i\mathfrak{g})\) is a global slice for the \(H\)-action on \(H^C\). Then this can be achieved by describing \(\exp_{H^C}(i\tilde{P}(O_0)) = P(\Omega_H) \cap \exp_{H^C}(i\mathfrak{g})\).

**Lemma 4.2.** \(\tilde{P}(O_0) = \mathfrak{h}\setminus \{(A,B,C) \in \mathfrak{h} : A^2 + B^2 = 3, C = 0\}\).

**Proof.** Let \((a,0,c) \in \{a^2 = f_1(c)\} \subset \partial O_1\) with \(c > 0\). From the proof of Lemma 4.1 it follows that \((a,0,c) \in O_0\). Since \(\tilde{P}(a,0,c) = (a\frac{\sinh(c)}{c},0,0)\) and

\[
\sqrt{f_1(0)} = \sqrt{3} \quad \text{and} \quad \lim_{c \to \infty} \sqrt{f_1(c)} \frac{\sinh(c)}{c} = \infty
\]

it follows that \((A,0,0) \in \tilde{P}(O_0)\) for all \(A > \sqrt{3}\).

For \(A > \sqrt{3}\) let \((a,0,c) \in O_0\) such that \(\tilde{P}(a,0,c) = (A,0,0)\). By the claim in Lemma 4.1 one has \(\rho_A(t) \in O_0\) for all \(t \geq c\). Moreover one sees that

\[
\tilde{P}(\rho_A(t)) = (A,0,C_A(t)) \quad \text{with} \quad \lim_{t \to \infty} C_A(t) = \infty.
\]

Then by \(\sigma\)-invariance of \(O_0\) and \(\sigma\)-equivariance of \(\tilde{P}\) it follows that \((A,0,C) \in \tilde{P}(O_0)\) for all \(C \in \mathbb{R}\), and \(A > \sqrt{3}\).

Now note that \(\tilde{P}(a,0,0) = (a,0,0,0)\) and \(f_0(0) = 3\), thus \((A,0,0) \in \tilde{P}(O_0)\) for all \(A < \sqrt{3}\) and arguing as above it follows that \((A,0,C) \in \tilde{P}(O_0)\) for all \(C \in \mathbb{R}\) and \(A < \sqrt{3}\).

Finally \(\rho_{\sqrt{3}}(0) \in \partial O_0\), thus \(\rho_{\sqrt{3}}(t) \in O_0\) for all \(t > 0\). It follows that \(t \mapsto \tilde{P}(\rho_{\sqrt{3}}(t))\) is injective for \(t > 0\) and since

\[
\lim_{t \to 0^+} C_{\sqrt{3}}(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} C_{\sqrt{3}}(t) = \infty
\]

one has \((\sqrt{3},0,C) \in \tilde{P}(O_0)\) if and only if \(C \neq 0\).

The statement follows from the invariance of \(O_0\) and the equivariance of \(\tilde{P}\) with respect to the group of rotations around the \(c\)-axis. \(\square\)

In the picture below one sees the boundary of \(O_0\) and \(O_1\) defined by \(f_0\) and \(f_1\) respectively as well as the level curve \(\rho_2\) in the upper half-plane \(\{b = 0, a \geq 0\}\) of \(T_e H\). Since \(O_0\) and \(O_1\) are invariant with respect to rotations around the \(c\)-axis this completely determine their shape and, by \(H\)-equivariance, that of \(\Omega_H\).
Proposition 4.3. The maximal domain $\Omega_H$ is neither holomorphically separable, nor holomorphically convex, its envelope of holomorphy is biholomorphic to $\mathbb{C}^3$. The maximal $\sigma$-invariant Stein subdomain of $\Omega_H$ is biholomorphic to 
\[ \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : (\text{Im} \ z_1)^2 + (\text{Im} \ z_2)^2 < 3 \} . \]

Proof. Note that the elements of $H$ with integer entries determine a discrete co-compact subgroup $\Gamma$ of $H$ and from [GH] it follows that $H^C/\Gamma$ is Stein (in fact it is easy to check that $H^C/\Gamma$ is biholomorphic to $(\mathbb{C}^*)^3$). Then [CIT, Prop. p. 543] applies to show that any holomorphically separable Riemann $H$-domain over $H^C$ is univalent. Moreover Lemma 4.1 implies that $P|_{\Omega_H} : \Omega_H \to H^C$ is not injective, therefore $\Omega_H$ is not holomorphically separable.

By a result of Loeb ([L, Th. p. 186]), a Stein $H$-invariant domain $U$ of $H^C$ is “geodesically” convex, i.e., it is convex with respect to all curves of the form $t \mapsto g \exp_{H^C}(it\xi)$, with $g \in U$ and $\xi \in \mathfrak{h}$. Since $H^C$ admits polar decomposition and $U$ is $H$-invariant, it is enough to consider curves of the form $\exp_{H^C}(i\eta) \exp_{H^C}(it\xi)$, with $\exp_{H^C}(i\eta) \in U$ and $\xi \in \mathfrak{h}$. Furthermore for a two step nilpotent Lie group one has

\[ \exp_{H^C}(i\eta) \exp_{H^C}(it\xi) = \exp_{H^C}(i\eta + it\xi - \frac{t}{2}[\eta, \xi]) = \]

\[ \exp_{H^C}(-\frac{t}{2}[\eta, \xi]) \exp_{H^C}(i(\eta + t\xi)) \]

and using $H$-invariance one more time we conclude that if $U = H \cdot \exp_{H^C}(iD)$, with $D$ a domain in $\mathfrak{h}$, is Stein then $D$ is convex in the usual affine sense.

Since $P(\Omega_H) = H \cdot \exp_{H^C}(i\bar{P}(O_0))$ and as a consequence of Lemma 4.2 the domain $\bar{P}(O_0)$ is not convex, it follows that $P(\Omega_H)$ is not Stein. Now $H^C$ is Stein and $P|_{\Omega_H}$ is locally biholomorphic, therefore by [R] there exists a
commutative diagram

$$
\begin{array}{ccc}
\Omega_H & \xrightarrow{j} & \hat{\Omega}_H \\
P \searrow & & \downarrow \hat{P} \\
& H^C & 
\end{array}
$$

where $\hat{\Omega}_H$ is the envelope of holomorphy of $\Omega_H$. Moreover $H$ acts on $\hat{\Omega}_H$ and all maps are $H$-equivariant. Furthermore $\hat{P}$ is injective by [CIT, Prop. p. 543] and if $\Omega_H$ is holomorphically convex then $j$ is surjective and consequently $\hat{\Omega}_H$ is biholomorphic to $P(\Omega_H)$, giving a contradiction. Hence $\Omega_H$ is not holomorphically convex.

Notice that $\hat{\Omega}_H \cong \hat{P}(\hat{\Omega}_H)$ contains $P(\Omega_H) = H \cdot \exp_{H^C}(i\hat{P}(O_0))$ and the convex envelope of $\hat{P}(O_0)$ is $\mathfrak{h}$, thus by the above arguments the envelope of holomorphy $\hat{\Omega}$ is biholomorphic to $H^C \cong \mathbb{C}^3$.

Finally the maximal convex $\sigma$-invariant subdomain of $\hat{P}(O_0)$ is $\{ (A, B, C) \in \mathfrak{h} : A^2 + B^2 < 3 \} = \hat{P}(O_2)$, where $O_2 := \{ (a, b, c) \in \mathfrak{h} : a^2 + b^2 < 3 \sinh^{-1}(c) \}$. One checks that $O_2 \subset O_1$, thus $\hat{P}|_{O_2}$ is injective and $H \cdot O_2$ is biholomorphic to $H \cdot \exp_{H^C}(i\hat{P}(O_2))$. Moreover one has

$$\exp_{H^C}( (A', B', C') + i(A, B, C) ) = \exp_H(A', B', C') \exp_{H^C}( i(A, B, C - \frac{1}{2}(A'B - AB') ) ).$$

It follows that

$$\exp_{H^C}^{-1}(H \cdot \exp_{H^C}(i\hat{P}(O_2)) = \{ (\text{Im} \, z_1)^2 + (\text{Im} \, z_2)^2 < 3 \},$$

where $(z_1, z_2, z_3) = (A' + iA, B' + iB, C' + iC)$ are natural complex coordinates of $h^C \cong \mathbb{C}^3$ and this yields the statement.

Remark. Since $\hat{P}$ is injective on $O_1$, the $H$-invariant domain defined by $O_1$ is holomorphically separable. As a matter of fact one may show that $\hat{P}(O_1) = \mathfrak{h} \setminus \{ (A, B, C) \in \mathfrak{h} : A^2 + B^2 \geq 3, C = 0 \}$ and analogous arguments as above show that such $H$-invariant domain is not holomorphically convex.
5. Generalized Heisenberg groups

Here we apply results of the previous section to generalized Heisenberg groups exhibiting additional examples of non-Stein maximal domains of existence for the adapted complex structure. We refer to [BTV] for the basic properties of generalized Heisenberg groups.

Let $G$ be a generalized Heisenberg group with Lie algebra $\mathfrak{g}$, consider its abelian subalgebra $\mathfrak{z} := [\mathfrak{g}, \mathfrak{g}]$ and the subspace $\mathfrak{v}$ orthogonal to $\mathfrak{z}$ with respect to the $G$-invariant metric $(\cdot, \cdot)$ of $G$. Then for all $V + Y \in \mathfrak{v} \oplus \mathfrak{z} = \mathfrak{g} \cong T_eG$ the unique geodesic tangent to $V + Y$ at 0 can be explicitly given by $t \mapsto \exp G \circ \varphi_G(t, V + Y)$ for a certain real-analytic map $\varphi_G : \mathbb{R} \times T_eG \to \mathfrak{g}$ (see [BTV, Th. p. 31]).

A straightforward computation shows that $\varphi_G$ extends holomorphically on $\mathbb{C} \times T_eG$ to $\mathfrak{g}^{\mathbb{C}}$ and analogous arguments as in the previous section imply that $\Omega_G := G \cdot O_G \subset T_G$ is the maximal the adapted complexification, where $O_G$ is the connected component of

$$\{ V + Y \in T_eG : \det(D\tilde{P}_G)_{V+Y} \neq 0 \}$$

containing 0 and $\tilde{P}_G : T_eG \to \mathfrak{g}$ is given by

$$V + Y \mapsto l(|Y|)V + (1 + |V|^2m(|Y|))Y.$$

Here $| \cdot |$ denotes the norm induced by $(\cdot, \cdot)$ and the real-analytic functions $l, m : \mathbb{R} \to \mathbb{R}$ are defined by

$$l(t) := \frac{\sinh(t)}{t}, \quad m(t) := \frac{t - \sinh(t) \cosh(t)}{2t^3}.$$

Now for $V + Y \in \mathfrak{v} \oplus \mathfrak{z} = T_eG$ with $Y \neq 0$ and $U + X \in T_{V+Y}T_eG \cong T_eG$ one has

$$(D\tilde{P}_G)_{V+Y}(U+X) = \frac{\partial}{\partial t} \tilde{P}_G((V+tU)+(Y+tX))|_{t=0} = l'(|Y|)\frac{(Y, X)}{|Y|}V + l(|Y|)U$$

$$+ \left( 2(V, U)m(|Y|) + |V|^2m'(|Y|)\frac{(Y, X)}{|Y|} \right) Y + (1 + |V|^2m(|Y|))X.$$

Note that the equation is written according to the splitting $\mathfrak{v} \oplus \mathfrak{z}$ and since $l(|Y|)$ never vanishes, the $\mathfrak{v}$-part vanishes if and only if

$$U = - \frac{(Y, X)l'(|Y|)}{|Y|l(|Y|)} V.$$
It follows that the central $3$-part also vanishes if and only if
\[ (1 + |V|^2 m(|Y|))X = |V|^2 \frac{(X,Y)}{|Y|} \left( 2 \frac{p'(|Y|)}{l(|Y|)} m(|Y|) - m'(|Y|) \right) Y. \]

In particular $Y$ and $X$ have to be proportional. Since both sides are homogeneous of degree $1$ in $X$, then $(D\varphi)_{(V+Y)}$ is singular if and only if
\[ (5) \quad 1 + |V|^2 m(|Y|) = |V|^2 |Y| \left( 2 \frac{p'(|Y|)}{l(|Y|)} m(|Y|) - m'(|Y|) \right) Y. \]

An analogous computation shows that $(D\tilde{P}G)_{V+Y}$ has maximal rank if $Y = 0$, thus equation (5) describes the singular locus of $D\tilde{P}G$. It is remarkable that this identity is independent of the fine structure of the generalized Heisenberg group, e.g., of its dimension or the dimension of its centre. In particular if $H$ is the $3$-dimensional Heisenberg group considered in the previous section, equation (5) determines the boundary of $O_H = \Omega_H \cap T_e H$. Using this fact we are now going to show that for a generalized Heisenberg group $G$ there exist many copies of $\Omega_H$ embedded as closed submanifolds in $\Omega_G$.

Let $G$ be a generalized Heisenberg group and choose non-zero elements $\bar{V}_1 \in \mathfrak{v}$ and $\bar{Y} \in \mathfrak{z}$. Then there exists an element $\bar{V}_2 \in \mathfrak{v}$ such that the closed subgroup $\text{exp}_G(\text{span}\{\bar{V}_1, \bar{V}_2, \bar{Y}\})$ is a totally geodesically embedded $3$-dimensional Heisenberg group (see [BTV, p. 30]). Denote by $I : H \to G$ such an embedding and note that since $\text{exp}_{G^C} : \mathfrak{g}^C \to G^C$ is a biholomorphism, $I$ extends to a holomorphic embedding $I^C : H^C \to G^C$ of the universal complexification of $H$ into the universal complexification of $G$ such that the diagram

\[
\begin{array}{ccc}
\mathfrak{h}^C & \xrightarrow{D_{I^C}} & \mathfrak{g}^C \\
\downarrow \exp_{H^C} & & \downarrow \exp_{G^C} \\
H^C & \xrightarrow{I^C} & G^C
\end{array}
\]

commutes. Now $I : H \to G$ is totally geodesic, thus $t \mapsto I \circ \exp_H \circ \varphi_H(t,v)$ is the unique geodesic of $G$ tangent to $DI(v)$ at $0$ for all $v \in \mathfrak{h}$. Then
\[ I \circ \exp_H \circ \varphi_H(t,v) = \exp_G \circ \varphi_G(t,DI(v)) \]
and by the identity principle
\[ (6) \quad I^C \circ \exp_{H^C} \circ \varphi_H(z,v) = \exp_{G^C} \circ \varphi_G(z,DI(v)) \]
for all $z \in \mathbb{C}$, since this holds for all $z \in \mathbb{R}$. The commutativity of the diagram
follows. For this note that being $I$ a group homomorphism then $DI : TH \rightarrow TG$ is $H$-equivariant, i.e., $DI(g \ast w) = I(g) \ast DI(w)$ for all $g \in H$ and $w \in TH$. In particular

$$P_G \circ DI(g \ast v) = P_G(I(g) \ast DI(v)) = I(g) \exp_{G^c} \circ \varphi_G(i, DI(v))$$

for all $g \in H$ and $v \in T_eH$. On the other hand using equation (6) one obtains

$$I^C \circ P_H(g \ast v) = I^C(g \exp_{H^c} \circ \varphi_H(i, v)) = I^C(g) I^C(\exp_{H^c} \circ \varphi_H(i, v)) = I(g) \exp_{G^c} \circ \varphi_G(i, DI(v)),$$

showing that the above diagram is commutative.

From the equivariance of $DI$ it follows that $DI(O_H) = I(H) \cdot DI(O_H) = I(H) \cdot DI(O_H)$ and since $DI$ is isometric and the boundary of $O_H$ is defined by equation (5) which also describes the singular locus of $\tilde{P}_G$ one has

$$DI(O_H) \subset O_G \cap \text{span}\{\tilde{V}_1, \tilde{V}_2, \tilde{Y}\}, \quad DI(\partial O_H) \subset \partial O_G.$$

Thus $DI(O_H) \cong I(H) \times DI(O_H)$ is closed in $O_G \cong G \times O_G$.

Furthermore $DI$ is injective and $P_H$, $P_G$ are locally biholomorphic where the adapted complex structure is defined (cf. Theorem 3.2), thus diagram (7) shows that $DI(O_H) \cong O_H$ is a closed complex submanifold of $O_G$. Finally by Proposition (4.3) the domain $O_H$ is neither holomorphically separable, nor holomorphically convex, thus one has

**Proposition 5.1.** Let $G$ be a generalized Heisenberg group. Then the maximal adapted complexification $\Omega_G$ is neither holomorphically separable, nor holomorphically convex.
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