MATHEMATICAL ANALYSIS I - 2015/2016 1st Term
Lecturers: Fabio Ciolli, Teresa D'Aprile
Tutor: Aleks Jevnikar

Program

Differential calculus for real functions: real and complex numbers; elementary real functions and their inverse: polynomial, exponential, logarithm, trigonometric etc.; concept of limit, limits of indefinite forms; continuity, properties of continuous functions, uniform continuity; derivatives, maxima and minima, the graph of a function; De L 'Hopital's Rule; Taylor expansions.
Integral calculus for real functions and numerical series: antiderivatives, integrals; improper integrals; series. Elementary differential equations of first and second order.
Introduction to multivariable calculus: continuity; differentiation, directional derivatives, gradient; higher order differentiations, Hessian matrix.

Textbooks

Tom M. Apostol: Calculus Vol.1, second edition; John Wiley & Sons, (1974).
Claudio Canuto, Anita Tabacco: Mathematical Analysis. Volumes I and II. Springer International Publishing, UNITEXT 84, (2015)
William F. Trench: Introduction to real analysis. Pearson free ed. (2012).

Detailed Program

(in **bold style** are listed some the main topics and theorems of the course)

Basic elements

Basic elements of numerical sets: Naturals, Integer, Rational and Real numbers ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$). Definition of rational numbers. Decimal alignments. Real numbers: non rationality of $\sqrt{2}$ (with proof, optional $\sqrt{2n}$). Properties of $\mathbb{Q}$: ordered commutative field; density; Archimedean property; ordering.
Complex numbers $\mathbb{C}$. Definition. Geometric interpretation. Modulus and argument. Complex exponentials. Roots of complex numbers.

Sets and elements of mathematical logic

Set-theoretical operations. Characteristic property defining a set. Power set. Venn diagrams. Cardinality. Connectives, Predicates, Quantifiers. Bernoulli inequality. Factorial and Newton's binomial expansion formula. Pascal (Tartaglia) triangle.
Ordered pairs and Cartesian product. Cartesian plane. Relations on the Cartesian plane.
Cardinality of an infinite set. Countable and uncountable infinite sets. $\mathbb{N}$, $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Q}$ are countable infinite sets. $\mathbb{R}$ is an uncountable infinite set.

The space $\mathbb{R}^n$

Bounded and unbounded intervals. Open and closed intervals. $\mathbb{R}$ as the real line and the extended real line $\mathbb{R}^* = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. Bounded sets in $\mathbb{R}$. Maximum and minimum of a bounded set in $\mathbb{R}$. Supremum and infimum of a set in $\mathbb{R}$. Properties of $\sup / \inf$. Extremes of a finite set. Completeness of a non-empty subset of $\mathbb{R}$. Definition of real numbers. Bounded sets in $\mathbb{R}^*$ and $\sup A = +\infty$.

Real functions of one real variable

Elementary functions. The absolute value, integer part and mantissa, polynomials, rational functions, exponential, the logarithmic and the trigonometric functions: basic properties, graph and related inequalities. The triangle inequality. Young inequality and Cauchy - Schwartz inequality. Principle of mathematical induction.
Functions. Domain, image (range), pre-image and graph. Sequence as a function on $\mathbb{N}$. Bounded functions. Subjectivity and injectivity. One-to-one function (bijection) and invertible function. Solving an equation defined by $f(x) = y$. Graph of the inverse function. Restriction of a function. Monotone

functions, (strictly) increasing and decreasing functions. Difference quotient and monotony. Composition of functions, domain and image of the composite function. Global and local max/min of a function.

Topology of $\mathbb{R}$ and $\mathbb{R}^n$

Neighbourhoods in $\mathbb{R}^n$. Norm and distance. Euclidean distance in $\mathbb{R}^n$. Internal, external and boundary points of a set in $\mathbb{R}^n$. Neighbourhoods of $+\infty$ and $-\infty$ in $\mathbb{R}^*$.
Accumulation points of a set in $\mathbb{R}^n$. Isolated points. Accumulation points of $\mathbb{N}$ and $\mathbb{Z}$ in $\mathbb{R}^*$. Open sets. Closure of a set. Dense sets. Characterization theorem of a closed set (with proof). Extremes of a closed and bounded set.

Limits

Limits of real functions and sequences: general definition using neighbourhoods; ε, δ, and n, M presentation. **Theorem of uniqueness of the limit (with proof)**. Left and right limit. Limit from above/below. **Theorem of permanence of sign in the limit (with proof)**. Property holding eventually for $x \to x_0 \in \mathbb{R}^*$. **Theorem of comparison of the limits of three functions (squeeze rule, with proof)**. Indeterminate forms: $\lim_{x\to\pm\infty} \frac{\sin(x)}{x} = 0$; $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.
Theorem algebras of limits (with proof), with values in $\mathbb{R}$. Example: $\lim_{x\to 0} \frac{\cos(x)-1}{x^2} = \frac{1}{2}$. Extension of the algebras of limits to the case of limits with values in $\mathbb{R}^*$. Indeterminate forms of algebraic type; $\lim_{x\to 0} |x|^\alpha \sin(1/x) = 0$, for $\alpha > 0$. **Theorem on the limit of a composite function (with proof)**. Variable substitution formula.
Limit of rational functions. Limits by rationalization. Limits of powers, exponential and logarithm. Indeterminate forms of exponential type $f(x)^{g(x)}$. **Theorem: limit of function by convergent sequences (Bridge-theorem, with proof)**. Examples of use. $\lim_{x\to\infty} \frac{x^\alpha}{a^x} = 0$ for all $\alpha \in \mathbb{R}$, $a > 1$, with proof. $\lim_{x\to\infty} |x|^\alpha a^x = 0$ for all $\alpha \in \mathbb{R}$, $a > 1$, with proof. $\lim_{x\to+\infty} \frac{|log_b x|^\alpha}{x^\beta} = 0$, for all $\alpha \in \mathbb{R}$, $\beta > 0$ and $b > 0$, $b \neq 1$, with proof. $\lim_{x\to 0^+} |\log_b x|^\alpha x^\beta = 0$, for all $\alpha \in \mathbb{R}$, $\beta > 0$ and $b > 0$, $b \neq 1$, with proof.

Limits of sequences

Monotone sequences. Subsequences. **Bolzano-Weierstrass Theorem (with proof)**. Limits of sequences and limits of functions. Some notable limits: $\lim_{n\to+\infty}(1+1/n)^n = e$, $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$, $\lim_{x\to 0} \frac{log(1+x)}{x} = 1$, $\lim_{x\to 0} \frac{e^x-1}{x} = 1$, $\lim_{x\to 0} \frac{a^x-1}{x} = \log a$, $\lim_{x\to 0} \frac{(1+x)^\alpha-1}{x} x = \alpha$. **Comparison of infinite** of factorial $n!$, exponential e^n and n^n: $\lim_{n\to+\infty} \frac{e^n}{n!} = 0$ and $\lim_{n\to+\infty} \frac{n!}{n^n} = 0$ (with proof). Sequences defined recursively.

Order of infinite and infinitesimal

Comparison of the infinities of power, logarithm and exponential type. The order of the infinity of the logarithm, compared to the one of the identical function, is not a real number, is positive and smaller than any positive real number. The order of the infinity of the exponential with a base greater than 1, compared to the one of the identical function, is not a real number, is positive and greater than any positive real number. For $a, b > 1$, $\alpha \in R^+$ and $\beta \in R^+$ and for $x \to +\infty$, the infinities of logarithm, power and exponential type are ordered as follow: $(\log_b x)^\alpha \prec x^\beta \prec a^x$.
Comparison of infinitesimals (infinities). **Equivalent infinitesimals (infinities)**. **Order of infinitesimal (infinity)**. If $f(x)$, $g(x)$ are infinitesimal for $x \to x_0$, and the order of f is greater than the order of g, we write $f(x) = o(g(x)) = g(x)o(1)$, for some $o(1)$ infinitesimal. **Order of infinitesimal (infinity) w.r.t. a referring or sample infinitesimal (infinite)**. **Approximation of functions** and sequences using the notation of little o of Landau. Approximations obtained by the notable limits. Examples of comparison of infinitesimals or infinities. "Big O" of Landau. Algebras of the infinitesimals, infinities and "little o" of Landau. Algebras of the order of infinitesimal or infinity.
Asymptotes to the graph of a function: vertical, horizontal and oblique. Approximation of a function by its asymptotes. Other notable limits: $\lim_{x\to 0} \frac{\arctan x}{x} = 1$, $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$.

Continuous functions

Continuous functions: definition and examples. The algebras of continuous functions. **Theorems: sign permanence and composition of continuous functions (with proof). Continuous functions on an interval**. Classes of continuous functions. **Points of discontinuity**: removable, jump or essential.

Continuous prolongation of a function. Theorem about the number and kind of discontinuity points of a continuous, monotone, function on an interval (with proof). **Theorem about the existence of zeros of a continuous function on an interval (with proof)**. The case of a strictly monotone continuous function (with proof). Corollary: existence of solution of the equation $f(x) = g(x)$, for f, g continuous on an interval. **Theorem: if f is continuous, the image of an interval is an interval (with proof)**.

Invertibility of continuous functions. Examples. Th (with proof) If f is continuous and invertible on an interval, then f is strictly monotone. Corollary: a continuous function on an interval is invertible iff is strictly monotone. Th (Continuity of the inverse) A continuous invertible function on an interval or closed bounded interval has continuous inverse function.

Uniformly continuous functions

Examples and counterexamples: $f(x) = x^2$ in uniformly continuous on $[-1, 1]$ whereas it fails to be uniformly continuous on $\mathbb{R}$. Lipschitz continuous functions. Examples and counterexamples: $f(x) = \sqrt{x}$ is uniformly continuous on $[0, +\infty)$ but it fails to be Lipschitz continuous on $[0, +\infty)$. Cantor's theorem: every continuous function from $[a, b]$ to $\mathbb{R}$ is uniformly continuous.

Sequentially compact and compact subspace in $\mathbb{R}$. **Th (with proof): For K compact subset of $\mathbb{R}$ and f continuous, $f(K)$ is compact and f admits** max **and** min **on K (Weierstrass). The case of $K = [a, b]$, closed interval.** Examples and counterexamples.

Differential Calculus

Differential Calculus of real functions of one real variable. **Tangent line to the graph of a function**. **Derivability (i.e. differentiability) of a function in a point**. Geometrical meaning of the derivative of a function in a point. **Best linear approximation**. **Th: if f is derivable in a point x_0 then it is continuous in x_0 (with proof)**. Vertical tangent and infinite derivative in a point. Calculation of the derivative: easy examples (constant function and monomials). Left/right derivative. Corner points and cusps. $n-$th derivative and notation.

Th (Algebras of the derivative): linearity of the derivative, derivative of the product and quotient of derivable functions in a point (with proof). Leibniz formula for the $n-$th derivative of the product $f \cdot g$. Th (Derivative of the composed function - "chain rule", with proof). Th (Derivative of the inverse function, with proof). Derivative of the elementary functions.

Calculus of the derivative using the definition or rules and theorems. Examples: $f_\alpha(x) = x^\alpha$, $\sin(\frac{1}{x})$, for $x \neq 0$ and $f_\alpha(0) = 0$, for $\alpha > 0$. Continuity, differentiability and regularity class $C^n(\mathbb{R})$, $C^\infty(\mathbb{R})$. **Extrema and critical points. Th (Fermat): if x_0 is a local extremum and f is derivable in x_0, then $f'(x_0) = 0$ (with proof)**. Examples and counterexamples. Search for local and global extrema of a function on a closed interval.

Th (Lagrange's Mean value theorem, with proof). Th (Rolle, with proof). Th (Cauchy, with proof). Consequences and applications (with proof): the one-side limit of the derivative equals the one-side derivative; if f is derivable on an open interval then its derivative f' do not admits jump discontinuity; relation between monotony and sign of derivative.

Lipschitz functions. Differentiability and Lipschitz functions: a function with bounded derivative on an interval is Lipschitz. De l'Hôpital Theorem (with proof); examples and counterexamples.

Taylor's theory of expansion polynomials

Taylor and Mac Laurin polynomials. **Th (Peano, with proof): existence, uniqueness and properties of Taylor polynomial**. Examples of elementary Mac Laurin polynomials. Mac Laurin polynomial of an even (odd) function.

Peano's and Lagrange's remainder formulas. Composition and linearity of expansions. Expansion of the derivative of f from the expansion of f, and vice versa. Calculation of Mac Laurin Polynomials of a composite function.

Application of Peano's theorem and Taylor expansion to the calculation of limits and to the search or the order of infinitesimal/infinite of a function. Obtaining a Taylor expansion in the point $x_0 \neq 0$ from the corresponding Mac Laurin expansion. Th (with proof) about a function having the derivative $f^{(k)}(x_0) = 0$ for $k = 1, \ldots n-1$ and$f^n(x_0) \neq 0$ and existence of an local extremum in x_0.

Convex functions of one variable

Convex combination of two points. Convex set. A function is convex iff its epigraph is a convex set. **Th (with proof) A convex function on an open interval is continuous and admits left and right derivative at any point of the interval**, as two increasing functions with $f'_-(x) \geq f'_+(x)$. Th (with proof) if a function f is derivable in an open interval, then f is (strictly) convex iff f' is (strictly) increasing and iff the tangent line at any point of the interval lays (strictly) below the graph. If f is 2-times derivable, f is convex iff $f'' \geq 0$; if $f'' > 0$ then f is strictly convex. Inflection point. **Th (with proof) Let f be a function on an interval and x_0 be an inflection point for f. If f is 2-times derivable in x_0, then $f''(x_0) = 0$.**

Differentiable functions of two variables

Topology in $\mathbb{R}^N$: usual norm of a point, distance between two points, open balls, accumulation point of a set, open sets. Functions of two or more variables: domain, limits and continuity.
The tangent plane to the graph of a function. Partial derivatives. Example of a function which fails to be continuous at $(0, 0)$ but the partial derivatives exist everywhere. Second-order partial derivatives and Schwarz' theorem. Definition of differentiable function. **Differentiability implies continuity (with proof).** A sufficient condition for differentiability (with proof). Derivative of the composite function $F(t) = f(g(t), h(t))$ with $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ and $g, h : I \subset \mathbb{R} \to \mathbb{R}$.
Directional derivative for a function of two variables. Differentiability implies existence of directional derivatives (with proof). Local maxima and local minima: first-order and second-order necessary condition (with proof), second-order sufficient condition.
Local maxima and local minima for a function of two variables. Second-order sufficient condition (with proof). Other tools to determine the behaviour of the function in small neighbourhoods around a critical point for cases where the Hessian determinant vanishes.

Primitive (antiderivative) of a function

Th (with proof): the absence of jump discontinuity is a necessary condition for the existence of the primitive. Th (with proof): existence of a family of primitives, obtained from a primitive, up to an additive real constant. Examples and counterexamples. **Indefinite integral. Primitives of elementary functions**. Existence and continuity of the primitives. Linearity of the indefinite integral. Integration by parts.
Integration by substitution: change of variable in the integration formula. Integration of rational functions. Decomposition of a polynomial with real coefficient in irreducible factors. **Decomposition of rational functions in linear sum of simple partial fractions**. Methods for determining the coefficients in the linear sum of simple partial fractions.
Integration of rational functions: primitives of the simple partial fractions: structure of the solution and types of primitives. Examples.
Primitives of $\frac{1}{x^2+px+q}$, for any p, $q \in \mathbb{R}$. Method of Hermite for multiples roots. Relevant substitutions for rational functions of some irrational or trigonometric functions. Examples.

Definite integrals – Riemann Theory

Partitions of a closed bounded interval I of the real line. Finer partition. Poset of the partition of a closed bounded interval. Lower sums and upper sums. Approximation of the sub-area of the graph of a function. **Definition of Riemann integral of a bounded function defined over a closed bounded interval**. The (algebraic) sub-area of the graph of a function as a definite integral.
The norm of a partition. **Examples of Riemann integrable function**: constant function, constant function on a open bounded interval. **The Dirichelet function on a bounded interval is not Riemann integrable. Th (with proof) a Criterion of Riemann integrability**: for every $\varepsilon > 0$ there exists a partition P_ε of the open, bounded interval I such that $S(P_\varepsilon, f) - s(P_\varepsilon, f) < \varepsilon$. **Th (with proof): continuous functions on a bounded, closed interval are Riemann integrable. Th (with proof): monotone functions on a bounded, closed interval are Riemann integrable. Th (with proof): bounded functions on bounded, closed interval with a finite number of discontinuity points, are Riemann integrable.** Generalization to a denumerable set of discontinuity points. Two functions defined on a bounded, closed interval that are equal up to a finite number of points, are both integrable, with the same value of the integral, or not integrable. Generalization to a denumerable set. Riemann integral sums. Th (without proof): a limit criterion for Riemann integrability.

Th (with proof): properties of the Riemann integral (linearity, positivity, positive and negative parts and absolute value, additivity w.r.t. the interval of integration). Th (with proof): integral mean values. Integral function. Th (with proof): Fundamental Theorem of integral calculus. Integral function and primitives.
Application of Riemann integration: the calculus of the area of subsets of the real plane; the length of an arc of curve, given as graph of a real variable function.

Ordinary differential equations

Introduction to ordinary differential equations and Cauchy problems. Examples. Simple harmonic motion.
Linear differential equations. **First-order equations with variable coefficients: general solution in the homogeneous and non-homogeneous case (with proof)**. Second order homogeneous equations with constant coefficients: characteristic equation; general solution.
Second order linear differential equations with constant coefficients: general solution in the homogeneous and non-homogeneous case. Determination of a particular solution of the non-homogeneous equation in some special cases.

Improper integral

1st fundamental example: $f(x) = x^\alpha$, α in $\mathbb{R}$, integrated over $(0,1]$ and $[1,+\infty)$.
Definition of an improper integral, both for unbounded function or unbounded domain, as a limit over the extremes of a Riemann integral. Geometrical meaning of the convergence of an improper integral. Relation between Riemann integrability and improper integrability: continuity and independence of the limits in the definition. **Th (Comparison test for improper integral, with proof). Corollary (order test for improper integrals)**. **2nd fundamental example**: $f(x) = \frac{1}{x(lnx)^\beta}$, for $\beta \in \mathbb{R}$, integrated over $[2,+\infty)$. **3nd fundamental example**: $f(x) = \frac{1}{x^\alpha(lnx)^\beta}$, for α, $\beta \in \mathbb{R}$, integrated over $[2,+\infty)$.
Absolute improper integrability. Th (with proof): Absolute improper integrability is sufficient, and non necessary, for simple integrability.

Numerical series

Definition and examples of summable series, e.g. Mengoli series and (generalized) geometric series; divergent series ($\sum_k \ln(1+\frac{1}{k})$. **Th (Necessary condition of (simple) convergence, with proof)**: if $\sum_k a_k$ is convergent then $a_k \to 0$ for $k \to \infty$. **Th (Cauchy Criterion for series, with proof): a series is converging iff its sequences of partial sums is Cauchy (i.e. fundamental)**. Example: the harmonic series is divergent. Numerical series and improper integrals. **Th (Integral Criterion for positive terms series, with proof)**. Example of an function integrable in the improper sense on the half line but non bounded.
Th (Comparison test for positive terms series, with proof).
Th (Leibniz Criterion for alternating sign series, with proof). **Th (Ratio test, with proof). Th (Root test, with proof)**.