Ecole Normale Supérieure de Lyon

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The Flag Variety and the Representation Theory

Master's thesis in Mathematics

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Introduction

This master's thesis is a survey about the relationship between the geometry of the flag varieties and the representation theory of the algebraic groups. The ground field will be always the field of the complex numbers.

The concept of flag comes from the linear algebra: given a finite dimensional vector space V, a flag is a maximal increasing sequence of vector subspaces:

$$V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n \,.$$

The first element V_0 will be the origin, V_1 a line, V_2 a plane containing the line V_1, and so on until the last element V_n that will be whole space V. The set of all the flags has a natural structure of algebraic variety: the local coordinates are the coefficients of the equations of the vector subspaces. Globally this space can be seen as a closed subvariety of the product of all the Grassmanians of the vector space, then it is a projective variety.

Let us call G the special linear group of V. The group G acts in a natural way on the flag variety, this action is regular and transitive. Given a flag, in a suitable base its stabilizer is the group B of the upper triangular matrices; we can thus identify the flag variety with the quotient G/B.

The group B is a closed connected resoluble subgroup of G, and it is maximal in respect of these proprieties: such a group is a called a Borel subgroup. One can prove that in any affine algebraic group all the Borel subgroup are conjugated, so the isomorphism classes of the quotient G/B is independent from the choice of the Borel subgroup B. We can now give the general definition of flag variety.

Definition 0.1 (The flag variety associated to an affine algebraic group)**.** *Let G be an affine algebraic group. The flag variety associated to G is the quotient G/B, where B is a Borel subgroup of G.*

We are interested in the groups G that are connected and reductive.

The first result is the Bruhat decomposition. We recall that we can associate to any reductive algebraic group its Weyl group. It is a finite group that detects many proprieties of the algebraic group. One characteristic, among the others, is that every element of the Weyl group has a well-defined length, which is a non-negative integer. If we let the group B act on the flag variety we obtain a decomposition into orbits, the Bruhat decomposition states that there is a bijective correspondence between the orbits and the elements of the Weyl group. Moreover each orbit is isomorphic to an affine space of dimension the length of the corresponding element of the Weyl group. Such a decomposition provides us the CW-structure of the flag variety and allow us to compute the dimension of all its de Rham cohomology groups. Finally, using the Bruhat decomposition, one can define an affine open covering of G/B, which is very easy to handle.

The second step is to study the vector bundles over the flag variety. We will define an equivalence of categories between the vector bundles over G/B and the B-modules. We will focus our attention over the line bundles. Each line bundle comes from a one dimensional representation of B, and these representations are essentially the characters of B.

The Borel-Weil theorem describes the global sections of the line bundles over the flag variety. It states that the space of global sections is an irreducible representation of G; namely if the character is dominant it is the dual of the representation associated to the character, if the character is not dominant the space of global sections is trivial. This theorem can be used also to prove that there exists a representation for every dominant character. Moreover it describes the divisor of a meromorphic section for each line bundle.

The Borel-Weil-Bott theorem is a generalization of the previous result, it describes all the cohomology groups of all the line bundles. This theorem says that the cohomology of a line bundle is non-trivial only for one degree, and it is possible to compute this degree using the Weyl group. There are some exceptions given by some line bundles called singular that have trivial cohomology. The classical proof of this result uses techniques of algebraic geometry, a (complicated) exposition of this proof is given in [Lur]. We will not discuss it. The strategy of our proof is to relate the statement of the Borel-Weyl-Bott theorem to a statement about the cohomology of the Lie algebras, and then to invoke the Kostant formula.

The Kostant formula describes the weight spaces of the cohomology of the maximal unipotent Lie subalgebras, it does not use any concept from algebraic geometry. The proof of this formula contains the main ideas of the classical proof of the Borel-Weil-Bott theorem. The key point (we describe it for the classical proof of the Borel-Weyl-Bott theorem) is to prove that the action of the simple reflections determines an isomorphism of degree plus or minus one between the cohomology of different line bundles. Then one uses the Borel-Weil theorem and some facts about the structure of the Weyl group to compute all the cohomology groups.

For all the basic results about the algebraic groups we refer to the book "Lie groups: an approach through invariants and representation theory" written by C. Procesi, in the text it is indicated as [Pro07]. This book provides a comprehensive introduction to Lie groups, Lie algebras and to algebraic groups. The proof of the Borel-Weil-Bott theorem was explained to us orally by our advisor O. Mathieu. We would like to thank him for introducing us to this subject and for his fascinating explications.

Contents

1 Notations, basic notions and examples

In this section we establish the notations that will be in force trough all the paper. We refer to [Pro07] for a complete discussion of all this notions and examples.

The ground field will be always the field of complex numbers $\mathbb{C}$. Let G be an affine algebraic group.

Definition 1.1 (Rational representations)**.** *A finite dimensional representation V of G is rational if and only if the map*
$$\rho : G \to GL(V)$$
that defines the action is regular.

An infinite dimensional representation is rational if and only if every vector is contained in a finite dimensional rational representation.

Proposition 1.2 (Example of rational representation)**.** *If G acts in a regular way on an affine algebraic variety X, then $\mathbb{C}[X]$ is a rational representation of G.*

We will consider only the rational representations of G. Now we give another key definition.

Definition 1.3 (Reductive groups)**.** *A group G is reductive if it does not contain any proper closed unipotent normal subgroup.*

The first foundamental fact about the reductive groups is the following.

Proposition 1.4. *If the characteristic of the ground field is 0, then a group G is reductive if and only if every representation splits in a direct sum of finite dimensional representions.*

Trough all the paper the letter G will denote an affine complex algebraic reductive and connected group.

An affine algebraic groups is always isomorphic to some closed subgroup of $GL(V)$, for some complex vector space V. Moreover all the simple representations of G can be obtained, after possibly tensoring by some power of te determinant, as quotients of tensor powers of V. This implies that if in a suitable base of V the groups G is self-adjoint, then G is reductive. Using this result we obtain a remarkable list of examples.

Proposition 1.5 (Examples of reductive groups)**.** *The groups*
$$SL(n, \mathbb{C}),\ SO(n, \mathbb{C}),\ Sp(n, \mathbb{C})$$
are affine complex algebraic connected and reductive groups.

We fix a Borel subgroup B of G, i.e. a maximal resoluble connected closed subgroup of G. For a closed subgroup of $GL(n, \mathbb{C})$ an example of Borel subgroup is G intersected the upper triangular matrices. We call U the subgroup (B, B), this is a maximal unipotent closed connected subgroup of G. A torus of G is a maximal closed commutative subgroup of G. We indicate with T the maximal torus inside B, we have the decomposition $B = T \ltimes U$.

We recall an important theorem about the resoluble groups.

Theorem 1.6 (Borel fixed point theorem)**.** *Let B be a resoluble group acting on a projective variety X, then there exists at least one fixed point.*

We call $\mathfrak{g}$ the Lie algebra of G, and respectively $\mathfrak{b}$, $\mathfrak{u}$ and $\mathfrak{t}$ the Lie algebras of B, U and T. We have two decompositions of Lie algebras: $\mathfrak{b} = \mathfrak{u} \oplus \mathfrak{t}$ and $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}$, where $\mathfrak{u}^-$ is the maximal unipotent subalgebra opposite to $\mathfrak{u}$. We call U^- the subgroup associated to $\mathfrak{u}^-$ and B^- the subgroup associated to $\mathfrak{b}^- := \mathfrak{u}^- \oplus \mathfrak{h}$.

If we let $\mathfrak{t}$ act on $\mathfrak{g}$, we find a decomposition into eigenspaces, namely

$$\mathfrak{g} = t \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \,.$$

The finite set Φ is a subset of the dual of $\mathfrak{t}$, it forms a root system. We denote by Φ^+ the eigenvalues of $\mathfrak{u}$ (the positive roots) and with Φ^- the eigenvalues of $\mathfrak{u}^-$ (the negative roots), one proves that $\Phi^- = -\Phi^+$. A positive root is simple if it can not be decomposed into a sum of two positive roots, the symbols α_i will denote the simple positive roots.

Definition 1.7 (The Weyl group of G)**.** *The Weyl group of G is denoted by W and is defined as follows:*

$$W := N_G(T)/T \,,$$

where $N_G(T)$ is the normalizer of T inside G.

This group acts on the root system, it plays a key role in all the theory. By abuse of notation, the same letter w will often denote at the same time an element of W and a representant of this element inside G. At each simple root α_i we can associate a simple reflection s_i inside W, these reflections generate W. Given w in W we define its length $l(w)$ as the smallest number of simple reflections one needs to write w; there exists a unique element w_0 that is the longest element of W, moreover $l(w_0) = \dim U$ and $B^- = w_0 B w_0^{-1}$. Finally one proves that $l(w)$ is the number of roots of Φ^+ that w sends in Φ^-.

2 The Bruhat decomposition for the special linear group

In this first section we want to state and prove the Bruhat decomposition in the case of the special linear group; in this case, as we have seen in the introduction, the associated flag variety is the variety of all the flag of a vector space.

We recall the definition of a flag. Let V be an n dimensional complex vector space.

Definition 2.1 (Flag of V)**.** *A flag $\mathcal{F}$ of V is a sequence of subspace of V*

$$\{0\} \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_i \subsetneq \cdots \subsetneq \mathcal{F}_{n-1} \subsetneq V\,,$$

such that

$$dim\, \mathcal{F}_i = i\,.$$

It is useful to use the bases of V in order to work on the flags, we give the following definition.

Definition 2.2 (Base adapted to a flag)**.** *A base $\{v_1, \cdots, v_n\}$ is adapted to the flag $\mathcal{F}$ if*

$$\mathcal{F}_i = Span_{\mathbb{C}} < v_1, \cdots, v_i >\,;$$

in symbol we write

$$\mathcal{F} = (v_1, \cdots, v_n)\,.$$

Clearly different bases may be adapted to the same flag: we are free to replace a vector v_i with a linear combinations of the previous ones.

We fix once for all a base $\{e_1, \cdots, e_n\}$ of V. Let W be the symmetric group of n elements (i.e. the Weyl group of the special linear group), for every w in W we define the flag

$$b_w := (e_{w(1)}, \ldots, e_{w(n)})$$

.

We introduce the special linear group. Let G be $SL(V)$ and let B be the subgroup of upper triangular matrices, B^- the subgroup of the lower ones and $T = B \cap B^-$ the diagonals (recall that we are working with a fixed base). *The group G acts transitively on the flags of V, the stabilizer of b_{Id} is B, hence every point of G/B represent a flag of V and we can call G/B the flags variety.*

The Bruhat decomposition is the decomposition of G/B in B^- orbits, such a decomposition has an explicit and useful description.

First we prove that the b_w are the unique fixed points of the action of T. In fact let $(v_1, \ldots, v_n)$ be a fixed point of T, we must have $tv_1 = \chi(t)v_1$ for some character χ and for every t in T, so v_1 is one of the e_i. Say $v_1 = e_{i_1}$. In the quotient space $V/ < e_{i_1} >$ the class of the e_i (for i different from i_1) form a base, in this base T is still diagonal, the spaced spanned by v_2 must be still stable for the action of T so we get that the class of v_2 is equal to to the class of e_{i_2} for some i_2. Since we are free to replace a vector with a combination of the previous ones we can suppose that v_2 is equal to e_{i_2}. By induction we obtain the statement for every v_i.

To obtain our goal we will define an invariant

$$c : G/B \to \{\text{flag of}\,\{1, \ldots, n\}\}\,.$$

A flag of $\{1, \ldots, n\}$ is defined as the flag for the vector space, just replacing subspace with subset and dimension with cardinality. The flag of $\{1, \ldots, n\}$ are in a bijective correspondence with W: given a w in W we can cook up the flag $\{w(1)\} \subset \{w(1), w(2)\} \cdots$, given a flag $\mathcal{F}$ we define a permutation $w(i) := \mathcal{F}_{i+1} \setminus \mathcal{F}_i$. The idea is to prove that $B^- b^w$ is exactly $c^{-1}(w)$, so c will be a complete invariant for the action of B^-.

First we establish some notation. Let $\mathfrak{P}(\mathrm{i})$ be the set of subset of cardinality i of $\{1, \ldots, n\}$, given an element I of $\mathfrak{P}(i)$ we order the element of I and we call I_j the j-th element of I. We define a partial order on $\mathfrak{P}(i)$, we we say that $I \leq J$ if $I_k \leq J_k$ for every k; given two elements of $\mathfrak{P}(i)$ they can be not comparable, for example if $n = 4$ and $i = 2$ take $\{1, 4\}$ and $\{2, 3\}$.

In order to define the function c we will use some results of standard monomial theory that we will prove in the next section.

Let U be an i dimensional subspace of V, let ω_i be a base of $\bigwedge^i U$, we write

$$\omega_i = \sum_{I \in \mathfrak{P}(\mathrm{i})} c_I e_I\,,$$

where e_I is $e_{I_1} \wedge \cdots \wedge e_{I_i}$. Let S be the subset of element I of $\mathfrak{P}(\mathrm{i})$ such that c_I is not zero. This set has a minimum because ω_i is a pure tensor, for a proof see the section 3 proposition 3.5. We define

$$c_i(W) := \text{minimum of}\, S\,.$$

Moreover suppose that U and U' are two subspace of V of dimension i and $i+1$ such that $U \subset U'$, we have

$$c_i(U) \subset c_{i+1}(U')\,.$$

To prove this remark that if ω_i is a base of U and v is a vector of $U' \setminus U$ then a base of U' is $v \wedge \omega_i$, so we can apply the proposition 3.7 of the section 3. We finally give our key definition.

Definition 2.3 (The function c)**.** *Let $\mathcal{F}$ be a flag of V, we define*

$$c(\mathcal{F}) := \{c_1(\mathcal{F}_1) \subset c_2(\mathcal{F}_2) \subset \cdots \subset c_i(\mathcal{F}_i) \subset \cdots \subset c_{n-1}(\mathcal{F}_{n-1})\}\,.$$

It follows directly from the definition that

$$c(b_w) = w \qquad \forall w \in W\,.$$

We can now state and prove the Bruhat decomposition.

Theorem 2.4 (The Bruhat decomposition for the special linear group)**.** *The function c defined above is a complete invariant for the action of B^- on G/B; in symbols:*

$$c^{-1}(w) = B^- b_w$$

for every w in W.

Proof. First we prove the inclusion

$$c^{-1}(w) \supseteq B^- b_w\,.$$

Let A be an element of B^- and $\mathcal{F} = A b_w$. We denote by A_j the j-th column of A, we have that

$$\mathcal{F} = (A e_{w(1)}, \ldots, A e_{w(n)}) = (A_{w(1)}, \ldots, A_{w(n)})\,.$$

A base of $\bigwedge^i \mathcal{F}_i$ is $\omega_i = A_{w(1)} \wedge \cdots \wedge A_{w(i)}$. The matrix A is lower triangular and all the elements of the diagonal are not zero, so $c_i(\omega_i) = \{w(1), \ldots, w(i)\}$ and then $c(\mathcal{F})$ is equal to w.

Now we prove the inclusion

$$c^{-1}(w) \subseteq B^- b_w\,.$$

Let $\mathcal{F}$ be a flag such that $c(\mathcal{F}) = w$. We must prove the following lemma

Lemma 2.5. *Let $\mathcal{F}$ be a flag, pose $w := c(\mathcal{F})$; there exists a base $\{v_1, \cdots, v_n\}$ adapted to $\mathcal{F}$ such that*

$$v_i = e_w(i) + \sum_{j<w(i)} a_{ij} e_j \, .$$

Proof. Take a base $\{v_1, \cdots, v_n\}$ adapted to $\mathcal{F}$, write

$$v_i = \sum_j a_{ij} e_j$$

and call A the matrix (a_{ij}). The data of the matrix A is equivalent to data of a base, so we will interchange the expressions "change A" and "change the base".

We know that we are free to add to the v_i a linear combination of the v_j with j smaller then i without changing the flag. Doing this elementary operation several times we will get the requested base. We construct a base such that $a_{jw(i)}$ is zero if $j < w(i)$ and not zero if $w(i) = j$ by induction on i.

For i equal to 1 since $c_1(v_1) = w(1)$ then a_{j1} is zero for $j < w(1)$ and $a_{w(1)w(1)}$ is different from zero. We can thus assume that $a_{w(1)j} = 0$ for all the j strictly bigger than $w(1)$.

We suppose the statement true for i smaller than t and we prove it for $i = t$. We will prove in the next paragraph that the coefficient of $v_1 \wedge \cdots \wedge v_t$ in $e_{w(1)} \wedge \cdots \wedge e_{w(t-1)} \wedge e_j$ is, up to a sign, equal to $a_{w(1)w(1)} \cdots a_{w(t-1)w(t-1)} a_{jw(t)} =: k$. We know by the inductive hypothesis that $a_{w(i)w(i)}$ is not zero for $i < t$ so k is equal to zero if and only if $a_{jw(t)}$ vanishes; since $c(\mathcal{F}) = w$ the coefficient k is zero if $j < w(t)$ and not zero if $j = w(t)$ so we have proved the lemma.

We compute the requested coefficient. We must compute the determinant of the minor of A formed by the line of indexes $\{w(1), \ldots, w(t-1), j\}$ and columns $\{w(1), \ldots, w(t)\}$, call this minor Γ. We relabel the indexes $\{1, \ldots, t-1\}$ with $\{i_1, \ldots, i_{t-1}\}$ in such a way that $w(i_p) < w(i_{p+1})$ for every p. Call i_s the coefficient such that $w(i_s)$ is the biggest before j; similar definition for i_k: it is the index such that $w(i_k)$ is the biggest before $w(t)$.

The matrix Γ is formed by square block, it is of the form

$$\begin{pmatrix} \alpha & B & C \\ * & \delta & F \\ * & * & \epsilon \end{pmatrix}$$

The first line of blocks goes from $w(i_1)$ to $w(i_s)$, the second from j to $w(i_k)$, the third from $w(i_k + 1)$ to the end. The first column of blocks goes from $w(i_1)$ to $w(i_s)$, the second from $w(i_{s+1})$ to t, the third from $w(i_{k+1})$ to the end. The idea is to prove that the blocks B,C, and F are all trivial and then to compute explicitly the determinant of the blocks α,δ, and ϵ.

The blocks B and C are all zero because of the induction and because $w(i_p) < w(i_{p+1})$ for all p. The block α is

$$\begin{pmatrix} a_{w(1)w(1)} & 0 & \cdots & 0 \\ * & a_{w(2)w(2)} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & a_{w(i_s)w(i_s)} \end{pmatrix}$$

because of the induction. The block δ is

$$\begin{pmatrix} 0 & 0 & \cdots & a_{jw(t)} \\ a_{w(s+1)w(s+1)} & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ * & * & a_{w(i_k)w(i_k)} & 0 \end{pmatrix},$$

the line j is like this because $j < w(i_{s+1})$, the column $w(t)$ because $w(i_p) < w(t)$ for every $p < k$. The block F is zero because of the induction and $j < w(t) < w(i_{k+1})$. The block ϵ is

$$\begin{pmatrix} a_{w(i_{k+1})w(i_{k+1})} & 0 & \cdots & 0 \\ * & a_{w(i_{k+2})w(i_{k+2})} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & a_{w(i_t)w(i_t)} \end{pmatrix},$$

for the same reasons of α. □

We can now prove the proposition. Let v_i be a base for $\mathcal{F}$ such that

$$v_i = e_w(i) + \sum_{j<w(i)} a_{ij} e_j \, ,$$

we define a matrix A in following way

$$A_{w(i)j} := a_{ij} \, ;$$

clearly Ab_w is equal to $\mathcal{F}$ and A is lower triangular. □

Definition 2.6 (Bruhat cells)**.** *We call $C(w)$ the orbit of b_w under the action of B^-.*

We can give a more explicit description of these orbits. First we choose a represent for every element of W, we choose the matrices of permutation (this choice is not canonical). If w and z are in W then $zb_w = b_{wz}$. We conclude that W acts simply transitively on the fixed points of T. Let U (respectively U^-) be the subgroup of strictly upper (respectively lower) triangular matrices. The stabilizer of b_w in B^- is

$$S(w) := wBw^{-1} \cap B^- \, ;$$

so $C(w)$ is isomorphic to $B^-/S(w)$ and hence

$$C(w) \cong wU^-w^{-1} \cap U^- =: U^w \, .$$

We have that U^- is isomorphic as a variety to $\mathbb{C}^{(n(n-1)/2)}$ so $C(w)$ is isomorphic to $\mathbb{C}^p$ for some p. This is a very important remark because show that the $C(w)$ give us the CW-structures of G/B!

We want a formula for the dimension of the $C(w)$. We define $l(w)$ as the smallest number of transposition $\{(1,2),(23),\ldots,(n-1,n)\}$ one needs to write w (we remark that these transposition generate the symmetric group). We claim that

$$dim\, C(w) = n(n-1)/2 - l(w) \, .$$

We can use this formula to compute the dimension of the de Rham cohomology groups of the flag variety. For example, the dimension of the cohomology groups of $SL(4,\mathbb{C})/B$ are:

$$1 \quad 3 \quad 5 \quad 6 \quad 5 \quad 3 \quad 1$$

3 Some results of standard monomial theory

In this section we prove some facts of standard monomial theory. These results are necessary to prove that the function c defined in 2.3 is well-defined. We will tackle the problems from several points of view.

We recall the notations. Let $\mathfrak{P}(\mathrm{i})$ be the set of subset of cardinality i of $\{1,\ldots,n\}$, given an element I of $\mathfrak{P}(i)$ we order the element of I and we call I_j the j-th element of I. We define a partial order on $\mathfrak{P}(i)$, we we say that $I \leq J$ if $I_k \leq J_k$ for every k; given two elements of $\mathfrak{P}(i)$ they can be not comparable, for example if $n = 4$ and $i = 2$ take $\{1,4\}$ and $\{2,3\}$.

3.1 Two dimensional case

We prove the proposition 3.5 in the two dimensional case. Let V be a vector space and $\{e_1, \cdots, e_n\}$ be a base of V. Let

$$v_1 = \sum_t a_t e_t \quad , \quad v_2 = \sum_t b_t e_t \, .$$

We write

$$v_1 \wedge v_2 = \sum_{s<t} c_{st} e_s \wedge e_t \, ,$$

where

$$c_{st} = a_s b_t - a_t b_s \, .$$

We define a partial order on the ordered couple saying that $(s,t) \leq (i,j)$ if and only if $s \leq i$ and $t \leq j$.

Proposition 3.1. *With the same notation as above there exist the smallest couple (s,t) among the couples such that c_{ij} is not zero.*

Proof. Let i, j, k, l be four indexes such that the couple (i,j) is not comparable with (k,l), without lost of generality we can assume that:

$$i < k < l < j \, .$$

We claim that if both c_{ij} and c_{st} are non zero then there is a non zero c_{st} such that (s,t) is smaller then both (i,j) and (k,l). The smallest couple in i, j, k, l is (i,k). If c_{ik} is not zero we are home. Assume that

$$c_{ik} = a_i b_k - a_k b_i = 0 \, . \tag{1}$$

We claim that

Lemma 3.2. *If c_{ik} is zero then*

$$c_{ij} c_{kl} - c_{il} c_{kj} = 0 \, .$$

Proof. Proof by explicit computation. We compute the first term

$$c_{ij} c_{kl} = (a_i b_j - a_j b_i)(a_k b_l - a_l b_k) = [a_i b_j a_k b_l + a_j b_i a_l b_k] - [a_i b_j a_l b_k + a_j b_i a_k b_l] \, .$$

Using 1 we have

$$a_i b_j a_l b_k + a_j b_i a_k b_l = a_k b_i (a_l b_j + a_j b_l) \, .$$

We compute the other term

$$c_{il} c_{kj} = (a_i b_l - a_l b_i)(a_k b_j - a_j b_k) = [a_i b_l a_k b_j + a_l b_i a_j b_k] - [a_i b_l a_j b_k + a_l b_i a_k b_j] \, .$$

Using 1 we have

$$a_i b_l a_j b_k + a_l b_i a_k b_j = a_k b_i (a_j b_l + a_l b_j)\,.$$

□

From the lemma follows the proposition. □

3.2 The Plucker relations

The previous computation can be generalized in an abstract way. Let V be a vector space, there exists a non degenerate pair

$$\bigwedge^n V^* \otimes \bigwedge^n V \to \mathbb{C}$$

$$(\phi_1 \wedge \cdots \wedge \phi_n, v_1 \wedge \cdots \wedge v_n) \mapsto det(\phi_i(v_j))\,;$$

so we can identify $\bigwedge V$ with $(\bigwedge V^*)^*$ and then we can define a structure of Hopf algebra on $\bigwedge V$.

For n bigger or equal then 2 we call Δ the composition of the comultiplication with the projection from $\bigwedge V \otimes \bigwedge V$ to $V \otimes \bigwedge^{n-1} V$; explicitly:

$$\Delta : \bigwedge^n V \to V \otimes \bigwedge^{n-1} V$$

$$e_I \mapsto \sum_{i \in I} e_i \otimes e_{I \setminus i}\,.$$

Now fix a vector ω in $\bigwedge^n V$, we define a map

$$\mu : V \otimes \bigwedge^{n-1} V \to \bigwedge^{n+1} V \otimes \bigwedge^{n-1} V$$

$$\alpha \otimes \beta \mapsto (\alpha \wedge \omega) \otimes \beta\,;$$

if ω is a pure tensor then $\mu(\Delta\omega)$ is zero, if it is not a pure tensor then the composition is not forced to be zero. Now let

$$\omega = v_1 \wedge \cdots \wedge v_n = \sum c_I e_I\,,$$

we compute explicitly $\mu\Delta\omega$, we get

$$\Delta(\omega) = \sum_I c_I \Delta(e_I) = \sum_I c_I \sum_{i \in I} e_i \otimes e_{I \setminus i}\,,$$

$$\mu(\Delta(\omega)) = \sum_I c_I \sum_{i \in I} e_i \wedge (\sum_J c_J e_J) \otimes e_{I \setminus i} = \sum_{I,J\ |I \setminus J| \geq 1} c_I c_J \sum_{i \in (I \setminus J)} \pm e_{J \cup i} \otimes e_{I \setminus i}\,;$$

As we have noticed before $\mu(\Delta(\omega)) = 0$; since the elements $e_{J \cup i} \otimes e_{I \setminus i}$ are linearly independent we can equal the coefficients to zero, so for every couple of set of indexes K and T we have the equation

$$\sum_{i \in K \setminus T} \pm c_{K \setminus i} c_{T \cup i} = 0\,.$$

Now we specialize this equation to the case

$$K = \{i, j, k\} \quad T = \{l\}\,,$$

with

$$i < k < l < j\,;$$

we get

$$-c_{ij} c_{kl} - c_{ik} c_{lj} + c_{jk} c_{il} = 0$$

and the lemma of the previous section follows.

3.3 General case

First we remark a basic fact of linear algebra.

Lemma 3.3 (Exchange lemma-first version). *Let F be a vector space and $I = \{I_1, \ldots, I_n\}$ and $J = \{J_1, \ldots, J_n\}$ two bases. For every index i there exist an index j such that*

$$I \cup \{J_i\} \setminus \{I_j\}$$

is still a base.

We give an easy generalization of this lemma.

Lemma 3.4 (Exchange lemma-second version). *Let F be a vector space and $I = \{I_1, \ldots, I_n\}$ and $J = \{J_1, \ldots, J_n\}$ two bases. Fix an integer k. We can replace at least one element among $\{I_k, I_{k+1}, \ldots I_n\}$ with an element of $\{J_1, J_2, \ldots, J_k\}$ obtaining a new base.*

Proof. This lemma follows from the previous one because of dimensional reason: if we can not replace an element of $\{I_k, I_{k+1}, \ldots I_n\}$ with an element J_s this means that J_s belong to the space spanned by $\{I_1, I_2, \ldots, I_{k-1}\}$; but $\{J_1, J_2, \ldots, J_k\}$ are k vectors linearly independent so they can not belong all to the space spanned by $\{I_1, I_2, \ldots, I_{k-1}\}$. □

These lemmas will be used several times.

Let V be a vector space of dimension n and $\{e_i\}$ be a base of V. Let

$$v_t = \sum_j a_{jt} e_j$$

for t from 1 to i. We write

$$v_1 \wedge \cdots \wedge v_i = \sum_{I \in \mathfrak{P}(\mathfrak{i})} c_I e_I \, ,$$

where e_I is $e_{I_1} \wedge \cdots \wedge e_{I_i}$. Let A be the matrix of coefficient a_{jt} and let A_k be the k-th line of A. The key remark is that c_I is the determinant of the matrix with lines the A_k's for k in I. Clearly c_I is not zero if and only if the set $\{A_i\}_{i \in I}$ is a set of linear independent vectors. In the next proof an element I of $\mathfrak{P}(\mathfrak{i})$ will denote at the same time a subset of $\{1, \ldots, n\}$ and the correspondent lines of the matrix A.

Proposition 3.5. *Let S be the subset of element of $\mathfrak{P}(\mathfrak{i})$ such that c_I is not zero. There exist a minimum and a maximum element in S.*

Proof. We just prove that there exist a minimum. Take two elements I and J in S, it is enough to prove that there exist a Z in S smaller then I and J. We give a definition.

Definition 3.6 (k-smaller). *For every k in $\{1, \ldots, n\}$ we say that I is k-smaller then J, in symbol $I <_k J$, if and only if the following conditions hold*

1. $\exists t < k \mid I_t < J_t$,
2. $I_s \leqslant J_s \forall s < k$,
3. $I_k > J_k$.

Moreover $i + 1$-smaller means smaller.

We prove that, gives two elements I and J in S there exists Z in S smaller then I and J. We can assume that I is k smaller then J. We prove the statement by decreasing induction on k. If $k = i + 1$ we are home. Now suppose the statement true for all the couple I and J such that I is $k + \epsilon$ smaller then J.

Using the second version of the exchange lemma we can replace one element among $\{I_k, I_{k+1}, \cdots, I_i\}$ with an element of $\{J_1, J_2, \cdots, J_k\}$. Call P the new element of S one obtains after this change, we will show that P is smaller then I and z smaller then J for z strictly smaller then k.

Let J_t be the element put inside I. If $t < k$ then the ordinate set of index P is of the form

$$(I_1, \ldots, I_s, J_t, I_{s+1}, \ldots I_{k-1}, \ldots),$$

where s is bigger then t because $I_t \leq J_t$. Remark that J_t is in the place $s + 1$.

First we confront P and I. The first s position are the same. For the position $s + 1$ we have that $J_t < I_{s+1}$ because of their positions in P. Moreover for the position after $s + 1$ we use the fact that $I_i < I_{i+1}$ for every i because the sets are ordinate, so we can conclude that $P < I$.

Now we confront P and J. The first s position are not changed so they are ok because $I <_k J$. In position s we have that $J_t < J_{s+1}$ because $s > t$. For the further positions we have that $P_i < J_i$ for i from $s + 1$ to k because $I_t < J_t < J_{t+1}$ for t from $s + 1$ to $k - 1$, so $P <_z J$ for a z strictly bigger then k.

By decreasing induction on k we can find an element T of S smaller then P and J so we are home.

If $t = k$ then P is of this form

$$(I_1, \ldots I_{k-1}, J_k, \ldots),$$

in this case P is $k + \epsilon_1$ smaller than I and $k + \epsilon_2$ smaller then J. We apply two times the inductive hypothesis. First we find an element $\tilde{Z}$ of S smaller then P and I, this element, being smaller then P, is $k + \epsilon$ smaller then J so there exist a Z in S smaller then I and J.

□

Let

$$\omega_i := v_1 \wedge \cdots \wedge v_i,$$

we call $c(\omega_i)$ the minimum of S. Let v be a vector of V such that $v \wedge \omega_i$ is not zero, recall that $c(\omega_i)$ and $c(v \wedge \omega_i)$ are both subset of $\{1, \ldots, n\}$, we claim the following proposition.

Proposition 3.7. *Keep notation as above. We have that:*

$$c(\omega_i) \subset c(v \wedge \omega_i).$$

Proof. As we have done for ω_i we write

$$v \wedge \omega_i = \sum_{I \in \mathfrak{P}(\mathrm{i}+1)} d_I e_I,$$

we call T the set of the I in $\mathfrak{P}(\mathrm{i}+1)$ such that d_I is not zero. Same definition for U: the subset of the i in $\{1, \ldots, n\}$ such that the coefficient of v in e_i is not zero.

An element of T is of the form $J \cup \{j\}$ where J is an element of S that does not contain j and j is in U. Fix an element $J \cup \{j\}$ of T, we will show that there exist an element of T smaller or equal to $J \cup \{j\}$ and containing $c(\omega_i)$: this is enough to prove the proposition. There are two cases. First j doesn't belong to $c(\omega_i)$, in this case $c(\omega_i) \cup \{j\}$ is the requested element. Now suppose that j belongs to $c(\omega_i)$, so $c(\omega_i) \cup \{j\}$ is not an element of T. For the first version of the exchange lemma we can cook up an element of S in the following way: remove j from $c(\omega_i)$ and add an element of J such that we do not get out S, call this new set I. Clearly I doesn't contain j so $I \cup \{j\}$ is in T and it is the requested element. □

4 Tori

4.1 Rigidity of Tori

A trous T is an algebraic group isomorphic to $(\mathbb{C}^*, \cdot)^n$. First we describe its subgroups. Let $X(T)$ be the group of characters of T, i.e. the group of algebraic homomorphism from T to $(C^*, \cdot)$. We have the morphism of evaluation:

$$ev : T \to Hom_{gr}(X(T), \mathbb{C}^*)$$

defined in the natural way:

$$ev(p)(\chi) := \chi(p)\,.$$

Clearly ev is a morphism of groups, moreover we have the following proposition.

Proposition 4.1. *The morphism ev is an isomorphism of algebraic groups.*

Proof. The injectivity is trivial, for the surjectivity we need coordinates. Let $t_1, \ldots, t_n$ be coordinates on T, we define $\chi_i(t_1, \ldots, t_n) := t_i$, the χ_i's are free generators of $X(T)$. Given an element f of $Hom_{gr}(X(T), \mathbb{C}^*)$ we claim that $ev(f(\chi_1), \ldots, f(\chi_n))$ is equal to f because they have the same value on the χ_i's. □

We define a canonical bijection between the subgroups of $X(T)$ and the subgroup of $Hom_{gr}(X(T), \mathbb{C}^*)$. Given a subgroup E of $Hom_{gr}(X(T), \mathbb{C}^*)$ we pose

$$Ker\, E := \bigcap_{f \in E} Ker\, f\,;$$

given a subgroup H of $X(T)$ we pose

$$Ann\, H := \{f \in E \,\text{s.t.}\, f\mid_H \equiv 1\}\,.$$

Clearly $Ann\, H$ and $Ker\, E$ are both subgroups, in general (for example in the case of vector space) they do not give a bijective correspondence, but in our case we can claim the following lemma.

Lemma 4.2. *With the same notation as above*

$$Ker\, Ann\, H = H \quad \textit{and} \quad Ann\, Ker\, E = E\,.$$

Proof. We will use the description of $X(T)$ as the group of monic monomial in the t_i's. For the first assertion is clear that

$$H \subset Ker\, Ann\, H\,.$$

To prove the reverse inclusion we will find a finite number of functions f_i in $Hom_{gr}(X(T), \mathbb{C}^*)$ such that the intersection of their kernels is is exactly H. We write $X(T)/H$ as product of finite many cyclic groups F_i. Every cyclic group can be embedded in $\mathbb{C}^*$ (the group $\mathbb{Z}$ with the exponential and the finite ones as roots of the unit). For each factor F_i we define an f_i as the projection of $X(T)$ on $X(T)/H$ composed with a map that embeds F_i and is trivial on all the other factors.

For the second assertion is clear that

$$E \subset Ann\, Ker\, E\,.$$

In order to prove the other inclusion we must prove that two subgroups E_0 and E_1 of $Hom_{gr}(X(T), \mathbb{C}^*)$ with the same kernel are equal (this statement is not true for example if $X(T)$ was a vector space instead that a group of characters). We argue by induction on n. For n equal to one we have that $E_0 =< t^k >$ and $E_1 =< t^s >$, since $\mathbb{C}$ is algebraically closed if k is different from s they have different kernels. If n is bigger then one let T_n be the subtorus of T with last coordinate equal to 1, we look at the group homomorphism given by the restriction

$$r : Hom_{gr}(X(T), \mathbb{C}^*) \to Hom_{gr}(X(T_n), \mathbb{C}^*) \,,$$

since $r(E_0)$ and $r(E_1)$ have the same kernel they are equal by inductive hypothesis. The kernel of r is given by the characters trivial on T_n, hence is $< t_n >$. Because of the first step the kernels of r restricted to E_0 and E_1 are equal. Since the image and kernel of r restricted to E_0 and E_1 are equal we conclude that E_0 and E_1 are equal.

□

Since $X(T)$ is isomorphic to $\mathbb{Z}^n$ we obtain that the subgroups of a torus are discrete. We now state an important theorem about the structure of the tori.

Theorem 4.3 (Rigidity of tori). *Let T_1 and T_2 be two tori and V a connected family of homomorphism, then V is trivial. In a more explicit way: let V be a connected variety, and suppose we have a map*

$$\alpha : V \times T_1 \to T_2$$

such that its restriction to $v \times T_1$ is an homomorphism of groups for every v in V, then α is constant in V: the map that associate at each v the homomorphism $\alpha\mid_v$ is constant.

Proof. The main idea is that a map between two torus correspond to a map between the character groups, so it can not be deformed. For a complete proof see [Bor91] page 116-117. □

One of the most important consequence of this theorem is the following proposition.

Proposition 4.4. *Let T be a torus inside an algebraic group G, let $N(T)$ ans $Z(T)$ be respectively its normalizer and its centralizer, then connected component of the identity of $N(T)$ is contained in $Z(T)$.*

Proof. Let N_0 be the connected component of the identity inside $N(T)$. Consider the map

$$\alpha : N_0 \times T \to T$$
$$(n, t) \mapsto ntn^{-1} \,.$$

If we restrict α to $\{Id\} \times T$ we get a trivial map, so applying the theorem of rigidity we obtain the statement. □

4.2 Fixed points of the a action of a Torus

Let T be a torus and V be a linear representation of T. Since T is abelian its action is diagonal so we can decompose V in eigenspaces

$$V = \bigoplus V_{\chi_i}$$

where the χ_i are characters of T. The fixed point of the action of T on $\mathbb{P}V$ are $\mathbb{P}V_{\chi_i}$, so if the V_{χ_i} are all one dimensional then the fixed points are isolated, otherwise the fixed points formed linear subspace of $\mathbb{P}V$ of positive dimension. Now we reduce the problem of studying the fixed points of the action of a torus to the study of the fixed points of the action of a generic 1-parameter subgroup.

Theorem 4.5. *Suppose that is given an action of T on $\mathbb{P}V$, then for a generic one-parameter subgroup ρ we have that*

$$Fix(T) = Fix(\rho)\,.$$

Proof. Call χ_i the eigenvalue of the action of T on V. For every couple of index i and j call K_{ij} the subgroup of T where χ_i is equal to χ_j. The generic one parameter subgroup of T will have an open dense subset outside all the K_{ij}. To prove this is enough to look the Lie algebra of G: the tangent spaces of the K_{ij} are contained in a finite number of hyperplanes, hence the generic 1 dimensional subspace of the Lie algebra can be integrated to a 1 parameter subgroup intersecting with an open dense subset outside the K_{ij}.

Now fix one generic one parameter subgroup of T

$$\rho : \mathbb{C}^* \to T\,.$$

Suppose that $[v]$ is not fixed by T, then v is outside the V_{χ_i} and we can write

$$v = v_1 + \cdots + v_k$$

where each v_i is an element of V_{χ_i} and k is bigger then one. The action of an element t of T is the following

$$t.v = \chi_1(t)v_1 + \cdots \chi_k(t)v_k\,,$$

now take an element $\rho(z)$ outside all the K_{ij}, then $\rho(z).v$ is not proportional to v and $t.[v]$ is different from $[v]$, so we can conclude that

$$Fix(\rho) = Fix(T)\,.$$

□

Fix a point $[v]$ on V we would like to compute

$$\lim_{z\to 0} \rho(z)[v]\,,$$

write $\chi_i(\rho(z)) = z^{m_i}$, for some m_i in $\mathbb{Z}$, recall that the m_i are all different, call χ_i the character with the smallest m_i; now write as before

$$v = v_{\chi_1} + \cdots + v_{\chi_k}\,,$$

then

$$\lim_{z\to 0} \rho(z)[v_{\chi_1}]\,.$$

We remark that

$$\lim_{z\to 0} \rho(z)[v]\,,$$

is a fixed point for the action of T for every $[v]$ in V.

We look the case when T is a maximal torus of a reductive group G and it acts on the Lie algebra of G with the adjoint representation. We recall the formula

$$dAd(e^{tv})(u)\mid_{t=0} = ad(v)(u)\,,$$

hence the differential of the character χ_i are the roots of the Lie algebra, and the eigenspace of T have all dimension 1.

5 Equivariant vector bundles

We want to tackle the following problem: given an action of an algebraic group G on a (projective or affine) variety X we want to find a linear representation V of G such that we can embed X inside V or $\mathbb{P}V$ in G equivariant way. Roughly speaking if we think X embedded in some space V or $\mathbb{P}V$ we want to extend the action of G to all V.

5.1 The affine case

First we tackle the problem when X is an affine variety. In this case we can always solve our problem. We consider the representation $\mathbb{C}[X]$. We first prove the following fact.

Lemma 5.1. *Let f be an element of $\mathbb{C}[X]$, then f is contained in a finite dimensional G module.*

Proof. Let ρ be the map from $G \times X$ to X given by the action. We have that

$$g.f(x) = f(g^{-1}x) = \rho^* f(g, x) = \sum_{i=1}^{k} u_i(g) v_i(x)\,;$$

the last equality comes from the fact that, given two algebraic varieties U and V, then $\mathbb{C}[U \times V]$ is isomorphic to $\mathbb{C}[U] \otimes_{\mathbb{C}} \mathbb{C}[V]$, and an isomorphism is given in the following manner

$$(\sum_i u_i \otimes v_i)(x, y) := \sum_i u_i(x) v_i(y)\,.$$

We can conclude that, if $\rho^* f = \sum_{i=1}^{k} u_i v_i$, then for every g the function $g.f$ is a linear combination of the v_i, namely

$$g.f = \sum_{i=1}^{k} u_i(g) v_i\,.$$

□

Now we fix a set of generators of $\mathbb{C}[X]$, this is a finite set so it spans a finite dimensional G module, call it M. There is a natural embedding of X in M^*:

$$ev : X \to M$$

$$x \mapsto ev_x\,,$$

and $ev_x(f) := f(x)$. This is an embedding because M contains the generators of $\mathbb{C}[X]$, moreover the map ev is G equivariant:

$$(g.ev_x)(f) = ev_x(g^{-1}.f) = f(gx) = ev_{gx}(f)\,.$$

5.2 The projective case

The projective case is more complicated and the answer is not always positive; we begin with some remarks. First we give the following definition

Definition 5.2. *An embedding*

$$\iota : X \to \mathbb{P}^N$$

is called non degenerate if and only if X is not contained in any proper subspace of $\mathbb{P}^N$.

Let ι be a proper embedding of X in some $\mathbb{P}^N$, we can consider the line bundle $\iota^*\mathcal{O}(1)$; call

$$V := \Gamma(X, \iota^*\mathcal{O}(1)),$$

the embedding of X in $\mathbb{P}^N$ is equivalent to the embedding of X in $\mathbb{P}(V^*)$ defined as follows:

$$x \mapsto ([s] \mapsto [s(x)]).$$

Our problem became: can we define an action of G on V such that the embedding is equivariant? First we recall that $\iota(x)$ is the hyperplane of V formed by all the section vanishing on x, so the embedding is equivariant if and only if, given a section s vanishing on x, the section gs vanishes on gx.

We give the following definition

Definition 5.3. *A line bundle L over a G variety X is equivariant if there is an action of G on L such that the projection is an equivariant morphism and the action is linear on the fibers.*

It is useful to reformulate our problem in term of G-equivariant vector bundles.

Proposition 5.4. *Let X be a G variety and*

$$\iota : X \to \mathbb{P}V^*$$

be a non degenerate embedding. Then V admits a structure of G module such that ι is G equivariant if and only if the line bundle $L := \iota^*\mathcal{O}(1)$ admits a structure of G equivariant vector bundle.

Proof. First recall that V is isomorphic to $\Gamma(X, L)$. If the bundle is equivariant we can define the following action on the section

$$g.s(x) = gs(g^{-1}x),$$

since p is equivariant then s is still a section. This action verifies the requested propriety: if $s(x) = 0$ then

$$g.s(gx) = g(s(x)) = g0 = 0$$

because g is linear on the fibers.

Now suppose that is given an action on V such that the embedding ι is equivariant. We define an action of G on L such that the bundle is equivariant. Let l be a point of L and x be $p(l)$, since the bundle is very ample then there exist a section $\tilde{s}$ such that $\tilde{s}(x) = l$. We define

$$g.l := (g.\tilde{s})(gx),$$

the map p is equivariant because $g\tilde{s}$ is a section, since G acts linearly on V then it acts linearly on the fibers; the definition does not depend on the choice of $\tilde{s}$ because $(gs)(x) = 0$ if and only if $s(gx) = 0$. $\square$

Let us provide an example.

Example 5.5. Consider the natural action of $G = SL(2, \mathbb{C})$ on $G/B \cong \mathbb{P}^1$. For $n > 0$ consider the very ample vector bundle $\mathcal{O}(n)$. In this case

$$V = \mathbb{C}[X, Y]_n,$$

where X and Y are the homogeneous coordinates of $\mathbb{P}^1$. We can lift the action of G to an action $\mathbb{C}[X,Y]_n$, namely we pose

$$(gp)(X,Y) = p(g^{-1}X, g^{-1}Y)\,,$$

and this action makes the bundle $\mathcal{O}(n)$ equivariant.

Instead consider $G = PSL(2,\mathbb{C})$, also in this case $G/B \cong \mathbb{P}^1$, but $PSL(2,\mathbb{C})$ doesn't act on $\mathbb{C}[X,Y]_1$. The group $PSL(2,\mathbb{C})$ acts just on the spaces $\mathbb{C}[X,Y]_{2k} = \Gamma(\mathbb{P}^1, \mathcal{O}(2k))$, thus only the bundles $\mathcal{O}(2k)$ admit a structure of $\mathbb{P}SL(2,\mathbb{C})$ equivariant bundle.

As the example suggest our problems admits always a solution of and only if the group is connected and simply connected. We follow [Ser54]. First we prove the following proposition, it is true also in the case of not simply connected groups.

Proposition 5.6. *Let G be a connected group, X a G variety and*

$$\iota : X \to \mathbb{P}V^*$$

a proper embedding, then we can define a projective action of G on $\mathbb{P}V^*$ such that ι is G equivariant.

Proof. First we need a lemma that uses some results of complex algebraic geometry.

Lemma 5.7. *For every g in G the line bundle g^*L is isomorphic to L.*

Proof. The group $\mathrm{H}^1(X, \mathcal{O}^*)$ is the Picard group of X and G acts via pull-back on it. We have the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0\,,$$

and since X is projective we have the exact sequence

$$0 \to Pic^0(X) = \mathrm{H}^1(X,\mathcal{O})/\mathrm{H}^1(X,\mathbb{Z}) \to \mathrm{H}^1(X,\mathcal{O}^*) \to \mathrm{H}^{1,1}(X,\mathbb{Z}) \to 0\,.$$

Let $[L]$ be the isomorphism class of L inside the Picard group. Since $\mathrm{H}^{1,1}(X,\mathbb{Z})$ is discrete and G is connected the orbit $G[L]$ is inside the coset $Pic^0(X)[L]$. We have thus an action

$$G \times Pic^0(X)[L] \to Pic^0(X)[L]\,.$$

Always because X is projective $Pic^0(X)$ is an abelian variety. Applying the rigidity of the abelian varieties we obtain the statement. For further details on the Picard group see [Voi02], for the rigidity of abelian varieties see [Bor91] page 116-117. □

Let $div(L)$ be the divisor of a meromorphic section of L and let us identify the non trivial global section of L with the meromorphic function f such that $div(f)+div(L) > 0$. The previous lemma means that for every g in G there exist a meromorphic function f_g on X, unique up to a constant, such that the map

$$\Gamma(X, g^*L) \to \Gamma(X, L)$$

$$s \mapsto f_g s$$

is an isomorphism of vector space. Moreover the pull back gives an isomorphism

$$\Gamma(X, L) \to \Gamma(X, g^*L)$$

$$s \mapsto g^* s\,.$$

Composing these two isomorphism we get a map

$$\rho : G \times \Gamma(X, L) \to \Gamma(X, L)\,.$$

This map in general is not an action. To explain the problem we define a map

$$\omega : G \times G \to \mathbb{C}^*$$

$$(g, h) \mapsto (f_g g^* f_h)/(f^* h^* f_{gh})\,,$$

the image of every couple is an invertible regular function on a projective variety, thus a non zero constant. But the map ω is not in general constant. The reason because ρ is not an action is that

$$\rho(g)\rho(h)\rho(h^{-1}g^{-1})s = \omega(g, h)s\,.$$

Since the image of ω is inside $\mathbb{C}^*$ we have that the induced map

$$\tilde{\rho} : G \times \mathbb{P}\Gamma(X, L) \to \mathbb{P}\Gamma(X, L)$$

is an action. □

In our original problem we are now free to replace X with a projective space $\mathbb{P}V$. The solution is the following.

Proposition 5.8. *Let G be a connected and simply connected group, let*

$$\rho : G \to \mathbb{P}GL(V)$$

be an action of G on $\mathbb{P}V$, we can always define an action $\tilde{\rho}$ of G on V such that ρ is the induced action.

Proof. Let

$$\pi : GL(V) \to \mathbb{P}GL(V)$$

be the projection, to find the needed action of G on V we must define a map

$$\tilde{\rho} : G \to GL(V)$$

such that $\pi \circ \tilde{\rho} = \rho$. If there was a section of π we will be home, but this is not the case. We use the Lie algebras, in this case $d\pi$ is just a projection

$$d\pi : \mathfrak{sl}(V) \oplus \mathbb{C}Id \to \mathfrak{sl}(V)$$

and there is a section, so we have found a representation

$$d\tilde{\rho} : Lie(G) \to \mathfrak{gl}(V)$$

such that $d\pi \circ d\tilde{\rho} = d\rho$, the problem is: can we integrate $d\tilde{\rho}$ to a representation $\tilde{\rho}$ of G? We can always integrate it if and only if G is simply connected. □

Remark 5.9. *If G is not simply connected, as $\mathbb{P}SL(2, \mathbb{C})$, there are some representations of the Lie algebra that we can not integrate to a representation of the group, and hence may happen that some line bundle doesn't admit a structure of G equivariant line bundle.*

Let us summarize all the previous discussion.

Theorem 5.10. *Let G be a connected group, X a G variety and*

$$\iota : X \to \mathbb{P}V$$

a proper embedding, there always exist a projective action of G on $\mathbb{P}V$ such that ι is a G equivariant. Moreover if G is simply connected there exist an action of G on V such that ι is G equivariant.

We are interested to apply this result to the case of the G projective variety G/B, where B is a Borel subgroup of G. In this case we have an explicit G equivariant embedding in a projective space. Call d the dimension of U and consider the adjoint action of G on $\bigwedge^d \mathfrak{g}$. Let l be the line $\bigwedge^d \mathfrak{u}$, we have a map

$$G \to \mathbb{P} \bigwedge^d \mathfrak{g}$$

$$g \mapsto g.[l]\,,$$

since the stabilizer of $\mathfrak{u}$ is B we obtain a G equivariant embedding

$$\iota : G/B \to \mathbb{P}V$$

where V is $\bigwedge^d \mathfrak{g}$.

6 The Bruhat decomposition

In this section we state and prove the Bruhat decomposition for a general linear reductive and connected group G. Our goal is to describe the orbits of the action of B^- on G/B. The flag variety is projective and it is a T variety, so we can apply the general results of the previous section.

6.1 The fixed points of the action of T on G/B

We recall some facts about the algebraic group without proofs. All the Borel subgroup are conjugated (see [Bor91] p.147) and the normalizer of a Borel subgroup is the group itself (see [Bor91] page 154). We define a bijective correspondence between the points of the quotient G/B and the Borel subgroups of G: at a point $[g]$ corresponds the Borel subgroup gBg^{-1}, at a Borel subgroup $\tilde{B}$ corresponds the unique class gB such that $\tilde{B} = gBg^{-1}$. The group G acts by left translation on G/B and by conjugation on the set of Borel subgroups, the bijective correspondence defined above is G-equivariant.

We want to study the fixed points of the action of T on G/B. We follow the strategy of [Pro07] page 358. We translate the problem into a Lie algebras problem. We think G/B as the variety of all the Borel subalgebra of the Lie algebra $\mathfrak{g}$ of G; we call $\mathfrak{b}$ the Lie algebra of B and at a point $[g]$ of G/B corresponds the Lie algebra $Ad(g)(\mathfrak{b})$. We claim that the fixed point of the action are the Borel subalgebra containing $\mathfrak{t}$. The key lemma:

Lemma 6.1. *Let Φ be the set of roots of $\mathfrak{g}$, a subset A of Φ has the following proprieties*

$$(S) \qquad \alpha, \beta \in A, \alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in A\,,$$

$$(B) \qquad \alpha \in A \Rightarrow -\alpha \notin A\,,$$

if and only if is a subset of $w(\Phi^+)$, where w is an element of the Weyl group W and Φ^+ is a fixed Weyl chamber.

Proof. We just list the main ingredients. The groups W is the group of automorphism of the root system. The roots in a Weyl chamber associated to a regular vector v are all the roots α such that $(v, \alpha) > 0$, the group W acts simply transitively on the Weyl chambers. Given two roots α and β such that $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root. $\square$

An easy consequence of this the lemma is the following proposition.

Proposition 6.2. *Given a subset A of Φ consider the following vector space*

$$\mathfrak{h} := \mathfrak{t} \bigoplus_{\alpha \in A} L_\alpha\,,$$

this subspace is a Lie subalgebra if and only if (S) is true, moreover it is solvable if and only if (B) is true.

Proof. The first assertion follows from the fact that $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$. For the second assertion suppose that (B) is not true, then the subalgebra associated to $\{\alpha, -\alpha\}$ is not solvable. If (B) is true, because of the lemma it is enough to prove that the Lie algebra associated to Φ^+ is a Borel subalgebra. Let v be a vector such that $(\alpha, v) > 0$ for every α in Φ^+, then $(\alpha + \beta, v) > (\alpha, v)$ for every α and β in Φ^+, since Φ^+ is a finite set and $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ the Lie algebra is solvable. $\square$

We conclude that the fixed points of the action of T are the Borel subalgebra containing $\mathfrak{t}$ and they are in a bijective correspondence with the Weyl chambers and with the elements of the Weyl group, hence they are a finite set. All this correspondence are equivariant for the action of W. We give a name to this special point.

Definition 6.3. *The point $[b_w]$ inside G/B it the point that corresponds to the Borel subgroup wBw^{-1} Sometime for short we will omit the brackets, or, when there is not any danger of confusion, we will call this point just w.*

We can see this point in a more concrete way. The Weyl group is defined as the normalizer of T quotiented by its centralizer. The centralizer is contained in B so the class of an element of W in G/B is well-defined.

6.2 The dense orbit

We recall that $U^- := (B^-, B^-)$ is the unipotent part of B^-, its Lie algebra is

$$\mathfrak{u}^- = \bigoplus_{\alpha \in \Phi^-} L_\alpha\,.$$

This group acts on G/B, we will see that the stabilizer of Id is trivial. The stabilizer of Id for the action of G is B, because B is equal to its normalizer, so the stabilizer for the action of U^- is $B \cap U^-$.

Lemma 6.4. *The group $B \cap U^-$ is finite.*

Proof. The Lie algebra of $B \cap U^-$ is

$$L(B) \cap L(U^-) = (\mathfrak{t} \oplus_{\alpha\in\Phi^+} L_\alpha) \cap \oplus_{\alpha\in\Phi^-} L_\alpha = \{0\}\,;$$

so the group is discrete, and a discrete subgroup of algebraic group is finite. □

Lemma 6.5. *A finite unipotent group is trivial.*

Proof. A finite group is always linearly reductive in characteristic 0 because we can find an invariant scalar product. The group is algebraic so we can choose a faithful representation V, being unipotent the group stabilize a vector v and since it is linearly reductive we can find a decomposition

$$V = \mathbb{C}v \oplus M\,.$$

By induction on the dimension the group acts trivially on M, but the representation is faithful so the group is trivial. □

We conclude that the stabilizer of Id for the action of U^- is trivial, so the action on b give an injection

$$i : U^- \hookrightarrow G/B\,.$$

We recall the universal propriety of the quotient.

Theorem 6.6 (Universal propriety of the quotient)**.** *Let G be a group, V a G variety and H the stabilizer of a point p of V. Then G/H has a structure of variety and it is isomorphic, as G algebraic variety, to the orbit of p.*

Proof. see [Pro07] page 175, [Bor91] page 98 or [Spr98] page 91. □

Using this theorem we have that the orbit of Id is isomorphic to U^-, call it $C(Id)$.

We compute the dimension of the orbit, we recall that

$$Lie(G) = Lie(B) \oplus Lie(U^-)\,,$$

so

$$dim\, C(Id) = dim\, U^- = dim\, G - dim\, B = dim\, G/B$$

and then $C(Id)$, being G/B connected, is a dense set.

We now prove that $C(Id)$ is open, this facts is due to the following general result.

Proposition 6.7. *An orbit is open in its closure.*

Proof. Let G an algebraic group acting on V, let p be a point of V and $O(p)$ its orbit. Let us consider the surjective application

$$f : G \to O(p)$$

$$g \mapsto gp\,.$$

The image of any regular map contains an open dense set of its closure because of a general result of algebraic geometry, see [Spr98] page 19. We apply this this fact to f, we get a point x in $O(p)$ which has a neighborhood U (for the topology of $\overline{O(p)}$) entirely contained in $O(p)$, since G acts transitivity on $O(p)$ every point has a neighborhood contained in $O(p)$ and hence $O(p)$ is open in $\overline{O(p)}$. □

Being $\overline{O(p)}$ equal to the whole space we have that $O(p)$ is open. All this discussions hold also if we replace B^- and U^- with B and U, one has just to conjugate everything for w_0. We summarize the discussion of this section.

Theorem 6.8 (Existence of an open dense U-orbit)**.** *There exist an open dense orbit for the action of U on G/B.*

6.3 The smaller cells

We want to study the other orbits of the action of B^- on G/B. We fix a one parameter subgroup ρ of T such that

$$Fix(T) = Fix(\rho)\,;$$

for every root α we have that

$$\alpha(\rho(z)) = z^{m_\alpha}\,,$$

Call $X(T)$ the group of characters of T and $X_*(T)$ the group of one parameter subgroup of T. We request ρ to verify the following propriety: for every negative root α we want that m_α is positive. Such a group exist because we have a perfect duality between the one parameter subgroup of T and the characters of T: the integer (ρ, χ) is the unique such that $\chi(\rho(z)) = z^{(\rho,\chi)}$. Then we have that $X(T)$ is isomorphic to the dual of $X_*(T)$. The simples root of Φ^- are $\mathbb{Z}$ linearly independent so we can define a linear map

$$l : X(T) \to \mathbb{Z}$$

such that $l(\alpha)$ is positive for every negative root and does not assume the same value on two different roots. This map correspond to the requested one parameter subgroup.

We can give a more clever description of $C(Id)$:

$$C(Id) = \{p \in G/B \quad \text{s.t.} \quad \lim_{z\to 0} \rho(z)p = Id\}\,.$$

We prove that this description agrees with the older one. First we see that, being U^- normal in B^-, then the orbit of Id under the action of U^- is T stable:

$$tuId = tut^{-1}tId = u'tId = u'Id\,.$$

If we take a point outside U^-Id we can not get inside this open set applying $\rho(\mathbb{C}^*)$, so $C(Id)$ is inside U^-Id. We show the other inclusion. Using the exponential map the action of T on U^-Id is isomorphic to the action of T on $\mathfrak{u}^-$, we identify Id with 0. Since the m_α are all positive for the negative roots we obtain that U^-Id is inside $C(Id)$.

Now we define the other Bruhat cells in the same way.

Definition 6.9 (The Bruhat cells). *The Bruhat cell associated to an element w of W is*

$$C(w) := \{p \in G/B \quad s.t. \quad \lim_{z\to 0} \rho(z)p = w\}\,.$$

Because of our previous analysis on the action of the one parameter group the Bruhat cell cover all the space G/B. For the same argument as above we have that

Proposition 6.10. *For every w in W*

$$C(w) = U^-w\,.$$

The point is that the stabilizer of the action is trivial if and only if w is the identity.

We can define a partial order on the element of the Weyl group using the Bruhat cells, namely we say that

$$w \prec w' \Leftrightarrow C(w) \subset \overline{C(w')}\,.$$

We can give another description of the Bruhat cell. Let $[v]$ be a point of G/B, as we have done in the previous section we can embed G/B in $\mathbb{P}V$ for some representation $V$of G, call v_w a non zero vector of the right corresponding to w in V, we write

$$v = c_{w_1}v_{w_1} + \cdots + c_{w_n}v_{w_n}\,.$$

At each point $[v]$ we can thus associated the following subset of the Weyl group

$$\mathcal{S}([v]) := \{w \in W \quad \text{s.t.} \quad c_w \neq 0\}\,,$$

and then we claim that

$$C(w) = \{p \in G/B \quad \text{s.t.} \quad \mathcal{S}(p) \prec w \quad \text{and} \quad w \in \mathcal{S}\}\,.$$

First we remark that if $\lim_{z\to 0}\rho(z)p = w$ then w must appear in $\mathcal{S}(p)$. Recall that the integer (ρ, χ_{w_i}) are all different because ρ is generic. Moreover we claim that if $w \prec w'$ then $(\rho, \chi_w) > (\rho, \chi_{w'})$. The reason is that $C(w')$ is dense in $\overline{C(w')}$ so what must happen is that $\lim_{z\to 0}\rho(z)p = w$ if and only if w' is not in $\mathcal{S}(p)$ and not vice versa.

We conclude that if p is in $C(w)$ then w is the maximum of the partially order set $\mathcal{S}(p)$.

6.4 The CW-complex structure of the flag variety

The cells $C(w)$ are all isomorphic to $\mathbb{C}^k$ for some k. The proof of the isomorphism between $C(w)$ and $\mathbb{C}^k$ is in [Bor91] page 182, we describe the main idea. In a group G we fix an unipotent subgroup U and a maximal torus T; for every root α of the Lie algebra of U we call U_α the unique closed subgroup that is fixed by the action of the connected component of the identity of

$Ker\,\alpha$ (here the roots are the characters of the action of T on $\mathfrak{u}$). The Lie algebra of U_α is L_α and U_α is isomorphic to the group $(\mathbb{C}, +)$. Moreover let $\alpha_1 \cdots \alpha_n$ be the roots of $\mathfrak{u}$, the map

$$f : \mathbb{C}^n \cong U_{\alpha_1} \times \cdots \times U_{\alpha_n} \to U$$

given by the multiplication is an isomorphism of algebraic variety. This proof works also in positive characteristic, over $\mathbb{C}$ is easier to use the exponential: being the lie algebra $\mathfrak{u}$ composed of nilpotent elements the exponential is an algebraic isomorphism between $\mathfrak{u}$ and U.

Because of this isomorphism the cells of the Bruhat decomposition are the cells of the CW-complex structure of G/B. The dimension of the i-th cohomology group of G/B is thus the number of element of the Weyl group of length i; the cell $C(Id)$ is the top dimensional cell of G/B.

7 The Borel-Weil theorem

7.1 Construction of the line bundle associated to a character

We construct a line bundle on G/B from a character χ of B.

First we remark that, being $\mathbb{C}^*$ commutative, the unipotent part of B is $(B,B) =: U$ and hence any character is trivial on U. Furthermore B is isomorphic (as algebraic variety) to TU and an isomorphism is given by the multiplication, hence χ is essentially a character on a torus. One can write any element b of B as a product tu with t in T and u in U. Let $t(b) = (t_1(b), \dots, t_n(b))$ be the coordinates of b on T. We have that

$$\chi(b) = \chi(t(b)) = t^m ,$$

where m is a fixed element of $\mathbb{Z}^n$ associated to χ.

Let $\mathbb{C}_\chi$ be the representation of B given by the character χ: the vector space is just $\mathbb{C}$ and the action is given by χ. The group B acts on the right on G and on the left on $\mathbb{C}_\chi$ so B acts on $G \times \mathbb{C}_\chi$ and we can consider the space

$$L_\chi := G \times \mathbb{C}_\chi / B$$

. We will prove that L_χ, that by now is just a set, is an algebraic variety and moreover is a line bundle on G/B.

The projection of $G \times \mathbb{C}_\chi$ on the first factor gives the projection

$$p : L_\chi \to G/B .$$

First we trivialize L_χ on $p^{-1}C(Id)$ (we know that $C(Id)$ is affine so L_χ, if it is a line bundle, must admit a trivialization on $C(Id)$). Given a point $[g]$ in $C(Id)$ there is a unique element of U^- such that $g = ub$, hence we define a map

$$f : p^{-1}C(Id) \to C(Id) \times \mathbb{C}$$

$$[(g,v)] = [(ub,v)] \mapsto (u, \chi(b)v) ,$$

the definition is well posed because $f([g,v]) = f([gb, \chi(b^{-1})v])$. If we let $U^w = wU^-w^{-1}$ act on $[w]$ in G/B we obtain again an open dense set isomorphic to U, we do an abuse of notation calling it U^w. The U^w form an open affine cover of G/B because of the Bruhat decomposition. We trivialize L_χ on U^w.

$$f_w : p^{-1}U^w \to U^w \times \mathbb{C}$$

$$[(g,v)] = [(wub,v)] \mapsto (wu, \chi(b)v) .$$

One verify that the f_w are a trivialization of L_χ.

We have not defined a complex structure on L_χ. We use the the f_w. Locally L_χ is isomorphic to $U^- \times \mathbb{C}_\chi$, we glue this chart with the $t_{w_1,w_2} = \frac{f_{w_1}}{f_{w_2}}$ and we obtain a complex structure for $\mathbb{C}_\chi$; moreover with this structure L_χ is a line bundle.

This strategy can be generalized. We follow [Spr98] page 95.

Proposition 7.1. *Let H be a closed subgroup of G such that the projection*

$$\pi : G \to G/H$$

admits local section, then for every H variety X the quotient $G \times_H X$ exists and the map

$$p : G \times_H X \to G/H$$

is a fibration of fiber X.

Proof. Given a section

$$\sigma : U \to G/H\,,$$

one defines

$$\begin{aligned} f : U \times X &\to p^{-1}U \times_H X \\ ([h], x) &\mapsto [(\sigma([h]), \sigma(h)^{-1}hx)] \end{aligned}$$

and

$$\begin{aligned} f^{-1} : p^{-1}U \times_H X &\to U \times X \\ [(g, x)] &\mapsto (p(g), g^{-1}\sigma(p(g))x)\,. \end{aligned}$$

These functions are well defined and one can use it to glue the $U \times H$ to construct the space $G \times_H X$. The space constructed in this way is a fibration with fiber X. □

To apply this general result to our case we need the following

Lemma 7.2. *The map*

$$\pi : G \to G/B$$

admits local section.

Proof. We use the open cover formed by the translation of the biggest cell of the Bruhat decomposition U^w defined above. Each U^w is isomorphic to U^-, moreover these isomorphisms are local sections for p because every element of $p^{-1}U^w$ can be written in a unique way as wub with u in U^w and b in B. □

We conclude that $G \times_B L_\chi$ exists and, since $\mathbb{C}_\chi$ is a linear representation, it is a line bundle.

Now we compute the transition functions t_{w_1,w_2} on $U^{w_1} \cap U^{w_2}$. (Remark that we are not giving an explicit description of the domain of the transition functions!) Let $[g]$ be a point of $U^{w_1} \cap U^{w_2}$, we can write in a unique way

$$g = w_1u_1b_1 = w_2u_2b_2\,,$$

so

$$t_{w_1,w_2}(g) = \chi(b_1b_2^{-1})\,.$$

We remark that $t_{w_1,w_2}(gb) = \chi(b_1bb^{-1}b_2^{-1}) = t_{w_1,w_2}(g)$.

We conclude our analysis finding a divisor D on G/B such that L_χ is isomorphic to $L(D)$.

We must study first the case $G = SL(2, \mathbb{C})$, in this case G/B is isomorphic to $\mathbb{P}^1$ and, fixed a point p, any divisor is equivalent to np for some n in $\mathbb{Z}$. We calaim the following proposition.

Proposition 7.3. *Let G be $SL(2, \mathbb{C})$, and χ the character of B given by*

$$\chi(\begin{pmatrix} t & * \\ 0 & \frac{1}{t} \end{pmatrix}) = t^m\,.$$

The line bundle L_χ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(mp)$.

Proof. The Weyl group of $SL(2, \mathbb{C})$ is composed by two elements: Id and w, we choose as representant of w the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Write $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if g is in the intersection of U^{Id} and U^w then both a and c are not zero. The Bruhat decomposition for such an element is the following

$$g = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} Id \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} -c & d \\ 0 & -\frac{1}{c} \end{pmatrix}.$$

We call s the section that is equal to 1 on the open affine set U^{Id}, we want to extend it to a meromorphic section and compute its order in $[w]$. In the intersection of the trivializing open sets take a point

$$g = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} = w \begin{pmatrix} 1 & 0 \\ -\frac{1}{k} & 1 \end{pmatrix} \begin{pmatrix} -k & 1 \\ 0 & -\frac{1}{k} \end{pmatrix};$$

then

$$s([g]) = [(g,1)] = [(w \begin{pmatrix} 1 & 0 \\ -\frac{1}{k} & 1 \end{pmatrix}, (-\frac{1}{k})^m)].$$

To obtain $s([w])$ we must let k tend to infinity, so s has a zero of order m in w. We conclude that the divisor of L_χ is mp. □

This analysis holds also in the case $G = PSL(2, \mathbb{C})$, because the center is inside B, the only difference is that m must be even to define a character on $PSL(2, \mathbb{C})$.

Now we will use this case to solve the general case. The key point is that the Bruhat cells are affine, so every line bundles restricted to a Bruhat cell is trivial and thus it admits a constant section. We fix a section s equal to one on Uw_0, where w_0 is the longest element of the Weyl group. We establish some notation: given a root α in the Lie algebra we call h_α the corresponding element in the Cartan subalgebra under the Killing duality, call e_α and f_α a base of the eigenspaces for α and $-\alpha$; t_α is the one parameter sub group of T given by the exponential of $\mathbb{C}h_\alpha$.

We write the complement of Uw_0 as the union of irreducible divisor Z_i, so the divisor of s will be a linear combination $\sum n_i Z_i$. The group B is irreducible so the closure of the B orbits are irreducible, we get that the Z_i are the closure of the B orbits of codimension 1. The codimension of Bw is the length of ww_0, so we are interested to the orbit of the element w such that ww_0 is a simple reflections; let w be such a point and Z the closure of its orbit, call α the root associated to the simple reflection ww_0. If we let B^- act on w we get an orbit of dimension 1, call it O. The sub Lie algebra of $\mathfrak{g}$ generated by $\{e_\alpha, h_\alpha, f_\alpha\}$ is isomorphic to $\mathfrak{sl}(2)$, let H be the subgroup associated. The group H is isomorphic either to $SL(2, \mathbb{C})$ or to $\mathbb{P}SL(2, \mathbb{C})$. This group acts transitively on O, moreover the closure of O is isomorphic to $H/(B \cap H)$. We restrict the bundle L_χ to $\bar{O}$, its divisor has degree (χ, t_α) for the computation done for $SL(2, \mathbb{C})$. Remark that w is a smooth point for both O and Z because it is inside an orbit.

The last step is to prove that D and Z intersect transversely, let π be the map given by the action of G on w, then $d\pi$ is a surjection from the Lie algebra of G to the tangent space of w, we decompose the Lie algebra of G in the following way

$$(\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^- | w(\alpha) \in \Phi^+} L_\alpha \oplus \bigoplus_{\alpha \in \Phi^+ | w(\alpha) \in \Phi^+} L_\alpha) \oplus (\bigoplus_{\alpha \in \Phi^- | w(\alpha) \in \Phi^-} L_\alpha) \oplus (\bigoplus_{\alpha \in \Phi^+ | w(\alpha) \in \Phi^-} L_\alpha),$$

the first bracket is the kernel of π, the second is isomorphic to the tangent space of D, the third to the tangent space of Z. We conclude that the following theorem holds.

Theorem 7.4 (Divisor associated to L_χ). *A divisor associated to L_χ is*

$$\sum_{\alpha \in R} (\chi, t_\alpha) Z_\alpha,$$

where R is the set of simple roots and Z_α is the closure of the orbit of the simple reflection associated to α under the action of B^-.

7.2 The space of global section and the Borel-Weil theorem

Given a character χ we define an action of G on L_χ

$$g.[h,v] = [gh.v]\,;$$

the definition does not depend from the representant, we remark that p is G-equivariant. The group B acts on $\Gamma := \Gamma(G/B, L_\chi)$ in the following way

$$(b.s)(x) := bs(b^{-1}x)\,;$$

we check that $b.s$ is still section

$$p(b.s)(x) = p(bs(b^{-1}x)) = bp(s(b^{-1}x)) = bb^{-1}x = x\,.$$

We can see the sections of L_χ in a simpler way, they are all the regular function

$$s : G \to \mathbb{C}$$

solving the following functional equation

$$s(gb) = \chi(b^{-1})s(g) \qquad \forall b \in B\, g \in G\,.$$

To prove this we see these functions as section of the trivial bundle $G \times \mathbb{C}_\chi$ over G. Because of the functional equations we can quotient the base and the bundle by B and they are still sections.

Let $U = (B,B)$ be the unipotent part of B and Γ^U be the global section invariant under the action of U. When an unipotent group acts on a vector space of positive dimension it always admits a non trivial fixed vector, so L_χ admits non trivial global section if and only if Γ^U is not trivial. Now suppose that s is a U invariant non trivial section, hence constant on Uw_0; using the computation and the notation of the last part of the section 7.1 we get that

$$div(s) = \sum_{\alpha \in R} (\chi, t_\alpha) Z_\alpha\,,$$

so an U invariant section is regular if and only if (χ, t_α) is bigger or equal than zero for every simple root α. We recall the following definition

Definition 7.5 (Domnninant characters)**.** *A character χ of G is dominant if and only if*

$$(\chi, t_\alpha) \geq 0\,,$$

for every simple root α.

If the character χ is dominant we prove that the bundle L_χ admits non trivial global section: we define s equal to 1 on Uw_0, we extend it to a meromorphic section, its divisor is positive so s is a global regular section. We have thus proved that the bundle L_χ admits global section if and only if χ is dominant, from now suppose that χ is dominant.

Proposition 7.6. *The vector space Γ^U is a B module.*

Proof. We have to prove that

$$u.b.s = b.s \qquad \forall s \in \Gamma^U, \forall b \in B, \forall u \in U\,.$$

We recall that U is a normal subgroup of B so

$$u.b.s = (ub).s = (b(b^{-1}ub)).s = b\tilde{u}.s = bs\,,$$

where $\tilde{u}$ is the element $b^{-1}ub$ of U. □

Proposition 7.7. *The dimension of* Γ^U *is one.*

Proof. Let s_1 and s_2 be two non trivial elements of Γ^U, we will show that they are linearly dependent. The function $f = s_1/s_2$ is a meromorphic function on G/B, since $C(Id)$ is dense there exist a point p in $C(Id)$ and a complex number z such that $f(p) = z$. As we have seen in the previous section $C(Id)$ is an U-orbit so, being the s_i's U invariant, f is constant on $C(Id)$. A function constant in a dense set is constant everywhere so

$$s_1 = zs_2\,.$$

□

We recall a general fact.

Theorem 7.8. *Given a representation V of an algebraic group G the number of irreducible component of V is equal to the dimension of the space V^U of the U invariants vectors.*

Proof. We give a sketch of the proof using the Lie algebras. Fix v in V, consider the map π given by the action of G on v, now the image of π is contained in a proper subspace of V if and only if the image of the differential is contained in a proper subspace, hence a representation of a group is irreducible if and only if is irreducible as a representation of the Lie algebra of the group. Take v in V^U. We recall the decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}^+$. The representation admits a base of eigenvectors for the action of $\mathfrak{t}$. If we apply repeatedly the elements of $\mathfrak{u}^-$ to v we obtain a stable subspace, call it $M(v)$. The point is to prove that $M(v)^U$ is $\mathbb{C}v$ and one get an irreducible subrepresentation of V for every one dimensional U invariant subspace of V. For the complete proof see [Pro07] page 343. □

We conclude that if $\Gamma(G/B, L_\chi)$ is not zero then it is an irreducible representation of G, we want to identify it. We know from the general theory that the representation of G are classified by the eigenvalues of T on the U invariant subspace, so let s the U invariant section and t an element of T, we know that

$$ts = \lambda(t)s\,,$$

we want to compute λ. Let w_0 be the longest element of the Weyl group, we compute $(ts)(w_0)$:

$$(ts)(w_0) = s(t^{-1}w_0) = s(w_0w_0^{-1}tw_0) = \chi(w_0t^{-1}w_0^{-1})s(w_0) = (-w_0\chi)(t)s(w_0)\,.$$

In this computation we have characterized the section of L_χ as complex valued function on G.

We know that the B orbit of w_0 is dense in G/B, since the space $\mathbb{C}s$ is U invariant we have that

$$ts = (-w_0\chi)(t)s\,.$$

We remark that given a representation V of weight χ its dual representation has weight $-w_0(\chi)$. To see this fix a basis of eigenvectors of T, call μ_i the eigenvalue, the dual base is is still a base of eigenvectors but the eigenvalues are $-\mu_i$. Changing the sign the highest vector became the lowest one, and one have to apply w_0 to the lowest to find the highest. We have thus proved the following theorem.

Theorem 7.9 (Borel-Weil). *The line bundle L_χ has non trivial global section if and only if χ is dominant, in this case*

$$\Gamma(G/B, L_\chi)$$

is an irreducible G module isomorphic to the dual of the representation of maximal weight χ.

8 The Kostant formula

In this section we work only on the Lie algebras. We fix a simple Lie algebra $\mathfrak{g}$ and a decomposition of $\mathfrak{g}$ of the kind

$$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{h} \oplus \mathfrak{u}\,,$$

where $\mathfrak{u}$ is a maximal unipotent subalgebra, $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{u}^-$ is the opposite maximal unipotent subalgebra.

We fix a dominant weight λ of $\mathfrak{g}$ and we call $L(\lambda)$ the irreducible representation of $\mathfrak{g}$ of maximal weight λ. The Cartan algebra $\mathfrak{h}$ acts on both $\mathfrak{u}$ and $L(\lambda)$ so, as we will see in details, it acts on $\mathrm{H}^s(\mathfrak{u}, L(\lambda))$; given a root μ we will describe

$$\mathrm{H}^s(\mathfrak{u}, L(\lambda))_\mu\,,$$

with this notation we indicate the vector space of all the elements ω of $\mathrm{H}^s(\mathfrak{u}, L(\lambda))$ such that $h.\omega = \mu(h)\omega$.

8.1 The key exact sequence

We consider the complex $\bigwedge \mathfrak{u}$, the boundary operator is

$$d_0 : \bigwedge^{i} \mathfrak{u} \to \bigwedge^{i-1} \mathfrak{u}$$

$$u_1 \wedge \cdots \wedge u_i \mapsto \sum_{s<t} (-1)^{(s+t+1)} [u_s, u_t] \wedge u_1 \wedge \cdots \hat{u_s} \cdots \hat{u_t} \cdots \wedge u_i$$

this is a complex of $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ module, the action is the adjoint one, namely

$$v.u_1 \wedge \cdots \wedge u_n = \sum_i (-1)^{i+1} [v, u_i] \wedge u_1 \wedge \cdots \hat{u_i} \cdots \wedge u_n\,.$$

We fix a simple root α_i, we call e_i a generator of the weight space of α_i and f_i a generator of the weight space of $-\alpha_i$. We call

$$\mathfrak{u}_i = \bigoplus_{\alpha \neq \alpha_i} \mathfrak{u}_\alpha\,,$$

this is an ideal of $\mathfrak{u}$, moreover we have the following decomposition

$$\mathfrak{u} = \mathfrak{u}_i \rtimes \mathbb{C}e_i\,,$$

and the algebra $\mathfrak{b}$ acts on $\mathfrak{u}_i$. Moreover also f_i acts on $\mathfrak{u}_i$. To prove this fact we recall that

$$[f_i, \mathfrak{u}_\alpha] \subset \mathfrak{u}_{\alpha - \alpha_i}\,,$$

so we have to prove that $\alpha - \alpha_i$ is a positive root different from α_i. Since α is different from α_i there exist at least one index j such that $\alpha - \alpha_i$ has a positive coefficient in α_j. Because of the general theory a root has either all non-negative coefficients in the simple roots or only non-positive coefficient, see [Pro07] pages 317-318. The coefficient of $\alpha - \alpha_i$ in α_j is positive then $\alpha - \alpha_j$ either it is a positive root or it is not a root. If it is not a root the action of α_i on $\mathfrak{u}_\alpha$ is trivial; if it is a root it must be different from α_i because $2\alpha_i$ is not a root, so we are home. We call

$$\mathfrak{a}_i := \mathbb{C}f_i \oplus \mathfrak{h} \oplus \mathbb{C}e_i\,,$$

this is a Lie algebra formed by a commutative factor plus a copy of $\mathfrak{sl}(2)$ generated as a vector space by e_i,f_i and h_i; because of the previous discussion $\mathfrak{u}_i$ is an $\mathfrak{a}_i$ module.

As we have done for $\mathfrak{u}$, we define the complex $\bigwedge \mathfrak{u}_i$, this is a complex of $\mathfrak{a}_i$ module. We have the following exact sequence of complexes

$$0 \to \bigwedge \mathfrak{u}_i \xrightarrow{\iota} \bigwedge \mathfrak{u} \xrightarrow{\pi} \bigwedge \mathfrak{u}_i \otimes_{\mathbb{C}} e_i \to 0\,,$$

where ι is the inclusion and π is the projection. The quotient makes sense because $\mathfrak{u}_i$ is an ideal of $\mathfrak{u}$ and it is a $\mathfrak{b}$ module: the action of $\mathfrak{u}$ on e_i is trivial, the action of $\mathfrak{h}$ on e_i is the adjoint one. We remark that the j-th element of the complex $\bigwedge \mathfrak{u}_i \otimes e_i$ is $\bigwedge^{j-1} \mathfrak{u}_i \otimes e_i$. The maps ι and π are maps of $\mathfrak{b}$ modules.

From the previous exact sequence we construct the following exact sequence of complexes:

$$0 \to \mathrm{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{u}_i, L(\lambda)) \otimes e_i^* \xrightarrow{\pi^*} \mathrm{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{u}, L(\lambda)) \xrightarrow{\iota^*} \bigwedge \mathrm{Hom}_{\mathbb{C}}(\mathfrak{u}_i, L(\lambda)) \to 0\,; \tag{2}$$

the differential is defined as follows

$$(d\omega)(\mathfrak{u}_1 \wedge \cdots \wedge u_n) :=$$

$$= \sum_{i<j} (-1)^{i+j+1} \omega([u_i, u_j] \wedge u_1 \wedge \cdots \hat{u_i} \cdots \hat{u_j} \cdots \wedge u_n) - \sum_i (-1)^{i+1} u_i \omega(u_1 \wedge \cdots \hat{u_i} \cdots \wedge u_n))$$

and

$$d(\omega \otimes e_i^*) := d(\omega) \otimes e_i^*\,.$$

The complexes and the maps are still complexes and maps of $\mathfrak{b}$ modules, the action is the following

$$b.\omega(u_1 \wedge \cdots \wedge u_n) := \omega(b.u_1 \wedge \cdots \wedge u_n) - b\omega(u_1 \wedge \cdots \wedge u_n)\,,$$

and on e_i^* is the dual of the action on e_i: the part on e_i of the complex is important because it changes the action of $\mathfrak{h}$. In the same way the third one is a complex of $\mathfrak{a}_i$ module. We compute the homology of these complexes.

We recall the standard way to compute the cohomology of a Lie algebra. Given a Lie algebra $\mathfrak{g}$ one has the Chevalley-Eilenberg complex $\bigwedge \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{U}(\mathfrak{g})$, where $\mathfrak{g}$ acts on the right on $\mathfrak{U}(\mathfrak{g})$. One applies the functor $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(\,-\,, L(\lambda))$ to the Chevalley-Eilenberg complex and one obtains $\mathrm{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}, L(\lambda))$ with the differential defined as above; the cohomology groups $\mathrm{H}^s(\mathfrak{g}, L(\lambda))$ are by definition the homology of this complex. We conclude that the jth homology groups of the complexes appearing in the sequence 2 are respectively $\mathrm{H}^{j-1}(\mathfrak{u}_i, L(\lambda)) \otimes e_i^*$, $\mathrm{H}^j(\mathfrak{u}, L(\lambda))$ and $\mathrm{H}^j(\mathfrak{u}_i, L(\lambda))$.

The complexes are complexes of $\mathfrak{b}$ modules, so $\mathfrak{b}$ acts on the homology groups, we will see that $\mathfrak{u}_i$ acts trivially on $\mathrm{H}^*(\mathfrak{u}_i, L(\lambda))$ and $\mathfrak{u}$ acts trivially on $\mathrm{H}^*(\mathfrak{u}, L(\lambda))$. The proofs are the same so we will do just the case of $\mathfrak{u}$. Fix an ω in $\mathrm{H}^j(\mathfrak{u}, L(\lambda))$ and an element u of $\mathfrak{u}$, the form is closed so

$$0 = d\omega(u \wedge u_1 \wedge \cdots \wedge u_j) = (u.\omega)(u_1 \wedge \cdots \wedge u_j) +$$

$$- \sum_{s<t} (-1)^{s+t+1} \omega([u_s, u_t] \wedge u \wedge u_1 \wedge \cdots \hat{u_s} \cdots \hat{u_t} \cdots \wedge u_j) + \sum_s (-1)^{s+1} u_s \omega(u \wedge u_1 \wedge \cdots \hat{u_s} \cdots \wedge u_j)\,;$$

we have to prove that

$$\sum_{s<t} (-1)^{s+t+1} \omega([u_s, u_t] \wedge u \wedge u_1 \wedge \cdots \hat{u_s} \cdots \hat{u_t} \cdots \wedge u_j) - \sum_s (-1)^{s+1} u_s \omega(u \wedge u_1 \wedge \cdots \hat{u_s} \cdots \wedge u_j) =$$

$$= (d\phi)(u_1 \wedge \cdots \wedge u_j)$$

for some ϕ in $\mathrm{Hom}(\bigwedge^{j-1} \mathfrak{u}, L(\lambda))$. The good ϕ is a contraction of ω:

$$\phi(u_1 \wedge \cdots \wedge u_j) := \omega(u \wedge u_1 \wedge \cdots \wedge u_j)\,.$$

Instead $\mathfrak{h}$ does not act trivially on the cohomology groups and e_i does not acts trivially on $\mathrm{H}^j(\mathfrak{u}_i, L(\lambda))$.

From 2 arises the following long exact sequence

$$\cdots \to \mathrm{H}^{j-1}(\mathfrak{u}_i, L(\lambda)) \xrightarrow{\delta} \mathrm{H}^{j-1}(\mathfrak{u}_i, L(\lambda)) \otimes e_i^* \to \mathrm{H}^j(\mathfrak{u}, L(\lambda)) \to \mathrm{H}^j(\mathfrak{u}_i, L(\lambda)) \xrightarrow{\delta} \cdots\,, \qquad (3)$$

we want to describe the connecting homomorphism δ. Fix an element ω in $\mathrm{Hom}(\bigwedge^j \mathfrak{u}_i, L(\lambda))$. We choose an element ϕ in $\iota^{*-1}(\omega)$ such that $\phi(\underline{u} \wedge e_i) = 0$ for every $\underline{u}$ in $\bigwedge^{j-1} \mathfrak{u}_i$. Take $d\phi$, by definition $\delta(\omega)$ is the class of an element of $\pi^{*-1}(d\phi)$. Such an element is $(e_i.\omega) \otimes e_i^*$ because

$$\pi^*((e_i.\omega) \otimes e_i^*)(u_1 \wedge \cdots \wedge u_j \wedge e_i) = e_i.\omega(u_1 \wedge \cdots \wedge u_j) = d\phi(u_1 \wedge \cdots \wedge u_j \wedge e_i)\,.$$

So we conclude that

$$\delta(\omega) = e_i.\omega \otimes e_i^*\,.$$

Because of the previous description of the connecting homomorphism, the long exact sequence 3 gives rise to a short exact one:

$$0 \to \mathrm{H}^{j-1}(\mathfrak{u}_i, L(\lambda))_{e_i} \otimes e_i^* \xrightarrow{\pi^*} \mathrm{H}^j(\mathfrak{u}, L(\lambda)) \xrightarrow{\iota^*} \mathrm{H}^j(\mathfrak{u}_i, L(\lambda))^{e_i} \to 0\,, \qquad (4)$$

where $\mathrm{H}^{j-1}(\mathfrak{u}_i, L(\lambda))_{e_i}$ means $\mathrm{H}^{j-1}(\mathfrak{u}_i)/(e_i.\mathrm{H}^{j-1}(\mathfrak{u}_i))$. This short exact sequence is a sequence of $\mathfrak{b}$ module and it will be crucial in the following discussion.

8.2 The Kostant formula

The Kostant formula describes the weight spaces of $\mathrm{H}^j(\mathfrak{u}, L(\lambda))$; we fix a weight μ and we use the sequence 4 to study its weight space. The idea is to take advantage of the fact that $\mathrm{H}^j(\mathfrak{u}_i, L(\lambda))$ is an $\mathfrak{sl}(2)$ module, namely it is a module for the Lie algebra generated by e_i,f_i and h_i: the non trivial factor of $\mathfrak{a}_i$.

We recall the general theory of the $\mathfrak{sl}(2)$ module. Let M be an $\mathfrak{sl}(2)$ module, let s be a representant of the non trivial element of the Weyl group of $\mathfrak{sl}(2)$ inside $SL(2)$. Call M^e the set of the e invariant and M_e the set of e covariant. We have the following classical theorem.

Theorem 8.1 (Proprieties of the $\mathfrak{sl}(2)$ module)**.** *Keep notation as above. Fix a weight μ of $\mathfrak{sl}(2)$. Then*

$$M^e_\mu \neq 0 \quad \Leftrightarrow \quad \mu(h) \geq 0\,,$$
$$(M_e)_\mu \neq 0 \quad \Leftrightarrow \quad \mu(h) \leq 0\,.$$

Moreover the action of s on M determines an isomorphism between M^e_μ and $(M_e)_{s(\mu)}$.

Remark that the choice of s is not unique so the isomorphism is not canonic.

In the case of $\mathrm{H}^j(\mathfrak{u}_i, L(\lambda))^{e_i}_\mu$ we have that

$$\dim\, \mathrm{H}^j(\mathfrak{u}_i, L(\lambda))^{e_i}_\mu \neq 0 \quad \Rightarrow \quad \mu(h_i) \geq 0\,.$$

Now we study $(\mathrm{H}^{j-1}(\mathfrak{u}_i, L(\lambda))_{e_i} \otimes e_i^*)_\mu$. First we have that

$$\dim\, (\mathrm{H}^{j-1}(\mathfrak{u}_i, L(\lambda))_{e_i})_\mu \neq 0 \quad \Rightarrow \quad \mu(h_i) \leq 0\,.$$

The presence of e_i^* twists the weights: a vector $v \otimes e_i^*$ belongs to $(\mathrm{H}^{j-1}(\mathfrak{u}_i, L(\lambda))_{e_i} \otimes e_i^*)_\mu$ if and only if v belongs to $(\mathrm{H}^{j-1}(\mathfrak{u}_i, L(\lambda))_{e_i})_{\mu+\alpha_i}$. This means that a weight space of weight α in $\mathrm{H}^{j-1}(\mathfrak{u}_i, L(\lambda))_{e_i}$ corresponds to a weight space of weight $\alpha - \alpha_i$ in $\mathrm{H}^{j-1}(\mathfrak{u}_i, L(\lambda))_{e_i} \otimes e_i^*$, we conclude that

$$\dim\, (\mathrm{H}^{j-1}(\mathfrak{u}_i, L(\lambda))_{e_i} \otimes e_i^*)_\mu \neq 0 \quad \Rightarrow \quad \mu(h_i) \leq -2\,.$$

We are now ready to study $\mathrm{H}^j(\mathfrak{u}, L(\lambda))_\mu$. Suppose that ω is a non trivial vector of weight μ, since the sequence 4 is an exact sequence of $\mathfrak{h}$ module at least one between the following possibilities is true:

1. $\iota^{*-1}(\omega)$ contains a non trivial vector of weight μ;
2. $\pi^*(\omega)$ is a non trivial vector of weight μ.

Because of the previous discussion the first possibilities is true if and only if $\mu(h_i) \leq -2$, the second if and only if $\mu(h_i) \geq 0$. We conclude that:

Proposition 8.2. *In the exact sequence 4 for every weight μ we have that:*

1. *if $\mu(h_i) \geq 0$ then ι^* is an isomorphism between $H^j(\mathfrak{u}, L(\lambda))_\mu$ and $H^j(\mathfrak{u}_i, L(\lambda))_\mu^{e_i}$;*
2. *if $\mu(h_i) \leq -2$ then π^* is an isomorphism between $H^j(\mathfrak{u}, L(\lambda))_\mu$ and $(H^{j-1}(\mathfrak{u}_i)_{e_i} \otimes e_i^*)_\mu$;*
3. *if $\mu(h_i) = -1$ then $H^j(\mathfrak{u}, L(\lambda))_\mu$ is trivial.*

We will use this result to understand the weight spaces of $\mathrm{H}^*(\mathfrak{u}, L(\lambda))$.

We need a twisted action of the Weyl group. Call ρ the weight $\sum \omega_i$, where the ω_i are the roots such that $\omega_i(h_j) = \delta_{ij}$. The equality $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ holds because the difference between the RHS and the LHS is stable under the action of the Weyl group.

We define the twisted action of the Weyl group in the following way:

$$w \cdot \lambda := w(\lambda + \rho) - \rho\,;$$

if w is a simple reflection the action turns to be

$$s_i \cdot \lambda = s_i(\lambda) - \alpha_i$$

because $s_i(\rho) = \rho - \alpha_i$. For more details see [Pro07] page 324. This action is important because $(s_i \cdot \mu)(h_i) \geq 0$ if and only if $\mu(h_i) \leq -2$, the proof is the following computation:

$$s_i \cdot \mu(h_i) = \langle s_i(\mu) - \alpha_i, \alpha_i\rangle = -\langle \mu, \alpha_i\rangle - 2 = -\mu(h_i) - 2\,,$$

where $\langle\ ,\ \rangle$ denotes the Cartan integers.

We give the following definition:

Definition 8.3 ((Twisted) Regular weight)**.** *A weight μ is regular if its stabilizer for the (twisted) action of the Weyl group is trivial.*

Remark 8.4. *A weight μ is regular for the classical action of the Weyl group if and only if $\mu(h_\alpha) \neq 0$ for every root α. It is regular for the twisted action of the Weyl group if and only if $\mu(h_\alpha) \neq -\rho(\alpha)$ for every root α; if α is simple $\rho(\alpha) = 1$.*

If the action of the Weyl group is the classical one then a dominant weight is regular if and only if the stabilizer of the highest weight vector in the correspondent irreducible representation is B; otherwise it is not regular if the stabilizer is bigger then B. In this and in the following section we will indicate in this way $w(-)$ the classical action and in this way $w\cdot -$ the twisted one. Regular will mean always regular for the twisted action.

We fix a representant of s_i inside G, we call it allways s_i. The action of s_i on $\mathrm{H}^j(\mathfrak{u}_i)$ gives an isomorphism of vector space between $\mathrm{H}^j(\mathfrak{u}_i)^{e_i}$ and $\mathrm{H}^j(\mathfrak{u}_i)_{e_i}$; as $\mathfrak{h}$ module a vector of weight μ became a vector of weight $s_i(\mu)$. Composing this action with the tensorization for e_i^* we get an isomorphism

$$f : \mathrm{H}^j(\mathfrak{u}_i)^{e_i} \to \mathrm{H}^j(\mathfrak{u}_i)_{e_i} \otimes e_i^*$$

that transform a vector of weight μ in a vector of weight $s_i \cdot \mu$ because the tensorization twists the weights of $-\alpha_i$. We remark that the choice of the representant of s_i is not unique, so f is not canonically defined.

We write the sequence 4 in two cases

$$0 \to \mathrm{H}^{j-1}(\mathfrak{u}_i)_{e_i} \otimes e_i^* \xrightarrow{\pi^*} \mathrm{H}^j(\mathfrak{u}) \xrightarrow{\iota^*} \mathrm{H}^j(\mathfrak{u}_i)^{e_i} \to 0$$

$$0 \to \mathrm{H}^{j}(\mathfrak{u}_i)_{e_i} \otimes e_i^* \xrightarrow{\pi^*} \mathrm{H}^{j+1}(\mathfrak{u}) \xrightarrow{\iota^*} \mathrm{H}^{j+1}(\mathfrak{u}_i)^{e_i} \to 0\,.$$

We define a map

$$\Phi : \mathrm{H}^j(\mathfrak{u}) \to \mathrm{H}^{j+1}(\mathfrak{u})$$

as the composition of the ι^* of the first sequence, the map f defined above and the π^* of the second sequence.

Proposition 8.5. *Keep notation as above and fix a weight μ. We have that:*

1. *if $\mu(h_i) \geq 0$ then Φ gives an isomorphism $H^j(\mathfrak{u}, L(\lambda))_\mu$ and $H^{j+1}(\mathfrak{u}, L(\lambda))_{s_i\cdot\mu}$;*
2. *if $\mu(h_i) \leq -2$ then Φ gives an isomorphism $H^j(\mathfrak{u}, L(\lambda))_{s_i\cdot\mu}$ and $H^{j+1}(\mathfrak{u}, L(\lambda))_\mu$;*
3. *if $\mu(h_i) = -1$ then $H^j(\mathfrak{u}, L(\lambda))_\mu$ is trivial.*

We recall that these isomorphisms are not canonic.

Proof. As we have seen before the tensorization for e_i^* determines an isomorphism between $\mathrm{H}^j(\mathfrak{u}_i)_\mu$ and $\mathrm{H}^j(\mathfrak{u}_i)_{\mu-\alpha_i}$. Suppose that $\mu(h_i) \geq 0$, because of the proposition 8.2 ι^* is an isomorphism between $\mathrm{H}^j(\mathfrak{u}, L(\lambda))_\mu$ and $\mathrm{H}^j(\mathfrak{u}_i, L(\lambda))_\mu^{e_i}$. The map f gives an isomorphism between $\mathrm{H}^j(\mathfrak{u}_i, L(\lambda))_\mu^{e_i}$ and $(\mathrm{H}^j(\mathfrak{u}_i, L(\lambda)) \otimes e_i^*)_{s_i\cdot\mu}$. The action of s_i on μ is twisted because of e_i^*. Now $\mu(s_i \cdot h_i) \leq 0$ so always because of the proposition 8.2 we get the statement.

The case $\mu(h_i) \leq -2$ is symmetric. The case $\mu(h_i) = -1$ follows directly form the proposition 8.2. □

We state the main result of this section.

Theorem 8.6 (Kostant formula)**.** *Let μ be a weight and λ a dominant weight for the classical action of the weight group, then the weight spaces*

$$H^j(\mathfrak{u}, L(\lambda))_\mu$$

is isomorphic to $\mathbb{C}$ if there exist an element w of the Weyl group such that $w\cdot\mu = \lambda$ and $l(w) = j$, otherwise it is trivial.

Proof. The proof will be done in three steps. First we collect all the informations we already have about the cohomology of $\mathfrak{u}$. In the second step we recall without proof the proprieties of the Weyl group. In the last step we use the proposition 8.5 and the proprieties of the Weyl group to describe all cohomology of $\mathfrak{u}$ starting from the "few" groups already known.

The already known informations about the cohomology of $\mathfrak{u}$ are the following.

Lemma 8.7 (The $\mathfrak{u}$-invariants). *The space*

$$H^0(\mathfrak{u}, L(\lambda))_\mu$$

has dimension one if μ is equal to λ, zero otherwise.

Moreover we have two bounds on the degrees of the non trivial cohomology groups:

1. the groups $\mathrm{H}^j(\mathfrak{u}, L(\lambda)$ are trivial for $j > \dim \mathfrak{u}$ because the length of Chevalley-Eilenberg complex is $\dim \mathfrak{u}$,
2. by definition the cohomology in negative degree is trivial.

Now we recall some proprieties of the Weyl group without proofs. We call a weight μ twisted dominant if $\mu(h_i) \geq -1$ for ever i, we call it twisted antidominant if $w_0 \cdot \mu$ is twisted dominant. Given a weight μ there exists an element w of the Weyl group such that $w \cdot \mu$ is twisted dominant, and an element w' such that $w' \cdot \mu$ is twisted antidominant. If w and w' are as above fix two reduced expressions $w = s_{i_1} \cdots s_{i_{l(w)}}$ and $w' = s'_{i_1} \cdots s'_{i_{l(w')}}$, one proves that for every j

$$s_{j+1} \cdots s_{i_{l(w)}} \cdot \mu(h_j) \leq -1$$

and

$$s'_{j+1} \cdots s'_{i_{l(w')}} \cdot \mu(h_j) \geq -1\,;$$

moreover $l(w) + l(w') = l(w_0)$. If a weight is regular then w and w' are unique. For proofs and further details see [Pro07] page 328.

Now we put together all the previous results. Suppose that there exist a w as in the statement of the Kostant formula. Because of the proprieties of the twisted action of the Weyl group and because of the proposition 8.5 we have that

$$\mathrm{H}^j(\mathfrak{u}, L(\lambda))_\mu \cong \mathrm{H}^0(\mathfrak{u}, L(\lambda))_{w\cdot\mu}\,.$$

Applying the proposition 8.7 we obtain the conclusion.

We tackle the other case. Fix an element w of the Weyl group such that $w \cdot \mu$ is twisted dominant. Because of the proprieties of the twisted action and of the proposition 8.5 we have that

$$\mathrm{H}^j(\mathfrak{u}, L(\lambda))_\mu \cong \mathrm{H}^{j-l(w)}(\mathfrak{u}, L(\lambda))_{w\cdot\mu}\,.$$

Suppose that $l(w)$ is equal to j and $w \cdot \mu$ is different from λ. We obtain the statement applying the proposition 8.7. Suppose that j is smaller than $l(w)$, then $j - l(w)$ is negative so the RHS has dimension zero.

Now suppose that j is bigger then $l(w)$. Take a w' such that $w' \cdot \mu$ is antidominant. We have that

$$\mathrm{H}^j(\mathfrak{u}, L(\lambda))_\mu \cong \mathrm{H}^{j+l(w')}(\mathfrak{u}, L(\lambda))_{w'\cdot\mu}\,;$$

in this case $j + l(w') = j - l(w) + l(w_0)$ is bigger than the dimension of $\mathfrak{u}$ so the RHS has dimension zero.

$\square$

9 The Borel-Weil-Bott theorem

The aim of this section is to prove the following foundamental theorem.

Theorem 9.1 (Borel-Weil-Bott). *The cohomology group*

$$H^i(G/B, L_\chi)$$

is isomorphic as G module to $L(w \cdot \chi)^*$ if and only if there exist an element w of length i of the Weyl group such that $w \cdot \chi$ is dominant. If such an element does not exist the group is trivial.

With the dot we denotes the twisted action of the Weyl group. We recall that a weight χ is dominant if and only if $\chi(h_i) \geq 0$ for every simple root. An element w as in the hypothesis of theorem exists if and only if χ is regular for the twisted action of the Weyl group. Moreover this element is unique. If λ is singular the theorem states that all the cohomology vanishes.

The strategy of our proof is to link the cohomology of the line bundles with the cohomology of the Lie algebras of B and U, then to apply the Kostant formula.

9.1 Another proof of the Borel-Weil theorem

In this section we give another proof of the Borel-Weil theorem. The proof of the Borel-Weil-Bott theorem will be a generalization of this one. We recall the Peter-Weyl theorem.

Theorem 9.2 (Peter-Weyl). *Let G be a complex reductive algebraic group, then*

$$\mathbb{C}[G] \cong \bigoplus_{U \in \hat{G}} U \otimes U^* \cong \bigoplus_{U \in \hat{G}} End_{\mathbb{C}}(U)^* \,,$$

where $\hat{G}$ is the collection of all the class of isomorphism of the finite dimensional representation of G.

Corollary 9.3. *Let G be complex reductive algebraic group simply connected, then*

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \in \Lambda^+} L(\lambda) \otimes L(\lambda)^* \cong \bigoplus_{\lambda \in \Lambda^+} End_{\mathbb{C}}(L(\lambda))^* \,,$$

where Λ^+ is the set dominant weights of the Lie algebra of G and $L(\lambda)$ is the irreducible representation associated to λ.

This theorem is classically stated for the compact topological group, for a complete discussion about these objects see [Hoc68]. The link between the Peter-Weyl theorem for Lie groups and for algebraic groups gives the Tannaka-Krein duality. For a complete discussion of this deep link see [Pro07].

When we study the variety G/B we can suppose that G is simply connected; otherwise let $\hat{G}$ be its universal cover, G is the quotient of $\hat{G}$ for a subgroup of the center, since the center is contained in every Borel subgroup we have that G/B is isomorphic to $\hat{G}/\hat{B}$.

As we have seen at the beginning of the section 7.2 the space of global section of L_χ is isomorphic to $\mathbb{C}[G] \otimes_B \mathbb{C}_\chi$; this second space is the space of B invariant elements of $\mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}_\chi$ so we have that

$$\mathrm{H}^0(G/B, L_\chi) \cong \mathrm{H}^0(B, \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}_\chi) \,.$$

Using the Peter-Weyl theorem we get

$$\mathrm{H}^0(G/B, L_\chi) \cong \bigoplus_{\lambda \in \Lambda^+} \mathrm{H}^0(B, L(\lambda)^* \otimes_{\mathbb{C}} L(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_\chi) \cong \bigoplus_{\lambda \in \Lambda^+} L(\lambda)^* \otimes_{\mathbb{C}} \mathrm{H}^0(B, L(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_\chi) \,;$$

the last isomorphism due to the fact that when B acts on $L(\lambda)^* \otimes_{\mathbb{C}} L(\lambda)$ it acts only on $L(\lambda)$.

The Borel-Weyl theorem states that

Theorem 9.4 (Borel-Weyl). *The group*

$$H^0(G/B, \mathbb{C}_\chi)$$

is isomorphic as G module to $L(\chi)^*$ if χ is dominant, it is trivial otherwise.

Since we have proved that

$$H^0(G/B, \mathbb{C}_\chi) \cong \bigoplus_{\lambda \in \Lambda^+} L(\lambda)^* \otimes_{\mathbb{C}} H^0(B, L(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_\chi)$$

we conclude that the the following result is equivalent to the Borel-Weil theorem:

Theorem 9.5 (Borel-Weil). *The G module $H^0(B, L(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_\chi)$ is isomorphic to the trivial module $\mathbb{C}$ if $\lambda = \chi$ and is zero dimensional otherwise.*

Proof. Since $B \cong T \ltimes U$ to look for the B invariant elements of a module is equal to look first for the U invariant elements and then for the T invariant elements among the U invariants, so we have that:

$$\mathrm{H}^0(B, L(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_\chi) \cong \mathrm{H}^0(U, L(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_\chi)^T .$$

Since U acts trivially on $\mathbb{C}_\chi$ we have that $\mathrm{H}^0(U, L(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_\chi)$ is isomorphic as T module to $\mathrm{H}^0(U, L(\lambda)) \otimes_{\mathbb{C}} \mathbb{C}_\chi$. Now we use a standard fact of representation theory:

$$\mathrm{H}^0(U, L(\lambda)) \cong L(\lambda)_\lambda \cong \mathbb{C}_\lambda .$$

The space $\mathbb{C}_\lambda \otimes_{\mathbb{C}} \mathbb{C}_\chi$ is one dimensional; it is H invariant if and only if $\lambda = \chi$ so we obtain the statement.

(Recall that the action of B on $\mathbb{C}_\chi$ is by definition the right one so $t.v = \mu(t)^{-1}v$.) □

We remark that this proof can be formulate in terms of Lie algebras, in fact since we are in characteristic zero we have that

$$\mathrm{H}^0(U, M) \cong \mathrm{H}^0(\mathfrak{u}, M) \qquad \text{and} \qquad \mathrm{H}^0(T, M) \cong \mathrm{H}^0(\mathfrak{t}, M)$$

for every module M; we will use this reformulation to apply the Kostant formula.

9.2 How to derive the composition of two functors

To prove the Borel-Weil-Bott theorem we will need to derive the composition of two functors several times so we recall quickly the general theory.

Let $\mathcal{F}$ be a covariant left exact functor. The right derived functor R^iF is defined in the following way: let $I^\bullet$ an injective resolution of M, then $R^iF(M)$ is the i-th homology group of $\mathcal{F}(I^\bullet)$. One proves that the derived functor is well-defined. There is another way to compute the derived functors, first we give a definition:

Definition 9.6 ($\mathcal{F}$-acyclic object and resolution). *An object M is $\mathcal{F}$-acyclic if $R^i\mathcal{F}(M) \cong 0$ for every $i > 0$*

Let $I^\bullet$ be a resolution of M, we say that $I^\bullet$ is $\mathcal{F}$-acyclic if I^k is an $\mathcal{F}$-acyclic object for every k.

We have that:

Theorem 9.7. *Let $I^\bullet$ be an $\mathcal{F}$-acyclic resolution of M, then $R^i\mathcal{F}(M)$ is isomorphic to the i-th homology group of $\mathcal{F}(I^\bullet)$.*

For a proof see [Voi02] page 101.

Now let $\mathcal{F}$ and $\mathcal{G}$ be two covariant left exact functors, we want to relate $R^i(\mathcal{F}\circ\mathcal{G})$ with the derived functors of $\mathcal{F}$ and $\mathcal{G}$. This task may be difficult, but we are interested just in two simple cases.

Lemma 9.8 (First case)**.** *Suppose that $\mathcal{G}$ is exact and for every object M there exist resolution $N^\bullet$ such that $\mathcal{G}(N^k)$ is an $\mathcal{F}$-acyclic object, then*

$$R^i(\mathcal{F}\circ\mathcal{G}) \cong R^i(\mathcal{F})\circ\mathcal{G}\,.$$

Proof. Since $\mathcal{G}$ is exact any resolution of M is $\mathcal{G}$-acyclic. Always because of the exactness of $\mathcal{G}$ we have that $\mathcal{G}(N^\bullet)$ is a resolution for $\mathcal{G}(M)$. For every k the object N^k is acyclic also for $\mathcal{F}\circ\mathcal{G}$ because N^k is acyclic for $\mathcal{G}$ and $\mathcal{G}(N^k)$ is acyclic for $\mathcal{F}$. Since $\mathcal{G}(N^\bullet)$ is an $\mathcal{F}$-acyclic resolution of $\mathcal{G}(M)$ we get the statement. □

Lemma 9.9 (Second case)**.** *Suppose that $\mathcal{F}$ is exact, then*

$$R^i(\mathcal{F}\circ\mathcal{G}) \cong \mathcal{F}\circ R^i(\mathcal{G})\,.$$

Proof. For every complex $I^\bullet$ the homology of $\mathcal{F}(I^\bullet)$ is isomorphic to $\mathcal{F}$ applied to the homology of $I^\bullet$ because $\mathcal{F}$ is exact. The statement follows. □

For further details and complete proofs see [Voi02] chapter I.4 .

9.3 The link between the cohomology of a Borel subgroup and the cohomology of the Lie algebra

We compare the cohomology of B to the cohomology of the Lie algebra $\mathfrak{u}$.

We consider the left action of U on itself, this gives rise to a rational action of U on $\mathbb{C}[U]$; we give to $\mathbb{C}[U]$ the structure of $\mathfrak{u}$ module differentiating this action.

Proposition 9.10. *The $\mathfrak{u}$ module $\mathbb{C}[U]$ is acyclic for the functur of the $\mathfrak{u}$ invariants.*

Proof. We must prove that

$$\mathrm{H}^i(\mathfrak{u},\mathbb{C}[U]) = 0$$

for every $i \geq 0$. We compute these groups using the Chevalley-Eilenberg complex. The group $\mathrm{H}^i(\mathfrak{u},\mathbb{C}[U])$ is the i-th homology group of the complex $\mathrm{Hom}_{\mathbb{C}}(\bigwedge^*\mathfrak{u},\mathbb{C}[U])$, and this complex is ismorphic to $\bigwedge^*\mathfrak{u}^*\otimes_{\mathbb{C}}\mathbb{C}[U]$. As variety U is ismorphic to $\mathbb{C}^n$ so its tangent bundle is trivial, namely it is isomorphic to $U\times\mathfrak{u}$. The algebraic De Rham complex of U is $\bigwedge^*\mathfrak{u}^*\otimes_{\mathbb{C}}\mathbb{C}[U]$ and, because of the Chevalley-Eilenberg theorem, the differential is the same of the differential of the Chevalley-Eilenberg complex; we conclude that

$$\mathrm{H}^i(\mathfrak{u},\mathbb{C}[U]) \cong \mathrm{H}^i(U,\Omega_U)\,.$$

The algebraic De Rham cohomology of $\mathbb{C}^n$ is trivial so we have proved the claim. □

We want to study the structure of B module of $\mathbb{C}[B]$ that arises from the left action of B on itself. We recall that B is isomorphic to $U \rtimes T$. Let (u,t) and (x,y) be elements of $U \rtimes T$, we remark that

$$(u,t).(x,y) = (utxt^{-1}, ty)\,,$$

so it is natural to define the following left action of B on U

$$(u,h).x = uhxh^{-1}\,.$$

If we consider U as a quotient of B for the left action of H this just the left action of B on B composed with the projection to the quotient. Moreover if we restrict the action of B to U we obtain the left action. We call π the projection from B to U and ι the immersion of U in B, these maps are regular. The map ι is a morphism of group but is not B equivariant, instead π is not a morphism of group but it is B equivariant.

We consider the structure of B module on $\mathbb{C}[U]$ given by the action defined above, we have the following proposition:

Proposition 9.11. *The module $\mathbb{C}[B]$ as left B module is isomorphic to*

$$\bigoplus_{\lambda \in X(T)} \mathbb{C}[U] \otimes_{\mathbb{C}} \mathbb{C}_\lambda\,,$$

where $X(T)$ is the group of characters of T.

Proof. First we consider the right action of T on B: $b.t := bt^{-1}$. (We write the right action on the right because there is a danger of confusion with the left action.) This gives rise to a rational action of T on $\mathbb{C}[B]$, namely $(f.t)(b) := f(bt)$. Being this action rational every vector is contained in finite dimensional submodule of $\mathbb{C}[B]$, this implies that $\mathbb{C}[B]$ can be decomposed in weight subspaces, namely:

$$\mathbb{C}[B] = \bigoplus_{\lambda \in X(T)} \mathbb{C}[B]_\lambda\,.$$

where

$$\mathbb{C}[B]_\lambda = \{f \in \mathbb{C}[B] | f.t = \lambda(t)f \quad \forall t\}\,.$$

The $\mathbb{C}[B]_\lambda$ are submodule for the left action of B because left and right actions commute. We want to prove that $\mathbb{C}[B]_\lambda$ is isomorphic to $\mathbb{C}[U] \otimes \mathbb{C}_\lambda$.

We define a map Φ from $\mathbb{C}[B]_\lambda$ to $\mathbb{C}[U] \otimes \mathbb{C}_\lambda$, we pose $\Phi(f) := \iota^* f \otimes 1$, we call Ψ its inverse, namely $\Psi(f \otimes 1) := \lambda \pi^* f$. (Remark that $\iota^*\lambda = 1$). We have to check that these maps are morphisms of B modules. For every t write $b = ut$ and call (x,y) another element of $U \rtimes T$. For Φ we have that

$$\Phi(b.f) = \iota^*(b.f) \otimes 1$$

and

$$\iota^*(b.f)(x) = b.f(x,1) = f(t^{-1}u^{-1}xt, t^{-1}) = \lambda(t)^{-1} f(t^{-1}u^{-1}xt, 1)\,;$$

on the other side

$$b.\Phi(f) = b.(\iota^* f \otimes 1) = (b.\iota^* f)\lambda(t)^{-1} \otimes 1$$

and

$$b.\iota^* f(x) = (\iota^* f)(t^{-1}u^{-1}xt) = f(t^{-1}u^{-1}xt, 1).$$

For Ψ we have that

$$\Psi(b.(f \otimes 1))(x,y) = \Psi(\lambda(t)^{-1}(b.f) \otimes 1) = (\lambda\pi^*(b.f))(x,y)\lambda(t)^{-1} = (b.f)(x)\lambda(y)\lambda(t)^{-1} =$$
$$= f(t^{-1}u^{-1}xt)\lambda(y)\lambda(t)^{-1} = b.(\lambda\pi^* f)(x,y) = \Psi(f \otimes 1)(x,y)\,.$$

$\square$

We remark that $\mathbb{C}[U] \otimes \mathbb{C}_\lambda$ is isomorphic to $\mathbb{C}[U]$ as U module, where the action of U is just the left one.

Proposition 9.12. *The B module $\mathbb{C}[U] \otimes_\mathbb{C} \mathbb{C}_\lambda$ are irreducible.*

Proof. We recall that in each finite dimensional irreducible B module the space of U invariant has dimension one. Given an infinite dimensional rational representation M each vector is contained in a finite dimensional irreducible submodule, hence if M is reducible the space of U invariant has dimension bigger than one then.

The U invariant of $\mathbb{C}[U]$ for the left action are the constant functions, hence is a one dimensional submodule. Since $\mathbb{C}[U] \otimes_\mathbb{C} \mathbb{C}_\lambda$ is isomorphic to $\mathbb{C}[U]$ as U module we get the statement. □

We have now a very careful description of $\mathbb{C}[B]$, first we put in evidence the following fact.

Corollary 9.13. *The module $\mathbb{C}[B]$ is acyclic for the functor of the $\mathfrak{u}$ invariant.*

Proof. A direct sum, also infinite, commute with the functor of the cohomology. □

One can go further in the analysis of $\mathbb{C}[B]$, one can prove that for every λ the module $\mathbb{C}[U] \otimes \mathbb{C}_\lambda$ is injective as $\mathfrak{u}$ module and it is the dual of the Verma modules of weight λ, so $\mathbb{C}[B]$ contains all the Verma modules.

Remark 9.14. *Almost all these results can be obtained using the functors of induction and restriction. See chapter I.3 of [Jan03].*

Now we need a key lemma, true for every kind of algebraic group.

Lemma 9.15. *Let G be an algebraic group and M a G module, then there exist an immersion of M inside $M_0 \otimes_\mathbb{C} \mathbb{C}[G]$, where M_0 is a trivial G module isomorphic to M as complex vector space.*

Proof. We first recall that $M \otimes_\mathbb{C} \mathbb{C}[G]$ is isomorphic to the space of regular function from G to M such that the image is contained in a finite dimensional subspace. The action of G gives an immersion

$$\iota : M \to M \otimes_\mathbb{C} \mathbb{C}[G]$$

$$v \mapsto (g \mapsto gv) .$$

Using the theory of Hopf algebras one proves that the G module $M \otimes_\mathbb{C} \mathbb{C}[B]$ is isomorphic to $M_0 \otimes_\mathbb{C} \mathbb{C}[G]$, where M_0 is isomorphic to M as a vector space but the action of G is trivial. See [Jan03] page 41. □

We can finally link the cohomology of B to the cohomology of $\mathfrak{u}$.

Proposition 9.16. *For every B module M and every integer p we have that*

$$H^p(B, M) \cong H^p(\mathfrak{u}, M)^T .$$

Proof. Let $\mathcal{F}$ be the functor that transform an U module in a $\mathfrak{u}$ module: the vector space $\mathcal{F}(M)$ is isomorphic to M, the maps do not change, but $\mathcal{F}$ gives to M the structure of $\mathfrak{u}$ module differentiating the action of U. Clearly M and $\mathcal{F}(M)$ are isomorphic as T module. Moreover $\mathcal{F}$ is an exact functor. In the statement there is a small abuse of notation, we should write

$$H^p(B, M) \cong H^p(\mathfrak{u}, \mathcal{F}(M))^T .$$

The functor $H^p(B,-)$ is the derived functor of $H^0(B,-)$. Since we are working in characteristic zero and U is normal in B, the functor $H^0(B,-)$ is isomorphic to the composition of $\mathcal{F}$, $H^0(\mathfrak{u},-)$ and $H^0(T,-)$. We have to prove that the derivation of this last composition is $H^p(\mathfrak{u},-)^T$.

First we remark that, being T composed only of semisimple elements, the functor of the T invariant is exact so, because of lemma 9.9, we have that

$$R^i[H^0(T,H^0(\mathfrak{u},\mathcal{F}(\,-\,)))] = H^0(T,R^i[H^0(\mathfrak{u},\mathcal{F}(\,-\,))]) .$$

Now we must prove that

$$R^iH^0(\mathfrak{u},\mathcal{F}(-)) = H^i(\mathfrak{u},\mathcal{F}(-)) .$$

The functor $\mathcal{F}$ is exact, so we are in the situation of the lemma 9.8. For any B module M we can cook up a resolution using the lemma 9.15. We know $\mathbb{C}[B]$ is acyclic for the $\mathfrak{u}$ invariants (actually we should say $\mathcal{F}(\mathbb{C}[B])$ is acyclic). The tensorization for a trivial module (also of infinite dimension) commute with the cohomology, thus $M_0 \otimes_{\mathbb{C}} \mathbb{C}[B]$ is acyclic for the functor of the $\mathfrak{u}$ invariants. Applying the lemma 9.8 we obtain the requested result. □

Remark 9.17. *Since the characteristic is zero the* $\mathfrak{t}$ *invariants are the same of the* T *invariants.*

9.4 The link between the cohomology of the Borel subgroup and the cohomology of the line bundles over the flag variety

Now we link the cohomology of B to the cohomology of the line bundles over G/B. We need to treat this argument in a more general framework: we will study G-equivariant vectors bundles (also of infinite dimension) on G/B.

First we define a functor $\mathcal{L}$ from the category of B module to the category of G equivariant vector bundles on G/B. For any B module M we pose

$$\mathcal{L}(M) := G \times_B M .$$

This functor is a generalization of the line bundles $L_\chi \cong \mathcal{L}(\mathbb{C}_\chi)$. This construction has been discussed in the section 7.1 proposition 7.1 or see [Spr98] page 95. The structure of G equivariant vector bundle is given by

$$g.[(x,v)] := [(gx,v)] .$$

If f is a morphism between two B module M and N we define

$$\mathcal{L}(f)([g,m]) := [g,f(m)] ,$$

the definition is well posed because f is a morphism of B module.. We need a closer look to G equivariant vector bundles. The fiber on the point $p := B/B$ of a G equivariant vector bundle is a B module because B is the isotropy group of p.

Proposition 9.18. *Let (E,π_E) and (F,π_F) be two G equivariant vector bundles on G/B and f a morphism of B modules from E_p to F_p, then f extends to a unique morphism of G-equivariant vector bundles between E and F.*

Proof. Let x be a point of E. Take an element g of G such that $gp = \pi_E(x)$. We define

$$f(x) := gf(g^{-1}x) ;$$

Since f is a morphism of B module the definition does not depend from the choice of g. On an affine neighborhood U of p the morphism is just

$$(Id, f) : U \times E_p \to U \times F_p \,.$$

The uniques is due to the fact that the morphisms between G-vector bundles have to be G equivariant, and G acts acts transitively on G/B. $\square$

Corollary 9.19. *Every G-equivariant vector bundles E on G/B is isomorphic to $\mathcal{L}(E_p)$.*

Corollary 9.20. *The functor $\mathcal{L}$ is an equivalence between the category of the B module and the category of the vector bundles over G/B; in particular it is an exact functor.*

We want now to prove that $\mathcal{L}$ transforms injective objects into acyclic object for the functor of global sections to apply the lemma 9.8; in order to do this we need to study the cohomology of the bundle $\mathcal{L}(\mathbb{C}[B])$. Call π la projection from G to G/B. We have that

Proposition 9.21. *The bundle $\mathcal{L}(\mathbb{C}[B])$ is isomorphic to $\pi_*(\mathcal{O}_G)$.*

Proof. Call j the inclusion of B in G and i the inclusion of p in G/B. Clearly $p = \pi(B)$. One has that

$$\pi_*(\mathcal{O}_G)_p = i^*\pi_*(\mathcal{O}_G) = \pi_* j^*(\mathcal{O}_G) = \pi_*(\mathcal{O}_B) = \mathcal{O}_B(B) = \mathbb{C}[B] \,.$$

We can now apply the corollary 9.19. $\square$

Proposition 9.22. *The cohomology groups*

$$H^p(G/B, \mathcal{L}(\mathbb{C}[B]))$$

are all trivial for p bigger than zero.

Proof. We compute the cohomology groups of $\pi_*\mathcal{O}_G$.

First we derive the functor $H^0(G/B, \pi_*(-))$, i.e. the composition of π_* and the functor of global section on G/B, using the lemma 9.8.

We prove that π_* is exact. Given any exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

we must apply the functor π_* and check that the new sequence is exact on every stalk. The map π is a fibration with affine fiber isomorphic to B, so given a sheaf $\mathcal{F}$ on G the stalk of $\pi_* F$ is isomorphic to $\mathcal{F}(B)$. Finally on each stalk we get the exact sequence:

$$0 \to \mathcal{F}(B) \to \mathcal{G}(B) \to \mathcal{H}(B) \to H^1(B, \mathcal{F}) \,.$$

We recall the Serre theorem: the cohomology of a quasi-coherent sheaf over an affine variety is always trivial in positive degree, for a proof see [Har77] page 215. In particular we obtain that $H^1(B, \mathcal{F})$ is zero because B is affine.

Since π_* is the right adjoint functor of π^*, then π_* transforms injective objects into injective objects.

We are now in the situation of lemma 9.8 so we get that:

$$R^p H^0(G/B, \pi_*(-)) \cong H^p(G/B, \pi_*(-)) \,.$$

for every p.

By definition of the direct image we have that

$$H^0(G/B, \pi_*(-)) = H^0(G, -)\,,$$

the derivation of the RHS is by definition $H^p(G, -)$, so for every p we have that

$$H^p(G/B, \pi_*(-)) = H^p(G, -)\,.$$

Being G affine $H^*(G, \mathcal{O}_G)$ is trivial in positive degree so we get the statement.

For further details about the theory of coherent and quasi coherent sheaves see [Har77]. □

Now we need to discuss the functor $\mathcal{F}$ defined as follow: given a B module M

$$\mathcal{F}(M) := M \otimes_{\mathbb{C}} \mathbb{C}[G]\,.$$

Proposition 9.23. *The functor $\mathcal{F}$ defined above is exact and the image of an injective object is still injective.*

Proof. The functor $\mathcal{F}$ is exact because we are tensorizing on $\mathbb{C}$.

Because of the Peter-Weyl theorem $\mathbb{C}[G]$ is the direct sum of finite dimensional G module, call these modules E_i (we are only interested in the fact that they are finite dimensional).

We prove the following fact: given an injective B module I and a finite dimensional B module M, the module $M \otimes_{\mathbb{C}} I$ is still injective. Since M is finite dimensional the vector space $Hom_B(M, V)$ has still a structure of rational B representation for every G module V, also of infinite dimension. To give an injection from $I \otimes_{\mathbb{C}} M$ to V is equivalent to give an injection from I to $Hom_B(M, V)$. Moreover the first injection admits a right inverse if and only if the second admits a right inverse. Since I is injective as B module also $I \otimes_{\mathbb{C}} M$ is.

The sum of countably many injective modules is still injective, so if I as injective B module also $\mathcal{F}(I)$ is.

□

We can finally link the cohomology of the vector bundles with the cohomology of B.

Proposition 9.24. *Given a B module M we have that*

$$H^p(B, \mathbb{C}[G] \otimes_{\mathbb{C}} M) \cong H^p(G/B, \mathcal{L}(M))$$

for every p.

Proof. The functors $H^0(B, \mathbb{C}[G] \otimes_{\mathbb{C}} -)$ and $H^0(G/B, \mathcal{L}(-))$ are isomorphic because of the discussion at the beginning of the section 7.2, so their derived functors are isomorphic.

Because of the previous proposition the derived functor of $H^0(B, \mathbb{C}[G] \otimes_{\mathbb{C}} -)$ is $H^p(B, \mathbb{C}[G] \otimes_{\mathbb{C}} -)$.

Given a B module M using the lemma 9.15 we can cook up a resolution $N^\bullet$ made of module of the form $V \otimes \mathbb{C}[B]$, where V is a trivial module. For every k the object $\mathcal{L}(N^k)$ is acyclic for the functor of global sections because both $\mathcal{L}$ and the cohomology commute with the infinite direct sums. Since $\mathcal{L}$ is exact applying the lemma 9.8 we conclude that

$$R^p[H^0(G/B, \mathcal{L}(-))] = H^p(G/B, \mathcal{L}(-))\,.$$

□

9.5 The proof of the Borel-Weil-Bott theorem

The proof of the theorem is now quite easy. Combining the isomorphisms 9.16 and the proposition 9.24 we get that

$$H^i(G/B, L_\chi) \cong H^i(\mathfrak{u}, \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}_\chi)^T .$$

Now we apply Peter-Weyl and we get

$$H^i(\mathfrak{u}, \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}_\chi)^T \cong \bigoplus_{\lambda \in \Lambda} L(\lambda)^* \otimes_{\mathbb{C}} H^i(\mathfrak{u}, L(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_\chi)^T .$$

because B acts trivially on $L(\lambda)^*$. Since $\mathfrak{u}$ acts trivially on $\mathbb{C}_\mu$ we have that as T module

$$H^i(\mathfrak{u}, L(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_\mu) \cong H^i(\mathfrak{u}, L(\lambda)) \otimes_{\mathbb{C}} \mathbb{C}_\mu .$$

Moreover we have the isomorphism

$$(H^i(\mathfrak{u}, L(\lambda)) \otimes_{\mathbb{C}} \mathbb{C}_\mu)^T \cong H^i(\mathfrak{u}, L(\lambda))_\mu .$$

We can now use the Kostant formula to describe $H^i(\mathfrak{u}, L(\lambda))_\mu$. Suppose that μ regular and let w be an element of the Weyl group such that $w \cdot \mu$ is dominant. We have that $(H^i(\mathfrak{u}, L(\lambda)) \otimes_{\mathbb{C}} \mathbb{C}_\mu)^T$ is trivial either if $\lambda \neq w \cdot \mu$ or $i \neq l(w)$. Putting everything together we get the main theorem.

Theorem 9.25 (Borel-Weil-Bott). *If λ is a regular weight let w be an element of the Weyl group such that $w \cdot \lambda$ is dominant, then the cohomology group*

$$H^i(G/B, L_\lambda)$$

vanishes if i is different from $l(w)$ and it is isomorphic as G module to $L(\lambda)^*$ if i is equal to $l(w)$. If λ is a singular weight then the cohomology

$$H^*(G/B, L(\lambda))$$

is zero.

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