Satake compactifications, Lattices and Schottky problem

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Thesis submitted for the degree of
Doctor of Philosophy

December 2013

To all teachers,
regardless of topic and level

Declaration of originality

This Ph.D. thesis was prepared under the supervision of Prof. N. Shepherd-Barron. It is mainly based on the joint work [CSB13], which will appear on Compositio Mathematica, and on my pre-print [Cod13]. Some of the results were also appeared in the essay that I submitted for the Smith-Knight and Rayleigh-Knight Prize 2013.

Sections 2, 3, 4 and 6 are background material, except that Lemma 3.1 is original. Section 7.2 is the main result of the joint work [CSB13]. Everything else is my own work except for known results that are signalled in the text.

Abstract

We prove some results about the singularities of Satake compactifications of classical moduli spaces. This will give an insight into the relation among solutions of the Schottky problem in different genera.

We denote Satake compactifications by a superscript script "S". The moduli space $\mathcal{A}_g$ lies in the boundary of $\mathcal{A}^S_{g+m}$ for every m. We will show that the intersection between $\mathcal{M}^S_{g+m}$ and $\mathcal{A}_g$ contains the m-th infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$. This implies that stable equations for $\mathcal{M}_g$ do not exist. In particular, given two inequivalent positive even unimodular quadratic forms P and Q, there is a curve whose period matrix distinguishes between the theta series of P and Q; we are able to compute its genus in the rank 24 case.

On the other hand, the intersection of $\mathcal{A}_g$ and Hyp^S_{g+m} is transverse: this enables us to write down many new stable equations for Hyp_g in terms of theta series. We have similar results for Prym's varieties.

Our work relies upon a formula for the first order part of the period matrix of some degenerations.

Acknowledgement

I would like to thank my advisor Nick Shepherd-Barron, without whom this work would not have been possible. He has introduced me to many different topics and his support has been essential.

During these years I have benefited of the expertise and generosity of Riccardo Salvati Manni. I owe a thank to Enrico Arbarello, who first introduced me to the theory of Riemann surfaces and somehow predisposed me to enjoy Fay's work. I am happy to thank my mentor Alfonso Sorrentino.

I had the pleasure and benefit of conversations with many mathematicians about the topics of my thesis, let me thank Gavril Farkas, Sam Grushevsky, Olivier Mathieu, Marco Matone, Oscar Randal-Williams, Filippo Viviani and Roberto Volpato. I received useful advices form my examiners Miles Reid and Burt Totaro.

I would like to thank all my collegues with whom I have shared this important experience, to name a few: Gabriele Benedetti, Gareth Ainsworth, Cangxiong Chen, Ruadhaí Dervan, Carmelo Di Natale, Marco Golla, Peter Herbrich, Sebastian Koch, Achilleas Kryftis, John Christian Ottem, Volker Schlue, Christina Vasilakopoulou and Joe Waldron. And also Andrea Fanelli.

My Ph.D. is funded by the DPMMS and the EPSRC.

Contents

1 Introduction

In our thesis we investigate some questions related to the Schottky problem and modular forms; the ground field will always be the field of complex numbers.

To a smooth complex genus g curve C one can associate its Jacobian variety $\mathrm{Jac}(C)$. This can be done in families, so we have the Jacobian morphism

$$\begin{array}{rccc} J\colon & \mathcal{M}_g & \to & \mathcal{A}_g \\ & C & \mapsto & \mathrm{Jac}(C) \end{array}$$

where $\mathcal{M}_g$ denotes the moduli space of smooth genus g curves and $\mathcal{A}_g$ the moduli space of g dimensional principally polarised abelian varieties. By Torelli's Theorem, the Jacobian morphism is injective. There is an important difference between the moduli stack and the associated coarse space. In this paper, we will mainly work on coarse spaces; in particular, all the Satake compactifications are coarse spaces.

The *Schottky problem* is to understand when an abelian variety is the Jacobian of a curve. This problem has many aspects; we are interested in the modular forms viewpoint. A degree g and weight k modular form is a section of the line bundle $L_g^{\otimes k}$ on the stack $\mathcal{A}_g$, where L_g is the determinant of the Hodge bundle. We would like to find modular forms vanishing on $\mathcal{M}_g$. In other words we are looking for the equations of $\mathcal{M}_g$ in $\mathcal{A}_g$. The same question makes sense for the locus of hyperelliptic curves Hyp_g.

The traditional approach is to define cusp modular forms using second order theta functions and then apply the Schottky-Jung relations: in this way one obtains an ideal S_g of equations for $\mathcal{M}_g$. In [vG84] is proved that, for $g > 4$, the ideal S_g cuts out a reducible subscheme and $\mathcal{M}_g$ is one of its irreducible components; in the more recent [vG13] some explicit examples are given.

There is a different and rather surprising way to construct modular forms: given an even unimodular positive definite lattice Λ, for every integer g, the associated theta series $\Theta_{\Lambda,g}$ is a weight $\frac{1}{2}\,\mathrm{rk}(\Lambda)$ modular form on $\mathcal{A}_g$ (see Definition 2.7). There is not a general way to express a theta series in term of theta functions. One of the aim of our work is to relate the arithmetic properties of the lattices to the behaviour of the theta series on $\mathcal{M}_g$ and Hyp_g.

In section 2.2, in order to give more structure to these theta series, we first consider the Satake compactification $\mathcal{A}_g^S$ of $\mathcal{A}_g$: this is a normal projective variety constructed using modular forms. The boundary of $\mathcal{A}_g^S$ is isomorphic to $\mathcal{A}_{g-1}^S$, thus we can construct the commutative ind-monoid

$$\mathcal{A}_\infty := \bigcup_{g\geq 0} \mathcal{A}_g^S\,,$$

where the monoid operation is the product of abelian varieties. We fix a lattice Λ and we consider the theta series all at once:

$$\Theta_\Lambda := \bigcup_{g\geq 0} \Theta_{\Lambda,g}\,;$$

in this way we obtain a *stable modular form* (see Definition 2.6). As discussed in section 2.2, theta series are also characters for the monoid $\mathcal{A}_\infty$.

In section 7.1, we consider the Satake compactification $\mathcal{M}_g^S$ of $\mathcal{M}_g$: this is the closure of $\mathcal{M}_g$ inside $\mathcal{A}_g^S$. Similarly, in section 8.1 we construct the Satake compactification Hyp_g^S of the hyperelliptic locus Hyp_g. The intersection of $\mathcal{M}_{g+m}^S$ (respectively Hyp_{g+m}^S) with $\mathcal{A}_g$ is, as set, $\mathcal{M}_g^S$ (Hyp_g^S) and we can construct the ind-monoids

$$\mathcal{M}_\infty := \bigcup_{g\geq 0} \mathcal{M}_g^S \qquad Hyp_\infty := \bigcup_{g\geq 0} Hyp_g^S \,.$$

These are sub-monoids of $\mathcal{A}_\infty$. Using lemma 2.3, we can show that the ideal of stable modular forms vanishing on them is generated as a vector space by differences of theta series

$$\Theta_\Lambda - \Theta_\Gamma \,.$$

The natural question is to describe these ideals in greater detail. To tackle this problem we give a precise description of the local structure of $\mathcal{M}_{g+1}^S$ at $\mathcal{M}_g$ and of Hyp_{g+1}^S at Hyp_g (cf. Sections 7.2, 7.3, 8.2 and 8.1). We will show how the behaviours of $\mathcal{M}_\infty$ and Hyp_∞ are completely different: compare Theorem 1.1 with Theorem 1.6, Corollary 1.2 with Theorem 1.9.

Our first result was obtained in [CSB13]:

Theorem 1.1 (=Theorem 7.9). *The intersection of $\mathcal{M}_{g+m}^S$ and $\mathcal{A}_g$ is not transverse, it contains the m-th infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$.*

The proof relies upon a construction of a degenerating family of curves and a formula for the first order part of its period matrix. Our first source about this topic was [Fay73]; however, its formula turns out to be false and a correction is provided in [Yam80]. In section 4 we review his result. After this, the proof is a local computation at the boundary of $\mathcal{M}_{g+1}^S$; it is carried out in section 7.2.

As pointed out in [CSB13], Theorem 1.1 has an important consequence for the Schottky problem:

Corollary 1.2 (=Corollary 7.6). *The ideal of stable modular forms vanishing on $\mathcal{M}_\infty$ is trivial.*

However, as shown below, it might be interesting to discuss further the behaviour of differences of theta series on $\mathcal{M}_g$.

Lemma 1.3 (=Lemma 7.8). *Given any two lattices Λ and Γ of the same rank, there exists an integer $\bar{g} = \bar{g}(\Lambda, \Gamma) > 0$ such that the modular form*

$$F_g = \Theta_{\Lambda,g} - \Theta_{\Gamma,g}$$

- *is zero on $\mathcal{M}_g$ for $g < \bar{g}$;*
- *it is not zero on $\mathcal{M}_{\bar{g}}$ and it cuts out a divisor of finite slope $s = s(\Lambda, \Gamma)$;*
- *it is not zero on $\mathcal{M}_g$ for $g > \bar{g}$ and cuts out a divisor of slope ∞.*

The classical, and for us motivating, Example is the Schottky form

Example 1.4. [Schottky form] The Schottky form is the difference of theta series

$$\Theta_{D_{16}^+} - \Theta_{E_8 \oplus E_8}\,.$$

Following the notation of Lemma 1.3, we have $\bar{g} = 5$ and $s = 8$. The proof has a long history, the main steps are taken in [Sch88], [Igu81] and [GSM11], see also Example 2.8. Moreover, the Schottky form vanishes on the hyperelliptic locus for every genus (cf. [Poo96]).

In [Cod13], we extend this result by considering rank 24 lattices.

Theorem 1.5 (= Corollary 11.2)**.** *Let Λ and Γ be two even positive definite unimodular lattices of rank 24 with the same number of vectors of norm 2, then the stable modular form*

$$F := \Theta_\Lambda - \Theta_\Gamma$$

vanishes on $\mathcal{M}_g$ for $g \leq 4$ and it cuts out a divisor of slope 12 on $\mathcal{M}_5$; with the notation of Lemma 1.3, $\bar{g} = 5$ and $s = 12$.

The key point of the proof is to show that, using the known facts about the slope of $\mathcal{M}_g$ for low values of g, the form F_{g+1} vanishes on $\mathcal{M}_{g+1}$ if and only if its restriction to $\mathcal{A}_g^S$ vanishes on $\mathcal{M}_g^S$ with multiplicity at least 2 (see Theorem 11.1). To prove this, first we show that the Fourier-Jacobi coefficients of F_{g+1} satisfy the heat equation (cf. Theorem 10.3). Then, we need the description of the boundary of the Satake compactification given in Theorem 7.14 and the theory of order two theta functions reviewed in section 3.

As a consequence of Theorem 1.5, we can also obtain explicit cusp forms on $\mathcal{A}_5$ (see Theorem 11.3). Similar results are obtained in [GV09] and [GKV10], however the terminology and the techniques are rather different. We do not consider rank 32 lattices, but we expect much more different behaviours.

We now turn our attention to the hyperelliptic locus. It is proved in [Poo96] that the Schottky form defined in example 1.4 is a *stable equation for the hyperelliptic locus*, i.e. it vanishes on Hyp_g for every g, so corollary 1.2 does not hold for Hyp_∞. In contra-position to Theorem 1.1, we can prove the following geometric result:

Theorem 1.6 (= Theorem 8.6)**.** *The intersection of $\mathcal{A}_g^S$ and Hyp_{g+1}^S is transverse.*

This means that the existence of stable equations for the hyperelliptic locus is entirely possible. In fact, in [Cod13] we are able to construct a number of explicit examples as follows. Given a lattice Λ, call $\mu(\Lambda)$ the norm of the shortest non-trivial vectors.

Theorem 1.7 (= Theorem 8.12)**.** *Let Λ and Γ be two even positive definite unimodular lattices of rank N and $\mu(\Lambda) = \mu(\Gamma) =: \mu$, if*

$$\frac{N}{\mu} \leq 8\,,$$

then

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is a stable equation for the hyperelliptic locus.

The proof relies upon Theorem 1.6 and the well known Criterion 8.10 based on projective invariants of hyperelliptic curves. The hypotheses on the lattices are quite restrictive, for example they imply that the rank is smaller than or equal to 48. However, this result holds for rank 32 lattices with $\mu = 4$ and in [Kin03] it is shown that there exist at least ten million of such lattices.

Using the description of the tangent space given Theorem 8.4, we can prove that also the stable modular forms discussed in Theorem 1.5 are stable equations for the hyperelliptic locus, although they do not satisfy the hypotheses of Theorem 1.7.

Theorem 1.8 (= Theorem 11.4). *Let Λ and Γ be two even positive definite unimodular lattices of rank 24 with the same number of vectors of norm 2, then*

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is a stable equation for the hyperelliptic locus.

Summarising, if we combine Theorem 8.2 and Theorem 1.7 we can give the following picture:

Theorem 1.9. *The ideal of stable equations for the hyperelliptic locus is generated by differences of theta series, there are more than* $10,000,000$ *of linearly independent differences of theta series vanishing on Hyp_∞.*

We actually show a bit more: any stable equation for the hyperelliptic locus is a linear combination of differences of theta series. An open question is whether the ideal of stable equations for the hyperelliptic locus is big enough to define Hyp_∞ in $\mathcal{A}_\infty$.

The n-gonal locus, for $n > 2$, behaves similarly to $\mathcal{M}_g$. In [SB13], there is a precise description of the tangent space to the boundary of the Satake compactification of the locus of n-gonal curves with total ramification. As a consequence, it is shown that the ideal of stable equations for the n-gonal locus is trivial.

We carry out a similar analysis for the Prym loci. We know that $\mathcal{M}_g$ lies in the closure of the Prym locus, so we can not have stable equations. Anyway, we can define the Satake compactification of the Prym locus and study its boundary.

In section 9.2 we study the locus $\mathcal{P}_g$ of Prym varieties arising from êtale double covers of curves. By proving some variational formulae, we obtain:

Theorem 1.10 (= Theorem 9.1). *The intersection of $\mathcal{P}^S_{g+m}$ and $\mathcal{A}_g$ contains the m-th infinitesimal neighbourhood of $\mathcal{P}_g$ in $\mathcal{A}_g$.*

Theorem 1.11 (= Theorem 9.5). *$\mathcal{P}^S_g$ contains the first infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$.*

Let us point out that both $\mathcal{M}_{g+1}^S$ and $\mathcal{P}_g^S$ contain the first infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$, this could be related to the Schottky-Jung relations. In section 9.3 we carry out a similar analysis for Prym varieties of double covers branched at two points.

In the title of [Cod13] we use the term "non-perturbative Schottky problem". This term derives from string theory; indeed, a non-perturbative bosonic string theory should take into account the spaces $\mathcal{M}_g$ all at once (see e.g. [BNS96]), so it should be constructed on $\mathcal{M}_\infty$. Moreover, stable modular forms could be non-perturbative partition functions. Usually, one fixes the genus and then consider the Schottky problem; in our work we look at how $\mathcal{M}_\infty$ and Hyp_∞ sit inside $\mathcal{A}_\infty$.

2 Stable modular forms

2.1 An introduction to ind-monoids

We recall some definitions about ind-varieties and we specialise them for ind-monoid. A reference is [Kum02] Chapter IV.

An ind-variety is a set X with a filtration X_n such that each X_n is a finite dimensional algebraic variety and the inclusion of X_n in X_{n+1} is a closed embedding. A line bundle L on X is the data of a line bundle L_n on each X_n compatible with the restriction. A section s is a collection of sections s_n compatible with the restriction. The ring of sections of L is thus defined as a projective limit in the category of graded rings

$$\mathcal{R}(X, L) := \varprojlim \mathcal{R}(X_n, L_n)\,.$$

An ind-monoid is an ind-variety M with an associative multiplication and an identity element 1_M. A multiplication is a family of maps

$$m_{g,h}\colon M_g \times M_h \to M_{g+h}$$

compatible with the restrictions. M is commutative if the multiplication is.

Definition 2.1 (Split monoid)**.** *Let M be a commutative ind-monoid and L a line bundle on M. We say that M is split with respect to L if the following two conditions hold:*

1. *For every g and h*
$$m^*_{g,h} L_{g+h} \cong pr_1^* L_g \otimes pr_2^* L_h$$
where pr_i are the projections;

2. *the ring of sections $\mathcal{R}(M, L)$ is generated as vector space by characters, where a section χ of L is a character if*
$$m^*_{g,h}\chi_{g+h} = \chi_g \chi_h \qquad \forall g, h\,.$$

Remark that the definition of character makes sense only if condition (1) holds. We write $\chi(\alpha\beta) = \chi(\alpha)\chi(\beta)$ instead of $\chi_{g+h}(\alpha\beta) = \chi_g(\alpha)\chi_h(\beta)$.

Lemma 2.2. *Let M be a commutative monoid, suppose it is split with respect to a line bundle L, then the characters are linearly independent.*

Proof. This proof is standard. We argue by contradiction. Take n minimal such that there exist n linearly dependent characters $\chi_1, \dots, \chi_n$. We can write

$$\chi_n = \sum_{i=1}^{n-1} \lambda_i \chi_i \qquad \lambda_i \in \mathbb{C}\,.$$

Take $\alpha \in M$ such that $\chi_1(\alpha) \neq \chi_n(\alpha)$, for any $\beta \in M$ we have

$$\sum_{i=1}^{n-1} \lambda_i \chi_i(\alpha)\chi_i(\beta) = \chi_n(\alpha)\chi_n(\beta) = \chi_n(\alpha)\left(\sum_{i=1}^{n-1} \lambda_i \chi_i(\beta)\right)$$

Since β is arbitrary we get

$$\sum_{i=i}^{n-1} \lambda_i(\chi_i(\alpha) - \chi_n(\alpha))\chi_i = 0\,.$$

The coefficient $\chi_1(\alpha) - \chi_n(\alpha)$ is non-zero, so we have written a non-trivial linear relation among fewer than n characters. This contradicts the minimality of n. $\square$

Lemma 2.3. *Let M be a commutative ind-monoid and N a submonoid. Suppose that M is split with respect to a line bundle L. Then the ideal I_N in $\mathcal{R}(M, L)$ of sections vanishing on N is generated by differences of characters*

$$\chi_i - \chi_j$$

Proof. Take s in I_N; since $\mathcal{R}(M, L)$ is generated as a vector space by characters we can write

$$s = \lambda_1\chi_1 + \cdots + \lambda_n\chi_n$$

where χ_i are characters and λ_i are constants. Restricting χ_i to N some of them might become equal. Up to relabelling the χ_i, we can fix integers $0 = m_0 < m_1 < \cdots < m_k = n$ and distinct characters $\theta_1, \ldots, \theta_k$ of N such that

$$\chi_i \mid_N = \theta_j \quad \iff \quad m_{j-1} < i \leq m_j$$

For $j = 1, \ldots, k$, let us define

$$\mu_j := \sum_{i=m_{j-1}+1}^{m_j} \lambda_i\,.$$

By hypothesis we know that

$$0 = s \mid_N = \sum_{j=1}^{k} \mu_j\theta_j\,.$$

By Lemma 2.2 we have $\mu_j = 0$ for every j, so

$$s = s - \sum_{j=1}^{k} \mu_j\chi_{m_j} = \sum_{j=1}^{k} \sum_{i=m_{j-1}+1}^{m_j} \lambda_i(\chi_i - \chi_{m_j})$$

and the differences $\chi_i - \chi_{m_j}$ vanish on N for $m_{j-1} < i \leq m_j$. $\square$

The previous argument actually shows that every element of the ideal can be written as a linear combination of differences of characters.

2.2 Satake compactification of the moduli space of abelian varieties

We recall some facts about modular forms and the Satake compactification of $\mathcal{A}_g$. General references are [BvdGHZ08], [Mum07] and [Fre83]. The Satake compactification was first defined in [Sat56].

The line bundle L_g of weight one modular forms on the stack $\mathcal{A}_g$ is the determinant of the Hodge bundle, it is ample.

Definition 2.4 (Siegel modular form). *A weight k and degree g Siegel modular form is a section of L_g^k on $\mathcal{A}_g$.*

The universal cover of $\mathcal{A}_g$ is the Siegel upper half space $\mathcal{H}_g$, the symplectic group $Sp(2g,\mathbb{Z})$ acts on $\mathcal{H}_g$ and

$$\mathcal{A}_g = \mathcal{H}_g / Sp(2g,\mathbb{Z})$$

A modular form can be also defined as a holomorphic function on $\mathcal{H}_g$ which transforms appropriately under the action of $Sp(2g,\mathbb{Z})$.

The Satake compactification $\mathcal{A}_g^S$ is a normal projective variety defined as follows

$$\mathcal{A}_g^S := \mathrm{Proj}(\bigoplus_{n\geq 0} H^0(\mathcal{A}_g, L_g^n))$$

The Siegel operator is a map of graded rings

$$\Phi : \bigoplus_{n\geq 0} H^0(\mathcal{A}_g, L_g^n) \to \bigoplus_{n\geq 0} H^0(\mathcal{A}_{g-1}, L_{g-1}^n) \tag{1}$$

defined as

$$\Phi(F)(\tau) := \lim_{t\to+\infty} F(\tau \oplus it)\,,$$

where τ is an element of $\mathcal{H}_g$. The Siegel operator is surjective for n even and larger than $2g$ ([Fre83] page 64), so it defines a stratification

$$\mathcal{A}_g^S = \mathcal{A}_g \sqcup \mathcal{A}_{g-1}^S = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \cdots \mathcal{A}_1 \sqcup \mathcal{A}_0\,.$$

In other words, the boundary $\partial\mathcal{A}_g^S$ is isomorphic to $\mathcal{A}_{g-1}^S$ and the Siegel operator is the restriction map from $\mathcal{A}_g^S$ to $\mathcal{A}_{g-1}^S$. Moreover, L_g restricts to L_{g-1}. Let us just recall the proof of the normality of $\mathcal{A}_g^S$.

Proposition 2.5 ([Fre83] page 118). *The variety $\mathcal{A}_g^S$ is normal.*

Proof. The Satake compactification is defined as the Proj of a ring; we are going to show that this ring is normal, which is actually a bit more than what we need. Take two degree g modular forms F and G of weight d and e such that their ratio satisfies a polynomial with coefficient in $\mathcal{R}(\mathcal{A}_g, L_g)$. We pull everything back to the Siegel upper half space $\mathcal{H}_g$, this is a smooth variety, in particular it is normal we thus conclude that F/G is a holomorphic function on $\mathcal{H}_g$. This ratio transform under the action of the symplectic group $Sp(2g,\mathbb{Z})$ as a modular form of weight $d-e$. Being F/G holomorphic $d-e$ must be positive, so we conclude that the ration F/G belongs to $\mathcal{R}(\mathcal{A}_g, L_g)$. $\square$

We can now define the commutative ind-monoid

$$\mathcal{A}_\infty := \bigcup_{g\geq 0} \mathcal{A}_g^S$$

the multiplication is the product of abelian varieties

$$\begin{array}{rccc} m_{g,h}: & \mathcal{A}_g^S \times \mathcal{A}_h^S & \to & \mathcal{A}_{g+h}^S \\ & ([X],[Y]) & \mapsto & [X\times Y] \end{array}$$

the identity element is $\mathcal{A}_0$. The line bundles L_g define a line bundle L on $\mathcal{A}_\infty$ of weight one stable stable modular form. We have

$$m_{g,h}^* L_{g+h} = pr_1^* L_g \otimes pr_2^* L_h\,.$$

Definition 2.6 (Stable modular forms)**.** *A weight k stable modular forms F is a section of L^k. More concretely, is a collection of weight k modular forms*

$$F = \bigcup_{g\geq 0} F_g$$

where F_g has degree g and

$$\Phi(F_{g+1}) = F_g$$

We can obtain stable modular forms from even positive definite unimodular lattices. Let us go trough all definitions. A lattice is a couple (Λ, Q), where Λ is a finitely generated free group and Q is a $\mathbb{Z}$-valued bilinear form on Λ. The rank of the lattice is the rank of Λ, elements of Λ are called vectors, the norm of a vector v is $Q(v,v)$. We always assume Q to be even (i.e. $Q(v,v)$ is even for every v), unimodular and positive definite. Often, we will denote a lattice just by Λ, omitting Q.

Definition 2.7 (Theta series)**.** *Let (Λ, Q) be an even unimodular positive definite lattice and g an integer, the associated theta series is*

$$\Theta_{\Lambda,g}(\tau) := \sum_{x_1,\ldots,x_g\in\Lambda} \exp(\pi i \sum_{i,j} Q(x_i,x_j)\tau_{ij})$$

where τ belongs to $\mathcal{H}_g$.

This is a weight $\frac{1}{2}\operatorname{rk}(\Lambda)$ and degree g modular form. The Siegel operator (1) acts as follows

$$\Phi(\Theta_{\Lambda,g+1}) = \Theta_{\Lambda,g}$$

so the union of all theta series

$$\Theta_\Lambda := \bigcup_{g\geq 0} \Theta_{\Lambda,g}$$

is a stable modular form. Given $X \in \mathcal{A}_g$ and $Y \in \mathcal{A}_h$, we have the factorisation property

$$\Theta_{\Lambda,g+h}([X \times Y]) = \Theta_{\Lambda,g}(X)\Theta_{\Lambda,h}(Y)\,,$$

which means that the **theta series are characters** for the monoid $\mathcal{A}_\infty$. Let us give an example.

Example 2.8 (Witt's lattices, [Igu81]). Using the same definition of the lattice E_8, for every integer k one can define the Witt lattices W_{8k}. The lattice W_{8k} has rank $8k$, it is equal to E_8 for $k = 1$ and to D_{16}^+ for $k = 2$. We have the following expansion

$$\Theta_{W_{8k},g}(\tau) = \sum_{\epsilon\,\text{even}} \theta[\epsilon]^{8k}(\tau)\,,$$

where the sum runs over all the even theta characteristics. In particular, the Schottky form defined in 1.4 can be written as

$$\sum_{\epsilon\,\text{even}} \theta[\epsilon]^{16}(\tau) - (\sum_{\epsilon\,\text{even}} \theta[\epsilon]^{8}(\tau))^2$$

It is proved in [Fre77] Theorem 2.2 that the ring of stable modular form is generated as a vector space by theta series, so $\mathcal{A}_\infty$ is split with respect to L.

Theorem 2.9 ([Fre77] Theorem 2.5). *The ring of stable modular forms $\mathcal{R}(\mathcal{A}_\infty, L)$ is the polynomial ring in the Theta series associated to irreducible lattices.*

We will just prove that the theta series are algbraically indepent, assuming that they generate the ring.

Proof. Let P be the vector space with basis the theta series associated to lattices. Given two lattices Λ and Γ, one can check that

$$\Theta_{\Gamma\oplus\Lambda} = \Theta_\Gamma\Theta_\Lambda\,,$$

so any theta series is a monomial in theta series associated to irreducible lattices and P is the polynomial ring in the theta series associated to irreducible lattices. We endow P with the co-multiplication $\mu(\Theta_\Lambda) = \Theta_\Lambda \otimes \Theta_\Lambda$, so $\mathrm{Proj}(P)$ is a commutative monoid.

We now argue by contradiction. Suppose that

$$\mathcal{R}(\mathcal{A}_\infty, L) = P/I$$

where I is some ideal. This mean that $\mathcal{A}_\infty$ is a submonoid of $\mathrm{Proj}(P)$, so we can apply Lemma 2.3 and show that I is generated by difference of theta series. To show that the ideal I is trivial, it is enough to prove that, given any two lattices Λ and Γ of rank N, their theta series are different for $g >> 0$. To see this we can take $g = N$ and look at the coefficients corresponding to Γ and Λ in the Fourier expansions of the theta series. $\square$

The highlight of this section is that we care about theta series because they are all the characters of the monoid $\mathcal{A}_\infty$. In [GV09] and [GKV10], the authors study the link between theta series and the vacuum to vacuum partition function of a string moving in the torus defined by the lattice.

3 The variety C-C

Let C be generic genus g curve, the surface $C-C$ is the image of the subtraction map

$$\begin{array}{cccc} \delta\colon & S^2C & \to & \mathrm{Km}(C) \\ & (a,b) & \mapsto & AJ(a-b) \end{array} \tag{2}$$

where S^2C is the second symmetric product of C, $\mathrm{Km}(C)$ is the Kummer variety of the Jacobian of C and AJ is the Abel-Jacobi map. Remark that the map is independent of the base point of the Abel-Jacobi map.

This variety is relevant to many questions and has been study for a long time. In our case, the results of this section will be important for the study of the boundary of the Satake compactification (cf. Section 7.3) and to understand theta series (cf. Section 11).

It is known that the map δ is an isomorphism outside the diagonal Δ and it contracts the diagonal to the origin 0, in symbols

$$\delta\colon (S^2C,\Delta) \to (C-C,0)$$

Moreover, the tangent cone of $C-C$ at 0 is the cone over the bicanonical model of C (cf. [Gae75]). We can say a bit more:

Lemma 3.1. *A neighbourhood of Δ in S^2 is isomorphic to a neighbourhood of the zero section of bitangent bundle of C. In other words, the noramilzation of the singularity $(C-C,0)$ is the cone over the bicanonical model of C.*

Proof. This result is a consequence of a classical result of Grauert ([Gra62] page 363, see also [CM03]). The normal bundle of Δ in S^2C is the bitangent bundle, so an isomorphism between a neighbourhood of the zero section and a neighbourhood of Δ exists at least locally. The obstructions to globalise this isomorphism lie in

$$H^1(C,(2n-1)K_C) \quad \text{for}\, n \geq 1$$

All these groups vanish but when $n=1$. In this case the obstruction is the class of the normal bundle exact sequence.

We first split the normal bundle exact sequence on $C\times C$. Call ι the involution swapping the two factors This involution preserves the diagonal, so we can split $T_{C\times C}\mid_\Delta$ into eigenspaces for ι. The one eignespace is the tangent space to the diagonal, the minus one is the normal space, so we have the requested splitting.

We now look at the quotient map

$$\pi\colon (C\times C,\Delta) \to (S^2C,\Delta)$$

and the decomposition

$$T_{C\times C}\mid_\Delta = T_\Delta \oplus N_{\Delta|C\times C}$$

since ι acts as the identity on the first component and as minus one on the second, we get

$$T_{S^2C}\mid_\Delta = T_\Delta \oplus N^2_{\Delta|C\times C} = T_\Delta \oplus N_{\Delta|S^2C}$$

□

On the Kummer variety of C we have the very ample divisor given by 2Θ, let us study its behaviour with respect to $C-C$. The first tool is a classical formula in the theory of theta functions. A standard reference is the second chapter of [Fay73], see also [MV10] appendix A and [Ber].

Fix a symplectic basis for the homology of C, then we have a corresponding basis $\{\omega_i\}$ for the holomorphic differentials, a period matrix τ and a basis $\{\frac{\partial}{\partial z_i}\}$ for the tangent space at the origin of the Jacobian of C.

Proposition 3.2. *Let s be a section of 2Θ on the Jacobian, for every couple of points a and b of C we have*

$$s(\tau, a-b) = E(a,b)^2[s(\tau,0)\omega(a,b) + \sum_{i,j} \frac{\partial^2 s}{\partial z_i \partial z_j}(\tau,0)\omega_i(a)\omega_j(b)$$

where E is the Prime form, $\omega(a,b)$ is the Szegö Kernel and everything is trivialised with respect to a choice of local co-ordinates z_a and z_b.

Proof. Because of the definition of the prime form, the ratio

$$\frac{s(\tau, a-b)}{E(a,b)^2}$$

is a section of the canonical bundle of S^2C. The space $H^0(S^2C, K)$ is generated by $S^2H^0(C,K)$ and the Szegö kernel, so

$$\frac{s(\tau, a-b)}{E(a,b)^2} = c_0\omega(a,b) + \sum_{i,j} c_{ij}\omega_i(a)\omega_j(b)$$

for some constant coefficients c_0 and c_{ij}. To compute the coefficients one fixes a and write out the Taylor expansion of both sides of the equation at $b=a$. The expansions of $\omega(a,b)$ and $E(a,b)$ are given in [Fay73] page 16-18. $\square$

Now, it is customary to define the linear system

$$\Gamma_{00} := \{s \in H^0(X, 2\Theta) \quad \text{s.t.} \quad \mathrm{mult}_{0_X}(s) \geq 4\}\,.$$

Its first property is the following:

Lemma 3.3. *The kernel of the map*

$$\delta^* : H^0(X, 2\Theta) \to H^0(S^2C, \delta^* 2\Theta)$$

is Γ_{00}. In particular, the base locus of Γ_{00} contains $C-C$.

Proof. The inclusion $\Gamma_{00} \subseteq \mathrm{Ker}(\delta^*)$ follows directly from Proposition 3.2. Let us prove the reverse inclusion. Take s in in the kernel of δ^* and consider the quadric

$$Q := \sum_{i,j} \frac{\partial^2 s}{\partial z_i \partial z_j}(\tau,0)\omega_i\omega_j$$

in $\mathbb{P}H^0(C,K)^\vee$. We know that $s(\tau,0)=0$, applying Proposition 3.2 we get

$$Q(a,b)=0 \quad \forall a,b\in C\,.$$

In particular, Q contains the canonical model of C. Polarizing the bilinear form and varying the local co-ordinates, we show that Q contains the secant variety of the canonical model of C.

Let us recall a general fact. Let X be a variety embedded in a projective space and not contained in any hyperplane. Then the secant variety of X has maximal embedded dimension along X, so if it were contained in a quadric Q then X would be contained in the singular locus of Q, which is linear, so Q must be trivial.

The previouse argument shows that Q si trivial so

$$\frac{\partial^2 s}{\partial z_i \partial z_j}(\tau,0)=0 \qquad \forall i,j$$

. ◻

Similar proofs can be found in [vGvdG86] Proposition 2.1 and [Wel86] Proposition 4.8.

The linear system Γ_{00} makes sense for every principally polarised abelian variety, and it has been used to tackle the Schottky problem:

Theorem 3.4 (Γ_{00}-conjecture, [vGvdG86] and [Wel86])**.** *On a Jacobian, the base locus of* Γ_{00} *is* $C-C$*; on a principally polarized abelian variety which is not the Jacobian of a curve, the base locus of* Γ_{00} *has dimension strictly lesser than two.*

Anyway, here we are not interested in this approach.

Lemma 3.5. *The line bundle* $\delta^*2\Theta$ *is isomorphic to the canonical bundle* K_{S^2C} *of* S^2C *and the map*

$$\delta^*: H^0(X,2\Theta)\to H^0(S^2C,K_{S^2C})$$

is surjective.

Proof. The first assertion is classical, it is equivalent to say that on $C\times C$ we have

$$\delta^*2\Theta = 2(K_1+K_2+\Delta)$$

where Δ is the diagonal and K_i is the pull-back of the canonical bundle via the i-th projection. This last assertion follows from Riemann's Theorem.

The second fact is because of dimensional reasons. The dimension of $H^0(X,2\Theta)$ is 2^g. The kernel of δ^* is Γ_{00} (cf. Lemma 3.3), so it is defined by the $\frac{1}{2}g(g+1)+1$ conditions

$$s(0)=0 \quad , \quad \frac{\partial^2 s}{\partial z_i \partial z_j}(\tau,0)=0 \quad \forall i,j\,.$$

These conditions are linearly independent for non-decomposable abelian variety (see the proof of Lemma 11 page 188 of [Igu72]), so the dimension of Γ_{00} is $2^g-\frac{1}{2}g(g+1)-1$. See also [vGvdG86] Proposition 1.1. ◻

We would like to finish this section by looking at the case in which C is a generic hyperelliptic curve. We define a map

$$\begin{array}{cccccc} \Psi: & C & \stackrel{f}{\to} & S^2C & \stackrel{\delta}{\to} & \mathrm{Km}(C) \\ & p & \mapsto & (p,\iota(p)) & & \\ & & & (a,b) & \mapsto & AJ(a-b) \end{array} \tag{3}$$

where ι is the hyperelliptic involution. Call W the divisor of Weierstrass points on C.

Lemma 3.6. *The line bundle $\Psi^*2\Theta$ is isomorphic to $2(K_C+W)$ and the map*

$$\Psi^*: H^0(X,2\Theta) \to H^0(C,2(K_C+W))$$

is not surjective; it has rank $2g$.

Proof. The line bundle $\delta^*2\Theta$ is isomorphic to $2(K_1+K_2+\Delta)$. The pull back via f of Δ is the locus defined by $p=\iota(p)$, so it is W. The pull back of K_1+K_2 is K_C.

Let us now prove the second assertion. We know, thanks to lemma 3.5, that δ^* is surjective. The pull back of $H^0(S^2C,K)$ to $C \times C$ is generated by $S^2H^0(C,K_C)$ and the Szegö kernel $\omega(a,b)$ (a symmetric differential with a pole of order two and no residue along the diagonal). In particular, $S^2H^0(C,K)$ is the pull back of the sections of 2Θ vanishing on the origin 0_X. Since 0_X is contained in $\Psi(C)$, the kernel of f^* is contained in $S^2H^0(C,K_C)$. We have now to compute the rank of

$$\begin{array}{cccc} f^*: & S^2H^0(C,K) & \to & H^0(C,2K) \\ & \omega_1 \otimes \omega_2 & \mapsto & \omega_1\iota^*\omega_2 \end{array}$$

Remark that $\omega_1\iota^*\omega_2$ is equal to $-\omega_1\omega_2$, so the rank of f^* is equal to the rank of the usual multiplication map from $S^2H^0(C,K)$ to $H^0(C,2K)$. This rank, being C hyperelliptic, is $2g-1$ (see [ACG11] page 223). $\square$

It would be interesting to generalize this picture to higher order even theta functions and to sub-linear systems of Γ_{00}. An important reference about this is [PP01], in this paper the authors investigate the links with the moduli of vector bundles over a curve. We think this could be also useful to understand the behaviour of theta series associated to high rank lattices.

4 A degeneration and its period matrix

This section is equivalent to the second section of [CSB13]. We construct a family of curves over a disc with singular central fibre and we compute the first order part of its period matrix.

Fix a generic genus g curve C, two points a and b and two local co-ordinates z_a and z_b. We want to construct a family of genus $g+1$ curves

$$\pi\colon \mathcal{C} \to \Delta$$

whose central fibre is $C/(a \sim b)$ and the other fibres are smooth. By Δ we denote a disc in the complex plane, the parameter on Δ is called t and the fibre over t is $\mathcal{C}_t$. Δ^* will be the pointed disc and $\mathcal{C}^*$ the family over it. We actually work on the germ $(\Delta, 0)$, so we assume the radius of Δ to be as small as we wish.

The local co-ordinate z_a (respectively z_b) gives an isomorphism between a neighbourhood U_a of a (U_b of b) with an open disc in the complex plane. We assume the radius of the disc to be the same for both a and b.

Given t, we denote by $\widetilde{U}_a^t$ the closed disc in U_a of radius $\frac{1}{2}|t|$. Let $\widetilde{U}_a$ be the closed subset of $C \times \Delta^*$ formed by the pairs (x,t) such that x belongs to $\widetilde{U}_a^t$. Same notations if we replace a with b.

Let W be the open subvariety of $C \times \Delta^*$ obtained by removing $\widetilde{U}_a$ and $\widetilde{U}_b$. Given two points (x,t) and (y,t) in W, we glue them if and only if

$$z_a(x)z_b(y) = t$$

(clearly, this includes checking that z_a is defined at x and z_b is defined at y).

This construction defines a flat family

$$\pi\colon \mathcal{C}^* \to \Delta^*$$

its flat limit is a family

$$\pi\colon \mathcal{C} \to \Delta$$

whose central fibre is $C/(a \sim b)$. The family is flat because the basis is smooth and one dimensional.

To show that the total space $\mathcal{C}$ is smooth we cover it with two open smooth surfaces: the first, let us call it W', is $(C \setminus (\overline{U}_a \cup \overline{U}_b)) \times \Delta$; the second is an affine surface S isomorphic to $XY = t$. It is worth remariking that W' is also an open subvariety of the trivial family $C \times \Delta$.

Let us point out that this construction can be turned, just by enhancing the notations, into a construction “with parameters”. By this we mean that, given any scheme V parametrizing a choice of points and local co-ordinates, we can construct a relative family π_V over $V \times (\Delta, 0)$, where $(\Delta, 0)$ is the germ of a disc at the origin. All the following computations will depend holomorphically on V.

We want to construct a symplectic basis for the homology depending, as differentiable function, on t. This basis will be affected by monodromy, as prescribed by the Picard-Lefschetz formula (see [ACG11] page 143). First, fix a symplectic basis $A_1, \dots, A_g, B_1, \dots, B_g$ for the homology of C with support disjoint from U_a and U_b. This basis extends to a family of cycles $A_1(t), \dots, A_g(t)$, $B_1(t), \dots, B_g(t)$ on $\mathcal{C}_t$, they are not affected by monodromy. The cycle $A_{g+1}(t)$ is the vanishing cycle, a representative is the boundary of the disc U_a. To define $B_{g+1}(t)$, we fix a path B in C joining a and b such that if x is in B and

$$z_a(x)z_b(y) = t$$

then y as well is in B. For every t, B defines a cycle $B_{g+1}(t)$ on $\mathcal{C}_t$.

Let ω be the relative dualizing sheaf $\omega_{\mathcal{C}/\Delta}$ on $\mathcal{C}$. If we restrict ω to a fibre $\mathcal{C}_t$ we get the dualizing sheaf of the fibre.

Lemma 4.1. *Up to shrink Δ, there exists sections $\omega_1, \dots, \omega_{g+1}$ of ω such that for every t they span the space of global sections of the dualizing sheaf of $\mathcal{C}_t$. Moreover, they are normalized with respect to the previouse basis of the homology, that is:*

$$\int_{A_i(t)} \omega_j(t) = \delta_{ij} \quad \forall\, i, j = 1, \dots, g+1$$

Proof. The function on Δ given by

$$\dim H^0(\mathcal{C}_t, \omega \mid_{\mathcal{C}_t})$$

is constatly equal to $g+1$, so we can apply Grauert's direct image Theorem to show that $\pi_*\omega$ is locally free. Up to shrink Δ, we can pick global sections of ω such that they span $H^0(\mathcal{C}_t, \omega_{\mathcal{C}_t})$ for every t; let us call them $\tilde{\omega}_1, \dots, \tilde{\omega}_{g+1}$.

To get the normalization, we remark that, since the $\tilde{\omega}_i(t)$ form a basis of the abelian differentials on $\mathcal{C}_t$, the matrix

$$\left(\int_{A_j(t)} \tilde{\omega}_i(t) \right)_{ij}$$

is invertible; its inverse is holomorphic in t and we can use it to obtain the normalized basis. $\square$

Let

$$\nu \colon C \to \mathcal{C}_0 = C/(a \sim b)$$

be the normalization map. The pull-back of $\omega_1(0), \dots, \omega_g(0)$ via ν is a basis for the abelian differentials on C dual to $A_1, \dots, A_g$. We denote this basis by $\omega_1, \dots, \omega_g$ and we call τ the associated period matrix. These forms are also a basis of the Kähler differentials on $\mathcal{C}_0$. The form $\omega_{g+1}(0)$ is a section of the dualizing sheaf but not a Kähler differential, its pull-back is a meromorphic form with opposite residues at a and b (see [ACG11] section X.2 for a general discussion).

The period matrix $T(t)$ of this family will not be univalued on the punctured disc, its expression with respect to this basis of the homology is of the form

$$T(t) = \begin{pmatrix} \tau & AJ(a-b) \\ {}^tAJ(a-b) & \frac{1}{2\pi i}\ln(t) + c_0 \end{pmatrix} + t \begin{pmatrix} \sigma(a, b, z_a, z_b) & \cdots \\ \vdots & c_1 \end{pmatrix} + O(t^2)$$

Let us discuss the zero order term. The logarithm is due to the monodromy. By $AJ(a-b)$ we mean the image of $a-b$ under the Abel-Jacobi, concretely

$$AJ(a-b) = \left(\int_B \omega_1, \cdots, \int_B \omega_g \right)$$

where B is the path on C joining a to b constructed above. The constants c_0 and c_1 are holomorphic functions of a, b, z_a and z_b, we are not going to specify them.

What is important to us is the explicit expression of the matrix $\sigma = \sigma(a, b, z_a, z_b)$. We have the following result:

Theorem 4.2 (Corollary 6 of [Yam80]). *Keep notations as above; then*

$$\sigma_{ij} = -2\pi i \left(\omega_i(a)\omega_j(b) + \omega_i(b)\omega_j(a) \right)$$

where the differentials are evaluated in term of dz_a and dz_b.

Our proof is closer is spirit to Fay's argument ([Fay73]) rahter than to [Yam80]; let us explain it. The number σ_{ij} is the coefficient of t in the entry (i, j) of the period matrix, so we have

$$\sigma_{ij} = \int_{B_j} \nu^* \frac{d}{dt} |_{t=0} \, \omega_i$$

The derivative $\frac{d}{dt} |_{t=0} \, \omega_i$ can be thought as a Lie derivative, or as the Gauss-Manin connection.

Lemma 4.3. *The meromorphic one form $\nu^* \frac{d}{dt} |_{t=0} \, \omega_i$ has zero residue at a and b.*

Proof. This form has exactly two poles and the sum of the residues is zero, so it is enough to prove that the residue at a vanishes.

Let γ be a homologically trivial path around a, e.g. the boundary of U_a. This cycle extends to family of cycles $\gamma(t) = A_{g+1}(t)$. We have

$$2\pi i \, \mathrm{Res}_a(\nu^* \frac{d}{dt} |_{t=0} \, \omega_i) = \int_\gamma \nu^* \frac{d}{dt} |_{t=0} \, \omega_i = \frac{d}{dt} |_{t=0} \int_{\gamma(t)} \omega_i(t)$$

The last integral is constant because the differentials $\omega_i(t)$ are normalized with respect to $A_j(t)$. $\square$

Let n be the node of $\mathcal{C}_0$, the functions z_a, z_b and t provide an embedding of a neighbourhood of n in $\mathcal{C}$ inside $\mathbb{C}^3$, they obey to the relation

$$z_a z_b = t$$

A set of generators for the relative dualizing sheaf around n is given by $\frac{dz_a}{z_b}$ and $\frac{dz_b}{b}$, we have a relation

$$z_a dz_b + z_b dz_a = 0$$

The relative one forms we are interested in, $\omega_1(t), \dots, \omega_g(t)$, are relative Kähler differentials, so they lay in the span of dz_a and dz_b. Using these two relations, we have an expansion around n

$$\omega_i(t) = \sum_{k\geq 0} a_k(t) z_a^k dz_a + \sum_{k\geq 0} b_k(t) z_b^k dz_a + \sum_{k\geq 0} c_k(t) z_b^k dz_b$$

where $a_k(t), b_k(t)$ and $c_k(t)$ are holomorphic functions on the disc.

We want evaluate $\nu^*\omega_i(0)$ at a in term of dz_a, to do this first we specialize $t = 0$, then we pull-back to C and we look at a neighbourhood of a, so $z_b = dz_b = 0$, last we divide by dz_a and we specilize $z_a = 0$. So we get

$$\omega_i(a) = a_0(0) + b_0(0)$$

In the same way, we have

$$\omega_i(b) = c_0(0)$$

Now, we want to describe the principal part of $\nu^* \frac{d}{dt} |_{t=0} \omega_i$ at a. To do this, we can make the change of variables

$$z_b = \frac{t}{z_a} \quad \text{and} \quad dz_b = -\frac{t}{z_a^2} dz_a$$

we have

$$\omega_i(t) = \sum_{k\geq 0} a_k(t) z_a^k dz_a + \sum_{k\geq 0} b_k(t) \frac{t^k}{z_a^k} dz_a - \sum_{k\geq 0} c_k(t) \frac{t^{k+1}}{z_a^{k+2}} dz_a$$

Applying $\frac{d}{dt}|_{t=0}$ we get

$$\frac{d}{dt} |_{t=0} \omega_i = \sum_{k\geq 0} \dot{a}_k(0) z_a^k dz_a + \dot{b}_0(0) dz_a + b_1(0) \frac{1}{z_a} dz_a - c_0(0) \frac{1}{z_a^2} dz_a$$

where the dot means the derivative with respect to t. By Lemma 4.3, the residue at a vanishes, so $b_1(0) = 0$. Summarizing, around a we have

$$\nu^* \frac{d}{dt} |_{t=0} \omega_i = -\omega_i(b) \frac{1}{z_a^2} dz_a + o(1)$$

In the same way, we obtain that the principale part of $\nu^* \frac{d}{dt} |_{t=0} \omega_i$ around b is

$$\nu^* \frac{d}{dt} |_{t=0} \omega_i = -\omega_i(a) \frac{1}{z_b^2} dz_b + o(1)\,.$$

This is the moment to recall a classical definition:

Definition 4.4 (Normalized differentials of the second kind η_p). *A normalized differential of the second kind η_p on C is a meromorphic differential with a single pole at p such that the two following independent conditions hold:*

- *the pole has order two and no residue;*
- *the integral $\int_{A_i} \eta_p$ is zero for every i.*

These differentials exist and they are unique up to a scalar. When a local co-ordinate around p is fixed, we denote by η_p the unique normalized differential of the second kind with leading coefficient one with respect to this co-ordinate. A comprhensive discussion can be found in [Spr57] pages 256-260.

Let us remark that

$$\int_{A_j} \nu^* \frac{d}{dt} |_{t=0} \omega_i = \frac{d}{dt} |_{t=0} \int_{A_j(t)} \omega_i(t) = \frac{d}{dt} |_{t=0} \delta_{ij} = 0$$

Moreover, the only poles of the differential $\nu^* \frac{d}{dt} |_{t=0} \omega_i$ are in a and b so

$$\nu^* \frac{d}{dt} |_{t=0} \omega_i = -\omega_i(a)\eta_b - \omega_i(b)\eta_a$$

The periods of this meromorphic form can be now computed with the following classical formula (cf. [Spr57] Corollary 10.6 page 260):

Theorem 4.5 (Riemann's bilinear relations for differentials of the second kind)**.** *Keep notations as above; we have*

$$\int_{B_j} \eta_p = 2\pi i \frac{\omega_j}{dz_p}(p)\,,$$

where z_p is a local coordinate around p such that the leading coefficient of η_p is 1, and $\int_{A_j} \omega_i = \delta_{ij}$.

This concludes the proof of Theorem 4.2. As discussed in [MV10], this formula is relevant for computing two-points functions in string theory.

Remark that, for every i and j, σ_{ij} is a section of $K_C^{\boxtimes 2}$ on the second symmetric power S^2C. Moreover, the σ_{ij} form a basis of $H^0(S^2C, K_C^{\boxtimes 2})$ so σ corresponds to the map

$$\sigma\colon S^2C \to \mathbb{P}H^0(S^2C, K_C^{\boxtimes 2})^\vee$$

given by the line bundle $K_C^{\boxtimes 2}$

We would like to point out that if we scale the local co-ordinates z_a and z_b by λ and λ^{-1}, for some non-zero complex number λ, the family $\mathcal{C} \to \Delta$ stays unchanged, so the period matrix as well should not change. Indeed, this is the case for the formula proved above. On the other hand, this is not true for the formulae given in the third chapter of [Fay73], hence they are false.

Before finishing, let us consider the hyperelliptic case. Assume that C is hyperelliptic and call ι the hyperelliptic involution. If we choose general points and co-ordinates the family will not be a family of hypelliptic curves. However, let us choose points and local co-ordinates that are conjugated under the hyperelliptic involution (in particular, we are not taking Weierstrass points). Under these hypotheses, the involution of the central fibre extends to an involution of the whole family. The quotient is a deformation of $\mathbb{P}^1$, which is trivial, so the family is a family of hyperelliptic curves. In this case, Theorem 4.2 reads

$$\sigma(p, \iota(p), z_p, \iota^* z_p)_{ij} = -4\pi i \frac{\omega_i \omega_j}{dz_p^2}(p)$$

This is, up to a constant, the same matrix we get in Proposition 5.1.

5 Schiffer variations and local Torelli Theorem

5.1 The general case

This section is not essential for the rest of the thesis. Using the ideas of section 4, we compute the period matrix of a family of smooth curves. This gives an explicit description of the differential of the Jacobian map

$$J\colon \mathcal{M}_g \to \mathcal{A}_g$$

and it can be useful for the local Torelli problem.

Let C be a generic genus g smooth curve; fix a point p and a local co-ordinate z_p. A Schiffer's variation at (p, z_p) of C is a particular family of smooth curves $\mathcal{C}$ over a disc Δ with central fibre isomorphic to C. We construct the fibre $\mathcal{C}_t$ over a point t of Δ by performing a local surgery around p. Fix a co-ordinate patch U_p around p where z_p makes sense. First, we remove a disc of radius $\frac{1}{2} \mid t \mid$ from U_p. Then, we glue the remaining annulus with another disc via the formula

$$z^*(t) = z_p + \frac{t}{z_p}\,,$$

where z^* is the co-ordinate on the new disc. For more details see [ACG11] page 175.

Fix a symplectic basis A_i, B_i for the homology of C with support disjoint from U_p; call τ the period matrix of C with respect to this basis. We extend this basis to a basis of $A_i(t), B_i(t)$ of the homology of $\mathcal{C}_t$ for every t.

Proposition 5.1 ([Pat63]). *The period matrix $T(t)$ of $\mathcal{C}_t$ is*

$$T(t) = \tau + t\sigma(p, z_p) + O(t^2)\,,$$

where σ is a holomorphic function of p and z_p defined as

$$\sigma(p, z_p)_{ij} = 2\pi i \frac{\omega_i(p)\omega_j(p)}{dz_p^2}\,.$$

Proof. We fix a basis $\omega_i(t)$ of the relative abelian differential dual to $A_i(t)$ as in Lemma 4.1.

By definition

$$dz^*(t) = dz - \frac{t}{z^2}dz$$

In the annulus around p we have

$$\omega_i(t) = f_i(z,t)(dz - \frac{t}{z^2}dz)\,,$$

where f_i is a holomorphic function on a disc around the origin. We apply the Gauss-Manin connection ∇ and specialise at $t = 0$ (i.e. we apply $\frac{d}{dt} \mid_{t=0}$). We get

$$\nabla\omega_i(0) = \frac{\partial f_i}{\partial t}(z,0)dz - f_i(z,0)\frac{1}{z^2}dz\,.$$

To compute the residue at p, take a small homologically trivial cycle γ around p, e.g. the boundary of U_p. Extend it to a family of homologically trivial cycles $\gamma(t)$ on $\mathcal{C}_t$. We have

$$\frac{1}{2\pi i}\operatorname{Res}_p \nabla\omega_i(0) = \int_\gamma \nabla\omega_i(0) = \frac{d}{dt}\,|_{t=0} \int_{\gamma(t)} \omega_i(t) = \frac{d}{dt}\,|_{t=0}\, 0 = 0$$

The integral of $\nabla\omega_i(0)$ along $A_j(0)$ is zero because of the same argument. We conclude that

$$\nabla\omega_i(0) = \frac{\omega_i}{dz_p}(p)\eta_p$$

where η_p is the normalized differential of the second kind, see Definition 4.4. The Proposition now follows from Riemann's bilinear relations 4.5. □

Recall that when we are considering automorphism-free curves, or we are dealing with stacks, we have the identifications

$$T_C\mathcal{M}_g = H^0(C, 2K_C)^\vee \quad ; \quad T_C\mathcal{A}_g = \operatorname{Sym}^2 H^0(C, K_C)^\vee\,.$$

The first is by Kodaira-Spencer theory; the second can be explained in term of the heat equation for theta functions (cf. [CvdG08]).

The differential of the Jacobian morphism, as explained in [ACG11] page 216-224, is given by the co-multiplication map $m^\vee$.

The matrix $\sigma(p, z_p)$ belongs to $T_C\mathcal{A}_g = \operatorname{Sym}^2 H^0(C, K_C)^\vee$; in a co-ordinate free fashion, it is given by

$$\sigma(p, z_p)(\omega) = 2\pi i \frac{\omega}{dz_p}(p)^2.$$

The projectivization of $\sigma(p, z_p)$ does not depend on z_p: it is the image of p under the map

$$\sigma\colon C \to \mathbb{P}H^0(C, K_C)^\vee \xrightarrow{\mathrm{Ver}_2} \mathbb{P}\operatorname{Sym}^2 H^0(C, K_C)^\vee\,.$$

On the moduli space of curves, we have

Theorem 5.2 ([ACG11] page 175)**.** *The tangent direction to a Schiffer variation of C at (p, z_p) is given by the image of p via the bi-canonical map*

$$C \xrightarrow{2K_C} \mathbb{P}H^0(C, 2K_C)^\vee = T_C\mathcal{M}_g$$

In particular, Schiffer variations generate the tangent space to the moduli space of curves.

We have a commutative diagram

$$\begin{tikzcd} C \arrow[r, "2K_C"] \arrow[d, "K_C"] & \mathbb{P}H^0(C, 2K_C)^\vee \arrow[d, "m^\vee"] \\ \mathbb{P}H^0(C, K_C)^\vee \arrow[r, "\mathrm{Ver}_2"] & \mathbb{P}\operatorname{Sym}^2 H^0(C, K_C)^\vee \end{tikzcd}$$

Provided that C is automorphism-free, this gives an alternative proof of the identification of the co-multiplication $m^\vee$ with the differential of the Jacobian map. In this way, we get also a classical Torelli Theorem:

Theorem 5.3 (Infinitesimal Torelli Theorem). *The differential of the Jacobian map at an automorphism-free curve is injective. Moreover, the affine cone over $\sigma(C)$ spans all $dJ(T_C\mathcal{M}_g)$.*

This result is equivalent to the classical Noether Theorem:

Theorem 5.4 (Max Noether, [ACGH85] page 117). *If C is not hyperelliptic, the multiplication map m is surjective.*

5.2 The hyperelliptic case

Let C be a hyperelliptic curve and ι be the hyperelliptic involution. If we perform a generic Schiffer variation the curve will not remain hyperelliptic; however our methods can still be applied.

Let p be a point which is not fixed by ι. If we perform simultaneously two variations, one at (p, z_p) and the other at $(\iota(p), \iota^* z_p)$, we obtain a family of hyperelliptic curves.

According to Proposition 5.1, the first order part of the period matrix is

$$\sigma(p, z_p) + \sigma(\iota(p), \iota^* z_p) = 2\sigma(p, z_p)$$

We thus have the following result:

Lemma 5.5. *For every p in C, the vector $\sigma(p, z_p)$ is tangent to the image of the hyperelliptic locus under the Jacobian morphism; i.e. it belongs to $T_C J(Hyp_g)$*

As a consequence of this result, we get the infinitesimal Torelli Theorem for hyperelliptic curves. Our approach is quite close to [OS80], see also [ACG11] pages 223-224.

Theorem 5.6 (Infinitesimal Torelli theorem for hyperelliptic curves). *Consider the Jacobian map*

$$J \colon \mathcal{M}_g \to \mathcal{A}_g$$

between stacks, or coarse moduli spaces with level structures. Its differential at a generic hyperelliptic curve is not injective: the image is the tangent space to the image of the hyperelliptic locus; in symbols:

$$dJ(T_C\mathcal{M}_g) = T_C J(Hyp_g)$$

Proof. Let C be a generic hyperelliptic curve. Its automorphism group G is the order two group given by the hyperelliptic involution. To compute the differential of the Jacobian map we first construct a Kuranishi family for C. To do this, we can pick $3g-3$ generic points on C and local co-ordinates and consider Schiffer variation with parameter t_i at the point p_i. Now, we have to compute the coefficient of t_i in the period matrix $T = T(t_1, \ldots, t_{3g-3})$. This is done in Proposition 5.1. Lemma 5.5 tell us that we will get a vector tangent to the image of the hyperelliptic locus.

We have now to worry about the action of G. If we introduce an appropriate level structure, G does not act anymore so we obtain the Theorem. If we consider stacks, it is enough to remark that the automorphism group of the Jacobian of C is equal to G, so we do not have to take the quotient. $\square$

Again, this result is equivalent to Max Noether's Theorem. As shown in [OS80], the previous result does not hold for coarse spaces. Indeed, dealing with coarse spaces we have to effectively mod out by G; and the co-ordinates t_i are not preserved. In particular, take $p_i = \iota(p_j)$, where ι is the hyperelliptic involution: in this case ι swaps t_i and t_j. The co-ordinate $t_i + t_j$ is preserved by ι; it gives us a tangent vector proportional to $\sigma(p_i, z_i)$. This vector, by Lemma 5.5, is tangent to the image of the hyperelliptic locus $J(Hyp_g)$. On the other hand, the co-ordinate $t_i - t_j$ gives a zero tangent vector; but it is not preserved by ι so to compute the differential we should look at the coefficient of $(t_i - t_j)^2$.

When p is a Weierstrass point and $\iota^* z_p = -z_p$, the tangent vector $\sigma(p, z_p)$ can be interpreted as follows:

Theorem 5.7 (Rauch's variational formula - [Fay73] page 47 or [May69]). *Let C be the curve defined by the equation*

$$y^2 = (x - p) \prod_{i=1}^{2g+1} (x - p_i)\,,$$

the tangent direction defined by the family

$$y^2(t) = (x - p - t) \prod_{i=1}^{2g+1} (x - p_i)$$

is $\sigma(p, z_p)$, where z_p is the local co-ordinate given by x.

6 Some general background

6.1 Tangent space and arcs

In this section we recall some general facts. Let X be a variety and p a point on X. Denote by Δ a small disc around the origin in the complex plain. An arc on X at p is (the germ of) a holomorphic map

$$\gamma\colon \Delta \to X$$

such that $\gamma(0) = p$. Taking the differential at the origin we obtain a line in the tangent space T_pX. More formally, we have a map

$$\begin{array}{rccc} d\colon & \mathrm{Arc}^\circ(X,p) & \to & \mathbb{P}T_pX \\ & \gamma & \mapsto & d\gamma(0)(T_0\Delta) \end{array}$$

Where $\mathrm{Arc}^\circ(X,p)$ is the space of arcs passing trough p with non-zero differential at the origin.

Definition 6.1. *We denote by T_pX^{arc} the subvector space of T_pX generated by tangent vectors to arcs on X passing trough p; in other words, the span of the affine cone over $d(Arc^\circ(X,p))$.*

Clearly, if p is smooth we have $T_pX = T_pX^{arc}$. We already have interesting examples by looking at rational surface singularities $(S,0)$. We start from the A_1 singularity

$$x^2 + y^2 + z^2 = 0$$

In this case the tangent vectors to these three arcs

$$(t, it, 0) \quad (t, 0, it) \quad (0, t, it)$$

generate the tangent space, so $T_0S = T_0S^{arc}$. To give an example where this is not the case we recall some general theory. Consider the minimal resolution of singularity

$$p\colon (\bar{S}, E) \to (S, 0)$$

call E_i the irreducible components of E. The exceptional divisors with the intersection pairing form a root system. We say that a vector $A = \sum a_iE_i$ is bigger than a vector $B = \sum b_iE_i$ if $a_i > b_i$ for every i. The *highest root* or *fundamental cycle* is the biggest vector $Z = \sum r_iE_i$ such that $Z^2 = 2$. The highest root exists and it is unique; all possible root lattices and their highest roots are depicted in figure 1.8 page 31 of [Ebe13]. A classical result is:

Theorem 6.2 ([Art66] Theorem 4)**.** *Let $(S,0)$ be a rational surface singularity and $Z = \sum r_iE_i$ its fundamental cycle; let f be a function on S vanishing at 0. Then the multiplicity of p^*f along E_i is at least r_i.*

Corollary 6.3. *Let $(S,0)$ be the E_8 singularity; then T_0S^{arc} is zero dimensional.*

Proof. Let f be a function vanishing at 0; its differential df must vanish at 0 as well, otherwise p^*f would vanish with multiplicity 1 on at least one of the exceptional divisor E_i, i.e. on one which is not contained in the hyperplane $df = 0$. This is impossible because all the coefficients of the fundamental cycle of E_8 are at least 2.

Now, let γ be an arc passing trough 0. By the previous remark we have

$$\frac{d}{dt} |_{t=0} \gamma^* f = 0$$

where t is a local co-ordinate on Δ. This means that $d\gamma(0)$ is zero for every arc γ, so $\mathrm{Arc}^\circ(S, 0)$ is empty and $T_0 S^{arc}$ is zero dimensional. $\square$

The arc space has a natural structure of topological space; a reference is [KA13].

6.2 Vanishing order and infinitesimal neighbourhoods

In this section we recall the notion of infinitesimal neighbourhoods. This is a local concept, so all varieties will be affine. We assume that the base field has characteristic zero, otherwise we should use symbolic powers of ideal.

Let X be an affine variety and Y a subvariety. Let I be the ideal of Y in X.

Definition 6.4 (m-th infinitesimal neighbourhood)**.** *The m-th infinitesimal neighbourhood of Y in X is the (non-reduced) subscheme of X defined by the ideal I^{m+1}.*

A function F on X vanishes on the m-th infinitesimal neighbourhood of Y if and only if it vanishes with order at least $m + 1$ along Y.

Let us put ourself in an ambient variety A, let Z and X be subvarieties of A and Y a subvariety of X. Then, Z contains the m-th infinitesimal neighbourhood of Y in X if and only if every function vanishing on Z with multiplicity $p > 0$, when restricted to X, vanishes with order at least $m + p + 1$ along Y; in particular the tangent space of Z contains the tangent space of Y.

Example 6.5. Let X be the line $y = 0$ in $\mathbb{C}^2$ and Y be the origin. The parabola $y = x^2$ contains the first infinitesimal neighbourhood of Y in X.

Let us remark that, to show that a function on X vanishes on Y with some multiplcity, it is enough to check it on an open dense subset of X which intersects Y non-trivially. This is indeed the content of the following Lemma in commutative algebra. (R is the ring of X, I some power of the ideal defining Y, J the ideal generated by the function and $\mathfrak{p}$ the ideal defining the complement of the open subset.)

Lemma 6.6. *Let R be a domain, I and J ideals and $\mathfrak{p}$ a prime ideal. Assume $\mathfrak{p}$ is not contained in I. If, localizing at $\mathfrak{p}$, we have*

$$I_{\mathfrak{p}} \subset J_{\mathfrak{p}}$$

then

$$I \subset J$$

7 The abscence of stable Schottky relations

7.1 Satake compactification and Schottky problem

The usual compactification of $\mathcal{M}_g$ is the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$. The boundary is composed by divisors δ_i, for $i = 0, \dots, \lfloor \frac{g}{2} \rfloor$. The general point of δ_0 represents a singular curve whose normalization is a genus $g-1$ curve, the general point of δ_i represents a curve whose normalization is the disjoint union of a genus i and a genus $g-i$ curve. This compactification can be seen as an orbifold and represents the functor of stable genus g curves. Our main reference is [ACG11]. In our thesis we focus on a different compactification. Recall that we have the Jacobian morphism

$$J\colon \mathcal{M}_g \to \mathcal{A}_g$$

By Torelli's Theorem, it is injective.

Definition 7.1 (Satake compactification)**.** *The Satake compactification $\mathcal{M}_g^S$ of $\mathcal{M}_g$ is the closure of $J(\mathcal{M}_g)$ inside $\mathcal{A}_g^S$.*

This compactification is a singular projective variety and it does not represent a functor.

In [Hoy63], it is shown that the degeneration of a Jacobian is still a Jacobian. This means that, set theoretically, $\mathcal{M}_g^S$ is equal to the union of all products $\mathcal{M}_{g_1} \times \cdots \times \mathcal{M}_{g_k}$ with $\sum g_i \leq g$. More specifically, since $\mathcal{A}_g^S$ contains as a scheme $\mathcal{A}_{g-1}^S$, $\mathcal{M}_g^S$ contains as a scheme $\mathcal{M}_{g-1}^S$; we are going to prove that the scheme structure of $\mathcal{M}_g^S$ along $\mathcal{M}_{g-1}^S$ is very rich.

As shown in [Nam73] Theorem 6, we can extend the Jacobian morphism

$$\begin{array}{rccc} J\colon & \overline{\mathcal{M}}_g & \to & \mathcal{A}_g^S \\ & C & \mapsto & \mathrm{Jac}(\hat{C}) \end{array} \tag{4}$$

where $\hat{C}$ is the normalisation of C. Since $\overline{\mathcal{M}}_g$ is reduced, the image of J is equal, as a scheme, to $\mathcal{M}_g^S$. The boundary divisors δ_i are mapped to $\mathcal{M}_i^S \times \mathcal{M}_{g-i}^S$ and δ_0 is mapped to $\mathcal{M}_{g-1}^S$.

On $\overline{\mathcal{M}}_g$, we have the Hodge bundle $\mathbb{E}_g$ and its determinant λ_g. The pull-back of L_g via J is λ_g; recall that the pull-back of an ample line bundle is semi-ample. We can consider the Stein factorization of J:

$$\overline{\mathcal{M}}_g \xrightarrow{\phi_{m\lambda_g}} \Sigma_g \xrightarrow{\nu} \mathcal{M}_g^S$$

where m is an integer big and divisible enough, so that $\phi_{m\lambda_g}$ is the Itaka map of λ_g, and (Σ_g, ν) is the normalization of $\mathcal{M}_g^S$. In [ACG11] page 435-437, it is shown that ν is bijective and the name "Satake compactification" is used for Σ_g. The interest of Σ_g is that it is a co-dimension 2 normal compactification of $\mathcal{M}_g$, so it can been used for instance to show that any holomorphic function on $\mathcal{M}_g$ is constant. On the other hand, it is not clear if Σ_g contains scheme theoretically Σ_{g-1}.

Definition 7.2 (Teichmüller modular forms). *A weight k and degree g Teichmüller modular form is a section of λ_g^k over $\overline{\mathcal{M}}_g$.*

Our main reference about Teichmüller modular forms is [Ich94]. The normality of $\mathcal{M}_g^S$ is equivalent to the surjectivity of the map

$$J^* \colon \mathcal{R}(\mathcal{A}_g^S, L_g) \to \mathcal{R}(\overline{\mathcal{M}}_g, \lambda_g)$$

in degree big and divisible enough. It is known that, in general, J^* is not surjective. The typical example is the product of all theta-nulls: as shown in [Tsu91], it admits a square root on $\overline{\mathcal{M}}_g$ but it does not on $\mathcal{A}_g^S$. Clearly, this example does not prevent J^* to be surjective when the degree is divisible by 2. The surjectivity of J^* is also relevant for string theory: partition functions belongs to $\mathcal{R}(\overline{\mathcal{M}}_g, \lambda_g)$, but usually they are written as restriction of elements of $\mathcal{R}(\mathcal{A}_g^S, L_g)$.

Remark 7.3. Let F_g be the pull-back of of a modular form from $\mathcal{A}_g^S$ to $\overline{\mathcal{M}}_g$. To compute the multiplicity of F_g on δ_0 we can see F_g as a modular form on $\mathcal{M}_g^S$ and compute its vanishing order on $\mathcal{M}_{g-1}$. The same is true if we replace b_0 with b_i and $\mathcal{M}_{g-1}$ with $\mathcal{M}_i \times \mathcal{M}_{g-1}$.

Since $\mathcal{M}_{g+1}^S$ contains $\mathcal{M}_g^S$ as a closed subscheme, we can construct the commutative ind-monoid

$$\mathcal{M}_\infty := \bigcup_{g \geq 0} \mathcal{M}_g^S \,,$$

this is a sub-monoid of $\mathcal{A}_\infty$, so we can apply the general result 2.3.

One of the main point of our work is that the structure of the boundary of the Satake compactification can tell us something about the Schottky problem. Our first result, which has been obtained in [CSB13], is the following:

Theorem 7.4 (= Theorem 1.1). *The intersection of $\mathcal{M}_{g+m}^S$ and $\mathcal{A}_g$ is not transverse, it contains the m-th infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$.*

The proof will be given in section 7.2, let us now draw some consequences about the Schottky problem. As explained in section 6.2, Theorem 7.4 means that if a modular form F_{g+m} on $\mathcal{A}_{g+m}$ vanishes with order at least k on $\mathcal{M}_{g+m}$, then its restriction F_g to $\mathcal{A}_g$ vanishes with order at least $k+m$ on $\mathcal{M}_g$.

Corollary 7.5. *Given a stable modular form F, if F_g vanishes on $\mathcal{M}_g$ with multiplicity exactly k, then F_{g+k} does not vanish on $\mathcal{M}_{g+k}$. In particular, for $g >> 0$ the modular form F_g does not vanish on $\mathcal{M}_g$.*

Corollary 7.6. *The ideal of stable modular forms vanishing on $\mathcal{M}_\infty$ is trivial.*

This last result is not in contradiction with the evident fact that $\mathcal{M}_\infty$ is not equal to $\mathcal{A}_\infty$: the line bundle of stable modular forms is not meant to be ample on $\mathcal{A}_\infty$.

Still differences of theta series might have an interesting behaviour. Let us recall the definition of slope given in [HM90]. The Picard group of $\overline{\mathcal{M}}_g$ over $\mathbb{Q}$ is freely generated by the determinant of the Hodge bundle λ and the boundary divisors δ_i. If an effective divisor D can be written as

$$D = a\lambda - \sum b_i\delta_i$$

with $a, b_i \geq 0$, its slope is defined as

$$s(D) := \max_i \frac{a}{b_i},$$

this could be a positive number or ∞. If D can not be written in this way then its slope is ∞. The slope of $\overline{\mathcal{M}}_g$ is the infimum of the slopes of effective divisors.

Let F_g be a Teichmüller modular form on $\overline{\mathcal{M}}_g$, call a its weight and b_i its multiplicity along δ_i. We can write in the Picard group of $\overline{\mathcal{M}}_g$

$$\sum b_i\delta_i + E = \{F_g = 0\} = a\lambda$$

where E is the closure of an effective divisor on $\mathcal{M}_g$. The slope of F_g is, by definition, the slope of E. More concretely, the slope of F_g is the weight divided by the smallest of the b_i. Let us state this formally.

Definition 7.7 (Slope of a modular form)**.** *The slope of a Teichmüller modular form F_g is the weight divided by the smallest of the b_i; where b_i is the multiplicity of F_g along δ_i.*

It is an open problem to find divisors of low slope, see [Far09] for an overview.

Lemma 7.8. *Given any two even positive definite unimodular lattices Λ and Γ of the same rank, there exists a positive integer*

$$\bar{g} = \bar{g}(\Lambda, \Gamma) > 0$$

such that the modular form

$$F_g = \Theta_{\Lambda,g} - \Theta_{\Gamma,g}$$

- *it is zero on $\mathcal{M}_g$ for $g < \bar{g}$, so it is a solution of the Schottky problem;*
- *it is not zero on $\mathcal{M}_{\bar{g}}$ and has finite slope $s = s(\Lambda, \Gamma)$;*
- *it is not zero on $\mathcal{M}_g$ for $g > \bar{g}$ and has slope ∞.*

Proof. Any difference of theta series is zero on $\mathcal{A}_0$. The finiteness of $\bar{g}$ is the non-trivial part, it is guaranteed by Corollary 7.5. We need to check the statements about the slope, and to do this we can apply the remark 7.3. For $g > \bar{g}$, F_g, as a modular form on $\mathcal{M}_g^S$, is not zero on $\mathcal{M}_{g-1}$, so $b_0 = 0$ and the slope is ∞. When $g = \bar{g}$, we know that $F_{\bar{g}}$ vanishes on $\mathcal{M}_{\bar{g}-1}$ so $b_0 \neq 0$. We need to check that $b_i \neq 0$ for $i > 0$, in other words that $F_{\bar{g}}$ vanishes on all components $\mathcal{M}_{\bar{g}-i} \times \mathcal{M}_i$. Take a point

$$\begin{pmatrix} \tau_i & 0 \\ 0 & \tau_{\bar{g}-i} \end{pmatrix} \in \mathcal{M}_i \times \mathcal{M}_{\bar{g}-i}$$

we argue as follows:

$$\begin{aligned} F_{\bar{g}}(\begin{pmatrix} \tau_i & 0 \\ 0 & \tau_{\bar{g}-i} \end{pmatrix}) = & \quad \Theta_{\Lambda,i}(\tau_i)\Theta_{\Lambda,\bar{g}-i}(\tau_{\bar{g}-i}) - \Theta_{\Gamma,i}(\tau_i)\Theta_{\Gamma,\bar{g}-i}(\tau_{\bar{g}-i}) \\ = & \quad \Theta_{\Lambda,\bar{g}-i}(\tau_{\bar{g}-i})F_i(\tau_i) = 0 \end{aligned}$$

where in the next to last equality we have used that $F_{\bar{g}-i}$ is zero on $\mathcal{M}_{\bar{g}-i}$ and in the last we have used that F_i is zero on $\mathcal{M}_i$ $\square$

The previous argument also shows that $F_{2\bar{g}+1}$ does not vanish on any boundary divisor of $\overline{\mathcal{M}}_{2\bar{g}+1}$. The values of $\bar{g}$ and s are known for lattices of rank 16 and 24. There exist only two lattices of rank 16, so the only case is the Schottky form discussed in the example 1.4. The rank 24 case is described in Theorem 1.5.

7.2 The failure of transversality

This section is equivalent to the third section of [CSB13]. It is devoted to the proof of Theorem 1.1; let us recall the statement:

Theorem 7.9. *The variety $\mathcal{M}^S_{g+m}$ contains the m-th infinitesimal neighbourhood of $\mathcal{M}^S_g$ in $\mathcal{A}^S_g$.*

Because of the discussion in section 6.2, after induction on m this result is equivalent to the following Theorem:

Theorem 7.10. *Let F_{g+1} a be a weight k modular form on $\mathcal{A}^S_{g+1}$ and F_g its restriction to $\mathcal{A}^S_g$. If F_{g+1} vanishes on $\mathcal{M}^S_{g+1}$ with multiplicity at least $p > 0$, then F_g vanishes on $\mathcal{M}^S_g$ with multiplicity at least $p+1$.*

Let us prove it. The problem being local, we can compute in local co-ordinates. Let J_{g+1} be the preimage of $\mathcal{M}_{g+1}$ in the Siegel upper half space $\mathcal{H}_{g+1}$. Let T be a matrix in $\mathcal{H}_{g+1}$ and write

$$T = \begin{pmatrix} \tau & z \\ {}^t z & t \end{pmatrix},$$

with τ in $\mathcal{H}_g$ and t in $\mathcal{H}_1$. To obtain the restriction F_g of F_{g+1} to $\mathcal{H}_g$ one writes $q := \exp(2\pi i t)$ and

$$F_{g+1}(T) = F_g(\tau) + f_1(\tau, z)q + O(q^2)$$

Let D be a degree p differential operator on $\mathcal{H}_{g+1}$ with constant coefficients. By hypothesis we have $DF_{g+1} = 0$ on J_{g+1}, we need to show that the restriction $(DF_{g+1})_g$ of DF_{g+1} to $\mathcal{H}_g$ vanishes with order at least two on J_g.

(We are using the clumsy notation $(DF_{g+1})_g$ to stress that first we apply D, then we restrict to $\mathcal{H}_g$: the other way round would not make sense.)

Let C be a generic genus g curve and τ one of its period matrices in J_g. Consider the family of genus $g+1$ curve

$$\pi \colon \mathcal{C} \to \Delta_t$$

constructed in section 4. Its period matrix $T(t)$ is in J_{g+1} for $t \neq 0$ and by hypothesis we have

$$DF_{g+1}(T(t)) \equiv 0$$

We want to compute the coefficient of t in $DF_{g+1}(T(t))$, so all the following will be modulo t^2.

We need to recall Fourier expansion of modular forms (cf. [Fre83] page 44). We have

$$F_{g+1}(T) = \sum_{X \in S_{g+1}} a_X(F_{g+1}) \exp(\pi i \sum_{i,j} X_{ij} T_{ij})$$

where S_{g+1} is the set of semi-positive definite $g+1$ by $g+1$ symmetric matrices with integer entries and even diagonal. The complex number $a_X(F_{g+1})$ is the Fourier coefficient of F_{g+1} with respect to X. By direct computation we have

$$DF_{g+1}(T) = \sum_{X \in S_{g+1}} a_X(F_{g+1}) P_D(\pi i X) \exp(\pi i \sum_{i,j} X_{ij} T_{ij})$$

where P_D is the homogeneous polynomial in the X_{ij}, with $i \leq j$, defining D:

$$D = P_D(\frac{\partial}{\partial T_{ij}}).$$

Let us go back to our computation. As shown in section 4, we have:

$$T(t) = \begin{pmatrix} \tau & AJ(a-b) \\ {}^t AJ(a-b) & \frac{1}{2\pi i} ln(t) + c_0 \end{pmatrix} + t \begin{pmatrix} \sigma & \cdots \\ \vdots & c_1 \end{pmatrix} + O(t^2)$$

To use the Fourier expansion, we need to remark that we have an injective map of sets

$$\begin{array}{ccc} S_g & \hookrightarrow & S_{g+1} \\ X & \mapsto & \bar{X} \end{array}$$

by adding to X a last column and a bottom row filled with zero. Denote by $\bar{S}_g$ the image of S_g in S_{g+1}. As a consequence of the semi-positivity, we have:

Lemma 7.11. *The set of matrices X in S_{g+1} such that $X_{g+1,g+1} = 0$ is equal to $\bar{S}_g$.*

As explained in [Fre83] page 45, this is crucial in restricting function from $\mathcal{H}_{g+1}$ to $\mathcal{H}_g$: for $\bar{X}$ in $\bar{S}_g$, we have

$$a_{\bar{X}}(F_{g+1}) = a_X(F_g)$$

More in general,

$$(DF_{g+1})_g(\tau) = \sum_{X \in S_g} a_X(F_g) P_D(\pi i \bar{X}) \exp(\pi i \sum_{ij} X_{ij} \tau_{ij})$$

Now, consider the partition

$$S_{g+1} = \bar{S}_g \sqcup R_{g+1} \sqcup C$$

where R_{g+1} is the set of X such that $X_{g+1,g+1} = 2$ and C is the complement. We compute the contribution to the coefficient of t in $DF_{g+1}(T(t))$ of each set separately.

For $\bar{S}_g$, we have

$$\frac{d}{dt}|_{t=0} \sum_{\bar{X}\in\bar{S}_g} a_{\bar{X}}(F_{g+1})P_D(\pi i\bar{X})\exp(\pi i\sum_{ij}\bar{X}_{ij}T_{ij}(t)) = \frac{d}{dt}|_{t=0}(DF_{g+1})_g(\tau+t\sigma) =: A$$

For R_{g+1}, we have

$$\frac{d}{dt}|_{t=0} \sum_{X\in R_{g+1}} a_X(F_{g+1})P_D(\pi iX)\exp(\pi i\sum_{ij}X_{ij}T_{ij}(t)) =$$

$$\exp(c_0)\sum_{X\in R_g} a_X(F_{g+1})P_D(\pi iX)\exp(\pi i(\sum_{i,j=1}^{g}\tau_{ij}X_{ij}+2\sum_{i=1}^{g}X_{i,g+1}AJ(a-b)_i)) =: \exp(c_0)B$$

where $AJ(a-b)_i$ is the i-th component of the Abel-Jacobi map. (Remark that for $D=1$ this is the first Fourier-Jacobi coefficient of F_{g+1} evaluated at $AJ(a-b)$.)

The contribution of C to the coefficient of t is trivial.

To summarize, we have

$$0 = \frac{d}{dt}|_{t=0}DF_{g+1}(T(t)) = A+\exp(c_0)B$$

Lemma 7.12. *Both A and B vanish.*

Proof. This can be shown by rescaling the co-ordinates z_a and z_b. Takes co-ordinates λ and μ on $\mathbb{C}^*\times\mathbb{C}^*$ and consider the action

$$(\lambda,\mu)(z_a,z_b)\mapsto(\lambda^{-1}z_a,\mu^{-1}z_b)$$

Under this action, all three terms A, B and c_0 will change, and we will have

$$0 = A(\lambda,\mu)+\exp(c_0(\lambda,\mu))B(\lambda,\mu)$$

By using the description of σ given in Theorem 4.2, one sees that

$$A(\lambda,\mu)=\lambda\mu A \quad , \quad B(\lambda,\mu)=B$$

If $B=0$ we have done, let us assume by contradiction that neither A nor B are zero. We have

$$c_0(\lambda,\mu)=\ln(-\frac{A}{B}\lambda\mu)$$

But a single valued logarithm of the function $\lambda\mu$ on the two torus does not exist. $\square$

In particular, we have shown that the derivative in the tangent direction σ of $(DF_{g+1})_g$ vanishes, in symbols

$$A = \frac{d}{dt}|_{t=0}(DF_{g+1})_g(\tau + t\sigma) = 0$$

To prove our Theorem we have to show that the $\sigma = \sigma(a, b, z_a, z_b)$ span all the tangent space $T_C\mathcal{A}_g$ when we vary the points a and b. Recall the expression of σ given in Theorem 4.2:

$$\sigma_{ij} = -2\pi i \frac{\omega_i(a)\omega_j(b) + \omega_i(b)\omega_j(a)}{dz_a dz_b}$$

If we identify $S^2H^0(C, K_C)$ with $H^0(S^2C, K_C^{\boxtimes 2})$, σ corresponds to the map

$$\sigma : S^2C \to \mathbb{P}S^2H^0(C, K_C)^\vee = \mathbb{P}T_C\mathcal{A}_g$$

given by the line bundle $K_C^{\boxtimes 2}$. Our claim now follows from this Lemma:

Lemma 7.13. *The image of S^2C via σ is not contained in any hyperplane.*

Proof. The embedding is given by a linear system. □

7.3 More on the local structure

In this section we first recall a description of the local structure of $\mathcal{A}_{g+1}^S$ along $\mathcal{A}_g$, which is due to Igusa (cf. [Igu67a], see also [BvdGHZ08]), then we describe the local structure of $\mathcal{M}_{g+1}^S$ along $\mathcal{M}_g$.

Take a general point X of $\mathcal{A}_g$, the local ring $(\mathcal{A}_{g+1}^S, X)$ is normal and it is isomorphic to the ring of convergent power series

$$\sum_{n=0}^{\infty} f_n(\tau, z)q^n ,$$

where now X is identified with $\mathbb{C}^g/\mathbb{Z}^g \oplus \tau\mathbb{Z}^g$, τ is in the Siegel upper half space $\mathcal{H}_g$, z belongs to $\mathbb{C}^g$, f_n is a section of $H^0(X, 2n\Theta)$ and q belongs to a small disc around zero in the complex plane. The image of a modular form F_{g+1} of degree $g+1$ in this local ring is the Fourier-Jacobi expansion of F_{g+1} at X. Let us be more explicit. For any element $T \in \mathcal{H}_{g+1}$ write

$$T = \begin{pmatrix} \tau & z \\ {}^t z & t \end{pmatrix} , \tag{5}$$

with t in $\mathcal{H}_1$ and τ in $\mathcal{H}_g$. Let $q := \exp(2\pi i t)$, the Fourier-Jacobi expansion of F_{g+1} is

$$F_{g+1}(T) = f_0(\tau) + \sum_{n\geq 1} f_n(\tau, z)q^n , \tag{6}$$

where $f_0 = \Phi(F_{g+1}) = F_g$ is the restriction of F_{g+1} to $\mathcal{A}_g$. The function f_n is the n-th Fourier-Jacobi coefficient of F_{g+1}, it belongs to $H^0(X, 2n\Theta)$. The normal bundle exact sequence of the inclusion of $\mathcal{A}_g$ in $\mathcal{A}_{g+1}^S$ is

$$0 \to T_X\mathcal{A}_g \to T_X\mathcal{A}_{g+1}^S \to H^0(X, 2\Theta)^\vee \to 0$$

Concretely, given a modular form F_{g+1} on $\mathcal{A}_{g+1}$, the tangent vectors in $T_X\mathcal{A}_g$ act on the restriction F_g of F_{g+1} to $\mathcal{A}_g$, the tangent vectors in $H^0(X, 2\Theta)^\vee$ act on the first Fourier-Jacobi coefficient of F_{g+1}. Moreover, if we slice $\mathcal{A}_g$ at X, the isolated singularity we get is the affine cone over the 2Θ embedding of the Kummer variety of X.

Let now X be the Jacobian of a generic genus g curve C, we are going to use notations and result from section 3. In particular recall we have the subtraction map (2)

$$\begin{array}{rccc} \delta\colon & S^2C & \to & \mathrm{Km}(X) \\ & (a,b) & \mapsto & AJ(a-b) \end{array}$$

whose image is denoted by $C - C$. We are now ready to write out the normal bundle exact sequence of the inclusion of $\mathcal{A}_g \cap \mathcal{M}^S_{g+1}$ in $\mathcal{A}^S_{g+1}$.

Theorem 7.14. *Let C be a generic genus g curve, we have the following exact sequence*

$$0 \to T_C\mathcal{A}_g \to T_C\mathcal{M}^S_{g+1} \to H^0(S^2C, K_{S^2C})^\vee \to 0$$

Proof. There are two things we have to prove, first

$$T_C(\mathcal{M}^S_{g+1} \cap \mathcal{A}_g) = T_C\mathcal{A}_g$$

this is Theorem 1.1, see also section 6.2.

To describe the co-kernel, first we have to show that, after blowing up $\mathcal{A}^S_{g+1}$ at $\mathcal{A}^S_g$, the proper transform of $\mathcal{M}^S_{g+1}$ meets the Kummer variety of X in $\delta(S^2C)$. This is proved in [Nam73] Theorem 6. Then, we apply Lemma 3.5. □

Let us point out that tangent cone of $\mathcal{M}^S_{g+1}$ at $\mathcal{M}_g$ is the cone over the 2Θ embedding of $C - C$, but neither the singularity nor its normalization are cone.

It is worth notice that the variety $\mathcal{M}^S_{g+1}$ does not represent the moduli functor at the boundary, so the exact sequence in Theorem 7.14 is not functorial.

To prove our results we have used tangent vectors to some explicit arcs, we would like to formalise this by using the ideas of sections 6.1. Fix a generic genus g curve, let V_C be the space parametrising pairs of distinct points on C with the choice of local co-ordinates. We have a morphism

$$\pi\colon V_C \to \mathrm{Arc}(\mathcal{M}^S_{g+1}, C)$$

mapping a vector (a, b, z_a, z_b) to the degenerating family studied in 4. Composing with d we get a map

$$d \circ \pi\colon V_C \to T_C(\mathcal{M}^S_{g+1})^{arc}$$

Theorem 7.15. *The tangent space $T_C\mathcal{M}^S_{g+1}$ is spanned by vectors tangent to arcs, i.e. it it equal to $T_C(\mathcal{M}^S_{g+1})^{arc}$*

Proof. To show the claim, it is enough to prove the following: let F_{g+1} be a modular form on $\mathcal{A}^S_{g+1}$, assume it vanishes at C and along $d \circ \pi(V_C)$; then it vanishes along $T_C\mathcal{M}^S_{g+1}$.

Concretely, vanishing along $d \circ \pi(V_C)$ means that, given any choice of points a and b and local co-ordinates z_a and z_b, we have

$$\frac{d}{dt}\,|_{t=0}\; F_{g+1}(T(t)) = 0$$

where $T(t)$ is the period matrix of the family studied in section 4.

Following the argument of section 7.2, Lemma 7.12 gives us the requested result. Indeed, the vanishing of A means that F_{g+1} vanishes along $T_C\mathcal{A}_g$. The vanishing of B means that the first Fourier-Jacobi coeffcient of F_{g+1} is zero when restricted to $C-C$. □

7.4 Quadrics via degenerations

Let F_{g+1} be a degree $g+1$ modular form, as explained in section 7.3 we can consider its Fourier-Jacobi expansion

$$F_{g+1}(T) = f_0(\tau) + f_1(\tau, z)q + O(q^2)$$

The function f_1 is the first Fourier-Jacobi coefficient of F_{g+1} and it is a section of 2Θ.

Lemma 7.16. *Suppose that F_{g+1} vanishes on $\mathcal{M}_{g+1}$, then for every curve C and points a and b we have*

$$f_1(\tau, AJ(a-b)) = 0\,,$$

where τ is the period matrix of C and AJ is the Abel-Jacobi map.

Proof. Let $T(t)$ be the period matrix of the degeneration studied in section 4. We have $F_{g+1}(T(t)) \equiv 0$, in particular $\frac{d}{dt}F_{g+1}(T(t)) \equiv 0$. Following computations and notations of section 7.2, we write

$$\frac{d}{dt}\,|_{t=0}\; F_{g+1}(T(t)) = A + \gamma_1 B\,.$$

Take as D just the identity, we have

$$B = f_1(\tau, AJ(a-b))\,.$$

The result now follows from Lemma 7.12. □

If we let a tend to b, by continuity we get $f_1(\tau, 0) = 0$. We now use Lemma 3.2:

$$f_1(\tau, a-b) = Q(a,b)E(a,b)$$

where Q is the second order part of f_1 and E is the prime form. (Actually, we should write $Q(dAJ(a), dAJ(b))$, where dAJ is the differential of the Abel-Jacobi map. This differential can be identified with the canonical map.)

We know that $E(a,b) \neq 0$ for every a different from b; we conclude by continuity that $Q(x,x) = 0$ for every x in C, i.e. Q is a quadric containing the canonical model of C. Let us summarise the result.

Theorem 7.17. *Let F_{g+1} be a degree $g+1$ modular form vanishing on $\mathcal{M}_{g+1}$; for every period matrix τ of a genus g curve C, let $Q(\tau)$ be the second order part of the first Fourier-Jacobi coefficient of F_{g+1} at τ, then $Q(\tau)$ is a quadric containing the canonical model of C.*

8 Stable equations for the hyperelliptic locus

8.1 Satake compactification of the hyperelliptic locus

In analogy with what we have done for the moduli space of curves, we construct the Satake compactification of the hyperelliptic locus:

Definition 8.1 (Satake compactification)**.** *The Satake compactification Hyp_g^S of the hyperelliptic locus Hyp_g is the closure of $J(Hyp_g)$ inside $\mathcal{A}_g^S$, where J is the Jacobian morphism.*

Again, Hyp_{g+1}^S contains Hyp_g^S as a scheme, so we can define the ind-monoid

$$Hyp_\infty := \bigcup_{g\geq 0} Hyp_g^S$$

The situation of the hyperelliptic locus is completely different from the moduli of curves. In [Poo96], it is shown that the Schottky form

$$\Theta_{D_{16}^+} - \Theta_{E_8\oplus E_8}$$

vanishes on Hyp_g for every g, so the ideal of stable equations for the hyperelliptic locus is not trivial. We can apply Lemma 2.3 to prove the following result

Theorem 8.2. *The ideal of stable modular forms vanishing on Hyp_∞ is generated by differences of theta series*

$$\Theta_\Lambda - \Theta_\Gamma\,.$$

Theorem 1.7 provides many new examples of differences of theta series vanishing on Hyp_∞.

From a geometric point of view, the existence of stable equations is unobstructed because of the following Theorem:

Theorem 8.3 (=Theorem 8.6)**.** *The intersection of $\mathcal{A}_g^S$ and Hyp_{g+1}^S is transverse. In other words, scheme theoretically, it is equal to Hyp_g^S.*

We will prove it in section 8.2; taking it for granted, we give a precise description of the tangent space at a generic genus g hyperelliptic curve C to Hyp_{g+1}^S. Let X be the Jacobian of C, recall the morphism (3)

$$\begin{array}{cccccc} \Psi\colon & C & \xrightarrow{f} & S^2C & \xrightarrow{\delta} & \mathrm{Km}(X) \\ & p & \mapsto & (p,\iota(p)) & & \\ & & & (a,b) & \mapsto & AJ(a-b) \end{array}$$

where ι is the hyperelliptic involution. The normal bundle exact sequence of $\mathcal{A}_g \cap Hyp_{g+1}^S$ in Hyp_{g+1}^S is the following:

Theorem 8.4. *Let C be a generic hyperelliptic curve of genus g, the following exact sequence holds*

$$0 \to T_C Hyp_g \to T_C Hyp^S_{g+1} \to P_C \to 0$$

where P_C is the image of the map

$$\Psi^* \colon H^0(X, 2\Theta) \to H^0(C, 2(K_C + W))$$

described in Lemma 3.6. In other words, P_C is the span of the affine cone over the 2Θ embedding of $\Psi(C)$.

Proof. Again, there are two things we have to prove, first

$$T_C Hyp^S_{g+1} \cap T_C \mathcal{A}_g = T_C Hyp_g$$

this is equivalent to Theorem 8.3.

To describe the co-kernel, we first show that, after blowing up $\mathcal{A}^S_g$ in $\mathcal{A}^S_{g+1}$, the proper transform of Hyp^S_{g+1} meets the Kummer variety of X in $\Psi(C)$. This is proved in [Nam73] Theorem 6, just remark that to obtain a generic irreducible nodal hyperelliptic curve we need to glue two points conjugated under the hyperelliptic involution. Then we apply Lemma 3.6. $\square$

In this case, if we slice the Hyp_g in Hyp^S_{g+1} at X and normalise, we really get a cone. To prove this we look at the resolution of the singularity provided by the Jacobian map. The fibre over X is C mod the hyperelliptic involution, so $\mathbb{P}^1$; the co-normal bundle has degree $4g$ and we can thus apply a classical result by Grauert ([Gra62] corollary page 363, see also [CM03]) which guarantees that the normalization of the singularity is a cone over the $4g$-Veronese embedding of $\mathbb{P}^1$.

In analogy with Theorem 7.15, we can generate this tangent space by considering arcs. Let U_C be the set of points and local co-ordinates on C such that the point is not fixed by the hyperelliptic involution ι. We have a map

$$\pi \colon U_C \to \mathrm{Arc}^\circ(Hyp^S_{g+1}, C)$$

mapping (p, z_p) to the family defined by $(p, \iota(p), z_p, \iota^* z_p)$. Again, we are using the construction of section 4.

Theorem 8.5. *The tangent space $T_C Hyp^S_{g+1}$ is spanned by vectors tangent to arcs, i.e. it it equal to $T_C(Hyp^S_{g+1})^{arc}$.*

8.2 Transversality

This section is devoted to the proof of the following result:

Theorem 8.6. *The intersection of $\mathcal{A}^S_g$ and Hyp^S_{g+1} is transverse. In other words, scheme theoretically, it is equal to Hyp^S_g.*

Let $I_{Hyp_{g+1}}$ be the ideal of modular forms on $\mathcal{A}_g$ vanishing on Hyp_g. The inclusion

$$\mathcal{A}_g^S \hookrightarrow \mathcal{A}_{g+1}^S$$

is induced by the Siegel operator Φ. We have to prove that the map

$$\Phi\colon I_{Hyp_{g+1}} \to I_{Hyp_g}$$

is surjective. For technical reasons, we will first prove the theorem on the finite cover defined by the level structure (4,8), the claim will follow because finite groups are linearly reductive in characteristic zero.

Let us recall a few facts about level structures, cf. e.g. [Fre83] II.6. The group $\Gamma(4,8)$ is a normal co-finite subgroup of $Sp(2g,\mathbb{Z})$. Call G the finite quotient. The moduli space $\mathcal{A}_g(4,8)$ is the quotient of the Siegel upper half space by $\Gamma(4,8)$. A point of $\mathcal{A}_g(4,8)$ represent a principally polarised abelian variety with extra structures. Among these extra data, we have an isomorphism ϕ between the subgroup of two torsion elements and $(\mathbb{Z}/2\mathbb{Z})^{2g}$. On $\mathcal{A}_g(4,8)$ there is the ample line bundle L of weight one modular forms, whose sections are holomorphic functions on $\mathcal{H}_g$ which transform appropriately under the action of $\Gamma(4,8)$. Using this line bundle, we can construct the Satake compactification $\mathcal{A}_g^S(4,8)$ of $\mathcal{A}_g(4,8)$. The boundary is composed by many irreducible components X_i, permuted by G.

For each components X_i, we have a Siegel operator Φ_i.

$$\Phi_i\colon H^0(\mathcal{A}_g(4,8),L^k) \to H^0(\mathcal{A}_{g-1}(4,8),L^k)$$

which realise an isomorphism between X_i and $\mathcal{A}_{g-1}^S(4,8)$. There is a component, say X_0, called the "standard component", where the Siegel operator is given by the usual formula

$$\Phi_0(F)(\tau) := \lim_{t\to\infty} F(\tau \oplus it)$$

The others Siegel operators are obtained by letting G act.

References about the hyperelliptic locus with level structure (4,8) are [Igu67b], [Tsu91] and [SM03]. We recall a few facts. The space $Hyp_g(4,8)$ inside $\mathcal{A}_g(4,8)$ is the preimage of Hyp_g under the quotient map. This space splits in many irreducible components Y_j permuted by G. Call $Hyp_g(4,8)^S$ the closure of $Hyp_g(4,8)$ in $\mathcal{A}_g^S(4,8)$. The intersection of $Hyp_g(4,8)^S$ with any of the X_i is, set-theoretically, equal to $Hyp_{g-1}(4,8)^S$. We shall show that the equality is true as schemes.

A way to specify an irreducible component Y_j is to fix a special fundamental system of Theta characteristics $\mathfrak{m} = \{m_0,\dots,m_{2g+1}\}$. This is a subset of $(\mathbb{Z}/2\mathbb{Z})^{2g}$ with some additional properties, see [SM03] for the definition. The relation between special fundamental system and irreducible components is the following. Call W the set of Weierstrass points of a hyperelliptic curve C. For any w in W, call AJ_w the Abel-Jacobi map with base point w. The set $AJ_w(W)$ is a subset of the 2-torsion subgroup of $\mathrm{Jac}(C)$. The choice of a special fundamental system of Theta characteristic $\mathfrak{m}$, determines the component $Y_i = Y_{\mathfrak{m}}$ of abelian varieties $(\mathrm{Jac}(C),\Theta,\phi)$ such that there exists a w in W for which

$\phi(AJ_w(W)) = \mathfrak{m}$. Call $Y^S_{\mathfrak{m}}$ its closure in $\mathcal{A}^S_g(4,8)$. Different choice of $\mathfrak{m}$ may determine the same component Y_i, this because of the freedom in the choice of the base point of the Abel-Jacobi map.

Fix a system of Theta characteristic $\mathfrak{m}$, so we have an irreducible component $Y_{\mathfrak{m}}$ of $Hyp_g(4,8)$. Let b be the sum of odd m_i in $\mathfrak{m}$. For every Theta characteristic m, the classical Thetanullerwerte θ_m is a well defined modular form on $\mathcal{A}_g(4,8)$ (but not on $\mathcal{A}_g$, this is the reason why we are using the level structure), see e.g. [Igu67b] or [SM03]. Our proof relies upon the following result.

Theorem 8.7 ([SM03] Theorem 1)**.** *The scheme $Y^S_{\mathfrak{m}}$ is ideal theoretically defined by the vanishing of Thetanullerwerte θ_{m+b} with $m = m_{i_1} + \cdots + m_{i_k}$, where $k \leq g$.*

We still need some more notations. As usual, we write a Theta characteristic as two vectors of size g. Call $\underline{0}$ the g dimensional zero vector. Define two $g+1$ dimensional Theta characteristics

$$p := \begin{bmatrix} \underline{0} & 0 \\ \underline{0} & 1 \end{bmatrix} \quad , \quad q := \begin{bmatrix} \underline{0} & 1 \\ \underline{0} & 1 \end{bmatrix}$$

For every g dimensional Theta characteristic $m = [\epsilon, \epsilon']$, let us define the $g+1$ dimensional Theta characteristic

$$\overline{m} := \begin{bmatrix} \epsilon & 0 \\ \epsilon' & 0 \end{bmatrix},$$

Moreover, for every special fundamental system of g dimensional Theta characteristic $\mathfrak{m}$, we pose

$$\overline{\mathfrak{m}} := p \cup q \cup \bigcup_{m \in \mathfrak{m}} \overline{m}\,.$$

This is a special fundamental system of $g+1$ dimensional Theta characteristics.

Let $I_{Hyp_{g+1}(4,8)}$ be the ideal of $Hyp_{g+1}(4,8)$ in $\mathcal{A}_{g+1}(4,8)$, by $I_{(Hyp_g(4,8),X_i)}$ we denote the ideal of $Hyp_g(4,8)$ in the boundary component X_i of $\mathcal{A}_{g+1}(4,8)^S$.

Lemma 8.8. *Scheme theoretically, the intersection of $Y^S_{\overline{\mathfrak{m}}}$ and X_0 is isomorphic to $Y_{\mathfrak{m}}$.*

Proof. By direct computation one sees that

$$\Phi_0(\theta \begin{bmatrix} \epsilon & 0 \\ \epsilon' & \delta \end{bmatrix}) = \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$$

for δ equal either to 0 or 1. Suppose that a g dimensional Theta characteristic m is of the form prescribed by theorem 8.7 for the special fundamental system $\mathfrak{m}$. The modular form θ_{m+b} vanishes on the irreducible component $Y_{\mathfrak{m}}$ of $Hyp_g(4,8)$, and the modular form $\theta_{\overline{m}+\bar{b}+p}$ vanishes on the irreducible component $Y_{\overline{m}}$ of $Hyp_{g+1}(4,8)$. We have

$$\Phi_0(\theta_{\overline{m}+\bar{b}+p}) = \theta_{m+b}\,,$$

so, because of theorem 8.7, the map

$$\Phi_0 : I_{Hyp_{g+1}(4,8)} \to I_{(Hyp_g(4,8),X_0)}$$

is surjective. □

Proposition 8.9. *The intersection of $Hyp_{g+1}(4,8)^S$ and X_i is transverse for every i.*

Proof. For $i=0$, the proposition is the previous lemma. For a general i, it is enough to notice that G acts transitively on the boundary components and preserves the hyperelliptic locus. $\square$

We have a G equivariant map

$$\bigoplus_i \Phi_i \colon I_{Hyp_{g+1}(4,8)} \to \bigoplus_i I_{(Hyp_g(4,8),X_i)}$$

This map is surjective because of the previous proposition. If we take G invariants, we get a map

$$\bigoplus_i \Phi_i \colon I_{Hyp_{g+1}} \to \bigoplus_i I_{Hyp_g}$$

which is still surjective because G is finite and the base field has characteristic zero. We obtain the theorem projecting onto one of the factor.

We would like to point out that this result agree with the period matrices computated in the previous sections. Indeed, the period matrix we get from the degeration described at the bottom of section 4 is, by Lemma 5.5, tangent to the hyperelliptic locus.

8.3 Projective invariants of hyperelliptic curves

In this section we introduce the projective invariants of a hyperelliptic curve and we prove the following well-known Criterion.

Criterion 8.10. *Let F_g be a weight n and degree g modular form. Restrict it to Hyp_g^S, suppose that it vanishes on Hyp_{g-1} with multiplicity at least k. If*

$$\frac{n}{k} < 8 + \frac{4}{g}$$

then F_g vanishes on Hyp_g.

This Criterion could be proved using the slope of the hyperelliptic locus, see [CH88] theorem 4.12. We will rather use projective invariants, references are [Igu67b], [Poo96], [AL02] and [Pas11] Chapter 2.

Let C be a smooth genus g hyperelliptic curve. Fix a two to one map π from C to $\mathbb{P}^1$, this morphism is unique up to projective transformations of $\mathbb{P}^1$, it ramifies at $2g+2$ points. A point p is called a Weierstrass point if it is a ramification point for π.

Definition 8.11 (Projective invariants)**.** *The projective invariants of C are the image of the Weierstrass points under π, considered up to permutations and projective automorphisms of $\mathbb{P}^1$*

Starting from $2g+2$ distinct points on $\mathbb{P}^1$, one can construct a smooth genus g hyperelliptic curve with the prescribed projective invariants. As an aside, let us recall that Thomæ's formula permits to write the cross-ratios of the projective invariants in term of second order theta functions evaluated at the period matrix.

Call $\mathcal{B}_g$ the moduli space of $2g+2$ points on $\mathbb{P}^1$, up to permutation and projectivity. This space is a GIT quotient, the semi-stable locus (i.e. the $2g+2$-tuples that $\mathcal{B}_g$ parametrises) consist of all the $2g+2$-tuples such that no more than $g+1$ points coincide. $\mathcal{B}_g$ can be defined as the Proj of the ring $S(2, 2g+2)$. This is the ring of symmetric functions in $2g+2$ variables, which are invariant under the natural action of $SL(2, \mathbb{C})$. See the references for more details. The discriminant Δ is an element of $S(2, 2g+2)$ of degree $4g+2$, it cuts the divisor D parametrising the $2g+2$-tuples of points where at least two entries coincide.

Because of the previous discussion, we have an isomorphism

$$f_g \colon Hyp_g \to \mathcal{B}_g \setminus D$$

mapping a curve to its projective invariants. Following [AL02], this isomorphism extend to a map

$$f_g \colon \overline{Hyp}_g \to \mathcal{B}_g$$

where $\overline{Hyp}_g$ is the Deligne-Mumford compactification of Hyp_g. This map is a birational isomorphism between the boundary divisor Ξ_0 and D, it contracts all the other boundary divisors of $\overline{Hyp}_g$ to subvariety of co-dimension greater than 1. The divisor Ξ_0 parametrises curves of compact type, i.e. curves obtained starting with a genus $g-1$ hyperelliptic curve C' and gluing two points conjugated under the hyperelliptic involution. Its image is the set of $2g+2$ points of the form $\{p_1, \ldots, p_{2g}, p, p\}$, the projective invariants of C' are $\{p_1, \ldots, p_{2g}\}$, the glued points are the preimages of p under π.

We can consider the rational inverse of f_g, call $\bar{\rho}$ the composition

$$\bar{\rho} \colon \mathcal{B}_g \overset{f_g^{-1}}{\dashrightarrow} \overline{Hyp}_g \xrightarrow{J} Hyp_g^S \hookrightarrow \mathcal{A}_g^S$$

this map is the geometric version of the Igusa morphism of projective invariants ρ defined in [Igu67b], which is a map of graded rings

$$\rho \colon \bigoplus_{n=0}^{\infty} H^0(\mathcal{A}_g, L_g^n) \to S(2, 2g+2)$$

whose kernel is exactly the ideal of modular forms vanishing on the hyperelliptic locus. The degree of ρ is $\frac{1}{2}g$.

The closure of the image of the divisor Ξ_0 in Hyp_g^S is Hyp_{g-1}^S, so the image of D under $\bar{\rho}$ is Hyp_{g-1}^S. We can now prove the Criterion.

Proof. (of Criterion 8.10) Suppose F_g vanishes with multiplicity at least k on Hyp_{g-1}^S. This means that $\bar{\rho}^* F_g$ vanishes with multiplicity at least k on D. In other words, Δ^k

divides $\rho(F_g)$. The degree of the discriminant in $S(2, 2g+2)$ is $4g+2$, the degree of $\rho(F_g)$ is $\frac{1}{2}gn$. Since, by hypothesis,

$$k(4g+2) > \frac{1}{2}gn$$

we obtain that $\rho(F_g)$ is equal to zero, so the claim. $\square$

8.4 Stable equations for the hyperelliptic locus

Motivated by Theorem 8.2, we are going to look for pairs of lattices Λ and Γ such that the stable modular form

$$F := \Theta_\Lambda - \Theta_\Gamma$$

vanishes on the hyperelliptic locus for every g, in other words such that it is a stable equation for the hyperelliptic locus.

We will need the following basic invariant of a lattice

$$\mu(\Lambda) := \min\{Q(v,v) \mid v \in \Lambda; v \neq 0\} = \min\{n \mid \mathcal{R}_n(\Lambda) \neq \varnothing\},$$

where $\mathcal{R}_n(\Lambda)$ is the set of vectors of Λ of norm $2n$.

First, we look for a necessary condition on Λ and Γ. If F is a stable equation for the hyperelliptic locus, for $g = 1$, it vanishes on $Hyp_1 = \mathcal{A}_1$, so

$$\Theta_{\Lambda,1} = \Theta_{\Gamma,1}.$$

Looking at the Fourier-Jacobi expansion, the previous equality means that the two lattices have the same number of vectors of any given norm. In particular, we have

$$\mu(\Lambda) = \mu(\Gamma).$$

Combining Theorem 8.6 and Criterion 8.10 we can prove the following:

Theorem 8.12. *Let Λ and Γ be two even positive definite unimodular lattices of rank N and $\mu(\Lambda) = \mu(\Gamma) =: \mu$. If*

$$\frac{N}{\mu} \leq 8,$$

then

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is a stable equation for the hyperelliptic locus. In other words, F_g vanishes on Hyp_g for every g.

Proof. The proof is by induction on g. The difference of two theta series vanishes on $\mathcal{A}_0$. Suppose the statement true for g, we want to apply Criterion 8.10 to F_{g+1}. Call $k := \frac{1}{2}\mu$, we need to prove that F_{g+1} vanishes at the boundary component Hyp_g^S with multiplicity at least k.

To stress why Theorem 8.6 is important, let us first give the proof when $k = 2$. The argument is local, take a generic point τ of Hyp_g^S. By induction we know that $F_{g+1}(\tau) = 0$, we want to prove that for every derivative D in $T_\tau Hyp_{g+1}^S$ we have $DF_{g+1}(\tau) = 0$.

Let us write out the n-th Fourier-Jacobi coefficient of a Theta series $\Theta_{\Lambda,g+1}$ (we keep the notations of section 7.3, in particular equations 5 and 6), we have

$$f_n(\tau, z) = \sum_{x_1,\dots,x_g\in\Lambda} \sum_{y\in\mathcal{R}_n(\Lambda)} \exp(\pi i \sum_{i,j} Q(x_i, x_j)\tau_{ij} + 2\pi i \sum_i Q(y, x_i)z_i) \tag{7}$$

In particular, if $\mathcal{R}_n(\Lambda)$ is empty then f_n is trivial.

Since $\mu = 4$, the first Fourier-Jacobi coefficient is trivial and the Fourier-Jacobi expansion of F_{g+1} looks like

$$F_{g+1} = F_g(\tau) + O(q^2)$$

so $DF_{g+1}(\tau) = DF_g(\tau)$. We can thus assume that D is tangent to $\mathcal{A}_g$. Now, we need to use theorem 8.6 to assume that D is tangent to Hyp_g. By inductive hypothesis F_g is zero on Hyp_g, so $DF_g(\tau) = 0$. This conclude the proof when $k = 2$. (Remark that we can not run the same argument for $\mathcal{M}_g$: there are plenty of tangent vectors to $\mathcal{M}_{g+1}^S \cap \mathcal{A}_g$ which are not tangent to $\mathcal{M}_g$)

For a general k the argument is pretty much the same, we just need to enhance the notations. Suppose F_{g+1} vanishes on Hyp_g^S with order at least s smaller than k, we want to prove it vanishes with order at least $s+1$. In the local ring (Hyp_{g+1}^S, τ), consider the ideal I of elements vanishing on Hyp_g^S. We know F_{g+1} belongs to I^s, we want to show that its class in I^s/I^{s+1} is trivial. The elements of I^s/I^{s+1} are symmetric s-linear forms on $T_\tau Hyp_{g+1}$, restricting them to $T_\tau(\mathcal{A}_g \cap Hyp_{g+1}^S) = T_\tau Hyp_g$ we get an exact sequence

$$H^0(\tau, 2s\Theta) \to I^s/I^{s+1} \xrightarrow{\Phi} \mathrm{Sym}^s(T_\tau Hyp_g^\vee)\,.$$

Where Φ is the restriction from $\mathcal{A}_{g+1}^S$ to $\mathcal{A}_g$. The class of F_{g+1} is in the kernel of Φ, because by inductive hypothesis F_g vanishes identically on Hyp_g. Moreover, it is zero in $H^0(\tau, 2s\Theta)$, because $s < k$, so the conclusion. (We have used theorem 8.6 to replace $T_\tau(\mathcal{A}_g \cap Hyp_{g+1}^S)$ with $T_\tau Hyp_g$.) $\square$

The hypothesis

$$\frac{\mathrm{rk}(\Lambda)}{\mu(\Lambda)} \leq 8 \tag{8}$$

is quite restrictive. Indeed, given any even unimodular lattice Λ, there is an upper bound

$$\mu(\Lambda) \leq 2\lfloor \frac{\mathrm{rk}(\Lambda)}{24} \rfloor + 2$$

where "$\lfloor\,\rfloor$" is the round down (see [CS99] section 7.7 Corollary 21); moreover $\mu(\Lambda)$ is even and the rank, since Λ is positive definite, is divisible by 8. We conclude that if an even unimodular positive definite lattice Λ satisfies hypothesis (8), then the only possibilities for the couple $(\mathrm{rk}(\Lambda), \mu(\Lambda))$ are (8,2),(16,2),(24,4),(32,4) and (48,6). All these lattices are **extremal**, which means

$$\mu(\Lambda) = 2\lfloor \frac{\mathrm{rk}(\Lambda)}{24} \rfloor + 2\,.$$

On the other hand, given two extremal lattices Λ and Γ, it is not true that $\Theta_{\Lambda,g} - \Theta_{\Gamma,g}$ vanishes on the hyperelliptic locus for every g. See [Oze88] for an example of 3 extremal lattices of rank 40 whose theta series are different for $g = 2$. In the proof of Theorem 8.12, we have not used the hypothesis $\mu(\Lambda) = \mu(\Gamma)$, however, as we have seen, hypothesis (8) and $\mathrm{rk}(\Lambda) = \mathrm{rk}(\Gamma)$ imply this fact.

There exist only two lattices of rank 16 and $\mu = 2$, their difference gives the Schottky form discussed in the introduction. There exists exactly one lattice of rank 8, E_8, and one lattice of rank 24 and $\mu = 4$, the Leech lattice, so in these cases we do not get any stable equation for the hyperelliptic locus

In [Kin03] Corollary 5, using a generalization of the mass formula, it is shown that there exist at least ten millions of lattices of rank 32 and $\mu = 4$ (in King's paper every lattice is tacitly assumed to be positive definite), however just 15 of them are known explicitly.

The situation for lattices of type (48,6) is not clear: believably, there exist many of them, see [Kin03] page 15, but there is not any lower bound and just 3 of them are known explicitly.

9 Prym varieties

9.1 Definitions and notations

Prym varieties arise from double covers $\pi\colon \widetilde{C} \to C$, unramified or branched at two points; general references are [BL92] Chapter 12, [ACGH85] Appendix C or [Far12].

Let g be the genus of C; the genus of $\widetilde{C}$ is $2g+1$ if the cover is étale, $2g$ otherwise. The Prym variety associated to π can be defined as follows:

$$\mathrm{Pr}(\widetilde{C}/C) := H^1(\widetilde{C}, K_{\widetilde{C}})^{\vee -}/H^1(\widetilde{C}, \mathbb{Z})^-$$

where "minus" indicates the minus one eigenspace of the involution ι of the cover. The differentials $H^1(\widetilde{C}, K_{\widetilde{C}})^-$ are called *Prym differentials.*

We describe a well-known way to compute the periods of a *Prym variety* (see e.g. Example 2.1 of [Far12]). Fix a symplectic basis $\tilde{A}_0$, $\tilde{B}_0$, $A_1^+, \dots, A_g^+$, $A_1^-, \dots, A_g^-$, $B_1^+, \dots, B_g^+$, $B_1^-, \dots, B_g^-$ for the homology of $\widetilde{C}$ with the following properties. The involution swaps A_i^+ with A_i^- and B_i^+ with B_i^-. The cycles $\tilde{A}_0$ and $\tilde{B}_0$ are present only in the unramified case; they are fixed by the involution. Call A_i and B_i the cycles $\iota_*(A_i^+)$ and $\iota_*(B_i^+)$, and A_0 and B_0 half of $\iota_*(A_0)$ and $\iota_*(B_0)$. These cycles form a symplectic basis for the homology of C. Let $\omega_i^{\pm}$ (respectively ω_i) be the corresponding normalised basis for the holomorphic differentials on $\widetilde{C}$ (C).

A symplectic basis for $H_1(\widetilde{C}, \mathbb{Z})^-$ is given by $A_i^+ - A_i^-, B_i^+ - B_i^-$, for i from 1 to g. The corresponding basis for $H^1(\widetilde{C}, K_{\widetilde{C}})^-$ is given by

$$w_i := \frac{1}{2}(\omega_i^+ - \omega_i^-)\,,$$

so the periods of Prym differentials are

$$\frac{1}{2}\int_{B_i^+ - B_i^-} \omega_j^+ - \omega_j^- \qquad i, j = 1, \dots, g\,.$$

We use the same notations for families. Given a one parameter family of double covers $\pi : \tilde{\mathcal{C}}_t \to \mathcal{C}_t$, we call $P(t)$ the corresponding family of Prym varieties; their periods are

$$P(t) = \frac{1}{2}\int_{B_i^+(t) - B_i^-(t)} \omega_j(t)^+ - \omega_j^-(t) \qquad i, j = 1, \dots g\,.$$

By abuse of notations, we denote by $P(t)$ at the same time the Prym varieties and the matrices of periods of the Prym differentials. Modulo t^2, the local expansion of the period around $t = 0$ is:

$$P(t) = \frac{1}{2}\int_{B_i^+(0) - B_i^-(0)} \Big(\omega_j(0)^+ - \omega_j^-(0)\Big) + \frac{1}{2}t\int_{B_i^+(0) - B_i^-(0)} \nabla\Big(\omega_j(t)^+ - \omega_j^-(t)\Big)\,|_{t=0}\,,$$

where ∇ is the Gauss-Manin connection.

9.2 The étale case

We denote by $\mathcal{P}_g$ the coarse moduli space of Prym varieties arising from étale double covers. The Satake compactification $\mathcal{P}_g^S$ is defined as the closure of $\mathcal{P}_g$ in $\mathcal{A}_g$ is its .

Pinching a non-trivial homological cycle We are going to prove the analogue result of 1.1. The intersection of $\mathcal{P}_{g+m}^S$ and $\mathcal{A}_g$ contains $\mathcal{P}_g$, we can give the following description.

Theorem 9.1. *The intersection of $\mathcal{P}_{g+m}^S$ and $\mathcal{A}_g$ contains the m-th infinitesimal neighbourhood of $\mathcal{P}_g$ in $\mathcal{A}_g$.*

The strategy of the proof is the same. We start with a 2-to-1 unramified cover $\widetilde{C} \to C$, where C is automorphism-free. Call g the genus of C. We pick points a and b and local co-ordinates on C and we pull them back to $\widetilde{C}$: call the preimages a^+, a^-, b^+ and b^-. We perform the construction of section 4 simultaneously on C and $\widetilde{C}$, so we get a family of covers $\widetilde{\mathcal{C}}_t \to \mathcal{C}_t$ over a disc Δ_t degenerating to $\widetilde{C}/(a^+ \sim b^+, a^- \sim b^-) \to C/(a \sim b)$. Call $P(t)$ the corresponding family of Prym varieties.

Fix bases for the homology as in section 9.1. There are two vanishing cycles on $\widetilde{C}_t$: $A_{g+1}^+(t)$ and $A_{g+1}^-(t)$. They are swapped by the involution.

The pull back of $w_1(0), \dots, w_{g-1}(0)$ is a basis for the Prym differentials of $\widetilde{C}$.

Proposition 9.2 ([FS86] Section 2)**.** *Keep notation as above, then*

$$P(0) = \begin{pmatrix} \mathrm{Pr}(\widetilde{C}/C) & AP(a-b) \\ {}^tAP(a-b) & \frac{1}{2\pi i}\ln(t) + c_0 \end{pmatrix}$$

where $AP(a-b)$ is the Abel-Prym map.

Proof. The logarithm is due to the monodromy of the integral of $w_g(t)$ over $B_g^+(t) - B_g^-(t)$: turning around the origin of the disc Δ_t, $B_g^\pm$ is increased by $A_g^\pm$: the integral of $w_g(0)$ on $A_g^+ - A_g^-$ is 1, so we can write the entry $P(t)_{g,g}$ as $\frac{1}{2\pi i}\ln(t)$ plus some holomorphic function.

Let us compute the rest of the first row. We have to integrate w_i on $B_1^+ - B_1^-$, for $i = 1, \dots, g-1$. To do this, we pull back w_i on $\widetilde{C}$, and take the difference of the integral from a^+ to b^+ and from a^- to b^-. This is nothing but the Abel-Prym map.

The biggest block of the matrix comes from pulling everything back to $\widetilde{C}$. □

Proposition 9.3. *The period matrix of $P(t)$ is*

$$P(t) = \begin{pmatrix} Pr(\widetilde{C}/C) & AP(a-b) \\ {}^tAP(a-b) & \frac{1}{2\pi i}\ln(t) + c_0 \end{pmatrix} + t \begin{pmatrix} \sigma & \cdots \\ \vdots & c_1 \end{pmatrix} + O(t^2)$$

where σ is a holomorphic function of the parameters given by

$$\sigma_{ij} = -2\pi i \left(w_i(a^+) w_j(b^+) + w_i(a^+) w_j(b^+) \right) .$$

The differentials are evaluated in term of the pull backs of dz_a and dz_b

Proof. The zero order term comes from the previous proposition. To compute σ one argues as in Theorem 4.2.

□

The tangent space to $P(0)$ at the origin is $H^0(C, K_C + \eta)^\vee$, where η is the theta characteristic of the cover $\pi : \widetilde{C} \to C$. In general, given an abelian variety X, we have

$$T_X \mathcal{A}_g = \operatorname{Sym}^2 T_0 X^\vee ,$$

so the tangent space $T_{\operatorname{Pr}(\widetilde{C}/C)} \mathcal{A}_{g-2}$ is isomorphic to $Sym^2 H^0(C, K_C + \eta)^\vee$.

In a co-ordinate free way, σ is given by

$$\sigma(w) = -2\pi i (\frac{w}{\pi^* dz_a}(a^+) \frac{w}{\pi^* dz_b}(b^+))^2 .$$

Recall that Prym differentials can be interpreted at the same time as differentials on $\widetilde{C}$ or as sections of a line bundle over C. We can interpret the projectivization of σ as the map

$$\operatorname{Sym}^2 C \to \mathbb{P} \operatorname{Sym}^2 H^0(K_C + \eta)^\vee$$

given by the line bundle $(K_C + \eta)^{\boxtimes 2}$.

The image of C via $K_C + \eta$ is, by definition, the Prym canonical model of the curve. If the Clifford index of C is bigger than or equal to 3, then this line bundle is very ample ([SV02] page 10).

Proposition 9.4. *Let C be a generic curve, varying the choice of a and b the image of σ span all $\mathbb{P}T_{\operatorname{Pr}(\widetilde{C}/C)} \mathcal{A}_{g-2}$.*

Proof. The map σ is given by a linear system. □

Now, we argue as in section 7.2. We suppose that a modular form F_{g-1} vanishes along $\mathcal{P}_{g-1}$ with multiplicity at least k, we take its Fourier expansion, we restrict its derivatives to P_t, and we can apply the Proposition 9.4 to prove Theorem 9.1.

Pinching a homologically trivial cycle As we will see, the moduli space $\mathcal{M}_g$ is contained in Pr_g^S (cf. [BL92] page 376); we want to prove the following Theorem:

Theorem 9.5. *The Satake compactification $\mathcal{P}_g^S$ contains the first infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$. In other words, let C be a point of $\mathcal{M}_g$, we have $T_C \mathcal{P}_g^S = T_C \mathcal{A}_g$.*

This result could be related to the Schottky-Jung relations. Indeed, these relations, morally, relate the vanishing on $\mathcal{M}_{g+1}$ to the vanishing on $\mathcal{P}_g$, and we know that both $\mathcal{M}_{g+1}^S$ and $\mathcal{P}_g^S$ contain the first infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$.

To prove the Theorem, we compute a variational formula for the periods of the following family of Prym varieties (for this construction see also [BL92] page 376).

First, we describe the central fibre. We start with a smooth curve C of genus g that is automorphism-free. We pick two points a and b on C and we call $\mathcal{C}_0$ the curve $C/(a \sim b)$. We construct an unramified 2-to-1 cover $\widetilde{\mathcal{C}_0} \to \mathcal{C}_0$ taking two copies of C, call them C^+ and C^-, and gluing a^+ with b^-, obtaining a node n_1, and a^- with b^+, obtaining a node n_2. Let $P(0)$ be the associated Prym.

Lemma 9.6. *The Prym variety $P(0)$ is isomorphic to the Jacobian of C.*

Proof. Keep notations as in section 9.1. On $\tilde{\mathcal{C}}_0$ we have $\int_{B_j^+} \omega_i^- = \int_{B_j^-} \omega_i^+ = 0$, so

$$\frac{1}{2} \int_{B_j^+ - B_j^-} \omega_i^+ - \omega_i^- = \int_{B_j} \omega_i$$

□

We construct a family of Pryms degenerating to $P(0)$. Pick local co-ordinates z_a and z_b on C. Using the procedure described in section 4, we obtain a family $\widetilde{\mathcal{C}}_t \to \mathcal{C}_t$ of covers degenerating to $\widetilde{\mathcal{C}}_0 \to \mathcal{C}_0$. Let $P(t)$ be the corresponding family of Prym varieties. This degeneration together with Lemma 9.6 show that $\mathcal{M}_g$ is contained in $\mathcal{P}_g^S$.

Fix bases for the homologies of $\widetilde{\mathcal{C}}_t$ and $\mathcal{C}_t$ as in section 9.1. Remark that both $A_0(t)$ and $\widetilde{A_0}(t)$ are vanishing cycles. Since $P(0)$ is isomorphic to the Jacobian of C, we denote by τ the period matrix of C and the period matrix of $P(0)$.

Proposition 9.7. *The period matrix of $P(t)$ is*

$$P(t) = \tau + t\sigma + O(t^2)\,,$$

where σ is a holomorphic function of a, b, z_a and z_b given by

$$\sigma_{ij} = -2\pi i \left(\omega_i(a)\omega_j(b) + \omega_i(b)\omega_j(a)\right)\,.$$

The differentials ω_i are a normalised basis for the abelian differentials of C. They are evaluated in term of dz_a and dz_b.

Proof. The zero order term has already been computed. The computation of σ goes as in Theorem 4.2.

□

The proof is as in section 7.2.

Schiffer variations We conclude with a variational formula for Schiffer variations (cf. section 5). Consider a 2-to-1 unramified cover $\pi : \widetilde{C} \to C$. Call $P(0)$ the period matrix of the associated Prym. Pick a point p on C and let $\{p^+, p^-\}$ be $\pi^{-1}(p)$. Perform simultaneously a Schiffer variation at (p, z_p), $(p^+, \pi^* z_p)$ and $(p^-, \pi^* z_p)$. We get a family of covers $\pi\colon \widetilde{\mathcal{C}}_t \to \mathcal{C}_t$ over a disc Δ_t.

Theorem 9.8. *The periods $P(t)$ of the family constructed above are*

$$P(t) = P(0) + t\sigma(p, z_p) + O(t^2)\,,$$

where

$$\sigma_{ij} = 2\pi i \frac{w_i(p) w_j(p)}{dz_p^2}$$

Proof. The proof is as in theorem 5.1. The only difference is that $\nabla\omega_i(0)$ will be a meromorphic differential with two poles of order two without residues at p^+ and p^-, and with appropriate leading coefficients. $\square$

The tangent space $T_{P(0)}\mathcal{A}_g$ is $\operatorname{Sym}^2 H^0(K_C+\eta)^\vee$, so σ is given by

$$\sigma(p,z_p)(\omega) = 2\pi i(\frac{w}{dz_p}(p))^2 \,.$$

The projectivization of $\sigma(p,z_p)$ does not depend on z_p, it is the image of p under the composition

$$C \xrightarrow{K_C+\eta} \mathbb{P}H^0(K_C+\eta)^\vee \xrightarrow{\mathrm{Ver}_2} \mathbb{P}\operatorname{Sym}^2 H^0(K_C+\eta)^\vee \,.$$

The moduli space of étale double covers is a finite cover of $\mathcal{M}_g$, so its tangent space is generated by Schiffer's variations. We have the following corollary

Corollary 9.9. *The tangent space of $\mathcal{P}_g$ at the Prym variety of the cover $\widetilde{C} \to C$ is generated by the affine cone over the image of the map*

$$C \to \mathbb{P}H^0(K_C+\eta)^\vee \xrightarrow{\mathrm{Ver}_2} \mathbb{P}\operatorname{Sym}^2 H^0(K_C+\eta)^\vee \,.$$

The line bundle $K_C+\eta$ is very ample if the Clifford index of C is greater than or equal to 3. An analysis of the Prym canonical model is carried out in [Deb89].

In the same spirit of [OS80] and Theorem 5.6, it might be possible to obtain a refined version of the local Torelli Theorem for Prym's varieties. A survey of the Torelli problem for Prym varieties is [SV02].

9.3 The ramified case

Let ${}^r\mathcal{P}_g$ be the coarse moduli space of Prym arising from double covers branched at two points. As usual, ${}^r\mathcal{P}_g^S$ will denote its closure in $\mathcal{A}_g^S$.

Pinching a non-trivial homological cycle We start with a two to one cover $\widetilde{C} \to C$ ramified at two points. We picks two points (distinct from the branch locus) and two local-co-ordinates on C, we pull them back on $\widetilde{C}$, and we perform the usual surgery. The construction and the computation go exactly as in 9.3, and we get the following theorem.

Theorem 9.10. *The intersection of ${}^r\mathcal{P}_{g+m}^S$ and $\mathcal{A}_g$ contains the m-th infinitesimal neighbourhood of ${}^r\mathcal{P}_g$ in $\mathcal{A}_g$.*

Pinching a trivial homological cycle As for the étale case, the moduli space $\mathcal{M}_g$ is a boundary component of ${}^r\mathcal{P}_g^S$. We want to study the singularity. Fix a genus g automorphism-free curve C and a point p. Take two copies of (C,p), call them (C^+,p^+) and (C^-,p^-). Let $\tilde{\mathcal{C}}_0$ the curve obtained gluing p^+ and p^-, call n the node. The curve $\tilde{\mathcal{C}}_0$ is a double cover of C. Fix a local co-ordinate z_p at p. Using this co-ordinate we construct a family $\widetilde{\mathcal{C}}_t$ degenerating at $\widetilde{\mathcal{C}}_0$. The involution on $\widetilde{\mathcal{C}}_0$ extends to an involution of the entire family, so $\widetilde{\mathcal{C}}$ is a double cover of a family $\mathcal{C}_t$ with central fibre isomorphic to C. (We have not an explicit description of $\mathcal{C}_t$.) Call $P(t)$ the corresponding family of Prym.

Lemma 9.11. *The Prym variety $P(0)$ is isomorphic to the Jacobian of C.*

Proof. Fix basis for the homology as in section 9.1, on the central fibre we have $\int_{B_i^-} \omega_j^+ = \int_{B_i^+} \omega_j^- = 0$, so

$$\frac{1}{2}\int_{B_i^+ - B_i^-} \omega_j^+ - \omega_j^- = \int_{B_i} \omega_j$$

□

Fix a basis for the homologies of $\widetilde{\mathcal{C}}_t$ and $\mathcal{C}_t$ as in section 9.1. We do not have any vanishing cycle. Since $P(0)$ is isomorphic to the Jacobian of C, we denote by τ the period matrix of C and the period matrix of $P(0)$.

Proposition 9.12. *The periods of the Prym differentials of the family $\widetilde{\mathcal{C}}_t \to \mathcal{C}$ are*

$$P(t) = \tau + t\sigma(p, z_p) + O(t^2)\,,$$

where σ is given by

$$\sigma(p, z_p)_{ij} = 2\pi i \frac{\omega_i(p)\omega_j(p)}{dz_p^2}\,.$$

The ω_i are normalized abelian differential on C.

Proof. Call

$$\nu : C^+ \sqcup C^- \to \tilde{\mathcal{C}}_0$$

the normalisation map. As usual, we follow the proof of Theorem 4.2. All differentials are evaluated with respect to dz_p. We have

$$\nu^*\omega_i^+(p^+) = \omega_i(p)\ ,\ \nu^*\omega_i^+(p^-) = 0\,,$$

so on C^+

$$\nu^*\nabla\omega_i^+ = 0$$

and on C^-

$$\nu^*\nabla\omega_i^+ = -\omega_i(p)\eta_p\,.$$

The statement follows from Riemann's bilinear relations 4.5. □

The matrix $\sigma(p, z_p)$ is the tangent vector to a Schiffer variation of C at (p, z_p), see proposition 5.1. To obtain a more general degeneration to $P(0)$, the only thing we can do is to perform some Schiffer's variations on C away from p, and pull them back to $\widetilde{C}$. The tangent vector we get in this way is a linear combination of Schiffer's variations, so we have the following result.

Theorem 9.13. *Let C be a generic point of $\mathcal{M}_g$, then*

$$(T_C^r \mathcal{P}_g^S)^{arc} = T_C \mathcal{M}_g \,,$$

where $(T_C^r \mathcal{P}_g^S)^{arc}$ is the subspace of $T_C^r \operatorname{Pr}_g^S$ spanned by vectors coming from arcs (cf. section 6.1).

This situation reminds the local Torelli theorem for hyperelliptic curves. The local Torelli problem in the ramified case is discussed in [MP12].

10 Lattices and heat equation

In this section we study when the Fourier-Jacobi coefficients of a theta series satisfies the heat equation. We will use the theory of degree one modular forms with harmonic coefficients. We first recall the necessary results and definitions.

Let (Λ, Q) an even positive definite unimodular lattice of rank N, given a degree d non-constant homogeneous polynomial

$$P\colon \Lambda \otimes \mathbb{R} \to \mathbb{C}$$

we define

$$\Theta_{\Lambda,P}(t) := \sum_{x\in\Lambda} P(x) \exp(\pi i Q(x,x)t)$$

where t is in the Siegel upper half space $\mathcal{H}_1$. The quadratic form Q turns the vector space $\Lambda \otimes \mathbb{R}$ into an Euclidean space, so we can define the Laplacian Δ. Explixcitely, take an ortonormal basis x_i, the Laplacian is

$$\Delta = \sum \frac{\partial^2}{\partial x_i^2}$$

A polynomial P is called *harmonic* if $\Delta P = 0$. The following result is classical:

Theorem 10.1 (Modular form with harmonic coefficients - [Ebe13] Theorem 3.1). *Keep notations as above; if the polynomial P is harmonic, then $\Theta_{\Lambda,P}$ is a cusp modular form of degree one and weight*

$$\frac{1}{2} rk(\Lambda) + d$$

Let

$$\mathcal{R}_n(\Lambda) := \{v \in \Lambda \mid Q(v,v) = 2n\},$$

call $r_n(\Lambda)$ the cardinality of $\mathcal{R}_n$ and

$$\mu(\Lambda) := \min\{Q(v,v) \mid v \in \Lambda,\, v \neq 0\} = \min\{2n \mid \mathcal{R}_n(\Lambda) \neq \varnothing\}$$

Theorem 10.2 (Heat equation for lattices). *Let $M(k)$ be the vector space of degree 1 and weight k cusp modular forms; if*

$$\frac{1}{2}\mu(\Lambda) > \dim M(\frac{1}{2}\operatorname{rk}(\Lambda) + 2)$$

then

$$r_n(\Lambda) 2n Q(v,v) = \operatorname{rk}(\Lambda) \sum_{y\in\mathcal{R}_n(\Lambda)} Q(y,v)^2$$

for all n and all v in Λ

The proof is a straightforward generalization of a classical argument due to Venkov, see [Ebe13] section 3.

Proof. Fix v in Λ, we define a degree 2 homogeneous polynomial on $\Lambda \otimes \mathbb{R}$ by

$$P_v(x) := Q(x,x)Q(v,v) - \operatorname{rk}(\Lambda)Q(x,v)^2$$

One can check by explicit computation that it is harmonic. Call $q := \exp(2\pi i t)$, we have a Fourier-Jacobi expansion

$$\Theta_{\Lambda,P_v}(q) = \sum_n \left(\sum_{y \in \mathcal{R}_n(\Lambda)} P_v(y) \right) q^n \,.$$

This is a degree $\frac{1}{2}\operatorname{rk}(\Lambda) + 2$ cusp modular form and its first $\frac{1}{2}\mu(\Lambda)$ Fourier-Jacobi coefficients are trivial. Recall that there exists a basis $f_1, \ldots, f_s$ of $M(k)$ such that the first i Fourier-Jacobi coefficients of f_i are trivial, and the $i+1$ coefficient is non-zero. Since

$$\frac{1}{2}\mu(\Lambda) > \dim M(\frac{1}{2}\operatorname{rk}(\Lambda) + 2)$$

this is enough to conclude that

$$\Theta_{\Lambda,P_v} = 0$$

In particular, for every n, we have

$$\sum_{y \in \mathcal{R}_n(\Lambda)} P_v(y) = 0$$

and this is what we wanted to prove. □

Let us see what this result means for theta series. The degree $g+1$ Theta series associated to (Λ, Q) is

$$\Theta_{\Lambda,g+1}(T) = \sum_{x_1,\ldots,x_{g+1} \in \Lambda} \exp(\pi i \sum_{i,j} Q(x_i,x_j)T_{ij}) \,,$$

where T is in the Siegel upper half space $\mathcal{H}_{g+1}$. Write

$$T = \begin{pmatrix} \tau & z \\ {}^t z & t \end{pmatrix} ,$$

with t in $\mathcal{H}_1$ and τ in $\mathcal{H}_g$. Let $q := \exp(2\pi i t)$, the Fourier-Jacobi expansion of $\Theta_{\Lambda,g+1}$ is

$$\Theta_{\Lambda,g+1}(T) = \Theta_{\Lambda,g}(\tau) + \sum_{n \geq 1} f_n(\tau,z) q^n \,,$$

where f_n is the n-th Fourier-Jacobi coefficient. By explicit computation we obtain

$$f_n(\tau,z) = \sum_{x_1,\ldots,x_g \in \Lambda} \sum_{y \in \mathcal{R}_n(\Lambda)} \exp(\pi i \sum_{i,j} Q(x_i,x_j)\tau_{ij} + 2\pi i \sum_i Q(y,x_i) z_i) \,.$$

Theorem 10.3 (Heat equation). *Let $M(k)$ be the vector space of degree 1 and weight k cusp modular forms, if*

$$\frac{1}{2}\mu(\Lambda) > \dim M(\frac{1}{2}rk(\Lambda)+2)$$

then n-th Fourier-Jacobi coefficient f_n of the theta series $\Theta_{\Lambda,g+1}$ satisfies the "heat equation"

$$2nr_n(\Lambda)2\pi i\frac{\partial f_1}{\partial\tau_{ij}}(\tau,0) = (1+\delta_{ij})\,\mathrm{rk}(\Lambda)\frac{\partial^2 f_1}{\partial z_i\partial z_j}(\tau,0)\,,$$

Proof. Fix two indexes i and j, by explicit computation we have

$$\frac{\partial^2 f_n}{\partial z_i\partial z_j}(\tau,0) = (2\pi i)^2\sum_{x_1,\ldots,x_g\in\Lambda}\;\sum_{y\in\mathcal{R}_n(\Lambda)} Q(y,x_i)Q(y,x_j)\exp(\pi i\sum_{i,j}Q(x_i,x_j)\tau_{ij})\,,$$

On the other hand

$$\frac{\partial f_n}{\partial\tau_{ij}}(\tau,0) = (1+\delta_{ij})2\pi i\sum_{x_1,\ldots,x_g\in\Lambda} Q(x_i,x_j)\exp(\pi i\sum_{i,j}Q(x_i,x_j)\tau_{ij})\,,$$

the coefficient $(1+\delta_{ij})$ is because on $\mathcal{H}_g$ we have $\tau_{ij}=\tau_{ji}$, so when we compute the derivative with respect to τ_{ij} we need to derive both τ_{ij} and τ_{ji}. The result now follows from Theorem 10.2, we just have to polarize the identity getting

$$2mr_n(\Lambda)Q(x_i,x_j) = \mathrm{rk}(\Lambda)\sum_{y\in\mathcal{R}_n(\Lambda)}Q(x_i,y)Q(x_j,y)$$

□

The dimension of the space of weight one and degree k cusp form is given by

$$\dim M(k) = \begin{cases} \lfloor\frac{k}{12}\rfloor - 1 & \text{if } k\equiv 2 \bmod 12\\ \lfloor\frac{k}{12}\rfloor & \text{else}\end{cases}$$

This means that Theorem 10.3 holds for all rank 24 and for all extremal lattices. Indeed, when a lattice is extremal we have

$$\mu(\Lambda) = 2\lfloor\frac{\mathrm{rk}(\Lambda)}{24}\rfloor + 2$$

and, since $\mathrm{rk}(\Lambda)$ is divisible by 8, we have

$$\lfloor\frac{\mathrm{rk}(\Lambda)}{24}\rfloor + 1 > \lfloor\frac{\mathrm{rk}(\Lambda)}{24}+\frac{1}{6}\rfloor$$

For the rank 24 case see also [MV10] page 16. The hypothesis in the theorem is sufficient but we do not know if it is necessary. One could look for higher order partial differential equations by taking more general harmonic polynomials.

11 Rank 24 lattices

In this section we deal with the weight 12 stable modular form

$$F := \Theta_\Lambda - \Theta_\Gamma$$

where Λ and Γ are rank 24 lattices **with the same number of roots**. By root we mean a vector of norm 2. Equivalently, the two lattices have the same Coxeter number. More specifically, we are dealing with the following 5 pairs of lattices

Lattices	$A_5^4 D_4$, D_4^6	$A_9^2 D_6$, D_6^4	$A_{11} D_7 E_6$, E_6^4	$A_{17} E_7$, $D_{10} E_7^2$	$E_8 D_{16}$, E_8^3
# roots	72	120	144	216	360

where a lattice of rank 24 is labelled by its root system (see e.g. [Ebe13] section 3 for more details). All rank 24 lattices satisfy the hypothesis of Theorem 10.3. Let us point out that the couple $E_8^3, D_{16}E_8$ corresponds to the modular form $\Theta_{E_8}(\Theta_{E_8 \oplus E_8} - \Theta_{D_{16}^+})$, so its behaviour must be the same of he Schottky form defined in 1.4. The others cases are not covered by previous results. On the other hand, if the number of roots of Λ is different from the number of roots of Γ, the modular form F is already non-zero on $\mathcal{M}_1 = \mathcal{A}_1$, so, from our point of view, these five are really the only interesting pairs of rank 24 lattices.

Let us now look at the behaviour of F_g on $\mathcal{M}_g$. We will first show the following:

Theorem 11.1. *Let Λ and Γ be two even positive definite unimodular lattices of rank 24 with the same number of roots, take $g \leq 4$, then the stable modular form*

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is zero on $\mathcal{M}_{g+1}$ if and only if its restriction F_g to $\mathcal{A}_g$ is zero with order at least two on $\mathcal{M}_g$.

Proof. One direction is a general fact, having nothing to do with lattices: Theorem 1.1 guarantees that if a modular form is zero with order at least k on $\mathcal{M}_{g+1}$, then its restriction to $\mathcal{A}_g$ is zero on $\mathcal{M}_g$ with order at least $k+1$.

For the other direction, we need to use all the hypotheses. We use the notion of slope defined in 7.7, which plays a similar role to Criterion 8.10.

We shall show below that F_{g+1} restricted to $\mathcal{M}_{g+1}^S$ vanishes with order at least two on $\mathcal{M}_g^S$; let us take it for granted. Then the pull-back of F_{g+1} to $\overline{\mathcal{M}}_{g+1}$ vanishes to order at least 2 along the divisor δ_0 (see Remark 7.3). Since $g \leq 23$, by Corollary 1.2 of [FP05] the slope of $F_{g+1}|_{\overline{\mathcal{M}}_{g+1}}$ equals the weight divided by its multiplicity along δ_0. The slope of $\overline{\mathcal{M}}_g$, for $g \leq 5$, is strictly bigger than 6 (cf. [HM90] Theorem 0.4), so the slope of $F_{g+1}|_{\overline{\mathcal{M}}_{g+1}}$ is smaller than the slope $\overline{\mathcal{M}}_{g+1}$ and thus F_{g+1} must vanish identically on $\mathcal{M}_{g+1}$.

Now, we show that F_{g+1} restricted to $\mathcal{M}^S_{g+1}$ vanishes with order at least two on $\mathcal{M}^S_g$. By inductive hypothesis we know it vanishes, we need to check that it vanishes along the tangent space. Let C be a general point of $\mathcal{M}_g$, call τ its period matrix. We use the description of the tangent space $T_C\mathcal{M}^S_{g+1}$ given in Theorem 7.14, let us recall it

$$0 \to T_C\mathcal{A}_g \to T_C\mathcal{M}^S_{g+1} \to H^0(S^2C, K_{S^2C})^\vee \to 0$$

Since F_g vanishes with order 2 on $\mathcal{M}_g$, we get that F_{g+1} vanishes along $T_C\mathcal{A}_g$.

We want to show that F_{g+1} vanishes along $H^0(S^2C, K)^\vee$. In other words, we need to show that the first Fourier-Jacobi coefficient f_1 of F_{g+1} is zero on $C - C$, or, more formally, f_1 is in the Kernel of δ^*, where the subtraction map δ is

$$\begin{array}{cccc} \delta: & S^2C & \to & \mathrm{Km}(C) \\ & (a,b) & \mapsto & AJ(a-b) \end{array}$$

Thanks to Lemma 3.3, this is equivalent to show that

$$f_1(\tau, 0) = 0 \quad , \quad \frac{\partial^2 f_1}{\partial z_i \partial z_j}(\tau, 0) = 0 \quad \forall i, j\,.$$

We have $f_1(\tau, 0) = F_g(\tau) = 0$. Now we use the heat equation 10.3: since $r_2(\Lambda) = r_2(\Gamma) =: r$ we have

$$(1 + \delta_{ij})24 \frac{\partial^2 f_1}{\partial z_i \partial z_j}(\tau, 0) = 2r2\pi i \frac{\partial f_1}{\partial \tau_{ij}}(\tau, 0)\,,$$

moreover

$$\frac{\partial f_1}{\partial \tau_{ij}}(\tau, 0) = \frac{\partial F_g}{\partial \tau_{ij}}(\tau)\,,$$

the right hand side is zero because F_g vanishes with order at least two on $\mathcal{M}_g$. □

The proof could work for a general g: the properties we have used about the slope of $\mathcal{M}_g$ are proven for low values of g, but conjectured for any g. Anyway, as shown below, the theorem can be applied only for $g \leq 4$.

Corollary 11.2. *Let Λ and Γ be two even positive definite unimodular lattices of rank 24 with the same number of roots, then the stable modular form*

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is zero on $\mathcal{M}_g$ for $g \leq 4$, and it cuts a divisor of slope 12 on $\mathcal{M}_5$.

Proof. Any difference of theta series is zero on $\mathcal{A}_0$. For $g \leq 3$, $\mathcal{M}^S_g$ is equal to $\mathcal{A}^S_g$, so applying theorem 11.1 we prove that F_g is zero on $\mathcal{M}_g$ for $g \leq 4$.

To show that F_5 does not vanish on $\mathcal{M}_5$ we show that F_4 does not vanish with order two on $\mathcal{M}_4$. The weight of F_4 is 12, $\mathcal{M}_4$ is defined by a form of weight 8, so F_4 vanishes with order 2 on $\mathcal{M}_4$ if and only if it is trivial on $\mathcal{A}_4$. This is not possible because $\Theta_{\Lambda,4}$

and $\Theta_{\Gamma,4}$ have different Fourier coefficients with respect to the quadratic form A_4, as shown in the table on page 146 of [BFW98].

To prove the statement about the slope, using the notations of 7.7 and the result of Lemma 7.8, it is enough to show that F_4 vanishes with order exactly 1 on $\mathcal{M}_4$, i.e. $b_0 = 1$. But we have already shown this above. □

As a by-product, we obtain some degree 5 cusp modular forms. For typographical reasons, we write $\Theta(\Lambda)$ rather than $\Theta_{\Lambda,g}$ and we specify the degree g before stating the equations.

Theorem 11.3. *The following degree 5 modular forms are non-trivial cusp forms*

$$\begin{aligned}
R_1 :=& \Theta(D_{16}E_8) - \Theta(E_8^3) - \frac{21504}{24}(\Theta(A_5^4 D_5) - \Theta(D_4^6)) \\
R_2 :=& \Theta(D_{16}E_8) - \Theta(E_8^3) - \frac{21504}{216}(\Theta(A_9^2 D_6) - \Theta(D_6^4)) \\
R_3 :=& \Theta(D_{16}E_8) - \Theta(E_8^3) - \frac{21504}{480}(\Theta(A_{11} D_7 E_6) - \Theta(E_6^4)) \\
R_4 :=& \Theta(D_{16}E_8) - \Theta(E_8^3) - \frac{21504}{-2520}(\Theta(A_{17} E_7) - \Theta(D_{10} E_7^2))
\end{aligned}$$

Proof. For this proof, we write $\Theta(\Lambda)_g$ rather than $\Theta_{\Lambda,g}$.

Corollary 11.2 tells us that a difference of theta series associated to rank 24 lattices with the same number of roots vanishes on $\mathcal{M}_4$, so it must be divisible by $\Theta(D_{16})_4 - \Theta(E_8^2)_4$. There is, up to a scalar, a unique weight 4 and degree 4 modular form, $\Theta(E_8)_4$, so we have a relation

$$\Theta(E_8)_4(\Theta(E_8^2)_4 - \Theta(D_{16})_4 = k(\Theta(\Lambda)_4 - \Theta(\Gamma)_4)$$

to compute the coefficient k we look at the Fourier coefficient A_4 (we use the table in [BFW98]). In this way we obtain 4 degree 5 cusp modular forms R_i (i.e. forms that vanish when restricted to $\mathcal{A}_4$). To show that the modular forms are not trivial we look at the A_5 coefficient. □

Similar formulæ are also obtained in [GV09] section 4.2.2, working on explicit generators of rings of modular forms. In [NV01], it is shown that the dimension of the vector space of degree 5 and weight 12 cusp forms has dimension 2, but with the known Fourier coefficients we can not understand if these forms span it.

Theorem 11.4. *Let Λ and Γ be two even positive definite unimodular lattices of rank 24 with the same number of roots, then*

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is a stable equation for the hyperelliptic locus.

Proof. The proof is pretty much the same of the proof of theorem 8.12 when $k = 2$. The only difference is that now the first Fourier-Jacobi coefficient f_1 is not trivial: we need to show that it vanishes when restricted to the normal bundle of Hyp_g in Hyp^S_{g+1}. This normal bundle is described in Lemma 8.4: we have to check that f_1 vanishes when restricted to points of the form $(\tau, p - \iota(p))$, where τ is the period matrix of a generic hyperelliptic curve C, p is a point of C and ι is the hyperelliptic involution.

To show this we argue as follows. First remark that

$$f_1(\tau, 0) = F_g(\tau) = 0$$

Then we apply the formula (3.2), trivializing everything with respect to co-ordinates z_p and $\iota^* z_p$ and recalling that

$$\frac{\omega}{dz_p}(p) = \frac{\omega}{\iota^* dz_p}(\iota(p))$$

we get

$$f_1(\tau, p - \iota(p)) = E(p, \iota(p))^2 \sum_{i,j} \frac{\partial^2 f_1}{\partial z_i \partial z_j}(\tau, 0)\omega_i(p)\omega_j(p)$$

Now the heat equation 10.3 comes into the game: since $r_2(\Lambda) = r_2(\Gamma) =: r$, we have

$$24 \sum_{i,j} \frac{\partial^2 f_1}{\partial z_i \partial z_j}(\tau, 0)\omega_i(p)\omega_j(p) = 2r2\pi i \sum_{i \geq j} \frac{\partial F_g}{\partial \tau_{ij}}(\tau)\omega_i(p)\omega_j(p) = (4r\pi i) dF_g(\tau)(p)$$

Let us explain the last equality: the fibre of the cotangent bundle of $\mathcal{A}_g$ at $Jac(C)$ is isomorphic to $\mathrm{Sym}^2 H^0(C, K_C)$, so $dF_g(\tau)$ is a quadric in $\mathbb{P}H^0(C, K_C)^\vee$ and we can evaluate it on the image of p under the canonical map.

The co-normal bundle of Hyp_g in $\mathcal{A}_g$ is given by the quadric vanishing on the image of C under the canonical map, so, since F_g vanishes on Hyp_g, we conclude that $dF_g(\tau)(p)$ is zero for every p in C. $\square$

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