Exercise 1. Sketch the graph of the following function

$$f(x) = \sqrt[3]{|x+3|(x^2-2)}\,,$$

specifying: domain, possible asymptotes, monotonicity, continuity, local and global maxima or minima, and non-derivability points (specifying the type, if any). Moreover, write the equation of the tangent line to the graph of f in the point $(0, f(0))$.

Solution

It is easy to see that $\operatorname{dom} f = \mathbb{R}$ so $\partial\operatorname{dom} f = \{-\infty, +\infty\}$. Moreover f is continuous over its domain being a composition of continuous functions.
Observe that $f = 0 \iff x = -3$ or $x = \pm\sqrt{2}$. Moreover $f(0) < 0 \iff -\sqrt{2} < x < \sqrt{2}$ and, according to the definition of the absolute value, we have

$$f(x) = \begin{cases} \left(-(x+3)(x^2-2)\right)^{\frac{1}{3}} & \text{if } x < -3 \\ \left((x+3)(x^2-2)\right)^{\frac{1}{3}} & \text{if } x \geq -3\,. \end{cases}$$

• Study at $\partial\operatorname{dom} f$:

$$f(x) \underset{x\to-\infty}{=} \left(-(x+3)(x^2-2)\right)^{\frac{1}{3}} = \left(-x^3-3x^2+2x+6\right)^{\frac{1}{3}} = -x\left(1+\frac{3}{x}-\frac{2}{x^2}-\frac{6}{x^3}\right)^{\frac{1}{3}}$$
$$= -x\left(1+\frac{3}{x}+o\left(\frac{1}{x}\right)\right)^{\frac{1}{3}} = -x\left(1+\frac{1}{3}\frac{3}{x}+o\left(\frac{1}{x}\right)\right) = -x-1+o(1) \underset{x\to-\infty}{\to} +\infty\,.$$

Hence $y = -x-1$ is the function of an oblique asymptote for f, when $x \to -\infty$. Similarly

$$f(x) \underset{x\to+\infty}{=} \left((x+3)(x^2-2)\right)^{\frac{1}{3}} = x\left(1+\frac{3}{x}+o\left(\frac{1}{x}\right)\right)^{\frac{1}{3}} = x\left(1+\frac{1}{3}\frac{3}{x}+o\left(\frac{1}{x}\right)\right)$$
$$= x+1+o(1) \underset{x\to-\infty}{\to} +\infty$$

and $y = x+1$ is the function of an oblique asymptote for f, when $x \to +\infty$.

• Monotonicity and critical points

$f'(x) = \dfrac{1}{3}\dfrac{\left(|x+3|(x^2-2)\right)'}{\left(|x+3|(x^2-2)\right)^{\frac{2}{3}}}$, hence

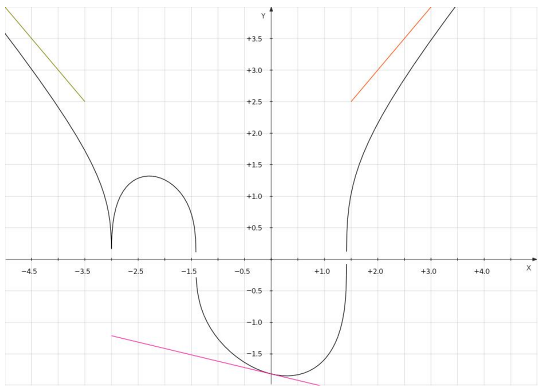
$$f'(x) = \begin{cases} \dfrac{-(x^2-2)-(x+3)(2x)}{3\left(-(x+3)(x^2-2)\right)^{\frac{2}{3}}} & \text{if } x < -3 \\ \dfrac{(x^2-2)+(x+3)(2x)}{3\left((x+3)(x^2-2)\right)^{\frac{2}{3}}} & \text{if } x > -3 \end{cases} = \begin{cases} \dfrac{-(3x^2+6x-2)}{3\left(-(x+3)(x^2-2)\right)^{\frac{2}{3}}} & \text{if } x < -3 \\ \dfrac{3x^2+6x-2}{3\left((x+3)(x^2-2)\right)^{\frac{2}{3}}} & \text{if } x > -3\,. \end{cases}$$

The derivative do not exist for $x = -3$, $-\sqrt{2}$ and $\sqrt{2}$ (in particular f' is not bounded for these values) and the sign of the derivative depends only on the numerator in both cases and it is as follow:

$f'(x) = 0 \iff x = x_1 = -1-\sqrt{\dfrac{5}{3}}$ and $x = x_2 = -1+\sqrt{\dfrac{5}{3}}$, i.e. x_1, x_2 are the critical points of f;

$3x^2+6x-2 > 0 \iff x < x_1$ or $x > x_2$ and $3x^2+6x-2 < 0 \iff x_1 < x < x_2$;

vice versa for $-(3x^2+6x-2)$.



Hence we have:
for $x < -3$, $f'(x) < 0$ and because for $x > -3$ $f'(x) > 0$ then $(-3,\ f(-3))$ is a local minimum for f;
for $x > -3$, we have $f'(x) > 0$ for $-3 < x < x_1$; $f'(x) < 0$ for $x_1 < x < x_2$; $f'(x) > 0$ for $x > x_2$. Then $(x_1, f(x_1))$ is a local maximum and $(x_2, f(x_2))$ is a local minimum for f.
Observe that $\lim\limits_{x\to -3^{\pm}} f'(x) = \text{“}\dfrac{7}{0^{\pm}}\text{”} = \pm\infty$ so $(-3, f(-3))$ is a cusp point for f.
Moreover $\lim\limits_{x\to \sqrt{2}^{\pm}} f'(x) = \text{“}\dfrac{22}{0^{+}}\text{”} = +\infty$ and $\lim\limits_{x\to -\sqrt{2}^{\pm}} f'(x) = \text{“}\dfrac{-22}{0^{+}}\text{”} = -\infty$ so $(\sqrt{2}, f(\sqrt{2}))$ and $(-\sqrt{2}, f(-\sqrt{2}))$ are vertical tangent points for f.
• Tangent line
$f'(0) = -\dfrac{2}{3\cdot 6^{\frac{2}{3}}}$ and $f(0) = \sqrt[3]{6}$ so the tangent line at $(0, f(0))$ is $y = -\dfrac{2}{3\cdot 6^{\frac{2}{3}}}\ x - \sqrt[3]{6}$.

Exercise 2. Considered the function

$$f(t) = \frac{\arctan(\log t)}{t\,(\log t - 1)^2}\,,$$

discuss the integrability and calculate, if possible, the integrals:

$$2.\text{a)}\quad \int_{e^2}^{e^3} f(t)\ dt\,; \qquad\qquad 2.\text{b)}\quad \int_{e}^{+\infty} f(t)\ dt\,.$$

Solution
The function $f(t)$ admits primitives as an elementary functions, obtained as follow.

$$\int \frac{\arctan(\log t)}{t\,(\log t - 1)^2}\,dt \underset{\ln t = x}{=} \int \frac{\arctan x}{(x-1)^2}\,dx \underset{\text{b.p.}}{=} -\frac{\arctan x}{x-1} + \int \frac{1}{(x-1)(x^2+1)}\,dx\,.$$

Here the first equality is obtained by the indicated substitution and the second one by integration by parts.
Now the rational function in the last integral decomposes as $\dfrac{1}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$ where, e.g. , we obtain the three constants by:
$A = \left.\dfrac{1}{x^2+1}\right|_{x=1} = \dfrac{1}{2}$, $B: \lim\limits_{x\to+\infty} \dfrac{x}{(x-1)(x^2+1)} = A + B \Rightarrow B = -\dfrac{1}{2}$ and
$C: \left.\dfrac{1}{(x-1)(x^2+1)}\right|_{x=0} = -1 = -A + C \Rightarrow C = -\dfrac{1}{2}$.
In a pure algebraic way, it is possible to obtain the constants A, B and C also by the relation

$$\frac{A}{x-1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1) + (Bx+C)(x-1)}{(x-1)(x^2+1)} = \frac{1}{(x-1)(x^2+1)}$$

equating the numerators in the last equation

$$A(x^2+1)+(Bx+C)(x-1)=1 \quad \Leftrightarrow \quad x^2(A+B)+x(-B+C)+A-C=1$$

and solving the 3×3 linear system obtained by comparing the polynomials at any power of x

$$\begin{cases} A+B=0 \\ -B+C=0 \\ A-C=1 \end{cases} \qquad \begin{cases} B=-A \\ C=-A \\ 2A=1 \end{cases} \qquad \begin{cases} A=\dfrac{1}{2} \\ C=B=-\dfrac{1}{2}\,. \end{cases}$$

Hence for the last integral above we have

$$\int \frac{1}{(x-1)(x^2+1)}\,dx = \frac{1}{2}\int \left(\frac{1}{x-1}-\frac{x+1}{x^2+1}\right)dx$$

and the primitives are given by

$$\begin{aligned}\int f(t)\,dt &= -\frac{\arctan x}{x-1}+\frac{1}{2}\left(\ln|x-1|-\arctan x-\frac{1}{2}\ln(x^2+1)\right)+C \\ &= -\frac{\arctan(\ln t)}{\ln t-1}+\frac{1}{2}\left(\ln|\ln t-1|-\arctan(\ln t)-\frac{1}{2}\ln(\ln^2 t+1)\right)+C\end{aligned}$$

• 2.a)

$\displaystyle\int_{e^2}^{e^3} f(t)dt$ is a definite Riemann integral and, using the primitive in t, we have

$$\int_{e^2}^{e^3} f(t)\,dt = \frac{3}{2}\arctan 2-\arctan 3+\frac{1}{4}\ln 2\,.$$

• 2.b)

$\displaystyle\int_{e}^{+\infty} f(t)\,dt$ is improper because the domain of integration is unbounded, at $t\to\infty$, and because $f(t)$ is unbounded for $t\to e^+$.

Using the order test for improper integrals of positive functions and fundamental examples, we have

$f(t) \underset{t\to+\infty}{=} \dfrac{\frac{\pi}{2}(1+o(1))}{t\ln^2 t}$, hence f has a convergent integral $\displaystyle\int_a^{+\infty} f(t)\,dt$ for any $a>e$.

For $t\to e$, observe that $\ln t = 1+\dfrac{t-e}{e}+o(t-e)$, so that $f(t)\underset{t\to e^+}{=}\dfrac{e\arctan(1)(1+o(1))}{(t-e)^2}$ and $\displaystyle\int_e^a f(t)\,dt$, for any real $a>e$, is divergent. Hence, by additivity, the improper integral $\displaystyle\int_e^{+\infty} f(t)\,dt$ is divergent. Similarly, it is possible to discuss 2.1) or 2.b) using the substitution and primitive in $x=\ln t$.

Exercise 3. Discuss the absolute and simple convergence of the following series:

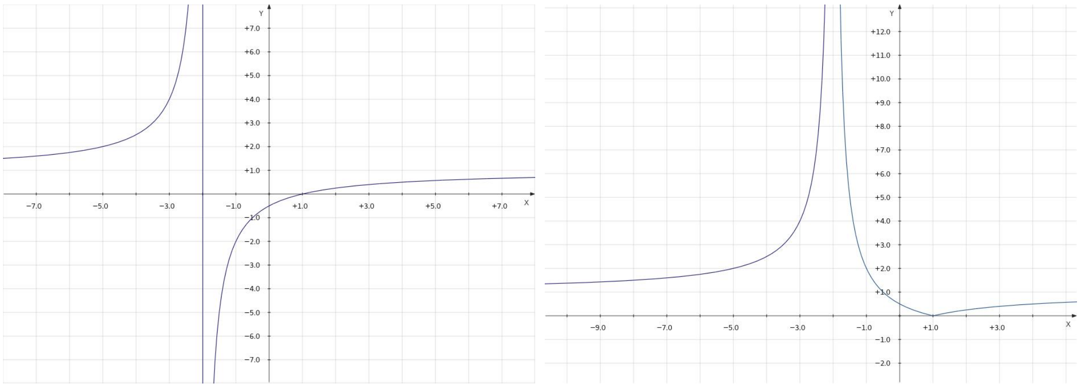
$$3.a)\quad \sum_{n=1}^{\infty}\left(\frac{n\tan(1+\frac{1}{n^2})}{n+1}\right)^n; \qquad\qquad 3.b)\quad \sum_{n=0}^{\infty}\left(\frac{x-1}{x+2}\right)^n \ln\left(\frac{n+1}{n+2}\right),\ \text{for } x\in\mathbb{R}\,.$$

Solution

• 3.a)

For $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n\tan(1+\frac{1}{n^2})}{n+1}\right)^n=\sum_{n=1}^{\infty}a_n$ we start discussing the Cauchy necessary condition, and we have

$$a_n=\left(\frac{n\tan(1+\frac{1}{n^2})}{n+1}\right)^n=\left(\frac{n\tan(1)(1+o(1))}{n+1}\right)^n=(\tan(1)(1+o(1)))^n \underset{n\to\infty}{\to} +\infty$$



$$x \mapsto q(x) \quad \text{and} \quad x \mapsto |q(x)|\,.$$

because $\tan(1) > \tan\left(\frac{\pi}{4}\right) = 1$. Moreover, $a_n > 0$ for any $n > 1$, hence we have absolute and simple divergence.

• 3.b)

Let $q = \dfrac{x-1}{x+2}$ so that we can write the series as $\displaystyle\sum_{n=0}^{\infty} q^n \ln\left(\frac{n+1}{n+2}\right)$ and observe that for $x = 1$ we have $q = 0$ so the series is trivial and for $x = -2$ we have that q and the series are not well defined. Moreover we have

$$\ln\left(\frac{n+1}{n+2}\right) = \ln\left(1 - \frac{1}{n+2}\right) < 0 \quad \text{for any} \quad n \geq 0\,.$$

Absolute convergence

We discuss $\displaystyle\sum_{n=0}^{\infty} \left|\frac{x-1}{x+2}\right|^n \left|\ln\left(\frac{n+1}{n+2}\right)\right| = \sum_{n=0}^{\infty} |q|^n \left|\ln\left(\frac{n+1}{n+2}\right)\right|$ for $x \in \mathbb{R}$. Here we observe that

$$\left|\ln\left(\frac{n+1}{n+2}\right)\right| \underset{n\to\infty}{=} \left|\ln\left(1 - \frac{1}{n+2}\right)\right| \underset{n\to\infty}{=} \frac{1}{n}\,.$$

Then, for $|q| \gtreqless 1$, that may be obtained, for example, by studying the graph of the functions $x \mapsto |q(x)|$ as in the figure, we have different characters of the series.

For $-\dfrac{1}{2} < x$ and $x \neq 1$ we have $0 < |q| < 1$ so that $\displaystyle\sum_{n=0}^{\infty} q^n \ln\left(1 - \frac{1}{n+2}\right)$ is absolutely convergent because, e.g. by comparison, $0 \leq |q|^n \ln\left(1 - \dfrac{1}{n+2}\right) < \dfrac{1}{n^2}$ eventually for $n \to \infty$.

For $x < -\dfrac{1}{2}$ we have $|q| > 1$ and $q^n \ln\left(1 - \dfrac{1}{n+2}\right) > n$ eventually for $n \to \infty$, so we have absolute divergence.

For $x = -\dfrac{1}{2}$ we have that $|q| = 1$ and the series reduces to $\displaystyle\sum_{n=0}^{\infty} \ln\left(1 - \frac{1}{n+2}\right)$ that is absolutely divergent because of the infinitesimal test, being $\ln\left(1 - \dfrac{1}{n+2}\right)$ equivalent to $-\dfrac{1}{n}$ for $n \to \infty$.

So the series is absolutely convergent for $-\dfrac{1}{2} < x$ and for this values of the parameter x is also simply convergent by the theorem of absolute convergence.

Simple convergence

By the theorem of absolute convergence, the series $\displaystyle\sum_{n=0}^{\infty} q^n \ln\left(\frac{n+1}{n+2}\right)$ is also simply convergent for $-\dfrac{1}{2} < x$, i.e. $|q| < 1$ but we have to discuss the convergence for the case $q \leq -1$.

For $x = -\dfrac{1}{2}$ we have $q = -1$ and defining $b_n := -\ln\left(1 - \dfrac{1}{n+2}\right) = \ln\left(1 + \dfrac{1}{n+1}\right) > 0$ for every $n \geq 0$, the series is

$$\sum_{n=0}^{\infty}(-1)^n \ln\left(1 - \frac{1}{n+2}\right) = \sum_{n=0}^{\infty}(-1)^{n+1}\left(-\ln\left(1 - \frac{1}{n+2}\right)\right)$$
$$= \sum_{n=0}^{\infty}(-1)^{n+1}\ln\left(1 + \frac{1}{n+1}\right) = \sum_{n=0}^{\infty}(-1)^n b_n$$

i.e. an alternating sign series that we discuss using Leibniz test:
i) $b_n = \dfrac{1}{n+1}(1 + o(1)) \to 0$, for $n \to \infty$, so the first condition is satisfied;
ii) b_n is decreasing for $n \to \infty$, in fact: n is strictly increasing (we use this symbol $\nearrow$), then $n+1 \nearrow$, then $\dfrac{1}{n+1}$ is decreasing ($\searrow$), then $1 + \dfrac{1}{n+1} \searrow$ and $\ln\left(1 + \dfrac{1}{n+1}\right) \searrow$.
We can also pass to the function $x \mapsto \ln\left(1 + \dfrac{1}{x+1}\right)$ and see that its derivative $-\dfrac{1}{(x+1)(x+2)}$ is negative for $x \to \infty$.
For $x < -\dfrac{1}{2}$ we have $q < -1$ and the series is $\sum_{n=0}^{\infty}(-1)^n\ |q|^n \ln\left(1 - \dfrac{1}{n+2}\right)$, with $|q| > 1$, and we use Leibniz test again. The sequence do not satiesfy the first condition, in fact $b_n \to +\infty$ for $n \to \infty$, so the Leibniz criterium may not be used. However, it is easy to se that, because $b_n \to +\infty$, the alternating sign makes the series indeterminate.

Exercise 4. Find the solution of the following Cauchy problem:

$$\begin{cases} y'' + 2y' + 2y = 5xe^x \\ y(0) = 0 \\ y'(0) = 1 \end{cases} .$$

We shall write the general integral of the equation as $y(x) = y_0(x) + y_p(x)$ where y_0 is the general integral of the associated homogeneous equation $y'' + 2y' + 2y = 0$ and y_p is a particular solution of the non-homogeneous equation $y'' + 2y' + 2y = g$, with source $g(x) = 5xe^x$.
The characteristic polynomial of the homogeneous equation is $\lambda^2 + 2\lambda + 2 = 0$, whose solutions are $\lambda_{1,2} = -1 \pm i$. Hence we have $y_0(x) = e^{-x}(A\cos x + B\sin x)$, with constants $A, B \in \mathbb{R}$.
Then, because the source term is of the form (polynomial(x)) $\cdot$ (exp(αx)), we may search for a particular solution of the form $y_p(x) = (ax + b)e^x$, being α not a root of the associated characteristic polynomial. So we have $y' = ae^x + y$ and $y'' = ae^x + y' = 2ae^x + y$. Substituting in the equation $y'' + 2y' + 2y = 5xe^x$ and equating we obtain $a = 1$ and $b = -\dfrac{4}{5}$, i.e. $y_p(x) = \left(x - \dfrac{4}{5}\right)e^x$.
Hence the general integral of the equation is $y(x) = e^{-x}(A\cos x + B\sin x) + \left(x - \dfrac{4}{5}\right)e^x$ with $A, B \in \mathbb{R}$.
Now we can fix the constant A and B imposing the given initial conditions:

$$\begin{cases} y(0) = A - \dfrac{4}{5} = 0 \Rightarrow A = \dfrac{4}{5} \\ y'(0) = -y(x) + e^x(-A\sin x + B\cos x) + e^x\left(1 + \left(x - \dfrac{4}{5}\right)\right)\Bigg|_{x=0} = B - \dfrac{4}{5} = 1 \Rightarrow B = \dfrac{9}{5} \end{cases},$$

and the solution of the Cauchy problem is given by

$$y(x) = e^{-x}\left(\frac{4}{5}\cos x + \frac{9}{5}\sin x\right) + \left(x - \frac{4}{5}\right)e^x .$$

Exercise 5. Compute the following limit:

$$\lim_{n\to+\infty}(n^4+\ln n)\left(\ln\left(1+\frac{1}{n^2}\right)-\frac{1}{n}\sin\left(\frac{1}{n}\right)\right).$$

$$\begin{aligned}
&\lim_{n\to+\infty}(n^4+\ln n)\left(\ln\left(1+\frac{1}{n^2}\right)-\frac{1}{n}\sin\left(\frac{1}{n}\right)\right)\\
&\quad=\lim_{n\to+\infty}(n^4+\ln n)\left(\left(\frac{1}{n^2}-\frac{1}{2n^4}+o\left(\frac{1}{n^4}\right)\right)-\frac{1}{n}\left(\frac{1}{n}-\frac{1}{6n^3}+o\left(\frac{1}{n^4}\right)\right)\right)\\
&\quad=\lim_{n\to+\infty}(n^4+\ln n)\left(\frac{1}{n^4}\left(-\frac{1}{2}+\frac{1}{6}\right)+o\left(\frac{1}{n^4}\right)\right)=\lim_{n\to+\infty}(n^4+\ln n)\left(-\frac{1}{3n^4}\right)(1+o(1))\\
&\quad=-\frac{1}{3}-\lim_{n\to+\infty}\frac{\ln n}{3n^4}=-\frac{1}{3}.
\end{aligned}$$