MATHEMATICAL ANALYSIS I - 2016/2017 1st term
Lecturers: Fabio Ciolli, Alfonso Sorrentino
Tutor: Aleks Jevnikar

Program
Differential calculus for real functions: real and complex numbers; elementary real functions and their inverse: polynomial, exponential, logarithm, trigonometric etc.; concept of limit, limits of indefinite forms; continuity, properties of continuous functions, uniform continuity; derivatives, maxima and minima, the graph of a function; De L'Hopital's Rule; Taylor expansions. Integral calculus for real functions: antiderivatives, Riemann integrals; improper integrals. Numerical series. Elementary differential equations of first and second order. Introduction to multivariable calculus: continuity, differentiation, directional derivatives, gradient; higher order differentiations, Hessian matrix.

Textbooks
Apostol, T. M., Calculus Vol.1, second edition; John Wiley & Sons, (1974).
Canuto C., Tabacco A., Mathematical Analysis I & II; Springer International Publishing, UNITEXT 84, (2015).
Trench, William F., Introduction to Real Analysis (2013). *Faculty Authored Books.* Book 7. http://digitalcommons.trinity.edu/mono/7. See also author page.

S Y L L A B U S

03.10.16 (Course presentation.) Basic elements of numerical sets: Naturals, Integers, Rationals and Reals $(\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R})$. Definition of rational numbers. Decimal alignments.

05.10.16 Sets. Set-theoretical operations. Characteristic property defining a set. Power set. Venn diagrams. Cardinality. Connectives, Predicates, Quantifiers.

06.10.16 Real numbers: non rationality of $\sqrt{2}$ (with proof *per absurdum*, optional $\sqrt{n}$). Properties of $\mathbb{Q}$ as an ordered commutative field. Density and Archimedean properties for $\mathbb{Q}$. The properties of $\mathbb{Q}$ extend to $\mathbb{R}$. Exercises: inequalities with absolute value and degree 2 polynomials.

07.10.16 Bounded and unbounded intervals. Open and closed intervals. Real line $\mathbb{R}$ and extended real line $\mathbb{R}^*$. Bounded sets in $\mathbb{R}$ and $\mathbb{R}^*$. Maximum and minimum (max and min) of a bounded set in $\mathbb{R}$. Supremum and infimum, (sup and inf) of a set in $\mathbb{R}$. Properties of sup and inf. The triangle inequality. Young inequality and Cauchy-Schwartz inequality.

10.10.16 Theorem of Completeness of $\mathbb{R}$ (without proof). Definition of real numbers. Definition of radicals. Principle of mathematical Induction. Bernoulli inequality. Factorial and Newton's binomial expansion formula. Pascal (Tartaglia) triangle. Cartesian product. Cartesian plane. Relation of (partial) order.

12.10.16 Equivalence relations. Relations on the Cartesian plane. Definition of a function, as a relation on the Cartesian plane. Domain, natural domain, restrictions, extensions; image (range) and graph of a function. Pre-image. Sequences as a functions on $\mathbb{N}$. Elementary functions: polynomials, rational functions, n^{th}-root, goniometric functions, integer part, sign function and mantissa; functions defined by cases.

13.10.16 Bounded functions. Surgectivity and injectivity. One-to-one function (bijection) and invertible function. Solving and equation defined by $f(x) = y$. Graph of the inverse function. Monotone functions, (strictly) increasing and decreasing functions. Difference quotient.

14.10.16 Example: invertibility of a family of functions depending on a parameter. Composition of functions, domain and image of the composite function: $\operatorname{dom}(g \circ f) := f^{-1}(\operatorname{im} f \cap \operatorname{dom} g)$, i.e. the pre-image under f of $\operatorname{im} f \cap \operatorname{dom} g$; $\operatorname{im}(g \circ f) := g(\operatorname{im} f \cap \operatorname{dom} g)$. Non-commutativity of the composition operation. Examples.

17.10.16 The exponential, the logarithmic and the trigonometric functions: definition, graph and properties. Complex numbers. $\mathbb{C}$ is a commutative field. Cartesian presentation. Geometric interpretation. Modulus and argument. Complex conjugate. Trigonometric presentation. De Moivre formula. Exponential presentation. Complex roots.

19.10.16 Fundamental Theorem of Algebra (without proof). Exercises: roots and equations in $\mathbb{C}$. Definition of an infinite set. Cardinality of an infinite set. Countable and uncountable infinite sets. $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Q}$ are infinite countable sets, by Cantor's diagonal argument. $\cup_{i \in \mathbb{N}} A_i$, for A_i an infinite countable set, is a countable infinite set. $\mathbb{R}$ is an uncountable infinite set (without proof). $|\mathbb{R}| = |I|$ for I any real interval. Neighbourhoods in $\mathbb{R}^n$. Euclidean norm and distance in $\mathbb{R}^n$.

20.10.16 Neighbourhoods of $\pm\infty$ in $\mathbb{R}^*$. Internal, external and boundary points of a set in $\mathbb{R}^n$. Accumulation points of a set in $\mathbb{R}^n$. Isolated points. Accumulation points of $\mathbb{N}$ and $\mathbb{Z}$ in $\mathbb{R}^*$. Open sets. Closure of a set. Dense sets. Theorem (Characterization of closed sets, with proof): if $A \subseteq \mathbb{R}^n$, A is closed $\iff \partial A \subseteq A \iff A' \subseteq A$ where A' is the derived set of A.

21.10.16 Theorem (Bolzano-Weierstass, with proof): An infinite, bounded subset of $\mathbb{R}^n$ admits at least an accumulation point. Extremes of a closed and bounded set. Global and local max/min of a function.

24.10.16 Limits of real functions and sequences: general definition by neighbourhoods; ε, δ and m, M presentation. Theorem (with proof): uniqueness of the limit. Examples of verification of the limit values, by the definition. Infinities and infinitesimals; a generic infinitesimal function is called "a little o" of Landau, in symbols $o(1)$.

26.10.16 Left and right limit. Limit from above/below. Theorem (with proof): permanence of sign in the limit. Property holding eventually for x $\to x_0$. Theorem (with proof): comparison of the limit of three functions (squeeze rule). Examples: $\lim_{x\to\pm\infty} \frac{\sin(x)}{x} = 0$; $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, i.e. $\sin(x) = x + xo(1)$, for $x \to 0$.

27.10.16 Theorem (with proof): algebras of limits, with values in $\mathbb{R}$. Example: $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$. Extension of the algebras of limits to the case of limits with values in $\mathbb{R}^*$.

28.10.16 Indeterminate forms of algebraic type. Example: $\lim_{x\to 0} |x|^\alpha \sin(\frac{1}{x}) = 0$, for $\alpha > 0$. Theorem (with proof): the limit of a composite function.

29.10.16 Variable substitution formula. Limit of rational functions. Limits by rationalization.

31.10.16 Theorem (Existence of limit for monotone functions, with proof): if $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is a monotone function, and $x_0 \in \mathbb{R}^* \cap X'$, then, in particular if f is monotone increasing and x_0 a left accumulation point for X, it holds $\lim_{x\to x_0^-} f(x) = \sup_{X\cap(-\infty,x_0)} f$. Application: existence of the limits of powers, exponential and logarithm. Indeterminate forms of exponential type $f(x)^{g(x)}$. Theorem (with proof): limit of function by convergent sequences (bridge-theorem).

02.11.16 Examples of use of the theorem by convergent sequences: non-existence of $\lim_{x\to\pm\infty} \sin x$, $\lim_{x\to 0} \sin\frac{1}{x}$ and $\lim_{x\to 0^+} x^\alpha \sin\frac{1}{x}$, for $\alpha < 0$. About limits of sequences: notation, theorems and main examples. $\lim_{x\to+\infty} \frac{x^\alpha}{a^x} = 0$ for all $\alpha \in \mathbb{R}$, $a > 1$ (with proof). $\lim_{x\to\infty} |x|^\alpha a^x = 0$ for all $\alpha \in \mathbb{R}$, $a > 1$ (with proof). $\lim_{x\to+\infty} \frac{|log_b x|^\alpha}{x^\beta} = 0$, for all $\alpha \in \mathbb{R}$, $\beta > 0$ and $b > 0$, $b \neq 1$ (with proof). $\lim_{x\to 0^+} |\log_b x|^\alpha x^\beta = 0$, for all $\alpha \in \mathbb{R}$, $\beta > 0$ and $b > 0$, $b \neq 1$ (with proof). Other special limits: $\lim_{x\to 0} \frac{\arctan(x)}{x} = 1$, $\lim_{x\to 0} \frac{\arcsin(x)}{x} = 1$.

03.11.16 Comparison of the infinite sequences: for $a > 1$, $a^n \prec n! \prec n^n$, i.e. $\lim_{n\to+\infty} \frac{a^n}{n!} = 0$ and $\lim_{n\to+\infty} \frac{n!}{n^n} = 0$ (with proof). Comparison of infinities and infinitesimals: definition by the limit of the quotient. Equivalent infinities (infinitesimals). Order of infinity (infinitesimal). If $f(x)$, $g(x)$ are infinities for $x \to x_0 \in \mathbb{R}^*$, and the order of f is greater than the order of g, we write $g(x) = o(f(x)) = f(x)o(1)$, for some $o(1)$ infinitesimal. Comparison of the infinities (and infinitesimals) of power, logarithm and exponential type. The order of the infinity of the logarithm, compared to the one of the identical function, is not a real number, is positive and smaller than any positive real number. The order of the infinity of the exponential with a base greater than 1, compared to the one of the identical function, is not a real number, is positive and greater than any positive real number. For $a, b > 1$, $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$ and for $x \to +\infty$, the infinities of logarithm, power and exponential type are ordered as follow: $(\log_b x)^\alpha \prec x^\beta \prec a^x$. "Big O" of Landau.

04.11.16 Order of infinity (infinitesimal) w.r.t. a referring or sample infinity (infinitesimal). Approximation of functions and sequences using the notation of the "little o" of Landau. Approximations obtained by the notable limits. Examples of comparison of infinitesimals or infinities. Algebras of the infinitesimals, infinities and "little o" of Landau. The Hyperreal set as an image of the set of orders of infinity and infinitesimal.

07.11.16 Asymptotes to the graph of a function: definition and examples. Vertical, horizontal and oblique asymptotes. Approximation of a function by its asymptotes. Theorem (Existence of the Napier's constant or Euler's number, with proof): $\lim_{n\to+\infty}(1 + \frac{1}{n})^n = e$. Theorem (with proof): $\lim_{x\to+\infty}(1 + \frac{1}{x})^x = e$.

09.11.16 Special limits from the definition of e (with proof): $(1 + \frac{\alpha}{x})^x = e^\alpha$ for $x \to +\infty$ and $\alpha \in \mathbb{R}$; $(1 + \alpha x)^{\frac{1}{x}} = e^\alpha$ for $x \to 0$ and $\alpha \in \mathbb{R}$; $\ln(1 + x) = x + o(x)$ for $x \to 0$; $e^x = 1 + x + o(x)$ for $x \to 0$; $a^x = 1 + \ln(a)x + o(x)$ for $x \to 0$ and $a \in \mathbb{R} \setminus \{0\}$; and $(1 + x)^\alpha = 1 + \alpha x + o(x)$ for $x \to 0$, $\alpha \in \mathbb{R}$. Stirling's formula. Exercises on limits and order of infinity or infinitesimal.

10.11.16 Sub-sequences: definition and examples. Theorem: $a_n \to \ell \in \mathbb{R}^* \iff$ any sub-sequence b_n of a_n, satisfies $b_n \to \ell$ (sketch of proof). Theorem (with proof): a bounded sequence with values in $\mathbb{R}$ admits a convergent subsequence. Fundamental (or Cauchy-) sequences. Theorem (Cauchy criterion for sequences, with proof): a sequence with values in $\mathbb{R}$ is convergent $\iff$ it is a Cauchy-sequence. Sequences defined by a recurrence relation: Definition and examples. Criterion for the possible value of the limit of a recurrence sequence.

11.11.16 Use of induction and theorems for limit of recurrence sequences. Fibonacci's sequence and golden ratio. Exercises on limit of sequences and recurrence sequence.

14.11.16 Continuous functions: definition and examples. The algebras of continuous functions. Theorems: sign permanence and composition of continuous functions. Function continuous on an interval. Classes of continuous functions. Points of discontinuity: removable, jump or essential kind. Continuous prolongation of a function.

16.11.16 (4 hours) Theorem (with proof): a continuous, monotone, function on an interval may have at most a denumerable infinity number of discontinuity points only of jump kind or removable kind, if at the extremes of the interval. Theorem (with proof): a continuous function f defined on a closed interval $[a, b]$ with $f(a)f(b) < 0$, admits a zero in (a, b); if the function is strictly monotone then the zero is unique. Corollary (with proof): existence of the solutions of the equation $f(x) = g(x)$, for f and g continuous functions on an interval. Theorem (with proof): if f is continuous on a interval I (bounded or unbounded) then $f(I)$ is an interval, i.e. f assumes all the values between $\inf_I f \geq -\infty$ and $\sup_I f \leq +\infty$.

17.11.16 Invertibility of continuous functions. Examples. Theorem (with proof): if f is continuous and invertible on an interval, then f is strictly monotone. Theorem (with proof): a continuous function on an interval is invertible $\iff$ is strictly monotone. Theorem (Continuity of the inverse, with proof): a continuous invertible function on an interval or closed bounded interval admits a continuous inverse function.

18.11.16 Sequentially compact subspace in $\mathbb{R}^n$. Theorem: a subspace in $\mathbb{R}^n$ is sequentially compact $\iff$ is compact $\iff$ is closed and bounded. Example: $[a,b]$, a bounded and closed interval, is compact. Theorem (with proof): if K is a compact subset of $\mathbb{R}$ and f a continuous over K, then f(K) is compact and f admits max and min on K (Weierstrass theorem). The case of K=[a,b], closed and bounded interval. Examples and counterexamples. Theorem: if f is a continuous and invertible function on a compact domain K, then f^{-1} is continuous and $\operatorname{dom} f^{-1} = f(K)$. Uniform continuity: definition and examples. Theorem (Heine-Cantor): if a function is continuous on a compact set then is uniformly continuous on this set. Theorem (Algebras of uniformly continuous functions): if f and g are uniformly continuous functions over a domain X then, for $\alpha \in \mathbb{R}$, αf and $f+g$ are uniformly continuous functions on X; if $g : X \to \mathbb{R}$ and $f : g(X) \to \mathbb{R}$ are uniformly continuous functions, then their composition $f \circ g : X \to \mathbb{R}$ is a uniformly continuous function on X. The point-by-point product of two uniform continuous functions is not, in general, a uniform continuous function.

21.11.16 Theorem (Extendibility by continuity to the closure): if f is a uniform continuous function on X, then f is extendible by continuity to the closure $\overline{X}$; moreover, f is uniform continuous on a bounded $X \iff f$ is extendible by continuity to $\overline{X}$. Theorem (Boundedness of a uniform continuous function): if f is uniform continuous on X then it is bounded on any bounded subset $A \subseteq X$. Corollary: a function f that admits vertical asymptotes is not uniform continuous. Definition: a function f is a Lipschitz function on $X \iff$ it admits bounded difference quotient for any $x, y \in X$. A Lipschitz function on X is uniform continuous on X. Sub-linear functions f on an half-line is uniform continuous and $f(x) = O(x)$, for $x \to +\infty$ (or $-\infty$). Theorem (uniform continuity on unbounded set): if f is continuous on an unbounded subset X and is uniform continuous on any bounded subset of X, then f is uniform continuous on X if one of the following is true: *a)* f admits an oblique asymptote; *b)* f is Lipschitz on an half-line; or *c)* f is periodic.

23.11.16 Example: $x \to \sin(x^2)$ is not uniform continuous on $[1, +\infty)$. Oscillation of a function: $\omega_A(f) := \sup_A f - \inf_A f$, if f is bounded in A; $+\infty$ otherwise. Proposition (Oscillation criterion for uniform continuity): a function f is uniform continuous on $X \iff$ for every $\varepsilon > 0$ exists $\delta > 0$: $\forall A \subseteq X$ with diam(A) $< \delta$ it holds $\omega_A(f) < \varepsilon$. Differential Calculus of real functions of one real variable. (Non-vertical) tangent line to the graph of a function. Differentiability (i.e. derivability) of a function in a point. Geometrical meaning of the derivative of a function in a point. Best linear approximation. Theorem (Derivability and continuity, with proof): if f is derivable in x_0 then it is continuous in x_0. Vertical tangent and infinite derivative in a point. Approximation of a derivable function in a point and error and best linear approximation of the function.

24.11.16 Calculation of the derivative: easy examples (constant function and monomials). Left/right derivative. Corner points and cusps. n-th derivative and notation. Theorem (Algebras of the derivative, with proof): linearity, product and quotient of derivable functions in a point. Leibniz formula for the n-th derivative of the product. Theorem (Derivative of the composed function - "chain rule", with proof). Theorem (Derivative of the inverse function, with proof). Derivative of the elementary functions.

25.11.16 Derivative of the elementary functions: x^α, e^x, $\sinh(x)$, $\cosh(x)$, $\ln|x|$, a^x and $\ln_a |x|$ for $a > 0$, $a \neq 1$, $\sin(x)$, $\cos(x)$, $\tan(x)$, $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$, $|x|$. f is even (odd) $\iff f'$ is odd (even). Calculus of derivatives using the definition or rules and theorems. Examples: $f_\alpha(x) = x^\alpha \sin(\frac{1}{x})$ for $x \neq 0$ and $f_\alpha(0) = 0$, for $\alpha \geq 0$. Continuity, differentiability (derivability) and regularity of class C^n and C^∞. Function $C^n(X)$, for $X \subset \mathbb{R}$ and $C^n(\mathbb{R})$, $C^\infty(\mathbb{R})$.

28.11.16 Extrema and critical points. Theorem (Fermat, with proof): if x_0 is a local extremum for f and f is derivable in x_0 then $f'(x_0) = 0$. Examples and counterexamples. Search for local and global extrema of a function on a closed interval. Theorem (Lagrange's Mean value th, with proof). Theorem (Rolle, with proof). Th (Cauchy, with proof). Corollary (of Lagrange th, with proof): the one-side limit of the derivative equals the one-side derivative.

30.11.16 Corollary (of Lagrange th, with proof): if f is derivable on an open interval then f' do not admits jump discontinuity; Theorem (relation between monotony and sign of f', with proof). Differentiability and Lipschitz functions: a function with bounded derivative on an interval is Lipschitz. Theorem (De l'Hôpital, with proof); examples and counterexamples. Taylor's theory of expansion (or approximating) polynomials.

01.12.16 Taylor Polynomials and Mac Laurin polynomials: definitions. Theorem (Peano, with proof): existence, uniqueness and properties of Taylor polynomial. Examples of elementary Mac Laurin polynomials. Mac Laurin polynomial of an even (odd) function. Peano's and Lagrange's remainder formulas. Composition and linearity of expansions.

02.12.16 Expansion of f' from the expansion of f, and vice versa. Calculation of Mac Laurin polynomials of a composite function. Application of Peano's theorem and Taylor expansion to the computation of limits.

05.12.16 Application of Taylor expansion to the computation or the order of infinitesimal/infinite of a function. Obtaining a Taylor expansion in the point $x_0 \neq 0$ from the corresponding Mac Laurin expansion. Theorem (with proof): about a function having the derivative $f^k(x_0) = 0$ for $k = 1, \ldots, n-1$ and $f^n(x_0) \neq 0$ and existence of a local extremum in x_0, for n an even or odd natural number. Convex functions of one variable: definition and examples. Convex combination of two points. Convex set. A function is convex $\iff$ its epigraph is a convex set.

07.12.16 Lemma (without proof): a geometrical characterization of (strictly) convexity. Theorem (with proof): A convex function on an open interval is continuous and admits left and right derivative at any point of the interval, as two increasing functions with $f'_+(x) \geq f'_-(x)$. Theorem (with proof): if a function f is derivable in an open interval, then f is (strictly) convex $\iff$ f' is (strictly) increasing and iff the tangent line at any point of the interval x_0 lays (strictly) below the graph of f at any point of the interval $x \neq x_=0$. If f is 2-times derivable, f is convex $\iff$ $f^{''} \geq 0$; if $f^{''} > 0$ then f is strictly convex. Inflection point. Theorem (with proof): Let f be a function on an interval, 2-times derivable in x_0 in the interval, with x_0 an inflection point for f; then $f^{''} = 0$.

12.12.16 Primitive (antiderivative) of a function and indefinite integral. Theorem (with proof): the absence of jump discontinuity is a necessary condition for the existence of a primitive. Theorem (Unicity, with proof): a family of primitives is obtained from a primitive, up to an additive real constant. Examples and counterexamples. Indefinite integral. Primitives of elementary functions. Existence and continuity of the primitives. Linearity of the indefinite integral. Integration by parts.Integration by substitution: change of variable in the integration formula. Relevant substitutions for rational functions of some irrational or trigonometric functions.

14.12.16 Integration of rational functions. Decomposition of a polynomial with real coefficient in irreducible factors. Decomposition of rational maps in linear sum of simple partial fractions. Methods for determining the coefficients in the linear sum of simple partial fractions. Integration of rational functions: primitives of the simple partial fractions: structure of the solution and types of primitives. Examples: Primitives of $\frac{ax+b}{x^2+px+q}$, for any a, b, p and q real numbers. Relevant substitutions for rational functions of some irrational or trigonometric functions. Examples.

15.12.16 Nonelementary antiderivatives: examples. Vector (linear), normed and metric spaces on $\mathbb{R}$. Scalar product, euclidean norm and distance. $\mathbb{R}^n$ as a normed and metric real vector space. Functions of two or more variables: domain, image, graph and geometrical interpretation.

19.12.16 Two variable function: level sets and curves. Topology in $\mathbb{R}^n$: open and closed ball, neighbourhoods, accumulation points. Definition of limits. Use of the bridge theorem and polar coordinates for the computation of limits. Continuity of two variable function: definition and examples. Use of general theorems for limits and continuity.

21.12.16 Use of the triangular and the Cauchy-Schwarz inequality for the computation of limits. Limits at the boundary points of a function defined by cases. Directional and partial derivatives; derivability and gradient at a point. Examples.

22.12.16 Differentiability of a n-variables function: definition example and counterexamples. Theorem (Relation between differentiability and continuity and derivability at a point, with proof) Differentiability at a point is a sufficient condition for continuity, derivability and existence of any directional derivatives of f at a point. Best linear approximation and tangent plane.

23.12.16 Condition $C^n(X)$ for function $f : X \to \mathbb{R}$, with $X \subseteq \mathbb{R}^n$ open set. Proposition: $f \in C^1(X) \Rightarrow f$ is differentiable in X. Higher order derivatives and differentials. Theorem (Schwarz, about mixed partial derivatives): if f is twice differentiable on X (or, stronger condition, $f \in C^2(X)$) then $\partial_i \partial_j f(\underline{x}) = \partial_j \partial_i f(\underline{x})$ for $i, j = 1, \ldots, n$. Second differential and Hessian matrix. Properties of the Hessian matrix as a quadratic form (in particular for case $n = 2$): symmetry, positive definiteness, semi-positive definiteness, eigenvalues. Theorem: a twice differentiable $f : X \to \mathbb{R}$ with $X \subseteq \mathbb{R}^n$ an open, convex set, is convex (strongly convex) $\iff$ is semi-positive definite (positive definite). $\underline{x} \in \operatorname{dom} f$ is a critical point iff $\nabla f(\underline{x}) = \underline{0}$. Theorem: $f : X \to \mathbb{R}$ with $X \subseteq \mathbb{R}^n$ and $\underline{x}$ a local max / min for f s.t. f is differentiable in $\underline{x}$, then $\underline{x}$ is a critical point for f. Theorem: Critical points, convexity, second differential and existence of local maxima and minima.

09.01.17 The vector ∇f is the maximum rate of change of the function at that point and is is orthogonal to the level curve (or set) at this point. Example and exercises about local maxima and minima of two variable functions: second differentiable case; use of the definition of local max/min and max/min at non regular points.

11.01.17 Riemann integration theory. Definite integrals. Partitions of a closed bounded interval of the real line and refinement of partitions. Poset of the partition of a closed bounded interval. Lower sums and upper sums of a bounded function defined over a closed bounded interval. Approximation of the sub-area of the graph of a function. Definition of Riemann integral. The (algebraic) sub-area of the graph of a function as a definite integral. Examples of Riemann integrable function: constant function, constant function on a open bounded interval. The Dirichelet function on a bounded interval is not Riemann integrable.

12.01.17 The norm of a partition. Theorem (Criterion of Riemann integrability, with proof): for every $\epsilon > 0$ exists a partition P_ϵ of the interval such that $S(P_\epsilon, f) - s(P_\epsilon, f) < \epsilon$. Theorem (with proof): continuous functions on a bounded, closed interval are Riemann integrable. Theorem (with proof): monotone functions on a bounded, closed interval are Riemann integrable. Theorem (with proof): bounded functions on bounded, closed interval with a finite number of discontinuity points, are Riemann integrable. Generalization to a denumerable set of discontinuity points. Two functions defined on a bounded, closed interval that are equal up to a finite number of points, are both integrable, with the same value of the integral or both not integrable. Generalization to a denumerable set. Riemann integral sums. Theorem (without proof): a limit criterion for Riemann integrability. Theorem (with proof): properties of the Riemann integral (linearity; positivity; the positive, the negative parts and the absolute value are Riemann integrable; additivity w.r.t. the interval of integration).

13.01.17 Theorem (with proof): integral mean values. Integral function. Corollaries (Fundamental Theorem of integral calculus; integral function and primitives, with proof). Application of Riemann integration: the calculus of the area of subsets of the real plane; the length of an arc of curve, given as graph of a real variable function.

16.01.17 Improper Integrals: 1st fundamental example: $f(x) = \int \frac{1}{x^\alpha} dx$, for $\alpha \in \mathbb{R}$, over $(0, 1]$ and $[1, +\infty)$. Definition of an improper integral, both for unbounded function or unbounded domain, as a limit over the extremes of a Riemann integral. Convergent and divergent improper integrals. Decomposition of an improper integral w.r.t. the domain of integration. Geometrical meaning of the convergence of an improper integral. Relation between Riemann integrability and improper integrability: continuity and independence of the limits in the definition.

18.01.17 Theorem (Comparison test for improper integral, with proof). Corollary (order test for improper integrals). 2nd fundamental example: $\int_2^{+\infty} f(x) = \frac{1}{x \ln^\beta(x)}\, dx$, for $\beta \in \mathbb{R}$. 3nd fundamental example: $\int_2^{+\infty} f(x) = \frac{1}{x^\alpha \ln^\beta(x)}\, dx$, for α, $\beta \in \mathbb{R}$. Absolute improper integrability: definition and examples. Theorem (with proof): absolute improper integrability is a sufficient condition, and non necessary, for simple integrability. Examples and exercises.

19.01.17 Miscellaneous exercises on improper integrability: examples of simply convergent, but not absolutely convergent, improper integrals. Numerical series: definition and examples of (easy) summable series: Mengoli series, geometric series, $\sum_{k=1}^{+\infty} \ln(1+\frac{1}{k})$. Theorem (Necessary condition of simple convergence, with proof): if $\sum_k^{+\infty} a_k$ is convergent then $a_k \to 0$ for $k \to +\infty$. Theorem (Cauchy Criterion for series, with proof): a series is converging $\iff$ its sequences of partial sums is Cauchy (i.e. fundamental). Example: the harmonic series is divergent.

20.01.17 Numerical series and improper integrals. Theorem (Integral Criterion for positive terms series, with proof). Examples: generalized harmonic series. Example of approximation of the sum of a series by improper integral. Example of an function integrable in the improper sense on the half line but non bounded. Theorem (Comparison test for positive terms series, with proof).

23.01.17 Theorem (Leibniz Criterion for alternating sign series, with proof). Theorem (Ratio test, with proof). Theorem (Root test, with proof). Miscellaneous exercises.

25.01.17 Introduction to ordinary differential equations (ODE): physical motivations; definitions and classification of ODE. First order ODE: normal form, solutions, general integral and integral curves, Cauchy problem. Equation with separable variables. Examples and miscellaneous exercises.

26.01.17 Linear first order ODE: homogeneous and non-homogeneous case. The variation of constant method for particular solution. Cauchy problem. Examples and miscellaneous exercises.

27.01.17 Linear second order ODE, with constant coefficients: homogeneous and non-homogeneous case. Linear independence of solutions. General integral. Characteristic polynomial. The variation of constants method and *ad hoc* method for particular solution. Cauchy problem. Motivation example: harmonic motion. Miscellaneous exercises.