Exercises in Differential and Riemannian Geometry

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These are three problem sheets proposed by M. Dafermos during the course in Differential and Riemannian geometry that he gave during the year 2012-13 at the University of Cambridge. Here, we collect some solutions. We thank Mihalis for giving us the opportunity to teach the example classes, and the students who patiently worked out the exercises with us.

1. Given an example of a smooth manifold which is not Hausdorff. Give an example which is not paracompact. Give an example which is paracompact but not second countable. From now on, manifolds are assumed Hausdorff and paracompact.

2. Let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifolds. Give the formal details of the definition of what it means for a continuous map $f : \mathcal{M} \to \mathcal{N}$ to be smooth at a point p. In particular, address the issue of exactly on what neighbourhoods of $\phi_\alpha(p)$ can the expressions $\tilde{\phi}_\alpha \circ f \circ \phi_\alpha^{-1}$ be defined, and what does it mean for $\tilde{\phi}_\alpha \circ f \circ \phi_\alpha^{-1}$ to be smooth "at" $\phi_\alpha(p)$.

3. Exhibit two complete atlases $\{(\phi_\alpha, \mathcal{U}_\alpha)\}$ and $\{(\tilde{\phi}_\alpha, \tilde{\mathcal{U}}_\alpha)\}$ on the same underlying topological space $\mathbb{R}^n$ such that the identity map

$$\mathrm{id} : (\mathbb{R}^n, \{(\phi_\alpha, \mathcal{U}_\alpha)\}) \to (\mathbb{R}^n, \{(\tilde{\phi}_\alpha, \tilde{\mathcal{U}}_\alpha)\})$$

is *not* a smooth map of manifolds. Show that the two manifolds are nonetheless diffeomorphic for your example, or prove otherwise.

4. Let $F : \mathcal{M} \to \mathcal{N}$ be smooth, let $f \in C^\infty(\mathcal{N})$. Let $F^*(f)$ denote $f \circ F$. Show that $F^*(df) = d(F^*(f))$. Now let $G : \mathcal{N} \to \mathcal{N}'$, and let $x \in \mathcal{M}$. Show that $((G \circ F)^*)_p = (G_*)_{F(p)} \circ (F_*)_p$. Let $\omega \in \Gamma(T^*\mathcal{N}')$ be a 1-form. Show that $(G \circ F)^*\omega = F^*(G^*(\omega))$.

5. Multilinear algebra. Show that the identification of $u^* \otimes v$ with the map sending $u \mapsto u^*(u)v$ extends by linearity an isomorphism $U^* \otimes V \cong \mathrm{Hom}(U, V)$. Show that the identification of $u^* \otimes v^*$ with the map sending $u \otimes v \mapsto u^*(u)v^*(v)$ extends to an isomorphism $U^* \otimes V^* \cong (U \otimes V)^*$. Show that these isomorphisms lead to a natural identification

$$\mathrm{End}(U) \cong (U^* \otimes U)^* \cong (\mathrm{End}(U))^*$$

where $\mathrm{End}(U)$ denotes $\mathrm{Hom}(U, U)$. Show that the image of $\mathrm{id} \in \mathrm{End}(U)$ in $(\mathrm{U}^* \otimes \mathrm{U})^*$ under this isomorphism, is the map $C : U^* \otimes U \to \mathbb{R}$ which takes $u^* \otimes u \mapsto u^*(u)$. Show that the image of $\mathrm{id} \in \mathrm{End}(U)$ in $(\mathrm{End}(U))^*$ is the map taking $L \in \mathrm{End}(U)$ to $\mathrm{tr}L$.

6. Let $\mathcal{M}$ be smooth of dimension m, and let $f_1, \ldots, f_d$ be a collection of smooth functions on $\mathcal{M}$. Let $\mathcal{N}$ denote the set where $f_1 = \cdots = f_d = 0$. Suppose $(df_i)_p$ span a subset of dimension d' in $T_p^*(\mathcal{M})$ for all $p \in \mathcal{N}$. (We assume d' to be constant, but d' need not equal d.) Show that $\mathcal{N}$ can be given the structure of a closed submanifold of M of dimension $m - d'$. The above applies to $\mathbb{S}^n$, where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is given by $f(x_1, ..., x_{n+1}) = -1 + \sqrt{\sum_{i=1}^{n+1} x_i^2}$. Show that the manifold structure defined above is the same as the structure defined via the projection maps to the coordinate hyperplanes.

7. Assuming the classical existence, uniqueness, and continuous dependence on parameters theorem for o.d.e.s, prove the following version on manifolds: Let $\mathcal{M}$ be a smooth manifold, and let $X \in \Gamma(TM)$ be a smooth vector field. Then for each $x \in \mathcal{M}$, there exists a unique maximal smooth curve $\gamma : (T_-, T_+) \to \mathcal{M}$, with $-\infty \leq T_- < 0 < T_+ \leq \infty$,such that $\gamma(0) = x$, and $\gamma'(t) = X$ for all $t \in (T_-, T_+)$, where $\gamma'(t)$ denotes $(\gamma_*)_t \left(\frac{\partial}{\partial t}\right)$. Moreover, if $T_+ < \infty$, then for every compact $K \subset \mathcal{M}$, there exits a $t_K < T_+$ such that $x[t_K, T_+) \cap K = \emptyset$. To remember the dependence on x, let us denote γ by γ_x, and T_+, T_- by $T_+(x), T_-(x)$. Finally, for every $x \in \mathcal{M}$, there exists an open subset $\mathcal{U}_x$ and an $\epsilon > 0$, such that $(T_-(\tilde{x}), T_+(\tilde{x})) \supset (-\epsilon, \epsilon)$ for all $\tilde{x} \in \mathcal{U}_x$, and such that the map $\phi : U \times (-\epsilon, \epsilon) \to \mathcal{M}$ defined by $\phi(\tilde{x}, t) = \gamma_{\tilde{x}}(t)$ is smooth.

8. Let G be a group, and $\mathcal{M}$ a manifold. Let $\mathrm{Diff}(\mathcal{M})$ denote the set of all smooth transformations of $\mathcal{M}$. (Recall that a smooth transformation is a diffeomorphism from $\mathcal{M}$ to itself.) Show that $\mathrm{Diff}(\mathcal{M})$ defines a group with composition as multiplication. Suppose there exists a group homomorphism $R : G \to \mathrm{Diff}(\mathcal{M})$. Given $x \in \mathcal{M}$, we define the isotropy group of x as the set of $g \in G$ such that $R(g)x = x$. Show that this defines a subgroup of G, denoted G_x. We say that R is properly discontinuous if the following are true:

1. For all $x, \tilde{x} \in \mathcal{M}$ such that there does not exist a g with $R(g)(x) = \tilde{x}$, there exist neighbourhoods $x \in \mathcal{U}$, $\tilde{x} \in \tilde{\mathcal{U}}$, with $R(g)(U) \cap \tilde{U} = \emptyset$, for all g.
2. G_x is finite for all $x \in \mathcal{M}$.
3. For all $x \in \mathcal{M}$, there exists a neighborhood $x \in \mathcal{U}$ such that $R(h)(U) \subset U$ for all $h \in G_x$, and $U \cap R(g)(U) = \emptyset$ for $g \notin G_x$.

Show the following: If $R : G \to \mathrm{Diff}(M)$ is properly discontinuous and injective, then the quotient space $\mathcal{M}/G$ (defined by the equivalence relation $x \sim \tilde{x}$ if there exists a $g \in G$ such that $R(g)x = \tilde{x}$) inherits the structure of a smooth manifold such that the quotient map $\pi : \mathcal{M} \to \mathcal{M}/G$ is smooth.

9. Take a thin strip of paper and attach the short ends to each other with the opposite orientation. Now exhibit this construction as a rank 1 vector bundle $E \to \mathbb{S}^1$, i.e. a vector bundle whose fibres have dimension 1. We call this the Möbius strip. We say that two vector bundles E', and E are equivalent if there exists a smooth $\phi : E \to E'$ such that $\phi(E_x) = E'_x$ and $\phi|_{E_x}$ is a linear isomorphism of E_x with $E_{x'}$. Show that the previous vector bundle is not equivalent to $\mathbb{S}^1 \times \mathbb{R}^1$.

10. Show that a Riemannian metric defines an equivalence, in the above sense, between the bundles $T^*\mathcal{M}$ and $T\mathcal{M}$. What about the converse?

11. Let $F : \mathcal{N}^n \to \mathcal{M}^m$ denote an immersion. Show that there exists a vector bundle E of rank $m - n$ over $\mathcal{N}$, and a smooth map $\tilde{F} : E \oplus T\mathcal{N} \to T\mathcal{M}$, of the form $\tilde{F} : e_p \oplus v_p \mapsto L_p(e_p) + (F_*)_p(v_p)$, where L_p maps linearly $E_p \to T_{F(p)}\mathcal{M}$, and such that $\tilde{F}$ is an isomorphism when restricted to the fibres. Show that this defines E uniquely up to equivalence. We call E the normal bundle of $\mathcal{N}$ defined by F . Show that in the case of Ex. 6, if $d = d'$, then the normal bundle of $\mathcal{N}$ in $\mathcal{M}$ is trivial.

12. Is $T\mathbb{S}^2$ equivalent to $\mathbb{S}^2 \times \mathbb{R}^2$? Is it diffeomorphic as a manifold?

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Hints and (sketches of) Solutions - Example sheet 1

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Linear Algebra review Given a vector space V, there are not, in general, canonical isomorphisms between V and its dual $V^\vee$. An isomorphism between a vector space and its dual is equivalent to a non-degenerate bilinear form on the space. On the other hand, there exists a canonical isomorphism between V and its bi-dual $V^{\vee\vee}$: the one sending a vector v to the map $(\eta \mapsto \eta(v))$.

Topology review A topological space is second countable if the topology has a countable basis, it is paracompact if every open cover has a locally finite refinement. The upshot is that paracompact spaces are arbitrary disjoint unions of second countable spaces.

Exercise 1 This is a list of pathological manifolds.

- Paracompact, 2nd countable, but not Hausdorff : a line with two origins.
- Hausdorff, paracompact, but not second countable: uncountable disjoint union of lines.
- Hausdorff, not paracompact, not second countable: Prüfer surface (see Wiki or Spivak appendix A) and the Long Line (see Wiki or Kobayashi and Nomizu page 166)

From now on, all the manifold are assumed to be Hausdorff, paracompact and second countable.

Let us remark that $\mathbb{R}^m$ is locally compact, so every manifold is locally compact.

Exercise 3 The idea is the following. Take a map F from $\mathbb{R}^n$ to $\mathbb{R}^n$ which is an isomorphism of topological spaces but it is not smooth (e.g. if $n = 1$ you can take x^3). Then take on the first copy of $\mathbb{R}^n$ an atlas and on the second the same atlas twisted by F. With this choice the identity is not a smooth map, but F is an isomorphism between the two manifolds.

Exercise 4 This exercise shows that the pull-back is functorial (i.e. $(G \circ F)^* = F^*G^*$). The situation is the following

$$M \xrightarrow{F} N \xrightarrow{G} N' \xrightarrow{f} \mathbb{R}$$

We can pull-back the function f, and we have

$$(G \circ F)^* f = F^* G^* f\,.$$

To prove the functoriality of the pull-back of one forms there are two possibilities. A first proof is a computation in local co-ordinates. Let us first prove that $F^*df = dF^*f$. Call $x = (x_1, \cdots, x_N)$ and $y = (y_1 \cdots, y_n)$ the co-ordinates.

$$d(F^*f) = \sum_{i,k} (\partial_{x_i} f)(\partial_{y_k} F_i) dy_k$$

$$F^*(df) = \sum_i \partial_{x_i} f dF_i = \sum_{i,k} (\partial_{x_i} f)(\partial_{y_k} F_i) dy_k$$

The other assertions follows from the previous computations working locally.

There second strategy is more intrinsic. A tangent vector v is a derivation, its push-forward is defined as $(F_*v)(f) := v(F^*f)$. The functoriality of the push-forward follows from the functoriality of the pull back of functions. A 1 form ω is a linear form on the tangent vectors, its pull-back is defined as $(F^*\omega)(v) = \omega(F_*v)$, so the functoriality of the pull-back for one forms follows from the functoriality of the push-forward.

Exercise 5 The main idea of this exercise is that there exists a canonical bilinear form on $End(U)$: the trace. We need to assume that the vector spaces are finite dimensional, otherwise the statements are false (find a counterexample!).

We show that the map from $U^\vee \otimes V$ to $Hom(U,V)$ is an isomorphism. Injectivity: take $F := \sum \eta_i \otimes v_i$, with η_i linearly independent, pick a vector u such that $\eta_i(u) = \delta_{1i}$, then $F(u) = v_i \neq 0$. To prove surjectivity, we remark that the vector spaces have the same dimension.

We want to describe the image of id under the chain of isomorphisms. Let us fix a basis u_i for U and the corresponding dual basis η_i for $U^\vee$ (i.e. $\eta_i(u_j) = \delta_{ij}$). We have that $id = \sum \eta_i \otimes u_i$ in $U^\vee \otimes U$, then we get $\sum u_i \otimes \eta_i$ inside $U \otimes U^\vee \cong End(U)^\vee$. Now write $L = \sum a_{ij}\eta_i \otimes u_j$, then

$$(\sum u_i \otimes \eta_i)(L) = \sum_{ij} a_{ij}\eta_i(u_j)\eta_j(u_i) = \sum a_{ii} = Tr(L)$$

Exercise 6 We work locally around a point p. First, let us assume $d = d'$. Let us complete $df_1, \dots df_d$ to a basis of $TM_p^\vee$, call the new elements $dg_1, \dots dg_k$. Because of the "Implicit Function Theorem", the vector -valued function $(f_1, \dots, f_d, g_1, \dots g_{m-d})$ define an isomorphism between a neighbourhood of p in M and one of the origin in $\mathbb{R}^m$, so the f_i are local co-ordinates and their zero locus is a submanifold. Equivalentely, consider the function $(f_1 \dots, f_d)$, the "IFT" gives (locally) a parametrisation of the preimage of 0 as function of $x_1, \cdots, x_d$.

Now, let's drop the assumption $d = d'$. Take a point p such that $f_1(p) = \cdots = f_d(p) = 0$. Suppose, without loss of genearality, that $df_1, \dots, df_{d'}$ are linearly independent around p, then they define (locally) a submanifold N. Let γ be an arc in N passing trough p, we need to prove that f_i is constant along γ for every $i > d'$, to get this is enough to express df_i as linear combination of the first d' differentials.

The structure defined via the projection is

$$(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, \sqrt{1 - \sum x_1^2})$$

which is exactly the same parametrisation given by the IFT.

Differential equations review All the statements of exercise 7 are true if you replace the manifold M with a neighbourhood U of the origin in $\mathbb{R}^n$. In this case the vector field X is just a differential operator $\sum f_i \frac{\partial}{\partial x_i}$, where f_i are smooth real-valued functions on U.

Exercise 7 Call S the set of smooth curves solving the differential equation. It is not empty, because, applying the classical existence theorem, we can construct a solution in a chart around p. Given two elements of S, they must agree on the intersection of their domains, because of a local argument. Finally, taking the "union" of all the elements of S one obtains a maximal solution γ.

To prove the second statement one argues by contraddiction. Take a sequence t_n in I_γ converging to T_+ such that $\gamma(t_n)$ converges to an element k of K; working locally around k one proves that the solution γ can be extended.

The last statement follows from its local analogue.

Exercise 8 We endow M/G with the quotient topology. The first hypothesis guarantees that M/G is Hausdorff.

Let us suppose that the stabiliser is always trivial. In this case the map R is a covering (see e.g the Example Sheet 2 Exercise 4 of last year Algebraic Topology Part II, or any book of (Algebraic) Topology). Using the fact that R is a local isomorphism, one can define an atlas on M/G.

If at some points x the stabiliser is not trivial, then it could be that the image of x in M/G is singular. Examples to keep in mind are the following ones

- The real line with the group $\{\pm 1\}$ acting by multiplication. The quotient is a smooth manifold with boundary.
- The complex plane with the group of roots of unity acting by multiplication. The quotient is a cone.
- The symmetric product of two Riemann surfaces, which is smooth.

If you are interested in quotient singularities, have a look to the (classical) McKay correspondence. It could be a good topic for a Part III seminar.

Exercise 9 Take two points a and b on $\mathbb{S}^1$ and consider the open cover formed by $U_1 := \mathbb{S}^1 \setminus a$ and $U_2 := \mathbb{S}^1 \setminus b$. Call V_1 and V_2 the two connected components of $U_1 \cap U_2$. In order to define a line bundle, we need to define the transition function on V_1 and V_2. The Möbius strip is obtained choosing the identity on V_1 and minus the identity on V_2.

Now, suppose by contradiction that the two line bundles are isomorphic. This means that we have a never vanishing section s of the Möbius strip. A section s is the datum of a smooth function f_1 on U_1 and a smooth function f_2 on U_2, such that $f_1 = f_2$ on V_1 and $f_1 = -f_2$ on V_2. We conclude that f_1 must change sign at some point, so it has at least two zeros (the same is true for f_2).

Exercise 10 Let g be the Riemannian metric. The map from TM to $TM^\vee$ is defined as follows

$$(x, v) \mapsto (x, (w \mapsto g(v, w)))\,.$$

Its inverse is defined as

$$(x, \eta) \mapsto (x, v)\,,$$

where v is the unique vector of T_xM such that $g(v, w) = \eta(w)$ for every w. This second map is well defined because of linear algebra reasons. It is clear that both are maps of vector bundles. In order to check that they are regular one has to use local co-ordinates.

The converse is not true, given an equivalence one gets a non-degenerate bilinear form on T_xM for every x (i.e. a section of $TM^\vee \otimes TM^\vee$), but a Riemannian metric must satisfy also a positivity and symmetry assumption. A counterexample is given by symplectic manifolds. A symplectic manifold is a pair (M, ω), where ω is a non degenerate closed alternating one forms. The form ω defines an equivalence between TM and $TM^\vee$, but it is far from being a Rimeannian metric.

Review quotient bundle Let M and N be two vector bundles, suppose an immersion ι (as vector bundles) of N in M is given. We define the quotient vector bundle $E := M/N$, its fiber at a point x is given by the quotient M_x/N_x. In a good trivialisation, the transition functions of M are in block form.

$$\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$$

where A is the transition function of N and B of E. (Take a book and have a look to a proper definition!)

As in all the abelian categories, we have an exact sequence

$$0 \to N \xrightarrow{\iota} M \xrightarrow{\pi} E \to 0$$

The quotient is unique up to isomorphism. The sequence splits if M is isomorphic to the direct sum of N and E, in order to split the sequence one can either find a right inverse for π or a right inverse for ι. (Have a look to the so called "five-lemma")

Exercise 11 To start with, let us assume the existence and prove the uniqueness. Composing the inverse of $\tilde{F}$ with the projection to E we obtain the normal bundle exact sequence

$$0 \to TN \xrightarrow{F_*} TM\mid_N \to E \to 0$$

The quotient of two vector bundles is unique up to isomorphism, so the (isomorphism class of the) normal bundle E is unique.

The previous argument proves also the existence of the vector bundle E, namely it is defined as the quotient of TM by F_*TN. Let us stress that this definition depends on M, N and on the embedding F. Now, we need to prove the existence of the map L, in other words we must show that the normal bundle exact sequence splits. To do this, let us fix a Riemannian metric g on M. Using the metric, we can define the orthogonal projection from TM to TN, this projection splits the sequence.

Keep notations as in exercise 7, when $d = d'$. It is enough to trivialise $E^\vee$, this bundle is the one generated by df_i. Consider the map

$$(x, \omega) \mapsto (x, (a_1, \cdots a_d))$$

where $(a_1, \cdots, a_d)$ is the unique vector of $\mathbb{R}^d$ such that $\sum a_i df_i = \omega$. One checks that this map is an isomorphism with the trivial bundle.

(To split the normal bundle exact sequence we used the fact that every differential manifold can be endowed with a Riemannian metric. To construct the metric one uses a partition of unity. This argument may not be refined: indeed the normal bundle exact sequence does not always split in categories where a partition of unity does not exist, e.g. complex manifolds.)

Exercise 12 The vector bundles $T\mathbb{S}^2$ is not trivial. If it was trivial, then we would have a never vanishing section, i.e. a never vanishing vector field on a sphere. It is known that a never vanishing vector field on a sphere does not exist. (The proof is classical, the idea is that such a vector field allows you to construct a homotopy between the identity and minus the identity, but these maps do not act in the same way on the cohomology)

The answer to the second question is no, but the proof is more complicated. You need some algebraic topology, in particular you need the second homotopy group and the correct definition of intersection of two oriented sub-manifolds. The proof goes more or less as follow.

Take a vector bundle E on S^2. The zero section $0 : S^2 \to E$ is an element of the second homotopy group $\pi_2(E)$. A vector bundle is homotopically equivalent to the base, so $\pi_2(E) = \mathbb{Z}$. Suppose that h is a diffeomorphism between two bundles E and E', then it must send generators of $\pi_2(E)$ to generaters of $\pi_2(E')$. This means that the image of the zero section of E is, up to the choice of the orientation, homotopically equivalent to the zero section of E'.

When the rank of E is two, it makes sense to speak about the self-intersection of the zero section, and the previous argument shows that it is (up to a sign) an invariant of the total space of the vector bundle as a manifold.

To compute the self intersection of the zero section we remark that all the sections are homotopically equivalent (you can rescale them), so it is enough to take a generic non trivial section and count its zeros (with sign). For the trivial bundle take a constant section and you get zero. For the tangent bundle consider the vector field generated by a rotation: you get two.

Bonus track Let X be a manifold. Suppose, for the sake of simplicity, that X is one dimensional. Let x_α be an atlas, call $g_{\alpha\beta}$ the changes of co-ordinates. Locally, the tangent bundle is trivialised by ∂_{x_α}, so we have natural co-ordinates given by (x_α, p_α), where p_α stands for the derivation $p_\alpha \partial_{x_\alpha}$ (these co-ordinates are "position" and "momentum"). The bundle TTX is a rank two vector bundle on TX, we claim that the transition function of this bundle is

$$\left(\begin{array}{c} \partial_{x_\alpha} \\ \partial_{p_\alpha} \end{array} \right) = \left(\begin{array}{cc} \partial_{x_\alpha} g & p \frac{\partial^2_{x_\alpha} g}{(\partial_{x_\alpha} g)^2} \\ 0 & \partial_{x_\alpha} g \end{array} \right) \left(\begin{array}{c} \partial_{x_\beta} \\ \partial_{p_\beta} \end{array} \right).$$

This means that the vertical space is always well defined. Instead, the horizontal one is defined only along the zero section (i.e. $p = 0$). To define the horizontal space at every point of the bundle one must fix a connection.

1. Let $\mathcal{M}$ be a smooth manifold, and ∇ a connection in $T\mathcal{M}$. Let ϕ_t define the 1-parameter group of local transformations of $T\mathcal{M}$ determining geodesic flow. Show that for any point $p \in \mathcal{M}$, ϕ_1 can be defined on an *open* set $\mathcal{U} \subset T\mathcal{M}$ containing 0_p. (It suffices to show that for the domain $\mathcal{U}$ of ϕ_1, there exists an open $\mathcal{W} \subset \mathcal{M}$ containing p, and $\epsilon > 0$, such that $\{V_q : |V_q| < \epsilon\}_{q\in\mathcal{W}} \subset \mathcal{U}$, where $|\cdot|$ denotes coordinate length.) We define the exponential map $\exp : \mathcal{U} \to \mathcal{M}$ by $\pi \circ \phi_1$, where $\pi : T\mathcal{M} \to \mathcal{M}$ denotes the natural projection. We call a connection ∇ geodesically complete if all solutions to $\nabla_T T = 0$ can be defined for all time, i.e. if the domain of exp can be taken as $T\mathcal{M}$. Show that compact manifolds are geodesically complete with respect to any connection ∇ in $T\mathcal{M}$ satisfying $\nabla_Y X = 0 \implies Yg(X,X) = 0$.

2. Let $\pi : E \to \mathcal{M}$ be a vector bundle. Show that there exists a connection in E. Let ∇, $\tilde{\nabla}$ denote two connections in E. Show that $\nabla - \tilde{\nabla}$ defines a section of $T^*\mathcal{M} \otimes E^* \otimes E$.

3. Let $\mathcal{A} \subset \Gamma(T\mathcal{M})$ denote a Lie subalgebra, i.e. a subspace of $\Gamma(T\mathcal{M})$ closed under the Lie bracket. Let $p \in \mathcal{M}$. Show that there is a submanifold $\mathcal{N}$ of $\mathcal{M}$, such that $T_q\mathcal{N}$ is spanned by $\mathcal{A}$ for all q in a neighborhood of p in $\mathcal{N}$.

4. Let $\mathcal{M}$ be a smooth manifold, let $\pi : E \to \mathcal{M}$ be a vector bundle and let ∇, $\tilde{\nabla}$ denote connections on E. For any curve γ from p_1 to p_2, let $T^{\nabla}_{p_1,p_2,\gamma}$ denote the parallel transport map with respect to ∇. Suppose $T^{\nabla}_{p_1,p_2,\gamma} = T^{\tilde{\nabla}}_{p_1,p_2,\tilde{\gamma}}$ for all p_1, p_2, γ. Show that $\nabla = \tilde{\nabla}$, i.e., "parallel transport determines the connection". Give sufficient conditions on a collection of maps $\{T_{p_1,p_2,\gamma}\}$ such that these arise as parallel transport maps from a connection ∇.

5. Consider $\mathbb{E}^n$ with its standard Euclidean connection ∇. Let $\mathcal{M}$ denote a submanifold. Define a connection $\tilde{\nabla}$ on $\mathcal{M}$ by $\tilde{\nabla}_\xi Y = \pi_{T_p\mathcal{M}} \nabla_\xi Y$, where $Y \in \Gamma(T\mathcal{M})$, $\xi \in T_p\mathcal{M}$, and $\pi_{T_p\mathcal{M}} : T_p\mathbb{E}^n \to T_p\mathcal{M}$ denotes the projection. Show in particular that this definition can be made sense of even though $Y \notin \Gamma(T\mathbb{E}^n)$. Show that $\tilde{\nabla}$ coincides with the Levi-Civita connection of the induced metric g on $\mathcal{M}$.

6. Now let $\mathbb{S}^n$ denote the n-sphere in $\mathbb{E}^{n+1}$. Compute the Christoffel symbols of its induced Riemannian connection in your favourite system of local coordinates. Describe geometrically parallel transport. Are there vector fields defined in open subsets $\mathcal{U}$ of $\mathbb{S}^n$, $V : \mathcal{U} \to T\mathcal{M}$ such that $\nabla_\xi V = 0$ for all ξ? Prove that $\mathbb{S}^n$ is not locally isometric to $\mathbb{R}^n$ using this information.

7. Let $\pi : E \to \mathcal{M}$ be a vector bundle, and let ∇ be a connection in E. Clearly, we can define the parellel transport map $T_{p_1,p_2,\gamma}$ not just for smooth curves γ, but for piecewise smooth curves. For $p \in \mathcal{M}$ consider the set $\{T_{p,p,\gamma}\}$, where γ ranges over all piecewise smooth closed curves from p to p. Show that this defines a subgroup $\mathcal{G}_p$ of the group of linear isomorphisms of V_p, and explain the composition law. This is known as the *holonomy group* at p. Show that if $\mathcal{M}$ is connected then $\mathcal{G}_p \cong \mathcal{G}_q$. Let ∇ now be the Levi-Civita connection of a Riemannian metric g. Show that $\mathcal{G}_p$ is a subgroup of the group of isometries of the tangent space.

8. Compute $\mathcal{G}_p$ for $\mathbb{R}^n$ and for $\mathbb{S}^n$ with their Levi-Civita connections. Produce a connected n-dimensional Riemannian manifold whose Levi-Civita connection gives a holonomy group bigger than that of $\mathbb{R}^n$ and smaller than that of $\mathbb{S}^n$. Produce a simply-connected such manifold.

9. Recall the vector bundle E defining the Möbius strip and let ∇ be a connection on E. What are the possibilities for $\mathcal{G}_p$?

10. Fill in the gaps in the proof of the first variation formula, i.e. in the proof that a curve $\gamma_0(t) = \gamma(t,0)$ extremizes length over variations $\gamma(t,s)$ with the same fixed points iff $\nabla_{\gamma_* \frac{\partial}{\partial t}} \gamma_* \frac{\partial}{\partial t} = 0$. In particular, justify why from

$$\int_a^b (\nabla_{\gamma_* \frac{\partial}{\partial t}} \gamma_* \frac{\partial}{\partial t}\big|_{s=0}, \gamma_* \frac{\partial}{\partial s}\big|_{s=0}) dt = 0$$

for all variations $\gamma(t,s)$, we can infer that $\nabla_{\gamma_* \frac{\partial}{\partial t}} \gamma_* \frac{\partial}{\partial t}\big|_{s=0} = 0$ for all t.

11. Let $(\mathcal{M}, g)$ be connected, and locally isometric to $\mathbb{R}^n$. Show that if γ is a topologically trivial closed curve, then $T_{p,p}\gamma = id$. More generally, show that there exists a well-defined group homomorphism

$$\phi : \pi_1(\mathcal{M}) \to \mathcal{G}_p$$

where π_1 denotes the fundamental group of $\mathcal{M}$.

Show that a Riemannian manifold $(\mathcal{M}, g)$ is locally isometric to $\mathbb{R}^n$ iff for all p, there exists a $\mathcal{U}$ containing p such that $T_{p,p,\gamma} = id$ for all $\gamma \subset \mathcal{U}$. Show that the condition $\gamma \subset \mathcal{U}$ can be weakened to the condition that γ is topologically trivial.

12. Let $(\mathcal{M}^n, g)$ be Riemannian. If $n = 2$, show that the condition $\nabla_\xi g(X,Y) = g(\nabla_\xi X, Y) + g(X, \nabla_\xi Y)$, together with the requirement that length extremizing curves γ, parametrized by arc length, should satisfy $\nabla_{\gamma'}\gamma' = 0$, determines ∇ to be Levi-Civita. Show that this is no longer the case for $n > 2$.

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Hints and (sketches of) Solutions - Example sheet 2

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Review of morphism of vector bundles Let $T: E \to F$ be a map of vector bundles over a manifold X. We can define a map

$$\begin{array}{cccc} T_*: & \Gamma(X,E) & \to & \Gamma(X,F) \\ & s & \mapsto & s \circ T \end{array}$$

The main point is that T_* is a $C^\infty(X)$-linear, i.e. $T_*(fs) = fT_*(s)$ for every function f in $C^\infty(X)$ and every section s of E.

Suppose you have a $C^\infty(X)$-linear map T_* as before, then one can show that there always exist a map T of vector bundle inducing T. The first step is to show, using bump functions, that for every point (x,p) in E there exist a section s of E such that $s(x) = p$. Then, you can define $T(x,p) := T_*(s)(x)$. Now, you need to show that T is well defined and it is a map of vector bundles.

If you work on complex manifolds, where bump functions do not exist, there is a refined version of this argument involving maps of $\mathcal{O}_X$-modules.

Exercise 2 Existence. On a local chart U^α, you can fix a trivialising frame e_i^α for E. A section s, locally, is given by $\sum f_i \alpha e_i^\alpha$, so a connection on U^α is given by

$$\nabla^\alpha(s) := \sum_i e_i^\alpha \otimes df_i^\alpha$$

Globally, we can define

$$\nabla(s) := \sum_\alpha \rho_\alpha \nabla^\alpha(s) = \sum_\alpha \rho_\alpha e_\alpha \otimes df_\alpha\,,$$

where ρ_α is a partition of unity. The only non-trivial thing one has to check is the Leibniz's rule. Indeed, connections form an affine space, so a sum of connection is not in general a connection, but a convex combination it is.

To solve the second point it is enough to check that $\nabla - \tilde{\nabla}$ is $C^\infty(X)$-linear, i.e. $(\nabla - \tilde{\nabla})(fs) = f(\nabla - \tilde{\nabla})(s)$ for every function f and section s.

Exercise 5 We want to give sense to $\nabla_\xi Y$. The problem is local, so let $\partial_1, \dots, \partial_N$ be a basis of the tangent space at p of $\mathbb{E}^N$ such that $\partial_1, \dots, \partial_n$ form a basis for $T_p\mathcal{M}$. Write

$$\xi = \sum_{i=1}^n c_i \partial_i \qquad Y = \sum_{i=1}^n f_i \partial_i$$

where c_i are numbers and f_i functions, we pose

$$\nabla_\xi Y := \sum_{i=1}^n c_i \frac{\partial f_i}{\partial x_i} \partial_i$$

We want to show that $\tilde{\nabla}$ is the Levi-Civita connection. It is enough to show that $\tilde{\nabla}$ is compatible with g and symmetric. This is a local problem. Call N the orthogonal (with respect to g) complement of $T\mathcal{M}$ in $T\mathbb{E}$. Every vector field X decompose as a sum $X_M + X_N$, with $g(X_M, X_N) = 0$, where $X_M = \pi(X) := \pi_{T_p\mathcal{M}}(X)$. A vector field X is tangent to $\mathcal{M}$ if $X_N = 0$. Suppose X and Y are tangent to $\mathcal{M}$ (and hence $[X, Y]$). The torsion τ^E of the Euclidean connection ∇^E is zero, so

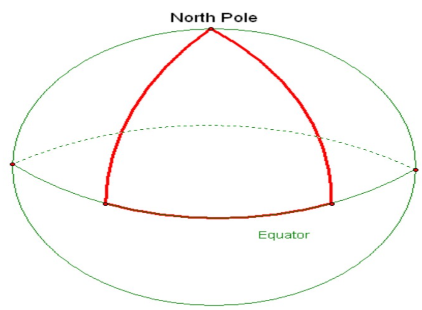
$$0 = \tau^E(X, Y) = \nabla_X(Y)_N + \nabla_X(Y)_M - \nabla_Y(X)_N - \nabla_Y(X)_M - [X, Y]$$

Apply π to the previous equation, since $[X, Y]$ is tangent to $\mathcal{M}$ we get

$$0 = \nabla_X(Y)_M - \nabla_Y(X)_M - [X, Y]$$

so $\tilde{\nabla}$ is symmetric. The compatibility follows from the fact that $g(X_M, X_N) = 0$

Exercise 6 Geometric picture Start with a tangent vector v at the north pole of a sphere. Let γ be the red triangle in the following picture. Call α the angle between the first and third segment.



The red triangle on the sphere has an angle sum greater than 180° since the angles at the equator are both 90°.

The parallel transport along γ rotates v of α, this means that v can not be extended to a parallel vector field. On the other hand, any tangent vector to the plane can be extend to a parallel vector field, just translate the vector. We can conclude that the plane is not locally isometric to the sphere (actually γ is not very local, but you can make it). Notice that the sum of the internal angles of γ is $\pi + \alpha$, so the existence of such a γ shows that the geometry on the sphere is not Euclidean.

Let us notice that the plan is conformally equivalent to the sphere: the stereographic projection is a conformal equivalence, i.e. it preserves the angles.

Bonus question Prove that the sum of the internal angles of a triangle in a plane in π (you do not need Gauss-Bonnet ...)

The computation Call ∇ the Levi-Civita connection on $\mathbb{S}^n$, ∇^E the standard connection on the Euclidean space, $\pi^\perp$ the orthogonal projection to the tangent space of $\mathbb{S}^n$.

Case $n = 1$. A one dimensional Riemannian manifold is always locally isometric to $\mathbb{E}^1$, the isometry is given by the arc-length parametrisation. Anyway, we work out the computation when $n = 1$.

We show that, with the usual parametrisation, the Christoffel's symbol of $\mathbb{S}^1$ is zero, so the parallel transport is as in the Euclidean space $\mathbb{E}^1$. A local parametrisation of $\mathbb{S}^1$ is given by $(x, y) = (cos(\theta), sin(\theta))$. The tangent space is spanned by $D_\theta = (-sin(\theta), cos(\theta))$. We have

$$\nabla_{D_\theta}(D_\theta) = \pi^\perp(\nabla^E_{D_\theta}(D_\theta) = \pi^\perp(-x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) = 0$$

So the unique Christoffel's symbol is zero.

Case $n = 2$. First, we look at the plane. Take a tangent vector $v := a\partial_x + b\partial_y$ at a point p, the constant vector field

$$X(x, y) = a\partial_x + b\partial_y$$

is a parallel extension of v.

If $\mathbb{S}^2$ were locally isometric to the plane, then every tangent vector at a point p could be extendend to a parallel vector field in a neighbourhood of p. Let us show that this is not the case. A parametrisation of $\mathbb{S}^2$ as submanifold of $\mathbb{E}^3$ is $(x, y, z) = (sin(\theta)cos(\phi), sin(\theta)sin(\phi), cos(\theta))$. A basis for the tangent space is

$$D_\theta = (cos(\theta)cos(\phi), cos(\theta)cos(\phi), -sin(\theta))$$

$$D_\phi = (-sin(\theta)sin(\phi), sin(\theta)cos(\phi), 0)$$

In this case, it is messy to carry out the computation done for $n = 1$. We first compute the metric on the $\mathbb{S}^2$, recall that the metric is nothing but the ordinary scalar product. The matrix we get is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & sin^2(\theta) \end{pmatrix}$$

where θ is on the first line and ϕ on the second. We compute the Christoffel's symbols using the formula

$$\Gamma^k_{ij} = \frac{1}{2}\sum_r g^{lr}(\partial_j g_{ri} + \partial_i g_{ri} + \partial_r g_{ij})$$

The only non-trivial one is

$$\Gamma^\theta_{\phi\phi} = sin(\theta)cos(\theta)$$

Pick a non-zero tangent vector v at a point $p := (\theta_0, \phi_0)$, we look for a parallel vector field

$$X(\theta, \phi) = F(\theta, \phi)\partial_\theta + G(\theta, \phi)\partial_\phi$$

such that $X(\theta_0, \phi_0) = v$. Being parallel means

$$\nabla_{D_\theta}(X) = \nabla_{D_\phi}(X) = 0$$

Now

$$\nabla_{D_\theta}(X) = (\partial_\theta F)\partial_\theta + (\partial_\theta G)\partial_\phi$$

and

$$\nabla_{D_\phi}(X) = (\partial_\phi F)\partial_\theta + (\partial_\phi G)\partial_\phi + (sin(\theta)cos(\theta)G)\partial_\theta\,.$$

First, we get that G is constant. Then, we have $\partial_\theta F = 0$ and $\partial_\phi F = -sin(\theta)cos(\theta)G$. Consider the integral on a closed path

$$\int_{[\theta,\phi]}^{[\theta+2\pi,\phi]} dF$$

this is zero because dF is exact, using the explicit expression of dF we get $G = 0$. So F is constant. We can assume that p to be the north pole, the vector field X is defined across p, so F must be zero.

We prove that $\mathbb{E}^n$ is not locally isometric to $\mathbb{S}^n$ by induction. The case $n = 2$ is done. Suppose by contradiction that $\mathbb{E}^n$ is locally isometric to $\mathbb{S}^n$. Then, let f be a local isometry from a neighbourhood of 0 in $\mathbb{E}^n$ to a neighbourhood of N in $\mathbb{S}^n$. Pick a hyperplane H in $T_0\mathbb{E}^n$. The exponential of H is a line, so a copy of $\mathbb{E}^{n-1}$, the exponential of $df(H)$ is a copy of $\mathbb{S}^{n-1}$. Because of the following formula

$$exp(df(v)) = f(exp(v))$$

we conclude that f is a local isometry between $\mathbb{S}^{n-1}$ and $\mathbb{E}^{n-1}$, which is impossible.

Notation We will write equivalently $T(a, b, \gamma)$ and $T_{a,b,\gamma}$ for the parallel transport.

Exercise 7 Let γ be a piece-wise regular path, suppose it is not smooth at $t \in [0, 1]$, then just define

$$T(\gamma(0), \gamma(1), \gamma) := T(\gamma(0), \gamma(t), \gamma\mid_{[0,t]}) \circ T(\gamma(t), \gamma(1), \gamma\mid_{[t,1]})$$

Let $\Gamma_p(\mathcal{M})$ be the groupoid of loops centred at p, the operation "$\cdot$" is the usual concatenation of paths (the one used to define the first homotopy group π_1). We define a map

$$\begin{array}{cccc} \phi : & \Gamma_p(\mathcal{M}) & \to & GL(E_p) \\ & \gamma & \mapsto & T(p, p, \gamma) \end{array}$$

This map is a morphism of groupoids, its image is, by definition, the holonomy group $\mathcal{G}_p$. It does depend on the connection.

Suppose that $\mathcal{M}$ is connected (recall that connected+locally path-connected = path-connected). Pick a path η from p to q, then the map

$$\begin{array}{cccc} F_\eta : & \mathcal{G}_p & \to & \mathcal{G}_q \\ & T(p, p, \gamma) & \mapsto & T(q, q, \eta \cdot \gamma \cdot \eta^{-1}) \end{array}$$

is an isomorphism of group.

Now, we restrict ourself to case $E = T\mathcal{M}$ and ∇ the Levi-Civita connection. We want to show that $\mathcal{G}_p$ is contained in $O(E_p, g_p)$. Let γ be a loop around p, take vectors v and w in E_p, and extend them to parallel vector fields $V(t)$ and $W(t)$ along γ (this is a small abuse of notation: $V(0) \neq V(1)$, and $V(1) = T(p, p, \gamma)(v)$). We have

$$\frac{d}{dt} g(V(t), W(t)) = g(\nabla_{\dot\gamma(t)}(V(t)), W(t)) + g(V(t), \nabla_{\dot\gamma(t)}(W(t))) = 0$$

The first equality is because the Levi-Civita connection is compatible with g, the second because V and W are parallel along γ. We conclude that

$$g(V(0), W(0)) = g(V(1), W(1))$$

so $T(p, p, \gamma)$ is an isometry of E_p.

Exercise 11 Let $\Gamma_p(\mathcal{M})$ be the group of loops centred at p, the operation is the usual concatenation of paths. We have a surjective homomorphism of groups

$$\begin{array}{cccc} \phi : & \Gamma_p(\mathcal{M}) & \to & \mathcal{G}_p \\ & \gamma & \mapsto & T_{p,p,\gamma} \end{array}$$

To show that this map factors through a map

$$\phi : \pi_1(\mathcal{M}) \to \mathcal{G}_p$$

we must show that if γ is homotpically equivalent to the constant path, then $\phi(\gamma) = Id$. To prove this, it is enough tho prove that if γ_s is a family of path from a to b, then T_{a,b,γ_s} does not depend on s. (It is enough because $T(\gamma(0), \gamma(1), \gamma) = T(\gamma(0), \gamma(t), \gamma \mid_{[0,t]}) T(\gamma(0), \gamma(1), \gamma \mid_{[t,1]})$, convince yourself)

We can arbitrarily subdivide the domain of γ_s, so we assume that the image of γ_s is contained in a co-ordinate patch U isometric to a ball in $\mathbb{E}^n$. Fix a tangent vector v at a. We want to show that T_{a,b,γ_s} does not depend on s. Extend v to parallel vector field X on U, we can do this because U is isometric to a ball in $\mathbb{E}^n$. Because of the definition of parallel vector field, $\nabla_{\dot\gamma_s}(X) = 0$ for every s. We conclude that $T_{a,b,\gamma_s}(v) = X(b)$, which does not depend on s.

Now, we want to prove that **if every contractible loop gives trivial parallel transport, then the manifold is locally isometric to $\mathbb{E}^n$.** Let U be a co-ordinate patch around a point p. Every tangent vector v can be extended to a parallel vector field X on U as follow. For any point z in U, define $X(z) := T_{p,z,\gamma}(v)$, where γ is any path from p to z contained in U. This definition does not depend on γ, because of the hypothesis and the fact that every path in U is topologically trivial. Fix a basis e_i of the tangent space $T_p\mathcal{M}$. Extend the basis to parallel vector fields X_i. At every point z, $X_i(z)$ are a basis for $T_z\mathcal{M}$ because the parallel transport is an isometry.

The Christoffel symbols associated to this local frame vanish identically, because the vector fields are parallel,

We want to show that the vector fields X_i can be integrated to a system of local co-ordinates, to do this we need to show that $[X_i, X_j] = 0$ for every i and j, see the proof of Frobenius' theorem proposition 1. We use the torsion tensor of the connection

$$\tau(X_i, X_j) = \nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j] = [X_j, X_i]$$

the last equality holds because the vector fields X_i are parallel. The torsion tensor of the Levi-Civita connection is zero, so the vector fields X_i commute.

Since the vector fields X_i can be integrated to local co-ordinate, we can use the formula

$$\Gamma^k_{ij} = \frac{1}{2} \sum_r g^{lr} (\partial_j g_{ri} + \partial_i g_{ri} + \partial_r g_{ij})$$

The symbols Γ^k_{ij} are zero for every i, j and k, so $\partial_k g_{ij} = 0$ for every i, j and k. The metric is costant, so we can choose a local frame of the tangent space such that g is constantly equal to the identiy. We conclude that $\mathcal{M}$ is locally isometric to the Euclidean space.

Exercise 8 The holonomy group of $\mathbb{E}^n$ is trivial, because of the exercise 11. Assume $n > 1$. The holonomy group of the sphere must be a subgroup of $SO(n)$. This because all paths are homotopic, so the map

$$\Gamma(\mathbb{S}^n) \xrightarrow{\phi} \mathcal{G}_p \xrightarrow{det} \{\pm 1\}$$

must be continuous, hence constant. (A similar argument shows that the holonomy is a subgroup of $SO(n)$ for every orientable manifold).

For $n = 2$, the holonomy group of $\mathbb{S}^2$ is the whole $SO(2)$. Indeed, the group $SO(2)$ is the group of rotations, and we can get a rotation of angle α considering a triangular path with two square angles and an angle of α. For $n > 2$, we still obtain that the holonomy group is $SO(n)$. The group $SO(n)$ is generated by rotation about axes. Call V the tangent space $T_N\mathbb{S}^n$, where N is the north pole. Fix an orthogonal basis $e_1, \ldots, e_n$ for V. For each i we have an injection

$$F_i : SO(n-1) \hookrightarrow SO(n)$$

The group $SO(n)$ is generated by the $F_i(SO(n-1))$, these are the "Euler angles". Call H_i the orthogonal complement of e_i. the exponential of H_i is a copy $\mathbb{S}_i$ of $\mathbb{S}^{n-1}$. The holonomy group of $\mathbb{S}_i$ is contained in $F_i(SO(n-1))$ and, by induction, it is equal to $F_i(SO(n-1))$.

The holonomy group of a product is the product of the holonomy groups (check it!) so an example is $\mathbb{E}^1 \times \mathbb{S}^2$. You can take $\mathbb{E}^2$ minus the origin and mod out by a finite group of rotation R. The quotient is a cone (minus the vertex: it is a singular point!) and the holonomy group is exactly R. In this last case the holonomy group is equal to the fundamental group, this is not always the case: see the circle $\mathbb{S}^1$ or the torus.

Exercise 8 - Easy case We can define the Möbius strip E as a quotient of $\mathbb{E}^2$. The Euclidean metric induces a metric on E, and we can consider the Levi-Civita connection. With this choice, the holonomy group is a subgroup of $O(\mathbb{R})$, which is $\mathbb{Z}_2$. If the holonomy group were trivial, then E would be orientable. We conclude that, for this connection, the holonomy group is $\mathbb{Z}_2$.

Exercise 10 You can find a complete proof of the first variational formula on Lee page 91. We sketch it. The simple case is when γ is smooth. In this case, just take $\partial_s = \partial_t$ and use g positive definite. Suppose γ is not smooth, with the help of a bump function, the previous argument shows that $\nabla_{\partial_t}\partial_t = 0$ on the regular intervals. We must shows that γ has no "corner". Again, using a bump function, we construct a ∂_s which is equal to $\Delta_i\dot{\gamma}$ for a fixed i and zero at the other corners. Using $g > 0$ we get $\Delta_i\dot{\gamma} = 0$. Since the velocities at the angles match up, and γ is a geodesic away from the corner, because of the uniqueness of the solutions of the ODE we get that γ is a smooth geodesic. (We need a Bootstrap argument for the regularity)

1 Question 3: Frobenius Theorem

Let $\mathcal{M}$ be a smooth manifold of dimension $m = h + k$. We say that $\mathcal{A}$ is a **smooth distribution** of dimension h inside $T\mathcal{M}$ if and only if it is a smoothly varying family of h-dimensional subspaces. This means that for each point $x_0 \in \mathcal{M}$ there exists a neighbourhood $\mathcal{U}_{x_0}$ and vector fields $X_1, \cdots, X_h$ in $\Gamma(T\mathcal{U}_{x_0})$ such that

- the X_i's are linearly independent,
- the vector space generated by the X_i's at a point $x \in \mathcal{U}_{x_0}$ is $\mathcal{A}_x$.

We shall call $(X_1, \cdots, X_h)$ a **local frame** for $\mathcal{A}$.

Definition 1. *Suppose we have a vector field X defined on some open subset $\mathcal{U}$ of $\mathcal{M}$. We say that X **belongs** to $\mathcal{A}$ (or to be more precise that X belongs to $\mathcal{A}\big|_{\mathcal{U}}$) and we write $X \in \mathcal{A}$ (or $X \in \mathcal{A}\big|_{\mathcal{U}}$) if and only if*

$$\forall x \in \mathcal{U}, \quad X_x \in \mathcal{A}_x. \tag{1}$$

Thus, in particular every element X_i of a local frame for $\mathcal{A}$ belongs to $\mathcal{A}$.

Definition 2. *We say that $\mathcal{A}$ is **involutive** (or that it is closed under Lie bracket) if and only if, given X and Y vector fields defined over an open set $\mathcal{U}$*

$$X \in \mathcal{A},\ Y \in \mathcal{A} \quad \Longrightarrow \quad [X, Y] \in \mathcal{A}. \tag{2}$$

Remark 1. *We point out that checking condition (2) for all the pairs of vector fields belonging to $\mathcal{A}$ is equivalent to checking it for every pair of elements in a local frame. So that $\mathcal{A}$ is involutive if and only if for each x_0 there exists a local frame $(X_i)_{1 \le i \le h}$ around it such that*

$$\forall 1 \le i, j \le h, \quad [X_i, X_j] \in \mathcal{A}. \tag{3}$$

Definition 3. *A distribution $\mathcal{A}$ is said to be **integrable** if and only if for each x_0 in $\mathcal{M}$ there exists an embedded submanifold $\mathcal{N}_{x_0} \hookrightarrow \mathcal{M}$ such that*

$$\bullet\ x_0 \in \mathcal{N}_{x_0}, \qquad \bullet\ \forall x \in \mathcal{N}_{x_0},\ T_x\mathcal{N}_{x_0} = \mathcal{A}_x.$$

Prove that every integrable distribution is involutive. The content of the question is to show the converse.

Theorem 1 (Frobenius)**.** *$\mathcal{A}$ is involutive if and only if it is integrable.*

Remark 2. *Applying the theorem to a Lie Group G with Lie algebra $T_eG = \mathfrak{g}$ we get that there is a correspondence*

$$\big\{\textit{Lie sub-algebras of } \mathfrak{g}\big\} \overset{1:1}{\longleftrightarrow} \big\{\textit{Lie sub-groups of } G\big\}. \tag{4}$$

We notice that every one-dimensional distribution $\mathcal{A}$ is involutive. If we also suppose that it is oriented we can find a vector field $X_{\mathcal{A}} \in \Gamma(\mathcal{M})$ such that $X_{\mathcal{A}} \in \mathcal{A}$. Thus in this case the theorem reduces to the standard theorem about local existence of integral curves for X, which asserts the existence of a flow map $\Phi^{X_{\mathcal{A}}} : (T^-(x_0), T^+(x_0)) \times \mathcal{U}_{x_0} \to \mathcal{M}$ around every point $x_0 \in \mathcal{M}$. We will use the shorthand $\Phi_t^{X_{\mathcal{A}}}(x) = \Phi^{X_{\mathcal{A}}}(t,x)$ when we want to consider the restriction $\Phi^{X_{\mathcal{A}}}\big|_{\{t\}\times\mathcal{U}_{x_0}}$.

Question 1. *When $\mathcal{A}$ is one-dimensional we also have local uniqueness of integral manifolds. Is this still true for higher dimensions?*

The proof of the Frobenius Theorem consists of two parts.

Proposition 1. *Suppose that for each $x_0 \in \mathcal{M}$ we have a local frame $(X_i)_{1\le i\le h}$ around x_0, such that*

$$\forall 1 \le i,j \le h, \quad [X_i, X_j] \equiv 0. \tag{5}$$

Then $\mathcal{A}$ is integrable.

Proof. Consider the flows Φ^{X_i} around x_0. We can suppose that they are defined on a common neighbourhood

$$(0, x_0) \in (-\varepsilon, \varepsilon) \times \mathcal{U}_{x_0} \subset \mathbb{R} \times \mathcal{M}. \tag{6}$$

Define the map

$$\begin{array}{rcl} F : (-\varepsilon,\varepsilon)^h & \to & \mathcal{M} \\ (t^1, \cdots, t^h) & \mapsto & \Phi^{X_h}_{t^h} \circ \cdots \circ \Phi^{X_1}_{t^1}(x_0). \end{array}$$

Condition (5) implies that the flows of X_i and X_j commute, namely

$$\Phi^{X_j}_{t^j} \circ \Phi^{X_i}_{t^i} = \Phi^{X_i}_{t^i} \circ \Phi^{X_j}_{t^j}. \tag{7}$$

We see that F is a smooth function since, for every $1 \le i \le h$, repeated use of Equation (1) yields partial derivatives at every point $(t_0^1, \cdots, t_0^h)$. Indeed,

$$\begin{array}{rcl} \dfrac{\partial F}{\partial t^i}\Big|_{(t_0^1,\cdots,t_0^h)} & = & d_{(t_0^1,\cdots,t_0^h)} F \left[\dfrac{\partial}{\partial t^i}\right] \\ & = & \dfrac{d}{dt^i}\Big|_{t^i=t_0^i} \Phi^{X_i}_{t^i}\Big(\overbrace{(\Phi^{X_h}_{t_0^h} \circ \cdots \circ \Phi^{X_1}_{t_0^1})}^{\Phi^{X_i}_{t_0^i} \text{ omitted}}(x_0)\Big) \\ & = & X_i(\Phi^{X_i}_{t_0^i}\Big(\overbrace{(\Phi^{X_h}_{t_0^h} \circ \cdots \circ \Phi^{X_1}_{t_0^1})}^{\Phi^{X_i}_{t_0^i} \text{ omitted}}(x_0)\Big)) \\ & = & X_i(F(t_0^1, \cdots, t_0^h)). \end{array}$$

If we look at F in a local chart we can use this argument to show that the partial derivatives of any order do exist, so that F is smooth.

Since $(X_i)_{1\le i\le h}$ is a local frame near x_0 we see that F is an immersion and hence, up to shrinking $(-\varepsilon,\varepsilon)^h$ to a smaller neighbourhood $(-\varepsilon_0,\varepsilon_0)^h$ of $0\in\mathbb{R}^h$, we have that $F:(-\varepsilon_0,\varepsilon_0)^h\hookrightarrow\mathcal{M}$ is an integral embedded submanifold for $\mathcal{A}$ near x_0. ☐

Proposition 2. *Suppose $\mathcal{A}$ is involutive. Then, for each $x_0\in\mathcal{M}$, there exists a local frame $(X_i)_{1\le i\le h}$ around it satisfying condition* (5)*.*

Proof. Since the statement is local we can suppose that we have a local frame for $\mathcal{A}$ on an open subset $\mathcal{U}$ of the Euclidean space $\mathbb{R}^{h+k}$. Denote by $\left(\frac{\partial}{\partial s^i}\right)_{1\le i\le h+k}$ the standard basis. Shrinking $\mathcal{U}$ even more and relabeling the variables if necessary we can assume that $\mathcal{A}\big|_{\mathcal{U}}$ is always transverse to the subspace

$$\mathcal{B}:=\operatorname{Span}\left\{\frac{\partial}{\partial s^i}\right\}_{h+1\le i\le h+k}.$$

- **Claim:** we have a unique local frame on $\mathcal{U}$ for $\mathcal{A}$ of the form

  $$\forall 1\le i\le h,\quad Y_i:=\frac{\partial}{\partial s^i}+\sum_{j=h+1}^{h+k}a_i^j\frac{\partial}{\partial s^j},\tag{8}$$

  for some smooth functions $a_i^j:\mathcal{U}\to\mathbb{R}$. In order to show uniqueness:

  – Prove that if for some i and some functions $b^j:\mathcal{U}\to\mathbb{R}$ the tangent vector $Z:=\frac{\partial}{\partial s^i}+\sum_{j=h+1}^{h+k}b^j\frac{\partial}{\partial s^j}$ belongs to $\mathcal{A}$, then you have $Z=Y_i$.

  In order to show existence:

  – write down X_i using the standard basis:

  $$X_i=\sum_{j=1}^{h+k}c_i^j\frac{\partial}{\partial s^j};\tag{9}$$

  – prove that $\left(c_i^j\right)_{1\le i,j\le h}$ is invertible, with inverse $\left(d_i^j\right)_{1\le i,j\le h}$;

  – show that $Y_i=\sum_{j=1}^h d_i^jX_j$ is a local frame of the form (8).

To end up the proof notice that, since $\left[\frac{\partial}{\partial s^i},\frac{\partial}{\partial s^j}\right]=0$,

$$\forall 1\le i,j\le h,\quad [Y_i,Y_j]\in\mathcal{B}.\tag{10}$$

On the other hand we know that $[Y_i,Y_j]\in\mathcal{A}$ and thus,

$$[Y_i,Y_j]\in\mathcal{A}\cap\mathcal{B}=\{0\}.\tag{11}$$

☐

Frobenius Theorem now is a corollary of these two proposition.

1.1 Addendum

Let us add some extra information about a dual formulation of Frobenius Theorem and how to prove that we have local uniqueness of integral manifolds. Observe that there is a bijective map between k-dimensional subspaces in $\Gamma(T\mathcal{M})$ and h-dimensional subspaces in $\Gamma(T^*\mathcal{M})$ sending each $\mathcal{A}$ into its annihilator $\operatorname{Ann}\mathcal{A}$. We recall that the annihilator at a point x_0 is defined as

$$(\operatorname{Ann}\mathcal{A})_{x_0} := \left\{\alpha \in T^*_{x_0}\mathcal{M} \mid \ker\alpha \supset \mathcal{A}_{x_0}\right\}. \tag{12}$$

The inverse correspondence will be denoted in the same way and it is given by

$$(\operatorname{Ann}\mathcal{F})_{x_0} := \{X \in T_{x_0}\mathcal{M} \mid \forall\alpha \in \mathcal{F}_{x_0},\ X \in \ker\alpha\}. \tag{13}$$

We ask now what is the image of involutive distributions under this bijection. Consider the following example. Suppose that $i : \mathcal{N} \hookrightarrow \mathcal{M}$ is an integral manifold for $\mathcal{A}$ around some point x_0. If α belongs to $\operatorname{Ann}(\mathcal{M})$, then

$$\alpha|_{\mathcal{N}} = i^*\alpha = 0. \tag{14}$$

This implies that

$$d\alpha|_{\mathcal{N}} = i^*(d\alpha) = d(i^*\alpha) = 0. \tag{15}$$

This leads us to the following definition.

Definition 4. *A k-dimensional distribution $\mathcal{F}$ of 1-forms is said to be **closed** if and only if*

$$\forall\alpha \in \mathcal{F}, \quad d\alpha|_{\operatorname{Ann}\mathcal{F}} = 0. \tag{16}$$

Proposition 3. *The annihilator correspondence restricts to a bijection between involutive h-dimensional distributions of tangent vectors and closed k-distributions.*

In order to prove this proposition one has to apply the identity

$$d\alpha[X,Y] = X\left(\alpha[Y]\right) - Y\left(\alpha[x]\right) - \alpha[[X,Y]]. \tag{17}$$

Definition 5. *We shall call a k-dimensional distribution $\mathcal{F}$ **integrable** if and only if its associated h-dimensional distribution $\operatorname{Ann}(\mathcal{F})$ is integrable.*

Theorem 2 (Dual formulation of Frobenius Theorem)**.** *A closed distribution of 1-forms is closed if and only if it is integrable.*

The analogue of condition (5) is the following property: each point $x_0 \in \mathcal{M}$ has a neighbourhood $\mathcal{U}_{x_0}$ and smooth functions $\{f^i\}_{1\le i\le k}$ defined on it such that $(df^i)_{1\le i\le k}$ is a basis for $\mathcal{F}$ on $\mathcal{U}_{x_0}$. Then, the analogue of Proposition 1 is simply the Implicit Function Theorem. The main difficulty is to prove a proposition corresponding to Proposition 2, where one needs to single out a proof for the existence of such a basis.

We end up by proving that the integral manifold is unique. By this we mean that if $\mathcal{N}$ and $\mathcal{N}'$ are two integral manifolds for $\mathcal{A}$ at x_0, then there exists a neighbourhood $\mathcal{U}_{x_0}$ of x_0 where we have

$$\mathcal{N} \cap \mathcal{U}_{x_0} = \mathcal{N}' \cap \mathcal{U}_{x_0}. \tag{18}$$

In order to show this, suppose that $\mathcal{N}$ is given through a set of equations

$$\mathcal{N} = \{f^i = c_0^i \mid 1 \leq i \leq k\}, \quad f^i : \mathcal{U}_{x_0} \to \mathbb{R}, \ \ (c_0^i) \in \mathbb{R}^k, \tag{19}$$

such that $(df^i)_{1 \leq i \leq k}$ is a basis for $\mathcal{F}$ on the whole $\mathcal{U}_{x_0}$. Then it is enough to prove that $f^i|_{\mathcal{N}'}$ is constant. This is true since $df^i \in \mathcal{F}$ on the whole neighbourhood $\mathcal{U}_{x_0}$ and thus also on $\mathcal{N}' \cap \mathcal{U}_{x_0}$.

Remark 3. *The way we constructed the integral manifold in the dual formulation shows that one actually gets a foliation of integral manifolds by varying the values (c^i) in $\mathbb{R}^k$. How can one adjust the proof we gave for the tangent vectors case to arrive to the same conclusion there?*

1 Question 2

Let $\pi : E \to \mathcal{M}$ be a vector bundle over a smooth manifold $\mathcal{M}$. Let us endow E with a connection ∇. This means that we have a map

$$\begin{array}{rcl} \nabla : T\mathcal{M} \times \Gamma(E) & \to & E \\ (X_x, s) & \mapsto & (\nabla_{X_x} s)_x \end{array}$$

such that

1. for each $s \in \Gamma(E)$ the restricted map $\nabla s : T\mathcal{M} \to E$ is smooth and linear in the fibers.
2. for each $(x, v) \in T\mathcal{M}$ the restricted map $\nabla_v : \Gamma(E) \to E_x$ satisfies the **Leibniz rule**
$$\forall f \in C^\infty(M), \quad (\nabla_v f s)_x = d_x f[v] s_x + f(x)(\nabla_v s)_x. \tag{1}$$

First of all let us point out that on every vector bundle, the set of all linear connections $\mathcal{C}(E)$ is not empty.

We do this in the following three steps.

1. If we have $E = \mathcal{M} \times \mathbb{R}^k$, a section of E is simply a smooth map $s : \mathcal{M} \to \mathbb{R}^k$ and we set
$$\nabla^0 s := ds : T\mathcal{M} \to T\mathbb{R}^k \simeq \mathbb{R}^k. \tag{2}$$
Verify that $\nabla^0 \in \mathcal{C}(E)$.
2. If $\Phi : E_1 \overset{\sim}{\to} E_2$ is an isomorphism of vector bundles, the map
$$\begin{array}{rcl} \mathcal{C}(E_2) & \to & \mathcal{C}(E_1) \\ \nabla & \mapsto & \Phi^*\nabla := \left(s \mapsto \Phi^{-1} \circ \nabla(\Phi)\right). \end{array}$$
is a bijection.
3. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $\mathcal{M}$ which trivialises E and let $\{\rho_\alpha\}_{\alpha \in I}$ be a partition of unity subordinate to it. From the previous two points we can find a collection of connections $\{\nabla^\alpha\}_{\alpha \in I}$, such that $\nabla^\alpha \in \mathcal{C}(E\big|_{U_\alpha})$. Verify that
$$\nabla := \sum_{\alpha \in I} \rho_\alpha \nabla^\alpha \tag{3}$$
belongs to $\mathcal{C}(E)$.

Having found out that we have many connections, we would also like to see how they are related to each other.

First remember that the space $\Gamma(E_0)$ of sections of a vector bundle E_0 has a structure of a $C^\infty(\mathcal{M})$-module. Simply set

$$\begin{array}{rcl} C^\infty(\mathcal{M}) \times \Gamma(E_0) & \to & \Gamma(E_0) \\ (f, s) & \mapsto & (fs)_p := f(p)s_p \end{array}$$

Call $\mathrm{Hom}(\Gamma(E_1), \Gamma(E_2))$ the set made of all the $C^\infty(\mathcal{M})$-linear maps $\mathcal{L}$ between E_1 and E_2. Namely, for any $f, \widetilde{f} \in C^\infty(\mathcal{M})$ and $s, \widetilde{s} \in \Gamma(E_1)$ we have

$$\mathcal{L}[fs + \widetilde{f}\widetilde{s}] = f\mathcal{L}[s] + \widetilde{f}\mathcal{L}[\widetilde{s}]. \tag{4}$$

Then, the following representation result holds.

Lemma 1. *Let E_1 and E_2 be two smooth vector bundles over $\mathcal{M}$. The map*

$$\begin{array}{rcl} \Gamma(\mathrm{Hom}(E_1, E_2)) & \to & \mathrm{Hom}(\Gamma(E_1), \Gamma(E_2)) \\ A & \mapsto & \left(\mathcal{L}^A[s]\right)_p := A_p[s_p] \end{array}$$

is a bijection.

Remark 1. *From the definition we see that a connection can also be defined as a map*

$$\nabla : \Gamma(E) \to \Gamma(T^*\mathcal{M} \otimes E) = \Gamma(\mathrm{Hom}(T\mathcal{M}, E)) \tag{5}$$

that satisfies the Leibniz rule. However ∇ does not come from a section of

$$\mathrm{Hom}(E, \mathrm{Hom}(T^*\mathcal{M}, E)) = E^* \otimes (T^*\mathcal{M} \otimes E) = T^*\mathcal{M} \otimes E^* \otimes E \tag{6}$$

because the $C^\infty(\mathcal{M})$-linearity fails.

Lemma 2. *Let ∇^0 and ∇^1 be two connections on E, then the map*

$$\nabla^1 - \nabla^0 : \Gamma(E) \to \Gamma(T^*\mathcal{M} \otimes E) \tag{7}$$

is $C^\infty(\mathcal{M})$-linear. Thus $\nabla^1 - \nabla^0 = \mathcal{L}^A$ for some $A \in \Gamma(E^* \otimes T^*\mathcal{M} \otimes E)$.

With this result we get an answer to Question 2. However we will end this section by giving some more precise information on the algebraic structure of $\mathcal{C}(E)$ and on the local representaion of its elements. In the following discussion we will abuse notation and denote A and $\mathcal{L}^A$ by the same symbol A.

Proposition 1. *Let E be a vector bundle on $\mathcal{M}$. Then, the map*

$$\begin{array}{rcl} \mathcal{C}(E) \times \Gamma(\mathrm{Hom}(T\mathcal{M}, \mathrm{End}(E))) & \to & \mathcal{C}(E) \\ (\nabla, A) & \mapsto & \nabla + A := (s \mapsto \nabla s + As). \end{array}$$

turns the space of connections in an affine space with associated vector space $\Gamma(T^*\mathcal{M} \otimes E^* \otimes E)$.

In particular if ∇^0 and ∇^1 belong to $\mathcal{C}(E)$,

$$\forall r \in \mathbb{R}, \quad \nabla^r := (1-r)\nabla^0 + r\nabla^1 \in \mathcal{C}(E). \tag{8}$$

Example 1. *When the bundle E is the trivial one $E = \mathcal{M} \times \mathbb{R}^k$ we have defined the connection ∇^0 which simply is the differential d of sections. In this case an element of $\mathrm{Hom}(T\mathcal{M}, \mathrm{End}(E))$ is a $k \times k$ matrix of one forms. Therefore any connection on E can be represented as*

$$d^A := \nabla^0 + A = d + A, \tag{9}$$

where $A \in \Gamma(\mathrm{Hom}(T\mathcal{M}, \mathrm{End}(E)))$. If also the (co)tangent bundle of $\mathcal{M}$ is trivialised by a frame $(\varphi^i)_{1\leq i\leq n}$, then

$$A = \sum_{i=1}^{n} \varphi^i \otimes A_i, \tag{10}$$

where $A_i : M \to \mathrm{End}(\mathbb{R}^k)$ are smooth functions.

Applying the discussion contained in the example we can give a nice local description of connections. Indeed, suppose we have trivialisations for E and $T\mathcal{M}$ on some open set $U \subset \mathcal{M}$. Then the restriction of a connection $\nabla \in \mathcal{C}(E)$ to U can be written as $\nabla\big|_U = d^A$ via these local trivialisations. The entries of the matrices A_i defined above are exactly the Christoffer symbols. Namely,

$$(A_i)^k_j = \Gamma^k_{ij}. \tag{11}$$

2 Question 4

2.1 From connection to covariant derivative and back

Every connection ∇ induces a covariant derivative along curves in $\mathcal{M}$. Let $\gamma : I_\gamma \to \mathcal{M}$ be a smooth curve and let $\pi_\gamma : \gamma^*E \to I_\gamma$ be the pullback of E through γ. In other words the fiber of γ^*E over $t \in I_\gamma$ is $E_{\gamma(t)}$. We define the covariant derivative along γ as the unique map

$$\frac{\nabla^\gamma}{dt} : \Gamma(\gamma^*E) \to \Gamma(\gamma^*E) \tag{12}$$

such that

1. satisfies the Leibniz rule

$$\frac{\nabla^\gamma}{dt}\Big|_{t=t_0} fs = f(t_0)\frac{\nabla^\gamma}{dt}\Big|_{t=t_0} s + \dot{f}(t_0)s_{t_0}; \tag{13}$$

2. satisfies a compatibility condition with ∇

$$\forall s \in \Gamma(E), \quad \frac{\nabla^\gamma}{dt}\Big|_{t=t_0} s\circ\gamma = \left(\nabla_{\dot\gamma(t)} s\right)_{\gamma(t)}. \tag{14}$$

One can check that this is indeed a good definition.

As before we have a simple local representation for $\frac{\nabla^\gamma}{dt}$ whenever we have a trivialisation for E. It is given by $\frac{d}{dt} + A^\gamma$, where $A^\gamma : (t_0-\varepsilon, t_0+\varepsilon) \to \mathrm{End}(\mathbb{R}^k)$ is a smooth path of $k \times k$ matrices. It is related to the matrix of one forms A for ∇ through the formula

$$A^\gamma(t) = A(\dot\gamma(t))_{\gamma(t)}. \tag{15}$$

We observe first of all that covariant derivatives determine the connection univoquely. This can be seen from the second property. Let $s \in \Gamma(E)$ and let $(x, v) \in T\mathcal{M}$. Take any curve $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ such that $(\gamma(0), \dot{\gamma}(0)) = (x, v)$. Then,

$$(\nabla_v s)_x = \frac{\nabla^\gamma}{dt}\Big|_{t=0} s \circ \gamma. \tag{16}$$

If we reverse the perspective, we can think of starting with a correspondence which assign to each smooth path $\gamma : I_\gamma \to \mathcal{M}$ a map D^γ satisfying Property 1. What further conditions must we impose so that the collection D^γ is induced by a connection ∇ on E, in other words $D^\gamma = \frac{\nabla^\gamma}{dt}$?. The easiest thing would be to ask the following two properties.

A) If $s \in \Gamma(E)$ and γ, σ are two paths such that $\gamma(0) = \sigma(0) = x$, then

$$\dot{\gamma}(0) = \dot{\sigma}(0) \quad \Longrightarrow \quad D^\gamma\Big|_{t=0} s \circ \gamma = D^\sigma\Big|_{t=0} s \circ \sigma. \tag{17}$$

B) For any point $x_0 \in \mathcal{M}$, there exists a coordinate neighbourhood (U, ϕ) of x_0 such that the following holds. For every $(x, v) \in T\mathcal{M}\big|_U$ call

$$\gamma_{x,v}(t) = \phi^{-1}\left(\phi(x) + t d_x\phi[v]\right). \tag{18}$$

In a down-to-earth language we use the coordinates in order to construct a family of curves which varies smoothly with the point x and the velocity v. Notice that

$$(\gamma_{x,v}(0), \dot{\gamma}_{x,v}(0)) = (x, v). \tag{19}$$

We then ask that, for every $s \in \Gamma(E\big|_U)$, and every $x \in U$ the map

$$\begin{aligned} L_x^s : \mathbb{R}^n &\to E_x \simeq \mathbb{R}^k \\ v &\mapsto D^{\gamma_{x,v}}\Big|_{t=0} s \end{aligned}$$

is **linear** and that $x \mapsto L_x^s$ is a **smooth** section of $\mathrm{Hom}(TM\big|_U, E\big|_U)$.

Then, we define $(\nabla_v s)_x := L_x^s[v]$. We see that ∇ is then a connection on U. By property 1 we also know that the covariant derivatives along curves induced by ∇ is exactly the family of map given, i.e. $D^\gamma = \frac{\nabla^\gamma}{dt}$. We want to conclude that two connections defined in this way on two overlapping coordinates neighbourhood patch together and gives a global connection on $E \to \mathcal{M}$. This follows from the fact that on the intersection of the two neighbourhood the two connections have the same covariant derivative and we showed before that covariant derivatives determines uniquely the connection. Thus, the two connections coincide on the intersection of the two neighbourhoods and we can glue them together.

2.2 From covariant derivative to parallelism and back

The notion of covariant derivative allows us to speak about parallel sections along curves.

Definition 1. *A section $s \in \Gamma(\gamma^* E)$ is said to be* ***parallel*** *if and only if*

$$\frac{\nabla^\gamma}{dt} s = 0. \tag{20}$$

Exploiting the local description of a covariant derivative we see that being parallel amounts in solving a non-autonomous linear equation in $\mathbb{R}^k$:

$$\frac{ds}{dt} = -A(t) s_t. \tag{21}$$

Thus we conclude that the space of parallel sections along a curve γ is a real vector space of dimension k. An explicit isomorphism is obtained sending a parallel section s to its value at some point $t \in I_\gamma$:

$$\mathrm{ev}_t : s \mapsto s_t \in E_{\gamma(t)}. \tag{22}$$

The composition of the isomorphisms for two different values $t_0, t_1 \in I_\gamma$ is called **parallel transport**

$$P^\gamma_{t_1,t_0} = ev_{t_1} \circ ev_{t_0}^{-1} : E_{\gamma(t_0)} \to E_{\gamma(t_1)}. \tag{23}$$

The fundamental properties of this family of linear isomorphisms are the listed below.

1. For any $t \in I_\gamma$, we have
$$P^\gamma_{t,t} = \mathrm{Id}_{E_{\gamma(t)}}\,. \tag{24}$$
2. For any $t_0, t_1, t_2 \in I_\gamma$, we have
$$P^\gamma_{t_2,t_1} \circ P^\gamma_{t_1,t_0} = P^\gamma_{t_2,t_0}. \tag{25}$$
3. If we have $t_0 \in I_\gamma$ and $e_0 \in E_{\gamma(t_0)}$, then $t \mapsto P^\gamma_{t,t_0} e_0$ is a smooth section.
4. P^γ is compatible with the covariant derivative along γ: for any $t_0 \in I_\gamma$ and $e_0 \in E_{\gamma(t_0)}$, $t \mapsto P^\gamma_{t,t_0} e_0$ is parallel. Namely,
$$\frac{\nabla^\gamma}{dt} P^\gamma_{t,t_0} e_0 = 0. \tag{26}$$

We see now that the parallel transport encodes all the information about the covariant derivative. Suppose that $s \in \Gamma(\gamma^* E)$ and that we want to compute $\frac{\nabla^\gamma}{dt}\Big|_{t=t_0} s$. Then,

$$\begin{array}{ccc} I_\gamma & \to & E_{\gamma(t_0)} \\ t & \mapsto & P^\gamma_{t_0,t} s_t \end{array}$$

is a smooth curve inside the vector space $E_{\gamma(t_0)}$. Thus

$$\frac{d}{dt}\Big|_{t=t_0} P^\gamma_{t_0,t} s_t \in T_{s_{t_0}} E_{\gamma(t_0)} \simeq E_{\gamma(t_0)} \tag{27}$$

and we are going to prove that, after the above identification,

$$\frac{\nabla^\gamma}{dt}\Big|_{t=t_0} s = \frac{d}{dt}\Big|_{t=t_0} P^\gamma_{t_0,t} s_t. \tag{28}$$

Indeed, take $(\sigma_i)_{1\le i\le n}$ a basis for $E_{\gamma(t_0)}$ and extend it to a frame of parallel sections on γ

$$s^\gamma_{i\ t} := P^\gamma_{t,t_0}\sigma_i. \tag{29}$$

Then, there exist functions $\{c^i : I_\gamma \to \mathbb{R}\}_{1\le i\le n}$ such that

$$s_t := \sum_{i=1}^n c^i(t) s^\gamma_{i\ t}. \tag{30}$$

Now you get

$$P^\gamma_{t_0,t} s_t = P^\gamma_{t_0,t}\left(\sum_{i=1}^n c^i(t) s^\gamma_{i\ t}\right) = \sum_{i=1}^n c^i(t) P^\gamma_{t_0,t} s^\gamma_{i\ t} = \sum_{i=1}^n c^i(t)\sigma_i. \tag{31}$$

Hence,

$$\begin{aligned}
\frac{\nabla^\gamma}{dt}\Big|_{t=t_0} s &= \frac{\nabla^\gamma}{dt}\Big|_{t=t_0} \sum_{i=1}^n c^i(t) s^\gamma_{i\ t} \\
&= \sum_{i=1}^n \dot{c}^i(t_0) s^\gamma_{i\ t_0} + \sum_{i=1}^n c^i(t_0) \frac{\nabla^\gamma}{dt}\Big|_{t=t_0} s^\gamma_i \\
&= \sum_{i=1}^n \dot{c}^i(t_0)\sigma_i \\
&= \frac{d}{dt}\Big|_{t=t_0} \sum_{i=1}^n c^i(t)\sigma_i \\
&= \frac{d}{dt}\Big|_{t=t_0} P^\gamma_{t_0,t} s_t.
\end{aligned}$$

From this we get that the parallel transport determines univoquely the covariant derivative.

Moreover we see that if we have a family of maps $P^\gamma_{t_1,t_0}$ satisfying Properties 1 and 2 given before (namely Equations (24) and (25)), we can use Equation (28) to define a collection of maps $D^\gamma : \Gamma(\gamma^* E) \to \Gamma(\gamma^* E)$:

$$D^\gamma\Big|_{t=t_0} s = \frac{d}{dt}\Big|_{t=t_0} P^\gamma_{t_0,t} s_t. \tag{32}$$

Then D^γ satisfies the Leibniz rule which we labeled as Property 1 in the previous subsection:

$$D^\gamma\Big|_{t=t_0} fs = f(t_0) D^\gamma\Big|_{t=t_0} s + \dot{f}(t_0) s_{t_0}. \tag{33}$$

We would like to complete the answer to Question 4 and find conditions that ensure that the collection of linear maps P^γ arises from a connection. In view of the preceding discussion this is equivalent to asking that the maps D^γ, that come from P^γ as described above, satisfy the two further conditions A and B from the previous subsection. A more elegant answer would be to translate those two properties into properties of the maps P^γ by using (28).

3 Question 1

Let us now specialise the discussion to the case in which $E = \mathcal{M}$. If $\gamma : I_\gamma \to \mathcal{M}$ is a smooth curve we have that $\dot\gamma$ is an element of $\Gamma(\gamma^* T\mathcal{M})$ and we also have a corresponding curve in $T\mathcal{M}$

$$\begin{array}{rcl} \widetilde{\gamma} : I_\gamma & \to & T\mathcal{M} \\ t & \mapsto & (\gamma(t), \dot\gamma(t)), \end{array}$$

called the **lift** of γ. Lifts of curves can be easily characterised inside the set of all curves with values in $T\mathcal{M}$ and they are preserved under differential of maps.

Lemma 3. *A smooth curve $\eta = (\gamma, v) : I_\eta \to T\mathcal{M}$ is a lift if and only if*

$$\forall t \in I_\eta, \quad \dot\gamma(t) = v(t) \in T_{\gamma(t)}\mathcal{M}. \tag{34}$$

If this is the case, we see that $\eta = \widetilde{\gamma}$.

Furthermore if we have a smooth map $F : \mathcal{M}_1 \to \mathcal{M}_2$ and $\eta : I_\eta \to T\mathcal{M}_1$ is a lift, then $dF \circ \eta : I_\eta \to T\mathcal{M}_2$ is a lift.

Since $\dot\gamma$ is a vector field over γ we can take its covariant derivative

$$\frac{\nabla^\gamma}{dt}\dot\gamma. \tag{35}$$

This can be interpreted as a generalisation the acceleration of a curve in the Euclidean space and leads us to give the following definition.

Definition 2. *A curve $\gamma : I_\gamma \to \mathcal{M}$ is called a geodesic if and only if it satisfies*

$$\frac{\nabla^\gamma}{dt}\dot\gamma = 0. \tag{36}$$

We write down this equation using local coordinates $y : U \to \mathbb{R}^m$. These give a trivialisation $u : \mathcal{M}\big|_U \to \mathbb{R}^m$ of $T\mathcal{M}\big|_U$. Its basic property is that if $y_\gamma = y \circ \gamma$, then

$$u^i(\dot\gamma) = \dot y^i_\gamma. \tag{37}$$

Using this trivialisation and the coordinates y on $\mathcal{M}$ we get local coordinates

$$(y, u) : T\mathcal{M}\big|_U \to \mathbb{R}^m \times \mathbb{R}^m \tag{38}$$

on $T\mathcal{M}$. The expression of the lift $\widetilde{\gamma}$ given by them is

$$(y,u)\circ\widetilde{\gamma} = (y_\gamma, \dot{y}_\gamma) = \widetilde{y_\gamma}. \tag{39}$$

If we write the geodesic equation on U using the Christoffer symbols (Γ_{ij}^k) associated to the given trivialisation we have

$$\frac{\nabla^\gamma}{dt}\dot{\gamma} = 0 \quad\iff\quad \frac{d}{dt}\dot{y}_\gamma^i + \sum_{j,k}\dot{y}_\gamma^j\Gamma_{jk}^i(y_\gamma)\dot{y}_\gamma^k = 0,\quad 1\le i\le m. \tag{40}$$

We can interpret this last equation as a differential equation for curves (z,w) in $\mathbb{R}^m\times\mathbb{R}^m$:

$$\frac{d}{dt}w^i + \sum_{j,k} w^j\Gamma_{jk}^i(z)w^k = 0,\quad 1\le i\le m. \tag{41}$$

If we also require that the curve (z,w) is a lift (this is the same as requiring that $(y,u)^{-1}\circ(z,w)$ is a lift by the lemma), we get the following first order differential equation

$$\begin{cases} \dfrac{dz^i}{dt} = w^i & 1\le i\le m, \\ \dfrac{dw^i}{dt} = -\displaystyle\sum_{j,k} w^j\Gamma_{jk}^i(z)w^k & 1\le i\le m. \end{cases} \tag{42}$$

We have seen in the previous example sheet that every first order differential equation comes from a vector field. If we push forward this vector field over $U\subset\mathcal{M}$ using $(y,u)^{-1}$ we get $X^U\in\Gamma\left(T(T\mathcal{M})\big|_{TU}\right)$ given by

$$X^{(U,y)} = \sum_i\left(u^i\frac{\partial}{\partial y^i} - \sum_{j,k}u^j\Gamma_{jk}^i(y)u^k\frac{\partial}{\partial u^i}\right). \tag{43}$$

The associated Cauchy problem yields us a smooth flow $\Phi^{(U,y)}: V^{(U,y)}\to TU$, where $V^{(U,y)}$ is an open neighbourhood of $\{0\}\times TU$ inside $\mathbb{R}\times TU$. If we have two overlapping local coordinates (U_α, y_α) and (U_β, y_β) the restriction of the flows to $T(U_\alpha\cap U_\beta)$ must coincide. Therefore, we can glue together all the local flows in order to get a global one Φ on $T\mathcal{M}$. This means that also the vector fields $X^{(U,y)}$ patch together (being determined by the corresponding flows) and yield a global vector field $X\in\Gamma(T(T\mathcal{M}))$ called the **geodesic vector field**.

Let us give the first properties of the geodesic flow. First of all, from the local equations, we see that X vanishes only at the zero section of $T\mathcal{M}$. Then we note that if we reparametrise a geodesic γ by a scalar factor c, we still have a geodesic $\gamma_c(t) := \gamma(ct)$. Indeed,

$$\frac{\nabla^{\gamma_c}}{dt}\Big|_{t=t_0}\dot{\gamma}_c(t) = c\frac{\nabla^{\gamma_c}}{dt}\Big|_{t=t_0}\dot{\gamma}(ct) = c^2\frac{\nabla^{\gamma_c}}{dt}\Big|_{t=t_0}\dot{\gamma}(ct) = c^2\frac{\nabla^{\gamma}}{dt}\Big|_{t'=ct_0}\dot{\gamma}(t'). \tag{44}$$

We can sum up all of this by saying that if $\gamma : (T^-(x,v), T^+(x,v)) \to \mathcal{M}$ is a maximal geodesic such that $\widetilde{\gamma}(0) = (x,v)$, then

$$\gamma_c : \left(\frac{1}{c}T^-(x,v), \frac{1}{c}T^+(x,v)\right) \to \mathcal{M} \tag{45}$$

is a maximal geodesic such that $\widetilde{\gamma_c}(0) = (x, cv)$. From this we also see that

$$\bullet\ T^-(x,cv) = \frac{1}{c}T^-(x,v), \quad \bullet\ T^+(x,cv) = \frac{1}{c}T^+(x,v).$$

We are now in position to give three proofs of the fact that Φ_1 is defined in a small neighbourhood of every $(x,0) \in T\mathcal{M}$.

1. We noticed before that $(x,0)$ is a rest point for the flow. In particular $(1,(x,0))$ is inside the domain of definition V of Φ, which is an open set. The fact that $(1,(x,0))$ is an interior point of V yields the conclusion.

2. For each $(x,0)$ there exist $\varepsilon, \delta > 0$ such that

$$\forall t \in \mathbb{R},\ |t| \leq \delta, \quad \Longrightarrow \quad \Phi_t \text{ is defined on } \{|v| < \varepsilon\}. \tag{46}$$

   The homogeneity property proved before implies that

$$\forall t \in \mathbb{R},\ |t| \leq 1, \quad \Phi_t \text{ is defined on } \{|v| < \varepsilon\delta\}. \tag{47}$$

3. We give only a sketch of a rough quantitative estimate that can be given. Consider a coordinate neighbourhood of x, $F : U \to B_r(0)$ with $F(x) = 0$. This will yield Christoffer symbols Γ^k_{ij}. Suppose we have a convenient uniform bound C on Γ^k_{ij}. This in turn will give a differential inequality

$$\frac{d}{dt}|\dot{\gamma}| \leq C|\dot{\gamma}|^2. \tag{48}$$

   Integrating this with some initial value ε and then integrating again to get a bound on γ we conclude that Φ_t is well defined on all the points $\{(y,v) \in T\mathcal{M}|_U \mid F(y) \in B_{r/2}(0),\ |v| < \varepsilon\}$ provided that t satisfies

$$|t| < \frac{1 - e^{\frac{Cr}{2}}}{\varepsilon C}. \tag{49}$$

This answer to the first part of the question.

Now we would like to investigate now if Φ is complete or not. In view of the previous discussion this is the same as asking if Φ_1 is defined on the whole tangent manifold. One natural assumption is to take $\mathcal{M}$ compact. However this is not enough, since the equation

$$\frac{dv^i}{dt} = -v^j\Gamma^i_{jk}(x)v^k \tag{50}$$

can blow up after a finite time even if the Γ_{ij}^k are uniformly bounded. If you want a concrete case, take the trivial bundle of rank one over S^1 and consider the connections $\nabla^a := d + a$, where a is a real number. Find explicitly the geodesic flow in this case. What happens when $a \neq 0$?

Completeness becomes true when together with compactness we also ask for a compatibility property between ∇ and some Riemannian metric g on $\mathcal{M}$:

$$\forall (x, v) \in T\mathcal{M}, \quad (\nabla_v X)_x = 0 \quad \Longrightarrow \quad v\big(g[X, X]\big) = 0. \tag{51}$$

Indeed, the metric induces a **kinetic energy function** on $T\mathcal{M}$:

$$\begin{aligned} \kappa : T\mathcal{M} &\to [0, +\infty) \\ (x, v) &\mapsto \frac{1}{2} g_x[v, v]. \end{aligned}$$

This function is an integral of motion for the geodesic flow, meaning that κ is constant along geodesics. Indeed,

$$\frac{\nabla^\gamma}{dt} \dot{\gamma} = 0 \quad \Longrightarrow \quad 0 = \dot{\gamma}\left(g[\dot{\gamma}, \dot{\gamma}]\right) = 2 \frac{d}{dt} \kappa(\widetilde{\gamma}). \tag{52}$$

The conclusion now follows from the fact that the level sets

$$\Sigma_c := \{\kappa = c\} \subset T\mathcal{M} \tag{53}$$

are compact if $\mathcal{M}$ is compact. Flows on compact manifolds are complete by the first example sheet.

4 Question 9

We recall how the Möbius strip was built. Let U_α and U_β be two open interval covering $\mathbb{R}/\mathbb{Z}$ and suppose that $U_\alpha \cap U_\beta = V_0 \sqcup V_1$, where V_0 and V_1 are two disjoint intervals. The transition map $G : U_\alpha \cap U_\beta \to \mathbb{R} \setminus \{0\}$ for the Möbius vector bundle $E \to \mathbb{R}/\mathbb{Z}$ is given by

$$G\Big|_{V_0} \equiv 1, \qquad G\Big|_{V_1} \equiv -1. \tag{54}$$

We would like to describe all the connections on E. From the fact that G is locally constant we have that the standard connections on U_α and U_β patch together and give a global connection ∇^0 on E. In other words if $s \in \Gamma(E)$, then

$$\nabla^0 s = \chi^{-1}\big(d(\chi(s))\big)^1, \tag{55}$$

where χ is either the trivialisation on U_α or on U_β which are related by G. From the discussion around Question 2, we know that any other connection is determined by some $a \in \Gamma(T^*\mathcal{M} \otimes E^* \otimes E) = C^\infty(\mathbb{R}/\mathbb{Z})$. Take now the curve $\gamma : [0, 1] \to \mathbb{R}/\mathbb{Z}$, such that $\gamma(t) := [t]$. The pullback bundle $\gamma^* E$ has a

[1] to be precise we mean that χ^{-1} acts on $d(\chi(s))$ after we compute it on a tangent vector.

trivialisation $F : \gamma^* E \to [0,1] \times \mathbb{R}$ such that $F \circ F^{-1}\big| : \{0\} \times \mathbb{R} \to \{1\} \times \mathbb{R}$ is simply the map $(0, v) \mapsto (1, -v)$. We see that the parallelism equation using the trivialisation F reduces to

$$\begin{cases} \dot{v} & = & -av, \\ v(0) & = & v_0. \end{cases} \tag{56}$$

If A is a primitive for a satisfying $A(0) = 0$, the solution of the previous problem is given by $v(t) = v_0 e^{-A(t)}$ and in particular $v(1) = v_0 B$, where $B := e^{-\int_0^1 a(t)dt} \geq 0$. Taking into account the isomorphism $F \circ F^{-1}$ we get

$$P^{\gamma}_{0,1} v = -Bv, \quad \forall v \in E_{[0]}. \tag{57}$$

So that we find

$$\left\{(-B)^k\right\}_{k \in \mathbb{Z}} \subset \mathcal{G}_{[0]}. \tag{58}$$

Actually you can prove that equality holds. This can be seen both as a consequence of Question 11 or by considering the covering $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and lifting E to $p^* E \to \mathbb{R}$. With an analogous argument as the one outlined before, one can now show that $p^* E$ has a global parallel section, so that the parallel transport along curves in $\mathbb{R}$ does not depend on the curve but only on the endpoints.

5 Question 12

We observed that if $E \to \mathcal{M}$ is a vector bundle, the space of connections $\mathcal{C}(E)$ is an affine space whose associated vector space is $\Gamma(T^*\mathcal{M} \otimes \mathrm{End}(E))$. We consider now an **inner product** g on E. By this we mean a section of the bundle $E^* \otimes E^*$, which is also symmetric and positive definite on each fiber. Using partition of unity we see that on E there are plenty of such inner products. Since for every x we have a vector space with inner product (E_x, g_x), we can speak of the antisymmetric transformations of this space and denote them by $\mathcal{A}(E_x, g_x)$. We collect all these linear maps into a vector bundle $\mathcal{A}(E, g) \to \mathcal{M}$.

Definition 3. *A trivialisation (U, χ) of E is said to be **orthonormal** if and only if the metric induced on the fibers of $U \times \mathbb{R}^k$ is the standard one.*

Proposition 2. *There exist local orthonormal trivialisations for (E, g) around every point on $\mathcal{M}$.*

Proof. Start with a local frame around some $x_0 \in \mathcal{M}$ and then apply the Gram-Schmidt orthonormalisation method in each point of the neighbourhood. We get an orthonormal frame which varies smoothly at each point since the Gram-Schmidt method is smooth. □

Remark 2. *We remark that elements of $\mathcal{A}(E_x, g_x)$ are represented by antisymmetric matrices in any orthonormal frame at x. By the previous proposition we know that, using orthonormal frames, we can represent sections of $\mathcal{A}(E, g) \to \mathcal{M}$ as sections of $U \times \mathcal{A}(\mathbb{R}^k, g_0)$ on every sufficiently small open set $U \subset \mathcal{M}$.*

Definition 4. *We shall say that* $\nabla \in \mathcal{C}(E)$ *is* ***compatible*** *with* g *if and only if*

$$\forall (x,v) \in T\mathcal{M},\ \forall s_1, s_2 \in \Gamma(E), \quad v(g[s_1,s_2]) = g_x[\nabla_v s_1, s_2] + g_x[s_1, \nabla_v s_2]. \tag{59}$$

We denote the space of compatible connections by $\mathcal{C}(E,g)$.

Proposition 3. *The set* $\mathcal{C}(E,g)$ *is not empty and it is an affine space with* $\Gamma(T^*\mathcal{M} \otimes \mathcal{A}(E,g))$ *as the associated vector space.*

Proof. The existence of a compatible connection follows as in Question 2, when one requires the local trivialisations to be orthonormal. Let $\nabla \in \mathcal{C}(E,g)$ and let $\widetilde{\nabla} \in \mathcal{C}(E)$. From Question 2 we know that there exists $A \in \Gamma(T^*\mathcal{M} \otimes \mathrm{End}(E))$ such that

$$\widetilde{\nabla} = \nabla^A = \nabla + A. \tag{60}$$

Using Equation (59) we see that

$$\nabla^A \in \mathcal{C}(E,g) \quad \Longleftrightarrow \quad A \in \Gamma(T^*\mathcal{M} \otimes \mathcal{A}(E,g)). \tag{61}$$

□

Suppose now that $E = T\mathcal{M}$. Then we can associate to every connection $\nabla \in \mathcal{C}(T\mathcal{M})$ a tensor $\tau^\nabla \in \Lambda^2 T^*\mathcal{M} \otimes T\mathcal{M}$ called the **torsion**:

$$\tau^\nabla[X,Y] = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{62}$$

In order to show that it is a tensor one needs to prove that

$$\forall f \in C^\infty(\mathcal{M}), \quad \tau^\nabla[fX,Y] = f\tau^\nabla[X,Y] = \tau^\nabla[X,fY]. \tag{63}$$

How does the torsion change under the $\Gamma(T^*\mathcal{M} \otimes \mathrm{End}(T\mathcal{M}))$ action?

Proposition 4. *Let* $\nabla \in \mathcal{C}(T\mathcal{M})$ *and* $A \in \Gamma(T^*\mathcal{M} \otimes \mathrm{End}(T\mathcal{M}))$. *Then*

$$\tau^{\nabla^A}[X,Y] = \tau^\nabla[X,Y] + A(X)[Y] - A(Y)[X]. \tag{64}$$

Finally, we also defined in Question 1 geodesics associated to a connection ∇ on $T\mathcal{M}$. Analogously to what we did for the torsion, we would like to investigate how geodesics change under the action of $\Gamma(T^*\mathcal{M} \otimes \mathrm{End}(T\mathcal{M}))$.

Proposition 5. *Let* $\nabla \in \mathcal{C}(T\mathcal{M})$ *and* $A \in \Gamma(T^*\mathcal{M} \otimes \mathrm{End}(T\mathcal{M}))$. *Then* ∇^A *and* ∇ *have the same geodesics if and only if*

$$\forall x \in \mathcal{M},\ \forall X, Y \in T_x\mathcal{M}, \quad A(X)[Y] + A(Y)[X] = 0 \iff A(X)[X] = 0. \tag{65}$$

Proof. Take $(x,v) \in T\mathcal{M}$ and consider a geodesic γ for both connections such that $\widetilde{\gamma}(0) = (x,v)$. Then,

$$0 = \frac{(\nabla^A)^\gamma}{dt}\Big|_{t=0} \dot{\gamma} = \frac{\nabla^\gamma}{dt}\Big|_{t=0} \dot{\gamma} + A(\dot{\gamma}(0))[\dot{\gamma}(0)] = A(v)[v]. \tag{66}$$

□

Endow $T\mathcal{M}$ with a metric g. Let ∇ be the Levi Civita connection associated to g. This means that $\nabla \in \mathcal{C}(T\mathcal{M}, g)$ and that its torsion is zero. It can be shown that this connection is unique. We know that length extremising curves parametrised by arc length are geodesics for ∇. Therefore, we conclude that a connection $\widetilde{\nabla}$ has the same geodesics as ∇ if and only if the geodesics for $\widetilde{\nabla}$ are length extremising curves parametrised by arc length. This leads us to the following result.

Proposition 6. *Let $\widetilde{\nabla} \in \mathcal{C}(T\mathcal{M}, g)$ and ∇ be the Levi Civita connection. We know that there exists $A \in \Gamma(T^*\mathcal{M} \otimes \mathcal{A}(T\mathcal{M}, g))$ such that $\widetilde{\nabla} = \nabla^A$. Then the following properties are equivalent.*

1. *The geodesics of ∇^A are length extremising curves parametrised by arc length.*
2. *The tensor A is antisymmetric in the first two variables*

$$\begin{aligned} A(X)[Y] &= -A(Y)[X], \quad \textit{or equivalently,} &(67)\\ A(X)[X] &= 0. &(68)\end{aligned}$$

3. *The torsion tensor of ∇^A is antisymmetric in the second and third variable*

$$\begin{aligned} g[\tau^{\nabla^A}(X)[Y], Z] &= -g[Y, \tau^{\nabla^A}(X)[Z]], \quad \textit{or equivalently,} &(69)\\ g[\tau^{\nabla^A}(X)[Y], Y] &= 0. &(70)\end{aligned}$$

4. $\tau^{\nabla^A} = 2A$,
5. $g[\tau^{\nabla^A}[\cdot,\cdot],\cdot]$ *and* $g[A(\cdot)[\cdot],\cdot]$ *are smooth 3-forms. In other words, if*

$$\Lambda^3 T^*\mathcal{M} \subset T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes T^*\mathcal{M} \tag{71}$$

is the vector bundle of antisymmetric trilinear forms on $T\mathcal{M}$, then

$$g[\tau^{\nabla^A}[\cdot,\cdot],\cdot],\ g[A(\cdot)[\cdot],\cdot] \in \Gamma(\Lambda^3 T^*\mathcal{M}). \tag{72}$$

We shall denote the set of connections satisfying any of these properties by $\mathcal{C}_0(T\mathcal{M}, g)$.

Let us now prove that $\mathcal{C}_0(T\mathcal{M}, g)$ contains only one element when the dimension is 2 and infinitely many when the dimension is bigger. To achieve this, we only have to compute the rank of the vector bundle $\Lambda^3 T^*\mathcal{M}$ in order to show the existence of a nonzero section B. Then the tensor A we are looking for will be univoquely determined by the relation

$$B[\cdot,\cdot,\cdot] = g[A(\cdot)[\cdot],\cdot]. \tag{73}$$

We need the following observation from linear algebra.

Lemma 4. *The vector space of trilinear antisymmetric forms on* $\mathbb{R}^k$ *has dimension* $\begin{pmatrix} k \\ 3 \end{pmatrix}$*.*

Proof. In order to compute the dimension of this space fix a basis of $\mathbb{R}^k$ and notice that a trilinear antisymmetric function is univoquely determined by its values on the ordered subsets of the basis made of three elements. The number of such subsets is $\binom{k}{3}$. □

Applying the next result to $\Lambda^3 T^*\mathcal{M}$, we answer completely to Question 12.

Proposition 7. *Let* $E \to \mathcal{M}$ *be a vector bundle. If the rank of* E *is zero, then* $\Gamma(E)$ *contains only the zero section. When the rank is bigger than zero, there are infinitely many elements in* $\Gamma(E)$*.*

Proof. Use local trivialisations and smooth functions with compact support. □

1. Give an example of a Riemannian manifold not geodesically complete, but so that every p and q can be joined by a length-minimizing geodesic. Give an example of a Riemannian manifold for which any two points p and q can be joined by a geodesic, but for which there exist two points $\tilde{p}$ and $\tilde{q}$ which cannot be joined by a length-minimizing geodesic.

2. Recall the definition of the Riemann curvature tensor

$$R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z - \nabla_{[Y,X]}Z. \tag{1}$$

Show the identities

$$R(X,Y)Z = -R(Y,X)Z,$$
$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$
$$g(R(X,Y)Z,W) = g(R(X,W)Z,Y),$$
$$g(R(X,Y)Z,W) = -g(R(X,Y)W,Z).$$

Show that given now a vector bundle $E \to \mathcal{M}$, and a connection ∇, the definition (**??**) still makes sense, and $R \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes E^* \otimes E)$. This is called the curvature tensor of the connection. Are the above identities (those that make sense, that is) still true?

3. Let $\mathbb{S}^n$ denote the standard n-sphere, and let $\mathbb{H}^n$ denote hyperbolic n-space. The latter is defined by the manifold $\{(x^1,\ldots,x^n) : (x^1)^2 + \cdots (x^n)^2 < 1\}$, with metric $g = -4((dx^1)^2 + \cdots (dx^n)^2)(1 - ((x^1)^2 + \cdots (x^n)^2))^{-1}$. Compute the Riemann curvature, sectional curvatures, Ricci curvature, and scalar curvature. Show that $\mathbb{H}^n$ is geodesically complete.

4. Let $\mathcal{N}$ denote a submanifold of codimension 1 of an n-dimensional Riemannian manifold $(\mathcal{M}, g)$. We define the second fundamental form, on a subset $\mathcal{U} \subset \mathcal{N}$, as follows: Let N denote a unit normal field on $\mathcal{U}$, i.e. a vector field defined along $\mathcal{U}$ such that $g(N,N) = 1$ and $g(N,T) = 0$ for all $T \in T_p\mathcal{N}$ for $p \in \mathcal{U}$. (There are two choices for N. Note which of the definitions that follow depend on the choice, and which do not.) For X, Y vector fields along $\mathcal{U}$, let $\tilde{X}$, $\tilde{Y}$, $\tilde{N}$, denote arbitrary extensions to a neighborhood $\tilde{\mathcal{V}}$ of $\mathcal{N}$ in $\mathcal{M}$, and define $B(X,Y) = -g(\nabla_{\tilde{X}}N, \tilde{Y})$. Show that this definition does not depend on the extensions. B is thus a covariant 2-tensor, i.e. an element of $\Gamma(T^*\mathcal{N} \otimes T^*\mathcal{N})$. Show moreover that B is symmetric.

5. Let $\mathcal{N}$, $(\mathcal{M}, g)$ be as above. Let B_{ij} denote (as usual) the components of the tensor B with respect to local coordinates on $\mathcal{N}$, and let $\bar{g}_{ij}$ denote the induced Riemannian metric on $\mathcal{N}$. Let $k_1 \ldots k_{n-1}$ denote the eigenvalues of B_{ij} with respect to g_{ij}. These are known as the *principal curvatures*. We call $\frac{1}{n-1}(k_1 + \cdots + k_{n-1}) = H$ the *mean curvature*. Show that $(n-1)H = g^{ij}B_{ij}$. We call $k_1 \cdot k_2 \cdots k_{n-1} = K$ the *Gauss curvature*. Show that if $n = 3$, $K = 2R$ where R is the scalar curvature of $(\mathcal{N}, \bar{g})$. This is the Theorema Egregium of Gauss. Derive a general relation relating the second fundamental form B_{ij} and the Riemann curvature tensor R^l_{kij}, valid in all dimensions $n \geq 2$.

6. For a connected Riemannian manifold, prove that metric completeness implies that every p and q can be joined by a length-minimizing geodesic by filling in the details to the following sketch: Let $\delta = d(p,q)$. Clearly, there exists a sequence of curves γ_i joining p and q such that $\lim L(\gamma_i) = \delta$. Extract a convergent subsequence of the γ_i. Argue that the limit of this sequence is the desired geodesic.

7. Now adapt the above argument to prove the following: Suppose $\mathcal{M}$ is compact. If $\pi_1(\mathcal{M})$ is nontrivial, then for each $[\gamma] \in \pi_1(\mathcal{M})$, there exists a $\tilde{\gamma} \in [\gamma]$ such that $\tilde{\gamma}$ is a closed geodesic, and $\tilde{\gamma}$ is length minimizing in its homotopy class, i.e. $L(\tilde{\gamma}) \leq L(\tilde{\gamma}')$ for all $\tilde{\gamma}' \in [\gamma]$. Does the existence of a closed geodesic in $(\mathcal{M}, g)$ imply that π_1 is non-trivial?

8. Now let $\mathcal{S}_1$ and $\mathcal{S}_2$ be smooth closed hypersurfaces in a connected Riemannian manifold $(\mathcal{M}, g)$. Suppose $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$, and suppose there exists an $\epsilon > 0$ and a compact set K, such that for all $p, q \in \mathcal{M} \setminus K$, $p \in \mathcal{S}_1$, $q \in \mathcal{S}_2$, we have $d(p,q) \geq \epsilon + d(\mathcal{S}_1, \mathcal{S}_2)$. Show that there exists a curve γ joining $\mathcal{S}_1$ and $\mathcal{S}_2$ which minimizes the distance between these two hypersurfaces. Show that γ is a geodesic orthogonal to both hypersurfaces. Now suppose conversely that γ is some geodesic connecting $\mathcal{S}_1$ and $\mathcal{S}_2$, which is orthogonal to both hypersurfaces. Show that γ locally extremizes the length functional for curves joining the hypersurfaces. Show by explicit example that γ is not necessarily a length minimizing curve. Investigate examples in Euclidean space.

9. Let $(\mathcal{M}, g)$ denote a Riemannian manifold, and let X, and Y be vector fields defined in a neighborhood of some point $p \in \mathcal{M}$, such that $[X, Y] = 0$ identically. For $t_0 \geq 0$, let A_{t_0} denote the parallel transport operator correponding to the curve $\gamma : [0, 4t_0] \to \mathcal{M}$ defined by

$$t \mapsto (\phi_t^X)(p), 0 \leq t \leq t_0$$

$$t \mapsto \phi_{t-t_0}^Y(\phi_{t_0}^X(p)), t_0 \leq t \leq 2t_0$$
$$t \mapsto \phi_{t-2t_0}^{-X}(\phi_{t_0}^Y(\phi_{t_0}^X(p))), 2t_0 \leq t \leq 3t_0$$
$$t \mapsto \phi_{t-3t_0}^{-Y}(\phi_{t_0}^{-X}(\phi_{t_0}^Y(\phi_{t_0}^X(p)))), 3t_0 \leq t \leq 4t_0,$$

where ϕ_t^X denotes the 1-parameter local group of transformations defined by X. We assume that t_0 is sufficiently small so the maps referred to above are defined. Note that $A_{t_0} : T_p\mathcal{M} \to T_p\mathcal{M}$. Show that

$$A_{t_0}(Z) = Id - t_0 R(X, Y)Z + o(t_0).$$

10. Recall the holonomy groups $\mathcal{G}_p$ of the previous example sheet. Use the above formula to show that for "generic" Riemannian metrics, $\mathcal{G}_p = SO(n)$. Justify your definition of genericity.

Differential Geometry Example Sheet 3 Errata

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January 21, 2013

Question 2 The third equation should be

$$g(R(X,Y)Z,W) = g(R(Z,W)X,Y). \tag{1}$$

Question 3 The expression for the metric in the hyperbolic space should be

$$g = \frac{4}{(1-((x^1)^2+\cdots+(x^n)^2))^2}((dx^1)^2+\cdots+(dx^n)^2). \tag{2}$$

Question 4 The definition of the bilinear form B should be

$$B(X,Y) = -g(\nabla_{\widetilde{X}}\widetilde{N}, \widetilde{Y}). \tag{3}$$

Question 5 In order to prove the Theorema Egregium of Gauss, you should assume that $(\mathcal{M}, g)$ is a flat manifold. Moreover for $n = 3$ the equation should read as

$$2K = R. \tag{4}$$

Question 7 The beginning of the third sentence should be changed in "*For each nonzero* $[\gamma] \in \pi_1(\mathcal{M})$ *there exists a* $\widetilde{\gamma} \in [\gamma]$ *such that ...*".

Question 8 You should also assume that $\mathcal{M}$ is complete.

Question 9 In the expansion for $A_{t_0}(Z)$ you should substitute t_0^2 for t_0, namely

$$A_{t_0}(Z) = \mathrm{Id} - t_0^2 R(X,Y)Z + o(t_0^2). \tag{5}$$

Differential Geometry Example Sheet 3 Solutions

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February 4, 2013

1 Question 2

Consider a vector bundle E over a smooth manifold $\mathcal{M}$ of dimension n and let ∇ be a linear connection on E. We can associate to ∇ its curvature tensor

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{1}$$

R is a section of the bundle $T^*\mathcal{M}\otimes T^*\mathcal{M}\otimes E^*\otimes E \to \mathcal{M}$. We will see that proving *Properties A to D* will correspond to check that the curvature is actually a section of particular smaller subbundles of $T^*\mathcal{M}\otimes T^*\mathcal{M}\otimes E^*\otimes E$.

Property A holds without any further assumption and it says that R is antisymmetric in the first two variables. This means that R is a section of $\Lambda^2 T^*\mathcal{M}\otimes E^*\otimes E$.

Suppose furthermore that E is endowed with an inner product g. In other words g is a section of the bundle $\mathrm{Sym}^2(E^*)\subset E^*\otimes E^*$ of symmetric bilinear forms and moreover it satisfies the positivity property

$$\forall x\in\mathcal{M},\ \forall Z_x\neq 0\in E_x,\quad g_x(Z_x,Z_x)>0. \tag{2}$$

We say that g and ∇ are **compatible** if and only if

$$X\big(g(Z,W)\big) = g\left(\nabla_X Z, W\right) + g\left(Z,\nabla_X W\right). \tag{3}$$

In textbooks, this is sometimes written omitting X as

$$dg(Z,W) = g\left(\nabla Z, W\right) + g\left(Z,\nabla W\right). \tag{4}$$

We can use g to lower one index of the curvature tensor and consider

$$R^*(X,Y,Z,W) := g(R(X,Y)Z,W), \tag{5}$$

which is now a section of $\Lambda^2 T^*\mathcal{M}\otimes E^*\otimes E^*$, since *Property A* translates into

$$\textit{Property A'}\quad R^*(X,Y,Z,W) = -R^*(Y,X,Z,W). \tag{6}$$

A calculation shows that compatibility of the metric yields *Property D*. This condition can also be rewritten as

$$\textit{Property D'}\quad R^*(X,Y,Z,W) = -R^*(X,Y,W,Z). \tag{7}$$

In other words R^* is antisymmetric also in the last two entries and therefore it is a section of $\Lambda^2 T^*\mathcal{M} \otimes \Lambda^2 E^*$.

In the following discussion, especially when we will deal with *Question 10*, it will be convenient to express the curvature as a linear map on the space of bivectors $\Lambda^2 T\mathcal{M}$. This goal can be achieved in two ways.

i) Define as $\mathrm{End}_g(E)$ the subbundle of $\mathrm{End}(E) = E^* \otimes E$ whose elements are g-antisymmetric endomorphisms of E. Namely,

$$L \in \mathrm{End}_g(E) \implies g(LZ_1, Z_2) = -g(Z_1, LZ_2). \tag{8}$$

Then, define $\mathfrak{R}^* : \Lambda^2 T\mathcal{M} \to \mathrm{End}_g(E)$ as the bundle map

$$\mathfrak{R}^* \left(\sum_i X_i \wedge Y_i \right)(Z) = \sum_i R(X_i, Y_i)Z. \tag{9}$$

ii) Using the metric tensor g we can put an inner product on the space of bivectors $\Lambda^2 E$. On simple bivectors $Z_1 \wedge W_1$, $Z_2 \wedge W_2$ it has the form

$$\widetilde{g}(Z_1 \wedge W_1, Z_2 \wedge W_2) := g(Z_1, Z_2)g(W_1, W_2) - g(Z_1, W_2)g(W_1, Z_2). \tag{10}$$

This inner product allows us to raise the last two indices of R^* and express the curvature as an operator $\mathfrak{R} : \Lambda^2 T\mathcal{M} \to \Lambda^2 E$. On simple bivectors the curvature operator is *defined* as follows

$$\widetilde{g}(\mathfrak{R}(X \wedge Y), W \wedge Z) = R^*(X, Y, Z, W). \tag{11}$$

We analyse now *Property C*. First of all we observe that, since we are swapping X with Z, we must necessarily have $E = T\mathcal{M}$. Then, *Property C* says that $\mathfrak{R} : \Lambda^2 T\mathcal{M} \to \Lambda^2 T\mathcal{M}$ is a **symmetric operator** with respect to the inner product $\widetilde{g}$ on bivectors introduced before. Hence we see that this condition is not implied by *Property D* and it gives a further restriction. Nevertheless one can ask:

Exercise Does *Property C* follow from the compatibility with the metric?

Also in this case we rewrite the property using R^*:

$$\textit{Property C'} \quad R^*(X, Y, Z, W) = R^*(Z, W, X, Y). \tag{12}$$

Finally let us deal with the second property of the list. Again we must require that $E = T\mathcal{M}$. A computation shows that if ∇ is torsion-free, *Property B* holds. We point out that the torsion tensor can be defined only for connections on $T\mathcal{M}$.

If we have both metric compatibility with some g and zero torsion, ∇ is the Levi Civita connection of g. In this case *Properties A,B and D* hold and yield as a purely algebraic consequence *Property C*, i.e. the symmetry of the curvature operator. Moreover, in the presence of a metric, we can use R^* in order to express *Property B* in the equivalent way

$$\textit{Property B'} \quad R^*(X, Y, Z, W) + R^*(Y, Z, X, W) + R^*(Z, X, Y) = 0. \tag{13}$$

We can call $F^{\mathcal{M}} \hookrightarrow \Lambda^2 T^*\mathcal{M} \otimes \Lambda^2 T^*\mathcal{M}$ the subbundle whose element are tensors η which satisfy *Property A', B', D'* (and hence also *C'*). Notice that this space does not depend on g.

Exercise The bundle $F^{\mathcal{M}}$ has rank $\frac{n^2(n^2-1)}{12}$, where n is the dimension of $\mathcal{M}$. For any metric g on $\mathcal{M}$ the following ones are sections of $F^{\mathcal{M}}$:

1. $\overline{g}(X,Y,Z,W) := \widetilde{g}(X \wedge Y, W \wedge Z)$,
2. the Riemann tensor R^*.

2 Question 3

Using stereographic projection we find coordinates $(x^i) : \mathbb{S}^n \to \mathbb{R}^n$ such that the metric is given by

$$g_{\mathbb{S}^n} = f^2_{\mathbb{S}^n}(r) g_0, \tag{14}$$

where

1. $r = \sqrt{\sum_i (x^i)^2}$ is the function giving the radial distance to the origin,
2. $f_{\mathbb{S}^n} : [0, +\infty) \to (0, +\infty)$ is the real function given by

$$f_{\mathbb{S}^n}(y) = \frac{2}{1+y^2}, \tag{15}$$

3. g_0 is the Euclidean metric on $\mathbb{R}^n$.

In the same way one defines the hyperbolic n-space $\mathbb{H}^n$ to be the open unit disc $B^n \subset \mathbb{R}^n$ with the metric

$$g_{\mathbb{H}^n} = f^2_{\mathbb{H}^n}(r) g_0, \tag{16}$$

where $f_{\mathbb{H}^n} : [0, 1) \to (0, +\infty)$ is the real function given by

$$f_{\mathbb{H}^n}(y) = \frac{2}{1-y^2}. \tag{17}$$

We see that both metrics are conformally flat, hence the spherical angles and the hyperbolic angles are the same as the Euclidean angles. In particular the spheres $\{r = R\}$ are perpendicular to the radial direction in each geometry. On Spivak's *A comprehensive introduction to differential geometry, Volume II, Chapter 7, Addendum II* you can find the computation of the curvature for metrics conformal to the flat one, with generic conformal factor f. There f can be any positive function on some subset of $\mathbb{R}^n$. In this way we give an answer to the first part of the question.

We will end this section by showing that $\mathbb{H}^n$ is complete. In order to prove this fact we observe that radii are length minimising curves in every *conformally flat rotationally symmetric* geometry. Hence they are geodesics up to reparametrisation. Let f be the conformal factor and suppose that it depends only on the radial distance r. Define $F(r) := \int_0^r f(r')dr'$ and introduce polar coordinates (r, θ). Let $\gamma = (r_\gamma, \theta_\gamma)$ be a curve from the origin to some point

$x_0 = (r_0, \theta_0)$ and let $\gamma_0(r) = (r, \theta_0)$ be the radial curve defined on $[0, r_0]$. Since the metric is conformally flat we see that the arc-length satisfies

$$|\dot\gamma(t)| \geq |\dot r_\gamma(t)| \left| \frac{\partial}{\partial r} \right|_{r=r_\gamma(t)} = f(r(t))|\dot{r_\gamma}(t)|. \tag{18}$$

Therefore

$$\begin{aligned} \ell(\gamma) = \int_I |\dot\gamma(t)|\, dt &\geq \int_I f(r(t))|\dot{r_\gamma}(t)| dt \\ &\geq \left| \int_I f(r(t)) \dot{r_\gamma}(t) dt \right| \\ &= \left| \int_I \frac{d}{dt} F(r(t)) dt \right| \\ &= F(r_0) \\ &= \ell(\gamma_0). \end{aligned}$$

Exercise Actually one can use F to exhibit normal coordinates around the origin. Just take the diffeomorphism $(r, \theta) \mapsto (F(r), \theta)$. Check that this is indeed a diffeomorphism and find the expression of the metric in the new coordinates.

In the hyberbolic case one finds

$$\begin{aligned} F_{\mathbb{H}^n} : [0, 1) &\to [0, +\infty) \\ r &\mapsto \log \frac{1+r}{1-r}. \end{aligned}$$

Therefore, all the closed balls for the hyberbolic metric centered at the origin are compact, since

$$\overline{B_{\mathbb{H}^n}(0, R)} = \{r \leq r_R\}, \tag{19}$$

where $r_R = F^{-1}(R) = \dfrac{e^R - 1}{e^R + 1} = \tanh \dfrac{R}{2} < 1$. The fact that $r_R < 1$ implies that $\{r \leq r_R\}$ is a compact subset of $\{r < 1\}$. This is exaclty the smooth manifold on which we are putting the hyperbolic metric $g_{\mathbb{H}^n}$.

3 Question 9

For the case of surfaces this question is simply a variation of the ideas at the basis of Gauss-Bonnet Theorem. Suppose $\gamma : [0, 1] \to S$ is a closed curve on a surface S and that γ bounds a region $R \subset S$. The element of the holonomy group associated to γ (or to R) will be a rotation in the tangent plane at $\gamma(0) = \gamma(1)$ of an angle $\Delta\theta(R)$. This is the angle between $Z(1)$ and $Z(0)$, where Z is any parallel vector field along γ. It is given by the formula

$$\Delta\theta(R) = \int_R K dA, \tag{20}$$

where K is the Gaussian curvature and dA is the area element. If we shrink R to a point $p \in S$ we get

$$\Delta\theta(R) = K(p)\Delta A(R) + o(\Delta A(R)), \tag{21}$$

where $\Delta A(R)$ is the area of the region R. Taking R as a small parallelogram whose sides are parallel to some commuting vector fields X and Y we get the formula contained in the Question.

Now we prove the general case. The following argument is entirely contained in the graduate project *On Parallel transport and curvature* by Raffaele Rani, which you can easily find on the web.

Let $\mathcal{M}$ be a Riemannian manifold of arbitrary dimension and let X and Y be two commuting fields defined near a point p. We can use their flows in order to construct a map

$$\begin{array}{rcl} f:[-t_0,t_0]\times[-t_0,t_0] & \to & \mathcal{M} \\ (x,y) & \mapsto & \Phi^X_x\big(\Phi^Y_y(p)\big). \end{array}$$

Then, for each $t \leq t_0$, we consider the parallelogram of side t parallel to X and Y. We collect all of them in a family of curves $H:[0,t_0]\times[0,1]\to\mathcal{M}$:

$$H(t,s) := \begin{cases} f(4st,0), & 0\leq s\leq 1/4, \\ f(t,(4s-1)t), & 1/4\leq s\leq 1/2, \\ f((3-4s)t,t), & 1/2\leq s\leq 3/4, \\ f(0,(4-4s)t), & 3/4\leq s\leq 1. \end{cases} \tag{22}$$

Hence, for fixed t, the curve $H^t := H(t,\cdot)$ goes around the parallelogram of side t as s goes from 0 to 1.

Take now $Z_0 \in T_p\mathcal{M}$ and define $A_t(Z_0) \in T_p\mathcal{M}$ as the vector obtained by parallel transporting Z_0 along the whole H^t. Observe that this is the same map considered in the Example Sheet 3, since the parallel transport does not depend on the parametrisation of the curve. Our goal will be to compute the expansion of $A_t(Z_0)$ in $t=0$ up to the second order:

$$\bullet\ A_0(Z_0) = ?, \quad \frac{d}{dt}\Big|_{t=0} A_t(Z_0) = ?, \quad \bullet\ \frac{d^2}{dt^2}\Big|_{t=0} A_t(Z_0) = ?. \tag{23}$$

Clearly $A_0(Z_0) = Z_0$, since H^0 is the constant path. In order to compute the first and second derivative we argue as follows.

Denote by $P^t_s : T_p\mathcal{M} \to T_{H^t(s)}\mathcal{M}$ the parallel transport operator along $H^t\big|_{[0,s]}$. Then, the extension of Z_0 to a vector field Z over $[0,t_0]\times[0,1]$ parallel along each H^t can be written as

$$\begin{array}{rcl} Z:[0,t_0]\times[0,1] & \to & T\mathcal{M} \\ (t,s) & \mapsto & P^t_s Z_0. \end{array}$$

We have the relations

$$\bullet\ Z(t,0) = Z_0, \quad \bullet\ Z(t,1) = A_t(Z_0), \quad \bullet\ \nabla_{\partial_s} Z = 0. \tag{24}$$

We would like to look for an analogous of $\frac{d}{dt}A_t(Z_0)$ as s changes. The expression $\frac{\partial}{\partial t}Z(t,s)$ is meaningless, since $Z(t_1,s)$ and $Z(t_2,s)$ live in $T_{H(t_1,s)}\mathcal{M}$ and $T_{H(t_2,s)}\mathcal{M}$ respectively, which might be different. We overcome this problem by using the covariant derivative along the t-direction. Consider the vector field $W(t,s) := \nabla_{\partial_t} Z(t,s)$ and notice that

$$\bullet\ W(t,0) = \frac{d}{dt}A_0(Z_0) = \frac{d}{dt}Z_0 = 0, \qquad \bullet\ W(t,1) = \frac{d}{dt}A_t(Z_0). \tag{25}$$

We now compute the variation of W in s using parallel transport. Indeed, observe that $\left(P_s^t\right)^{-1} W(t,s) \in T_{f(t,0)}\mathcal{M} = T_p\mathcal{M}$ belongs to a fixed vector space (namely $T_p\mathcal{M}$), so that we can take its derivative in s. In order to compute this partial derivative, remember the formula

$$\frac{d}{ds}\left(P_s^t\right)^{-1} = \left(P_s^t\right)^{-1}\nabla_{\partial_s}. \tag{26}$$

Exercise Prove identity (26) by writing a vector field along a curve in term of a parallel basis.

Using the previous relation and the definition of curvature one finds

$$\begin{aligned}
\frac{d}{ds}\left(P_s^t\right)^{-1}W(t,s) &= \left(P_s^t\right)^{-1}\nabla_{\partial_s}\nabla_{\partial_t}Z(t,s) \\
&= \left(P_s^t\right)^{-1}\Big(R_{H(t,s)}(\partial_t,\partial_s)Z(t,s) + \nabla_{\partial_t}\nabla_{\partial_s}Z(t,s)\Big) \\
&= \left(P_s^t\right)^{-1} \circ R_{H(t,s)}(\partial_t,\partial_s) \circ P_s^t\left(Z_0\right) \\
&= R_{t,s}(Z_0),
\end{aligned}$$

where we have defined

$$R_{t,s} := \left(P_s^t\right)^{-1} \circ R_{H(t,s)}(\partial_t,\partial_s) \circ P_s^t \in \mathrm{End_g}(\mathrm{T_p}\mathcal{M})\,. \tag{27}$$

Integrating the previous relation between 0 and 1, we get

$$\left(P_1^t\right)^{-1}W(t,1) - \left(P_0^t\right)^{-1}W(t,0) = \left(\int_0^1 R_{t,s}ds\right)(Z_0) \tag{28}$$

and finally

$$\frac{d}{dt}A_t(Z_0) = W(t,1) = P_1^t\left(\int_0^1 R_{t,s}ds\right)(Z_0) \tag{29}$$

Let us write down $R_{t,s}$:

$$R_{t,s} = \begin{cases} 0 & 0 \le s < 1/4 \text{ or } 3/4 < s \le 1, \\ t\left(P_s^t\right)^{-1} \circ R_{H(t,s)}(X,Y) \circ P_s^t & 1/4 \le s \le 3/4. \end{cases} \tag{30}$$

Therefore,

$$\frac{d}{dt}A_t(Z_0) = tP_1^t\left(\int_{1/4}^{3/4}\left(P_s^t\right)^{-1} \circ R_{H(t,s)}(X,Y) \circ P_s^t ds\right)(Z_0). \tag{31}$$

Taking derivatives we see that

- $\left.\dfrac{d}{dt}\right|_{t=0} A_t(Z_0) = 0,$
- $\left.\dfrac{d^2}{dt^2}\right|_{t=0} A_t(Z_0) = P_1^0\left(\displaystyle\int_{1/4}^{3/4} R_{H(t,0)}(X,Y)ds\right)(Z_0) = 2R_p(X,Y)Z_0$

These identities prove the expansion

$$A_t(Z_0) = Z_0 + t^2 R_p(X,Y)Z_0 + o(t^2). \tag{32}$$

4 Question 10

From an intuitive point of view we would like to prove that we can always perturb a little an arbitrary $(\mathcal{M}, g)$ and get another Riemannian manifold with $\mathcal{G}_p = SO(n)$ at some point $p \in \mathcal{M}$ (and hence at any point). We will see that it is enough to fix $\mathcal{M}$ and $p \in \mathcal{M}$ and vary only the metric in an arbitrary small neighbourhood of the point. Thus we consider the set

$$\mathrm{Met}(\mathcal{M}) := \{g \in \Gamma(\mathrm{Sym}^2(T^*\mathcal{M})),\ g \text{ positive definite}\} \tag{33}$$

and we are going to endow it with a sufficiently strong topology τ (namely Hausdorff). We would like to prove that the subset

$$B := \{g \in \mathrm{Met}(\mathcal{M}) \mid \mathcal{G}_p = SO(n)\} \tag{34}$$

is *topologically big*. The easiest way to define a topological size is to say that a set is big if it has an **open and dense** subset. The previous question helps us finding such a subset. We need first the following nontrivial fact.

Theorem 1. *The holonomy group at* p *is a Lie subgroup of* $SO(n)$*. In particular it is a smooth submanifold of* $SO(n)$*.*

From this important result we see that $\mathcal{G}_p$ is the whole special orthogonal group if and only if it contains a neighbourhood of the identity e. Indeed, suppose that $\mathcal{G}_p$ contains a neighbourhood U of the identity. Then, if $h \in \mathcal{G}_p$, the set $h \cdot U$, which is a neighbourhood of h, is contained in $\mathcal{G}_p$. This fact shows that $\mathcal{G}_p$ is an open subgroup of $SO(n)$. However, since

$$SO(n) \setminus \mathcal{G}_p = \bigcup_{h \notin \mathcal{G}_p} h \cdot \mathcal{G}_p, \tag{35}$$

we see that also its complement is an open set. Since $SO(n)$ is connected this implies that $\mathcal{G}_p = SO(n)$.

Now we observe that checking if $\mathcal{G}_p$ is a neighbourhood of the identity e is the same as checking

$$T_e\mathcal{G}_p = T_e SO(n) \tag{36}$$

because of the *Inverse Function Theorem*. The vector space $T_e SO(n)$ corresponds to the g-antisymmetric linear maps. In *Question 2* we used for it the notation $\mathrm{End}_g(T_p\mathcal{M})$. Then, the results contained in *Question 9* imply

$$\mathfrak{R}^*_p(\Lambda^2_p\mathcal{M}) \subset T_e\mathcal{G}_p \subset \mathrm{End}_g(T_p\mathcal{M}), \tag{37}$$

where $\mathfrak{R}^*$ is the curvature operator defined in *Question 2*. Therefore if the linear map $\mathfrak{R}^*_p$ associated to some metric g is surjective, we can conclude that $g \in B$. Since $\Lambda^2_p\mathcal{M}$ and $\mathrm{End}_A(T_p\mathcal{M})$ have the same dimension this is the same as asking for $\mathfrak{R}^*_p$ to be an isomorphism.

On the other hand since $\mathfrak{R}_p$ and $\mathfrak{R}^*_p$ are related by some form of metric duality we can require equivalently that $\mathfrak{R}_p$ is an isomorphism. Since $\mathfrak{R}_p$ is an endomorphism of $\Lambda^2 T\mathcal{M}$, we can check this property by looking at the determinant of $\mathfrak{R}_p$. Hence we consider the following real function on $\mathrm{Met}(\mathcal{M})$:

$$\begin{aligned} D_p : \mathrm{Met}(\mathcal{M}) &\to \mathbb{R} \\ g &\mapsto \det \mathfrak{R}_p. \end{aligned}$$

All the previous discussion can be summarised in

$$\{D_p \neq 0\} \subset B. \tag{38}$$

Thus, the goal for the rest of this section will be to show that $\{D_p \neq 0\}$ is open and dense with respect to the topology τ. In order to achieve this result we will need τ with the following two properties

i) D is continuous,

ii) for any $u \in \Gamma(\mathrm{Sym}^2(T^*\mathcal{M}))$

$$\lim_{\varepsilon \to 0} g + \varepsilon u = g. \tag{39}$$

As far as the first property is concerned, we notice that, taking local coordinates around p, $D_p(g)$ becomes a polynomial in the coefficient of the curvature tensor. These in turn are polynomials in the Taylor expansion of the coefficients of the metric up to second order at p. From this observation we see that, in order to achieve the first condition, τ must control the derivative of the metric g up to second order.

We will postpone the actual construction of the topology until the end of this section. Now let us point out as the two conditions we gave guarantee that $\{D_p \neq 0\}$ is open and dense.

On the one hand openness stems from the fact that the preimage under a continuous map of an open set is open.

On the other hand, in order to show that $\{D_p \neq 0\}$ is dense, it is sufficient to find a section u such that the set

$$\{\varepsilon \mid D_p(g + \varepsilon u) \neq 0\} \subset \mathbb{R} \tag{40}$$

has $\varepsilon = 0$ in its closure. The second condition previously imposed on τ guarantees that this is enough.

Let us prove the existence of such an u. First of all consider a coordinate chart $(x^k) : (U, p) \to (\mathbb{R}^n, 0)$ around p. Let g_{ij} and R^*_{iklj} the coordinate expression of g and R^* respectively. For the sake of notation we will denote partial derivatives of a function f in the chart by

$$\frac{\partial}{\partial x^k} f = f_{,k}. \tag{41}$$

Suppose now that in the coordinates (x^k) the metric g is Euclidean up to the first order at p. Namely, we are asking that

$$\bullet\ g_{ij}(0) = \delta_{ij}, \qquad \bullet\ g_{ij,k}(0) = 0. \tag{42}$$

Observe that such coordinates always exist, since normal coordinates around p satisfy these two properties.

Take $\eta^* \in F_0^{\mathbb{R}^n}$ an element in the fibre over the origin of the subbundle of $T^*\mathbb{R}^n \otimes T^*\mathbb{R}^n \otimes T^*\mathbb{R}^n \otimes T^*\mathbb{R}^n$ introduced in *Question 2*. Let η^*_{iklj} be the coordinates of η^* with respect to the standard basis. As we did when we discussed *Question 2* we can associate to η^* an operator $H : \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n$, using the duality induced by the flat metric δ_{ij}. We shall assume that $\det H \neq 0$. We claim that such an η^* can always be found. We start with the identity map $\mathrm{Id} : \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n$. Then we take the duality isomorphism in the opposite direction to get a form $\mathrm{id}^* \in \mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$ whose coordinates expression is

$$\mathrm{id}^*_{iklj} = \delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}. \tag{43}$$

From one of the exercises in *Question 2* we know that id^* belongs to $F_0^{\mathbb{R}^n}$ and thus the claim is proved.

We now define u locally around p in the (x^k) coordinates as

$$u_{ij} := -\frac{1}{3}\sum_{k,l} \eta^*_{iklj} x^k x^l \tag{44}$$

and away from p we cut u to zero using a bump function. Since η^* satisfies *Property C'*, we know that u is indeed a symmetric bilinear form. The local expression for $g^\varepsilon := g + \varepsilon u$ is

$$g^\varepsilon_{ij} = g_{ij} - \frac{\varepsilon}{3}\sum_{k,l} \eta^*_{iklj} x^k x^l, \tag{45}$$

so that all the metrics g^ε are Euclidean up to the first order at p and

$$g^\epsilon_{ij,kl}(0) = g_{ij,kl}(0) - \frac{\varepsilon}{3}(\eta^*_{iklj} + \eta^*_{ilkj}). \tag{46}$$

In the following lemma we describe how to compute the curvature R^* in coordinates for these kind of metrics.

Lemma 1. *Suppose that $\overline{g}_{ij}$ is a metric on $\mathbb{R}^n$ which is Euclidean up to the first order at the origin. Then its curvature tensor satisfies*

$$\begin{aligned} 2\overline{R}^*_{iklj}(0) &= \overline{g}_{il,kj}(0) + \overline{g}_{kj,il}(0) - \overline{g}_{ij,kl}(0) - \overline{g}_{kl,ij}(0) & (47)\\ &= \mathcal{A}_{j,l}\left(\overline{g}_{il,kj}(0) - \overline{g}_{kl,ij}(0)\right), & (48)\end{aligned}$$

where $\mathcal{A}_{j,l}$ is the antisymmetrisation operator in the indexes j and l.

Exercise Prove the Lemma by computing the Christoffel symbols first.

We can apply the lemma to g^ε. Using identity (46), we get

$$2R^{*\varepsilon}_{iklj}(0) = 2R^*_{iklj}(0) - \frac{\varepsilon}{3}\mathcal{A}_{j,l}\left(\eta^*_{ikjl} + \eta^*_{ijkl} - \eta^*_{kijl} - \eta^*_{kjil}\right). \tag{49}$$

Applying *Property B'* to the second summand we find

$$\mathcal{A}_{j,l}\left(\eta^*_{ikjl} + \eta^*_{ijkl} - \eta^*_{kijl} - \eta^*_{kjil}\right) = -6\eta^*_{iklj}. \tag{50}$$

Therefore we have found the relation

$$R^{*\varepsilon}_{iklj}(0) = R^*_{iklj}(0) + \varepsilon\eta^*_{iklj}. \tag{51}$$

Switching to the operator formulation we rewrite this identity as

$$\mathfrak{R}^\varepsilon_p = \mathfrak{R}_p + \varepsilon H. \tag{52}$$

We can use this formula to compute the determinant:

$$\begin{aligned}\det\left(\mathfrak{R}_p^{\varepsilon}\right) &= \det\left(\mathfrak{R}_p + \varepsilon H\right)\\ &= \det(H)\varepsilon^{\binom{n}{2}} + \ldots + \det(\mathfrak{R}_p).\end{aligned}$$

Hence $D_p(g^{\varepsilon})$ is a polynomial in the variable ε whose leading coefficient is nonzero because of the assumptions we made on H. Since the polynomial is nonzero, its zeroes are isolated and therefore the statement about density has been proven.

Finally it remains to check that we can find τ as claimed. Let us choose

i) a natural number r bigger than or equal to 2;

ii) a finite collection of charts for $\mathcal{M}$, $\overrightarrow{\phi} = (\phi_a : U_a \to V_a)_{a\in A}$, where $U_a \subset \mathcal{M}$ and $V_a \subset \mathbb{R}^n$;

iii) a finite collection of compact sets $\overrightarrow{K} = (K_a)_{a\in A}$ such that

$$\bullet\ K_a \subset U_a, \qquad \bullet\ \mathcal{M} = \bigcup_{a\in A} K_a. \tag{53}$$

Define the following distance between elements g_1 and g_2 of $\mathrm{Met}(\mathcal{M})$

$$d^r_{(\overrightarrow{\phi},\overrightarrow{K})}(g_1, g_2) := \max_{a\in A}\left\{\sup_{x\in\phi_a(K_a)}\left|(\phi_a^{-1})^* g_1 - (\phi_a^{-1})^* g_2\right|_r(x)\right\}, \tag{54}$$

where $|\cdot|_r$ takes into account the derivatives up to order r:

$$|h|_r(x_0) := \sum_{|\alpha|=0}^{r}\left|\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} h(x_0)\right|. \tag{55}$$

Exercise Check that $d^r_{(\overrightarrow{\phi},\overrightarrow{K})}$ induces a topology with the required properties.

Hint and (sketch of) solutions - Example sheet 4

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Exercise 1 You can take any proper open geodesically convex subset of a complete manifold. A good example is a finite interval inside a line, or any ball inside an Euclidean space.

For the second example, you can take a sphere minus a point, or a flat torus minus a point.

Review on metric space Let (T, d) be a metric space. T is complete if every Cauchy sequence converges. Let K be a subspace of T. K is totally bounded if, for every positive ϵ, there exists a finite collection of points k_i of K such that K is contained in the union of the balls of centre t_i and radius ϵ. The following fact holds: *If T is complete and K is closed, then K is compact if and only if it is totally bounded.* This property is needed in the proof of the following lemma

Lemma 0.1. Let (M, g) be a finite dimensional Riemannian manifold. Call D the distance induced by g. If M is complete with respect to D, then, for every point p and every positive number R, the closed ball of centre p and radius R is compact.

Proof. Fix p, and call B_R the closed ball $B(p, R)$. Call $I \subset \mathbb{R}_{>0}$ the set of R such that B_R is compact. If some r is in I, then I contains the interval $(0, r]$, this because a closed subset of a compact set is compact. We are going to prove that I is non-empty, open and close.

Non-empty. Take a local chart U around p, this is isomorphic (as a topological space) to an open subset of $\mathbb{R}^n$. If R is small enough, then B_R is contained in U, so B_R is compact and R is in I.

We will need the following auxiliary fact (which does not hold in a generic metric space). Let R and δ be positive numbers, and call ∂ the boundary of $B(p, R)$.

$$B(p, R+\delta) \subset B(p, R) \cup \bigcup_{q \in \partial} B(q, \delta). \tag{1}$$

The proof is as follow. Pick a point x of $B(p, R+\delta)$, let γ be a path from p to x. The length of γ, $L(\gamma)$, is equal to $R + \delta + \epsilon$. Because of the definition of D, we can not assume that ϵ is zero, but we can take it as small as we wish. The distance is continuous, so there exists a t such that $D(p, \gamma(t)) = R$, i.e. $\gamma(t) \in \partial$, call $a := \gamma(t)$. The length of the path γ restricted to $[t, 1]$ is $\delta + \epsilon$, so $D(a, x) \leq \delta + \epsilon$. We have shown that

$$B(p, R+\delta) \subset B(p, R) \cup \bigcup_{q \in \partial} B(q, \delta + \epsilon)$$

for every positive ϵ. The intersection of infinitely many closed set is closed, so we get the claim.

Open. Suppose that R is in I, we prove that, for some δ, $R+\delta$ is in I as well, so $(0, R+\delta)$ is in I and I is open. The boundary ∂ of B_R is compact, so we can pick a (small) r such that the ball $B(q,r)$ is compact for every q in ∂. We cover ∂ with a finite number of balls $B(q_i, \frac{r}{2})$. Call Z the union of $B(p,R)$ and the balls $B(q_i, r)$, we want to show that Z contains the ball $C := B(p, R+\frac{r}{2})$. Take x in $C \setminus B_R$, because of 1 there exists q in ∂ with $D(q,x) \leq \frac{r}{2}$. By construction, there exists q_i with $D(q_i, q) \leq \frac{r}{2}$, we conclude that $D(q_i, x) \leq r$, so x is in Z. The set Z is a finite union of compact sets, so it is compact. The ball C is a closed subset of a compact set, so it is compact, we conclude that $R+\frac{r}{2}$ is in I.

Closed. Take a sequence R_n in I converging to R. Fix N such that

$$| R - R_N | < \epsilon \,.$$

Cover $B(p, R_N)$ with a finite number of balls B_j of radius ϵ. Since the boundary ∂ is compact, there exists a finite collection q_i of points on the boundary such that the balls $B(q_i, \frac{\epsilon}{2})$ cover ∂. Because of 1, the ball $B(p,R)$ is contained in the union of the balls B_j and $B(q_i, \epsilon)$, so $B(p,R)$ is totally bounded. **Now we use that M is complete** and we conclude that $B(p,R)$ is compact. (Warning: we can not assume that the ball $B(q_i, \epsilon)$ are compact) □

General idea for exercises from 6 to 8 We consider some space of "regular" paths P and the functional given by the length

$$L : P \to \mathbb{R}_{>0}$$

If L has a (local) minimum, using variational formulae we show that this minimum, up to reparemetrization, is a geodesic with the requested properties. There might be regularity issues, according to the space we are dealing with. A good discussion about space of paths is in the book *"Riemannian Geometry and Geometric Analysis", 2nd edition, by J. Jost*, section 5.4 .

To show that this minimum exists, we will need to construct some (finite dimensional) subspace K of P. First, we will show that for every γ in P there exists an η in K such that $L(\eta) \leq L(\gamma)$. Then, we will show that, for a suitable topology, K is compact and L is continues. Another possibility is to use Ascoli-Arzela' theorem.

Exercise 6 Fix points p and q on M, call R their distance. To solve our problem we can replace M with a ball of centre p and radius bigger than R (say $2R$), so we can assume that M is compact.

Let V be the space of paths γ from p to q such that γ is defined on the interval [0,1] and it is piece-wise C^∞. A (local) minimum of L is a smooth geodesic because of the first variational formula. Consider the functional given by the energy:

$$\begin{array}{rccc} E: & V & \to & \mathbb{R}_{>0} \\ & \gamma & \mapsto & \int_I g(\dot\gamma, \dot\gamma) \end{array}$$

We could endow V with a distance which makes both L and E continues, but actually we do not need it. This metric is discussed in the book suggested before.

Lemma 0.2. Let γ be a path defined on an interval $I = [0, l]$, then

$$L(\gamma)^2 \leq lE(\gamma)$$

Proof. Holder's inequality says

$$\left(\int_I \sqrt{g(\dot{\gamma}, \dot{\gamma})} \cdot 1\right)^2 \leq \int_I g(\dot{\gamma}, \dot{\gamma}) \int_I 1$$

the RHS is equal to $E(\gamma)l$. $\square$

For every positive constant C, let

$$V_C := \{\gamma \,\text{s.t.}\, E(\gamma) \leq C\}$$

If we take C to be $R^2 + \epsilon$, for some positive ϵ, then for every γ in V there exists an η in V_C with $L(\eta) \leq L(\gamma)$. So it is enough to look for the global minimum of L on V_C, instead than on V. We still need a smaller space of paths.

We know that for every x in M, there exists a number $\rho(x)$ such that any point of the ball $B(x, \rho(x))$ can be joined to x via a unique geodesic. Since all balls are compact and M is compact as well, we can find a positive number ρ such that for every point x and every couple of points a and b in the ball $B(x, \rho)$ there exists a unique geodesic from a to b contained in the ball.

Let $S = \{t_1, \dots, t_k\}$ be a subset of I, with $t_i < t_{i+1}$, $t_1 = 0$ and $t_k = 1$. Call $\bar{S}$ the subset of V_C of paths which are geodesic away from the points t_i. This set could be empty, or very big.

We now fix an S such that

$$\mid t_i - t_{i+1} \mid \leq \frac{\rho^2}{C}$$

for every i. With this choice, we will see that $\bar{S}$ is quite nice. We construct a "regularising" operator

$$R : V_C \to \bar{S}$$

as follow. Given a path γ, because of the lemma we have

$$D(\gamma(t_i), \gamma(t_{i+1})) \leq \rho$$

so there exist a unique geodesic from $\gamma(t_i)$ to $\gamma(t_{i+1})$. We define $R(\gamma)$ to be the union of the geodesics from $\gamma(t_i)$ to $\gamma(t_{i+1})$ for every i. (In particular, this shows that $\bar{S}$ is not empty.) Inside every ball, the geodesic is a length-minimize path, so

$$L(R(\gamma)) \leq L(\gamma)\,.$$

Now, we need to find a minimum for L in $\bar{S}$. A path γ in $\bar{S}$ is uniquely determined by the values of $\gamma(t_i)$. Let us denote by M^k the k-fold product of M. We have an injective map

$$\begin{array}{ccc} \bar{S} & \hookrightarrow & M^{k-2} \\ \gamma & \mapsto & (\gamma(t_2), \dots, \gamma(t_{k-1})) \end{array}$$

In particular, $\bar{S}$ can be identified with the set

$$\{(x_2, \dots, x_{k-1}) \mid D(x_i, x_{i-1}) \leq \rho\} \tag{2}$$

where $x_1 = p$ and $x_k = q$. $\bar{S}$ inherits a topology from M^{k-2}, with this topology it is a closed subset of M^{k-2}. Since M^{k-2} is compact, $\bar{S}$ itself is compact. The functional L is continue on $\bar{S}$, so it has a global minimum.

Exercise 7 The solution of this exercise is similar to the solution of exercise 6. We just point out the difference.

Fix a class $[\gamma]$ in $\pi_1(M, p)$. Pick a C^∞-representative γ, replacing M with $B(p, L(\gamma))$ we can assume that M is compact.

We define V_C as before, but we take $p = q$ and we ask that every loop has the homotopy type of γ.

We can define S and $\bar{S}$ in the same way. We have to check that the homotopy type of $R(\gamma)$ is the same of γ for every loop γ. Call γ_i the restriction of γ to the interval $[t_i, t_{i+1}]$. We need to show that the image of γ_i is contained in a simply connected set, so it can be deformed to the geodesic joining $\gamma(t_i)$ and $\gamma(t_{i+1})$. To get this, we need that $L(\gamma_i) \leq \rho$, so the condition

$$\mid t_i - t_{i+1} \mid \leq \frac{\rho^2}{C}$$

is enough.

The description 2 of $\bar{S}$ does not work any more. We need to show that $\bar{S}$ is closed in M^{k-2}. Given a sequence γ_n in S which converges, we just need to show that the limit has the same homotopy time of γ_n. Which is doable.

An example of simply connected Riemannian variety with closed geodesic is the sphere. In this case, every geodesic is closed and contractible. Another interesting example is the flat torus: there are many closed geodesics, but none of them is contractible.

Exercise 8 To show the existence of γ we argue as in exercise 6. Now, V_C is the space of paths with starting point in S_1 and ending point in S_2. Looking for the minimum, we can restrict our attention to paths with starting point in $K \cap S_1$ and ending point in $K \cap S_2$. With this choice, $\bar{S}$ is a subset of $K \times M^{k-2} \times K$, which is compact because both K and M are. So the global minimum exists.

We now have to write down carefully the first variational formula for γ: since the start and end points are moving, we have a boundary term in the integral. Consider a vector field $v(t)$ along γ, this defines a variation with parameter s, we have

$$\frac{d\gamma_s}{ds}(0) = \text{usual terms } + g(v(0), \dot{\gamma}(0)) + g(v(1), \dot{\gamma}(1))$$

Usually both $v(0)$ and $v(1)$ are zero, because the starting and ending points are fixed. In our case, they are vectors tangent to S_1 and S_2 respectively. Using bump functions one shows that

$$g(v(0), \dot{\gamma}(0)) = 0$$

for every vector $v(0)$ tangent to S_1, so γ is orthogonal to S_1. The same holds for S_2. This computation shows as well that if γ is a geodesic orthogonal to S_1 and S_2, then

$$\frac{d\gamma_s}{ds}(0) = 0$$

for every variation γ_s.