Exercise 1. Sketch the graph of the following function

$$f(x) = |2x-1|\, e^{\frac{1}{x-3}}\,,$$

specifying: domain, possible asymptotes, monotonicity, continuity, local and global maxima or minima, corner points (if any) and non-derivability points.

Solution

It is easy to see that $\operatorname{dom} f = \mathbb{R}\setminus\{3\}$ so $\partial\operatorname{dom} f = \{-\infty,\, 3,\, +\infty\}$. Moreover f is continuous over its domain being a composition of continuous functions.

Observe that $f(x) = 0 \iff x = \dfrac{1}{2}$. Moreover $f(x) \geq 0$ for every $x \in \operatorname{dom} f$ and, according to the definition of the absolute value, we have

$$f(x) = \begin{cases} (1-2x)e^{\frac{1}{x-3}} & \text{if } x < \dfrac{1}{2} \\[2ex] (2x-1)e^{\frac{1}{x-3}} & \text{if } x \geq \dfrac{1}{2}\,. \end{cases}$$

• Study at $\partial\operatorname{dom} f$:

$$f(x) \underset{x\to-\infty}{=} (1-2x)\left(1 + \frac{1}{x-3} + o\left(\frac{1}{x}\right)\right) = 1 - 2x - 2 + o(1) = -2x - 1 + o(1) \underset{x\to-\infty}{\to} +\infty\,,$$

hence $y = -2x - 1$ is the function of an oblique asymptote for f, when $x \to -\infty$. Similarly

$$f(x) \underset{x\to+\infty}{=} (2x-1)\left(1 + \frac{1}{x-3} + o\left(\frac{1}{x}\right)\right) = 2x - 1 + 2 + o(1) = 2x + 1 + o(1) \underset{x\to+\infty}{\to} +\infty\,.$$

and $y = 2x + 1$ is the function of an oblique asymptote for f, when $x \to +\infty$.

At point $x = 3$ we have the following asymptotic behaviours:

$$\lim_{x\to3^-} f(x) = \text{“}\ 5(1+o(1))\, e^{\frac{1}{0^-}}\ \text{”} = 0^+ \text{ and } \lim_{x\to3^+} f(x) = \text{“}\ 5(1+o(1))\, e^{\frac{1}{0^+}}\ \text{”} = +\infty;$$

so $(3, 0)$ is an asymptotic point for the function f for $x \to 3^-$ and $x = 3$ is a vertical asymptote for the function f for $x \to 3^+$.

• Monotonicity and critical points

For $x > \dfrac{1}{2}$ we have

$$f'(x) = e^{\frac{1}{x-3}}\left(2 - \frac{2x-1}{(x-3)^2}\right) = \frac{e^{\frac{1}{x-3}}}{(x-3)^2}[2(x-3)^2 - 2x + 1] = \frac{e^{\frac{1}{x-3}}}{(x-3)^2}[2x^2 - 14x + 19]\,,$$

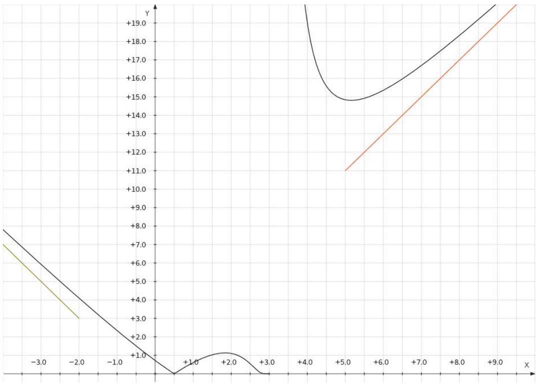
then $f'(x) \geq 0 \iff \dfrac{1}{2} \leq x \leq x_1 = \dfrac{7-\sqrt{11}}{2} \approx 2$ or $x \geq x_2 = \dfrac{7+\sqrt{11}}{2} > 5$ and $f'(x) \leq 0 \iff x_1 \leq x < 3$ and $3 < x \leq x_2$.

For $x < \dfrac{1}{2}$ we have

$$f'(x) = e^{\frac{1}{x-3}}\left(-2 - \frac{1-2x}{(x-3)^2}\right) = -\frac{e^{\frac{1}{x-3}}}{(x-3)^2}[2(x-3)^2 - (1-2x)]$$

then $f'(x) < 0$ for every $x < \dfrac{1}{2}$.

Hence $(x_1, f(x_1)$ is a local maximum for f and $(x_2, f(x_2)$ is a local minimum for f.



Moreover, because of the Lagrange Theorem, it is easy to see that $f'_+\left(\frac{1}{2}\right) = \lim_{x\to\frac{1}{2}^+} f'(x) = 2^{-\frac{2}{5}}$ and $f'_-\left(\frac{1}{2}\right) = \lim_{x\to\frac{1}{2}^-} f'(x) = -2^{-\frac{5}{2}}$ so that the point $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right)$ is a corner point and a local minimum, and the global one, of f.

Exercise 2. For any $\alpha \in \mathbb{R}$, the following function is given

$$f(x) = \begin{cases} |\alpha| \arctan\left(\dfrac{1}{x}\right), & x > 0 \\ \\ (\alpha+1)x^2 - 2x + 1\,, & x \leq 0\,. \end{cases}$$

1. Find the values of α (if any) such that f result to be continuous or differentiable or globally invertible on its domain;
2. Fix $|\alpha| = 1$ and write the equation of the tangent line to the graph of the (local) inverse function f^{-1} of f, in the point $\left(\frac{\pi}{4}, f^{-1}\left(\frac{\pi}{4}\right)\right)$.

Solution

1. Regularity

Start observing that $f \in C^\infty(\mathbb{R}\setminus\{0\})$. Then, by definition, f is continuous in $x = 0 iff |\alpha|\frac{\pi}{2} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) = 1$, hence $\iff \alpha = \pm\frac{\pi}{2}$.

Then we have:

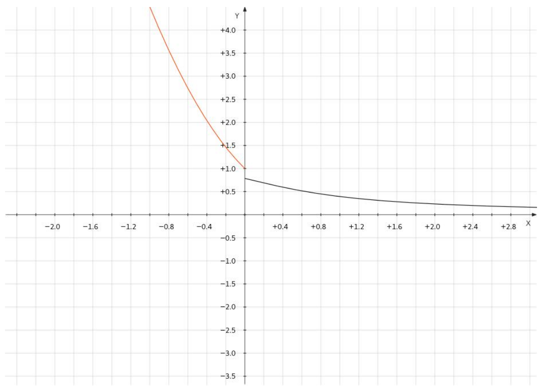
$$f'(x) = \begin{cases} |\alpha|\dfrac{1}{1+\frac{1}{x^2}}\left(-\dfrac{1}{x^2}\right) = -\dfrac{|\alpha|}{1+x^2}\,, & x > 0 \\ \\ 2(\alpha+1)x - 2\,, & x < 0\,. \end{cases}$$

So f is derivable in $x = 0$ *iff* f is continuous in $x = 0$, i.e. $\alpha = \pm\frac{\pi}{2}$ and also $\lim_{x\to 0^+} f'(x) = -|\alpha| = \lim_{x\to 0^-} f'(x) = -2$, hence $\alpha = \pm 2$. Since the two conditions are not compatible, the function f is not differentiable at $x = 0$ for any $\alpha \in \mathbb{R}$.

1. Global invertibility

Observe that the two branches of the function are continuous and for $\alpha = 0$ the branch for $x > 0$ is not invertible because is constant equal to zero and for any $\alpha \in \mathbb{R}\setminus\{0\}$ and $x > 0$ the function is positive and strictly monotone decreasing (see the derivative above) at zero for $x \to +\infty$. Moreover, for $x \leq 0$ we have that $f(0) = 1$ hence, for injectivity, the graph of this branch of the parabola has to be strictly monotone decreasing to the point $(0, 1)$.

Then, is easy to see that for global invertibility we have to require:



a) monotonicity for $x \leq 0$, i.e. that the vertex of the parabola $x = \dfrac{1}{\alpha+1}$ has to be at $x \geq 0$, so that $\alpha > -1$, and
b) for injectivity close to $x = 0$, that $|\alpha|\dfrac{\pi}{2} = \lim_{x\to 0^+} f(x) \leq f(0) = 1$.
The conditions a) and b) on α give for the global invertibility: $-\dfrac{2}{\pi} \leq \alpha \leq \dfrac{2}{\pi}$. Observe that the case $\alpha = -1$, when the parabola branch reduces to $y = -2x + 1$ is not admitted because of condition b). For example, graph of the function is as in the figure below, case $\alpha = 0.5$.
2. Tangent line
Observe that $|\alpha| = 1$ is not a globally invertible case, then we can have more then one value $x \in \mathbb{R}$ such that $f(x) = \dfrac{\pi}{4}$ and we have to check all the possibilities.
Consider $x > 0$ then, because of local invertibility of f, we have

$$y_0 = \arctan\frac{1}{x_0} = \frac{\pi}{4} \iff \frac{1}{x_0} = 1 \iff x_0 = 1\,.$$

On the other side, for $x \leq 0$, we have: if $\alpha = -1$, then $-2x + 1 = \dfrac{\pi}{4}$ i.e. $x = \dfrac{1}{2}\left(1 - \dfrac{\pi}{4}\right) > 0$ so no solution is possible; for $\alpha = 1$ we have $2x^2 - 2x + 1 = \dfrac{\pi}{4}$ and the two solutions are positive so also in this case no solution is possible.
Then the required computation concerns only the point (x_0, y_0). The tangent line at the point (y_0, x_0) has equation $x = x_0 + \left(f^{-1}\right)'(y_0)(y - y_0)$ where $\left(f^{-1}\right)'(y_0) = \dfrac{1}{f'(x_0)}$ and

$$f'(x_0)\big|_1 = \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)\Bigg|_1 = -\left.\frac{1}{1+x^2}\right|_1 = -\frac{1}{2}$$

so the equation is $x = -2y + \left(1 + \dfrac{\pi}{2}\right)$.
Observe that is also possible to obtain such an equation by inverting the function f: it is easy to see that $x = f^{-1}(y) = \dfrac{1}{\tan y}$ for $x > 0$.

Exercise 3. Consider the integral

$$I_\alpha = \int_2^{+\infty} f_\alpha(x)\,dx\,, \quad \text{for} \quad f_\alpha(x) = \frac{\ln^{3\alpha-5}(\sqrt{x}+1)}{(x+3)^\alpha} \quad \text{and} \quad \alpha \in \mathbb{R}\,.$$

1. Discuss the integrability of I_α in the improper sense, for any $\alpha \in \mathbb{R}$;
2. Fix $\alpha = 2$ and calculate (if any) the value of I_2;
3. Fix $\alpha = 2$ and calculate the order of infinite/infinitesimal for $x \to 0^+$ of the function

$$f_2(x) - \frac{1}{9}\left(\sin(\sqrt{x})\cos(\sqrt{x}) - \arcsin\left(\frac{x}{2}\right)\right)\,.$$

Solution

1. The functions f_α are continuous and bounded for every bounded interval of the integration domain; so the integral has to be studied in the improper sense only because of the unboundedness of the domain of integration, for $x \to +\infty$. Observe that for every $\alpha \in \mathbb{R}$, f_α is a positive function on $[2, +\infty)$, hence absolute and simple convergence is the same.
In this case we have for every $\alpha \in \mathbb{R}$, the functions may be approximated for $x \to +\infty$ as

$$f_\alpha(x) = \frac{\ln^{3\alpha-5}(\sqrt{x}+1)}{(x+3)^\alpha} = \left(\frac{1}{2}\right)^{3\alpha-5}\frac{\ln x}{x^\alpha}(1+o(1))$$

Then, using to the infinitesimal comparison test and referring to the fundamental examples of improper integration, we have:
for $\alpha > 1$, I_α converges;
for $\alpha = 1$, we have $3\alpha - 5 = -2$ so I_α converges; and
for $\alpha < 1$, I_α diverges.

2. For $\alpha = 2$ we have

$$I_2 = \int_1^{+\infty} \frac{\ln(\sqrt{x}+1)}{(x+3)^2}dx \underset{x=t^2}{=} \int_{\sqrt{2}}^{+\infty} \frac{\ln(t+1)}{(t^2+3)^2}2tdt \underset{\text{b.p.}}{=} -\left.\frac{\ln(t+1)}{t^2+3}\right|_{\sqrt{2}}^{+\infty} + \int_{\sqrt{2}}^{+\infty}\frac{1}{(t^2+3)(t+1)}dt.$$

The rational function in the last integral decomposes as follow

$$\frac{1}{(t^2+3)(t+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+3}$$

and $A = \left.\frac{1}{t+1}\right|_{-1} = \frac{1}{4}$; $B : \lim_{t\to+\infty} t\frac{1}{(t^2+3)(t+1)} = 0 = \lim_{t\to+\infty}\left(t\frac{A}{t+1} + t\frac{Bt+C}{t^2+3}\right) \Rightarrow B = -A = -\frac{1}{4}$ and $C : \left.\frac{1}{(t^2+3)(t+1)}\right|_{t=0} = \frac{1}{3} = A + \frac{C}{3} \Rightarrow C = \frac{1}{4}$.
With these coefficients we have

$$\begin{aligned}\int \frac{1}{(t^2+3)(t+1)}\,dt &= \frac{1}{4}\ln|t+1| - \frac{1}{4}\int\frac{t-1}{t^2+3}\,dt \\ &= \frac{1}{4}\left\{\ln|t+1| - \frac{1}{2}\ln(t^2+3) + \frac{1}{\sqrt{3}}\int\frac{1}{\left(\frac{t}{\sqrt{3}}\right)^2+1}dt\right\} = \frac{1}{4}\left\{\ln\frac{|t+1|}{\sqrt{t^2+3}} + \arctan\frac{t}{\sqrt{3}}\right\}\end{aligned}$$

where in the last integral we used $s = \frac{t}{\sqrt{3}}$. Hence

$$\begin{aligned}I_2 &= \frac{1}{4}\left.\left\{-\frac{4\ln(t+1)}{t^2+3} + \ln\frac{|t+1|}{\sqrt{t^2+3}} + \arctan\frac{t}{\sqrt{3}}\right\}\right|_{\sqrt{2}}^{+\infty} \\ &= \frac{\pi}{2} + \frac{4\ln(\sqrt{2}+1)}{5} - \frac{\ln(\sqrt{2}+1)}{\sqrt{5}} - 2\arctan\sqrt{\frac{2}{3}}\,.\end{aligned}$$

3. We have

$$\frac{\ln(\sqrt{x}+1)}{(x+3)^2} - \frac{1}{9}\left(\sin(\sqrt{x})\cos(\sqrt{x}) - \arcsin\left(\frac{x}{2}\right)\right)$$

$$\underset{x\to 0^+}{=} \left(x^{\frac{1}{2}} - \frac{1}{2}x + \frac{1}{3}x^{\frac{3}{2}} + o\left(x^{\frac{3}{2}}\right)\right)\frac{1}{9}\left(1 - \frac{2}{3}x + o(x)\right) +$$

$$- \frac{1}{9}\left(\left(x^{\frac{1}{2}} - \frac{1}{3!}x^{\frac{3}{2}} + o\left(x^{\frac{3}{2}}\right)\right)\left(1 - \frac{1}{2}x + \frac{1}{4!}x^2 + o(x^2)\right) - \left(\frac{1}{2}x + o(x^2)\right)\right)$$

$$= \frac{1}{9}\left(x^{\frac{1}{2}} - \frac{1}{2}x + x^{\frac{3}{2}}\left(\frac{1}{3} - \frac{2}{3}\right) - x^{\frac{1}{2}} + \frac{1}{2}x + x^{\frac{3}{2}}\left(\frac{1}{2} - \frac{1}{3!}\right) + o\left(x^{\frac{3}{2}}\right)\right)$$

$$= \frac{1}{9}\left(x^{\frac{3}{2}}\left(-\frac{1}{3} + \frac{1}{2} + \frac{1}{3!}\right)(1 + o(1))\right) = \frac{1}{27}x^{\frac{3}{2}}(1 + o(1)).$$

Hence the function is an infinitesimal of order $\frac{3}{2}$ w.r.t. $x \to 0$.

Exercise 4. Find the local minimum and local maximum points (if any) of the following function:

$$f(x,y) = e^{x^2} + y^4 - 1, \quad (x,y) \in \mathbb{R}^2.$$

Solution

The function f is defined on $\mathbb{R}^2$ and is of class $C^\infty(\mathbb{R})$. Then the critical points are obtained by the vanishing of the gradient, as

$$\nabla f(x,y) = \left(2xe^{x^2}, 4y^3\right) = (0,0) \quad \Longleftrightarrow \quad (x,y) = (0,0)$$

so the only critical point is the origin $(0,0)$.

Observe that $f(0,0) = 0 < f(x,y)$ for every $(x,y) \in \mathbb{R}^2$, in particular for (x,y) in a neighbourhood of $(0,0)$ the function f may be approximated by

$$f(x,y) = 1 + x^2 + o(x^3) + y^4 - 1 = (x^2 + y^4)(1 + o(1)) \quad \text{for } (x,y) \to (0,0),$$

hence the origin is a strict minimum for the function f.

Observe that in this case the discussion using the Hessian matrix is not useful to determine the nature of the critical point: in fact we have

$$H_f(0,0) = \left.\begin{pmatrix} e^{x^2}(2+4x^2) & 0 \\ 0 & 12y^2 \end{pmatrix}\right|_{(x,y)=(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

so $\det(H_f(0,0)) = 0$ and the nature of the stationary point $(0,0)$ cannot be determined by $H_f(0,0)$ only.

Exercise 5. Find all complex numbers z which satisfy the following equation and express them in the Cartesian form $a + \mathrm{i}b$:

$$z^6 + \mathrm{i}z^3 = 0.$$

Solution

We have

$$z^6 + \mathrm{i}z^3 = 0 \iff z^3(z^3 + 1) = 0 \iff z = 0 \quad \text{or} \quad z^3 = -i.$$

Hence $z = 0$ is a root and for $z^3 = -i$, writing in exponential form $z = \rho e^{i\theta}$, we have $z^3 = \rho^3 e^{i\theta} = e^{-i\frac{\pi}{2}}$ with solutions $\rho = 1$ and $\theta_k = \frac{-\frac{\pi}{2} + 2k\pi}{3}$ for $k = 0,1,2$, i.e. $\theta_0 = -\frac{\pi}{6}$, $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{7\pi}{6}$.

Hence we have four roots that in exponential form are

$$z = 0, \quad z_0 = e^{-i\frac{\pi}{6}}, \quad z_1 = e^{i\frac{\pi}{2}} \quad \text{and} \quad z_2 = e^{i\frac{7\pi}{6}},$$

and in Cartesian form read as

$$z = 0, \quad z_0 = \frac{\sqrt{3}}{2} - i\frac{1}{2}, \quad z_1 = i \quad \text{and} \quad z_2 = -\frac{\sqrt{3}}{2} - i\frac{1}{2}.$$