Università di Tor Vergata – Dipartimento di Ingegneria Civile ed Ingegneria Informatica

Exercises in Mathematical Analysis I

Alberto Berretti, Fabio Ciolli

1 Fundamentals

1.1 Polynomial inequalities

Solve the following inequalities for $x \in \mathbb{R}$:

Ex. 1. $(x^3 - 3x + 2)(x - 4) > 0.$ $\Big[x < -2,\ x > 4\Big]$

Ex. 2. $(1 - x)(x - 3)(x + 2) < 0.$ $\Big[-2 < x < 1,\ x > 3\Big]$

1.2 Rational inequalities

Solve the following inequalities for $x \in \mathbb{R}$:

Ex. 3. $\dfrac{x^2 + x - 2}{x^2 - 10x + 21} < \dfrac{x - 1}{x - 3} + 3\dfrac{x + 1}{x - 7}.$ $\Big[x < 0,\ 3 < x < 5,\ x > 7\Big]$

Ex. 4. $\dfrac{x + 12}{x + 8} - \dfrac{x - 6}{x^2 + 2x - 48} \geq \dfrac{3x - 3}{x - 6}.$ $\Big[-8 < x < 6\Big]$

Ex. 5. $\dfrac{-9x^2 - 12x - 4}{2x^2 - 5x + 2} < 0.$ $\Big[x < -\dfrac{2}{3},\ -\dfrac{2}{3} < x < \dfrac{1}{2},\ x > 2\Big]$

Ex. 6. $\dfrac{(x - a)(x - b)}{x^2 - a^2} \geq 0,\ a > b > 0.$ $\Big[x < -a,\ b \leq x < a,\ x > a\Big]$

1.3 Irrational inequalities

Solve the following inequalities for $x \in \mathbb{R}$:

Ex. 7. $2x - 3 > \sqrt{4x^2 - 13x + 3}.$ $\Big[x \geq 3\Big]$

Ex. 8. $x - 8 < \sqrt{x^2 - 9x + 14}.$ $\Big[x \leq 2,\ x \geq 7\Big]$

Ex. 9. $\sqrt{x - 1} - \sqrt{x - 2} < 2.$ $\Big[x \geq 2\Big]$

Ex. 10. $\sqrt{x + 2} < 8 + \sqrt{x - 6}.$ $\Big[x \geq 6\Big]$

Ex. 11. $\sqrt{3x - 8} > \sqrt{5x + 3} + \sqrt{x + 6}.$ $\Big[\text{no solution}\Big]$

Ex. 12. $\sqrt{x - 1} \leq x - 2.$ $\Big[x \geq \dfrac{5 + \sqrt{5}}{2}\Big]$

Ex. 13. $\sqrt{x-1} \geq -100 - x$. $\left[x \geq 1\right]$

Ex. 14. $\dfrac{\sqrt{x-2}}{\sqrt{x}-4} < 1$. $\left[2 \leq x < 16\right]$

Ex. 15. $\sqrt[3]{|x+8|} > 1$. $\left[x < -9,\ x > -7\right]$

Ex. 16. $\sqrt{4-|x+3|} < 2$. $\left[-7 \leq x \leq 1\right]$

Ex. 17. $\sqrt[3]{4-|x+3|} < 2$. $\left[\mathbb{R}\right]$

Ex. 18. $\sqrt{4-|x+2|} < 2-|x|$. $\left[\dfrac{-5+\sqrt{17}}{2} < x < 1\right]$

Ex. 19. $\sqrt{3-|4x+2|} < 1-2|x|$. $\left[0 < x \leq \dfrac{1}{4}\right]$

1.4 Absolute value inequalities

Solve the following inequalities for $x \in \mathbb{R}$:

Ex. 20. $|\,|x-1|-1\,| \geq 2$. $\left[\{x \leq -2\} \cup \{x \geq 4\}\right]$

Ex. 21. $|x-2|-|x| < 3$. $\left[\mathbb{R}\right]$

Ex. 22. $|\,|x-2|-|x|\,| \leq 3$. $\left[\mathbb{R}\right]$

Ex. 23. $|x^2-2x-4| \geq |x|+2$. $\left[\left\{x \leq -2\} \cup \{x \geq \dfrac{3+\sqrt{33}}{2}\right\} \cup \left\{\dfrac{3-\sqrt{17}}{2} \leq x \leq 2\right\}\right]$

Ex. 24. $|x-2|+|x| < 3$. $\left[\left\{-\dfrac{1}{2} < x < \dfrac{5}{2}\right\}\right]$

Ex. 25. $\left|\dfrac{x-2}{x-3}\right| - |x-2| < 2$. $\left[\{x < 1+\sqrt{3}\} \cup \{x > 2+\sqrt{2}\}\right]$

1.5 Exponential and logarithmic inequalities

Solve the following inequalities for $x \in \mathbb{R}$:

Ex. 26. $4^{x+1}6^{3x-2} < 8^x$. $\left[x < \dfrac{2\log 3}{\log 108}\right]$

Ex. 27. $3 \cdot 5^{2(2x-7)} - 4 \cdot 5^{(2x-7)} + 1 > 0$. $\left[x < \dfrac{7}{2} - \dfrac{\log 3}{2\log 5},\ x > \dfrac{7}{2}\right]$

Ex. 28. $\log_3(2x^2-7x+103) > 2$. $\left[\mathbb{R}\right]$

Ex. 29. $\log_5(x^2 - 7x + 11) < 0.$ $\left[2 < x < \frac{7-\sqrt{5}}{2},\ \frac{7+\sqrt{5}}{2} < x < 5\right]$

Ex. 30. $\log_{10}(x + 4)^2 > \log_{10}(13x + 10).$ $\left[-\frac{10}{13} < x < 2,\ x > 3\right]$

Ex. 31. $2^{2x} - 5 \cdot 2^x + 4 < 0.$ $\left[0 < x < 2\right]$

Ex. 32. $\frac{6}{2^x - 1} + \frac{3}{2^x + 1} > \frac{2}{2^x - 1} + 5.$ $\left[0 < x < 1\right]$

Ex. 33. $|\log_{10}(3x + 4) - \log_{10} 7| < 1.$ $\left[-\frac{11}{10} < x < 22\right]$

1.6 Trigonometric inequalities

Solve the following inequalities for $x \in \mathbb{R}$:

Ex. 34. $2\sin^2 x - \cos x - 1 > 0.$ $\left[\frac{\pi}{3} + 2k\pi < x < \pi + 2k\pi,\ \pi + 2k\pi < x < \frac{5}{3}\pi + 2k\pi,\ k \in \mathbb{Z}\right]$

Ex. 35. $\cos 2x + 3\sin x \geq 2.$ $\left[\frac{\pi}{6} + 2k\pi \leq x \leq \frac{5}{6}\pi + 2k\pi,\ k \in \mathbb{Z}\right]$

Ex. 36. $3\tan^2 x - 4\sqrt{3}\tan x + 3 > 0.$ $\left[-\frac{\pi}{2} + k\pi < x < \frac{\pi}{6} + k\pi,\ \frac{\pi}{3} + k\pi < x < \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right]$

Ex. 37. $\log_a\left(\frac{1}{2} - |\sin x|\right) < 0,\ a > 1.$
$\left[-\frac{1}{6}\pi + 2k\pi < x < \frac{1}{6}\pi + 2k\pi, \frac{5}{6}\pi + 2k\pi < x < \frac{7}{6}\pi + 2k\pi,\ k \in \mathbb{Z}\right]$

Ex. 38. $3\cos x + \sin^2 x - 3 > 0.$ [not possible]

Ex. 39. $4\cos\left(x + \frac{\pi}{6}\right) - 2\sqrt{3}\cos x + 1 \geq 0.$ $\left[-\frac{7}{6}\pi + 2k\pi \leq x \leq \frac{\pi}{6} + 2k\pi,\ k \in \mathbb{Z}\right]$

Ex. 40. $\left|\frac{\cos 2x}{\sin x}\right| \leq 1.$ $\left[\frac{\pi}{6} + 2k\pi \leq x \leq \frac{5}{6}\pi + 2k\pi,\ \frac{7}{6}\pi + 2k\pi \leq x \leq \frac{11}{6}\pi + 2k\pi,\ k \in \mathbb{Z}\right]$

Ex. 41. $\left|\frac{\tan 2x}{\cot x}\right| < 1.$ $\left[k\pi < x < \frac{\pi}{6} + k\pi,\ \frac{5}{6}\pi + k\pi < x < \pi + k\pi,\ k \in \mathbb{Z}\right]$

1.7 Boundedness of numerical sets

Study the boundedness of the following numerical sets, expressing for any of them sup, inf, max and min by verifying the definition

Ex. 42. $A = \left\{\frac{1}{n^2 + 1},\ n \in \mathbb{N}\right\}.$ $\left[\inf A = 0,\ \max A = 1\right]$

Ex. 43. $A = \left\{\frac{(-1)^n}{n^2 + 2},\ n \in \mathbb{N}\right\}.$ $\left[\min A = -\frac{1}{3},\ \max A = \frac{1}{2}\right]$

Ex. 44. $A = \left\{\frac{x+2}{x-3},\, x \in \mathbb{R},\, x > 3\right\}$. $\left[\inf A = 1,\ \sup A = +\infty\right]$

Ex. 45. $A = \left\{\frac{x+2}{x-2},\, x \in \mathbb{R},\, x < 2\right\}$. $\left[\inf A = -\infty,\ \sup A = 1\right]$

Ex. 46. $A = \left\{\frac{nm}{n^2+m^2},\, (n, m) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0)\}\right\}$. $\left[\min A = 0,\ \max A = \frac{1}{2}\right]$

Ex. 47. $A = \left\{\frac{nm}{n^2+m^2},\, (n, m) \in \mathbb{N} \setminus \{0\}\right\}$. $\left[\inf A = 0,\ \max A = \frac{1}{2}\right]$

Ex. 48. $A = \left\{\frac{n+m}{n-m},\, n, m \in \mathbb{N},\, n \neq m\right\}$. $\left[\inf A = -\infty,\ \sup A = +\infty\right]$

Ex. 49. $A = \left\{\frac{n}{m} + \frac{m}{n},\, n, m \in \mathbb{N} \setminus \{0\}\right\}$. $\left[\inf A = 2,\ \sup A = +\infty\right]$

Study the boundedness of the following numerical sets, expressing for any of them sup, inf, max and min

Ex. 50. $A = \left\{\frac{3n+1}{n+2},\, n \in \mathbb{N} \setminus \{0\}\right\}$. $\left[\min A = \frac{4}{3},\ \sup A = 3\right]$

Ex. 51. $A = \left\{\frac{1}{1+2^{-n}},\, n \in \mathbb{N} \setminus \{0\}\right\}$. $\left[\min A = \frac{2}{3},\ \sup A = 1\right]$

Ex. 52. $A = \left\{\frac{2n}{n!+1},\, n \in \mathbb{N} \setminus \{0\}\right\}$. $\left[\inf A = 0,\ \max A = \frac{4}{3}\right]$

Ex. 53. $A = \left\{\frac{\log n!}{n!},\, n \in \mathbb{N}\right\}$. $\left[\min A = 0,\ \max A = \log\sqrt{2}\right]$

Ex. 54. $A = \left\{\frac{n}{\sin(1+n\pi/2)},\, n \in \mathbb{N}\right\}$. $\left[\inf A = -\infty,\ \sup A = +\infty\right]$

Ex. 55. $A = \left\{\frac{\sqrt{n} - \sqrt{n+2}}{n^2},\, n \in \mathbb{N} \setminus \{0\}\right\}$. $\left[\min A = -\frac{2}{1+\sqrt{3}},\ \sup A = 0\right]$

Ex. 56. $A = \left\{\left|(-1)^n \frac{n}{n+3} - \frac{1}{5}\right|,\, n \in \mathbb{N}\right\}$. $\left[\min A = \frac{1}{5},\ \sup A = \frac{6}{5}\right]$

Ex. 57. $A = \left\{\left|n^2 + \sin(\frac{n\pi}{2})\right|,\, n \in \mathbb{N}\right\}$. $\left[\min A = 0,\ \sup A = +\infty\right]$

Ex. 58. $A = \left\{\sin\left(\frac{(2n+1)\pi}{2}\right) 2^{1/(n+1)},\, n \in \mathbb{N}\right\}$. $\left[\min A = -\sqrt{2},\ \max A = 2\right]$

Establish if the following numerical sets are bounded; find sup, inf, max and min, if they exist

Ex. 59. $A = \left\{\frac{1}{1+2n},\, n \in \mathbb{N},\, n \geq 1\right\}$. $\left[\inf A = 0,\ \max A = \frac{1}{3}\right]$

Ex. 60. $A = \left\{x \in \mathbb{R} : \frac{x}{x+1} > \frac{1}{2}\right\}$. $\left[A = (-\infty, -1) \cup (1, +\infty);\ \inf A = -\infty,\ \sup A = \infty\right]$

Ex. 61. $A = \left\{x \in \mathbb{R} : \sqrt{x^2 - 2x} < \frac{1}{2}x\right\}$. $\left[\min A = 2,\ \sup A = \frac{8}{3}\right]$

Ex. 62. $A = \{x \in \mathbb{R} : \sqrt{\log(\sin x)} \in \mathbb{R}\}$. $\left[A = \{\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}\};\ \inf A = -\infty,\ \sup A = +\infty\right]$

Ex. 63. $A = \{x \in \mathbb{R} : 1 \le 3^{2x+1} < 9\}$. $\left[\min A = -\frac{1}{2},\ \sup A = \frac{1}{2}\right]$

Ex. 64. $A = \left\{x \in \mathbb{R} : 5 < \frac{1}{5}^{3x-3} \le 25\right\}$. $\left[\min A = \frac{1}{3},\ \sup A = \frac{2}{3}\right]$

Ex. 65. $A = \left\{1 - \frac{(-1)^n}{n},\ n \in \mathbb{N} \setminus \{0\}\right\}$. $\left[\min A = \frac{1}{2},\ \max A = 2\right]$

Ex. 66. $A = \left\{\begin{array}{ll} \frac{4}{2n+1}, & n \in \mathbb{N},\ n \text{ even} \\ 2 - \frac{1}{n+1}, & n \in \mathbb{N},\ n \text{ odd} \end{array}\right\}$. $\left[\inf A = 0,\ \max A = 4\right]$

Ex. 67. Define an infinite set using a non-monotone sequence such that 0 and 1 will be the inf and sup of the set respectively.

Ex. 68. Find inf e sup of the areas of the surfaces of the rectangles with perimeter equal to $4a$, for a a positive real number, different from zero.

1.8 Domain of functions

Determine the domain of the following functions and study the boundedness of such sets. Then trace a qualitative graph of the functions themselves.

Ex. 69. $f(x) = \sqrt{x^2 - 1}$.

Ex. 70. $f(x) = \sqrt{\frac{1-x}{x+2}}$.

Ex. 71. $f(x) = \sqrt[4]{\frac{|1-x|}{x+2}}$.

Ex. 72. $f(x) = \log_{1/2}(1 - |x|)$.

Ex. 73. $f(x) = \sqrt[6]{\log_{1/3}(2 - |x|)}$.

Ex. 74. $f(x) = \sqrt{\log_2(x^2 - 2x - 5) - 1}$.

Ex. 75. $f(x) = \sqrt{\log_3(2x + 2) - \log_3 x}$.

Ex. 76. $f(x) = \sqrt{\log_3(\frac{x+2}{x})}$.

Ex. 77. $f(x) = \sqrt{\log_3(x+1) - \log_9(x+2) + 1}$.

Ex. 78. $f(x) = 2^{(x+2)/(x^2-3x-4)}$.

Ex. 79. $f(x) = \log_5(6^{2x} - |4 \cdot 6^x - 1|)$.

Ex. 80. $f(x) = \cos\left(\frac{2x-1}{x+1}\right)$.

Ex. 81. $f(x) = \sqrt{\cos\left(\frac{2x-1}{x+1}\right)}$.

Ex. 82. $f(x) = \left(\cos\left(\frac{2x-1}{x+1}\right) - \frac{1}{2}\right)^{1/4}$.

Ex. 83. $f(x) = \frac{1}{\sin x + \cos x}$.

Ex. 84. $f(x) = 2\log_3(\sin x + 2\cos x)$.

Ex. 85. $f(x) = \log_3(\sin x + 2\cos x)^2$.

Ex. 86. $f(x) = \log_3^2(\sin x + 2\cos x)$.

Ex. 87. $f(x) = \arccos\left(\frac{x+1}{x-1}\right)$.

Ex. 88. $f(x) = \arcsin\left(\frac{x+1}{|x|-1}\right)$.

Ex. 89. $f(x) = \left(\log_4(\sin x)\right)^{1/2}$.

Ex. 90. $f(x) = \left[2^{\sqrt[4]{1-\log_7(x^2+x)}} - (x^2+x)\right]^{1/2}$.

Ex. 91. Indicated by D the domain of any function of the exercises in the paragraph 1.8, determine the set of the interior points $\mathring{D}$ of D and the set of its boundary points ∂D. Moreover, say if such sets are oper or closed and study their boundedness.

Ex. 92. Determine the set of the images (range) for any function of the exercises in the paragraph 1.8, and the set of the accumulation points of such sets.

Ex. 93. Given two functions $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$, show the following implications:

1. f, g increasing $\Longrightarrow f + g$ increasing;
2. f, g decreasing $\Longrightarrow f + g$ decreasing;
3. f increasing and g strictly increasing $\Longrightarrow f + g$ strictly increasing;
4. f decreasing and g strictly decreasing $\Longrightarrow f + g$ strictly decreasing.

Ex. 94. Establish under which conditions the following implication is true:

$$f, g \text{ increasing (or decreasing)} \Longrightarrow f \cdot g \text{ increasing (or decreasing)}.$$

Ex. 95. Furnish an example such that the result of the exercise 94 is, in general i.e. without further hypothesis, false.

Ex. 96. Show that if $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is invertible, then

$$f \text{ increasing (decreasing)} \Longrightarrow f^{-1} \text{ increasing (decreasing)}.$$

Ex. 97. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be such that $0 \notin f(A)$ and increasing. Determine if $\dfrac{1}{f}$ is increasing or decreasing.

Ex. 98. Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ two injective functions. Is the function $f + g$ invertible?

Ex. 99. Let $f : X \to Y$ and $g : V \to W$ and let moreover $f(X) \cap V \neq \emptyset$. If f and g are invertible functions, is the composition $f \circ g$ an invertible function?

Ex. 100. Furnish three different examples of functions $f : X \to X$ such that $f \equiv f^{-1}$.

1.9 Invertibility of functions

Study the invertibility of the following functions in their natural definition set.

Ex. 101. $f(x) = 2^x + x$.

Ex. 102. $f(x) = -x + \log_{1/2} x$.

Ex. 103. $f(x) = x^2 + \log_3(1 + x)$.

Ex. 104. $f(x) = \dfrac{5^x}{1 + 5^x} + x^3$.

Ex. 105. $f(x) = x|x| + 1$.

Ex. 106. $f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x > 1 \\ x + a & \text{if } x \leq 1 \end{cases}$ al variare di $a \in \mathbb{R}$.

Ex. 107. $f(x) = \begin{cases} x^2 + ax & \text{if } x \leq 0 \\ -\dfrac{1}{x} & \text{if } x > 0 \end{cases}$ for any $a \in \mathbb{R}$.

Ex. 108. $f(x) = \begin{cases} x^3 & \text{if } |x| \geq 1 \\ ax & \text{if } |x| < 1. \end{cases}$ for any $a \in \mathbb{R}$.

Ex. 109. Let $f : X \to Y$ and $g : V \to W$ be two invertible functions such that it is well defined the composed function $g \circ f$. Call f^{-1} and g^{-1} their inverses respectively, show that

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Verify that the following functions are invertible; then determine the inverse of any of them, specifying its domain.

Ex. 110. $f(x) = x|x| + x$.

Ex. 111. $f(x) = x(x-2),\ x \leq 0$.

Ex. 112. $f(x) = \log_{1/2}(1 - x^3)$.

Ex. 113. $f(x) = \dfrac{3^{x+1}}{1 + 3^{x+1}}$.

Ex. 114. $f(x) = \sqrt{e^{2x} + e^x + 1}$.

Ex. 115. $f(x) = \sin^3\left(\dfrac{x^2}{x^2+1}\right),\ x \leq 0$.

Ex. 116. $f(x) = \arccos(\log_2 x)$.

Ex. 117. $f(x) = \tan(x^3 + 1), \quad \dfrac{\pi}{2} < x^3 + 1 < \dfrac{3}{2}\pi$.

Ex. 118. $f(x) = \arctan(x^3 + 1)$.

Ex. 119. $f(x) = \arcsin(\sqrt{x^2 + 1}), \quad x < 0$.

2 Complex numbers

2.1 Elementary properties of complex numbers

Determine $\bar{z}$, $\operatorname{Im}(z)$ e $|z|$ in the following cases:

Ex. 120. $z = 3 - 4i$. $\left[3 + 4i,\ -4,\ 5\right]$

Ex. 121. $z = (2 - i)(-3 + 2i)$. $\left[-4 - 7i,\ 7,\ \sqrt{65}\right]$

Ex. 122. $z = (1 - i)(3 - 7i)$. $\left[-4 + 10i,\ -10,\ 2\sqrt{29}\right]$

Ex. 123. $z = (2 - 3i)^2$. $\left[-5 + 12i,\ -12,\ 13\right]$

Ex. 124. $z = \left(\dfrac{1 - 2i}{2}\right)^2$. $\left[-\dfrac{3}{4} + i,\ -1,\ \dfrac{5}{4}\right]$

Ex. 125. $z = \dfrac{2}{3 - i}$. $\left[\dfrac{3 - i}{5},\ \dfrac{1}{5},\ \sqrt{\dfrac{2}{5}}\right]$

Ex. 126. $z = \dfrac{2 - 3i}{1 - i}$. $\left[\dfrac{5 + i}{2},\ -\dfrac{1}{2},\ \sqrt{\dfrac{13}{2}}\right]$

Ex. 127. $z = (1 - 2i)^3$. $\left[-11 - 2i,\ 2,\ 5\sqrt{5}\right]$

Ex. 128. $z = \dfrac{(1 - i)^3}{2 - i}$. $\left[\dfrac{-2 + 6i}{5},\ -\dfrac{6}{5},\ 2\sqrt{\dfrac{2}{5}}\right]$

Ex. 129. $z = \dfrac{(1 + 2i)^4}{(1 - i)^2}$. $\left[12 + \dfrac{7}{2}i,\ -\dfrac{7}{2},\ \dfrac{25}{2}\right]$

Compute the absolute value and argument of z in the following cases:

Ex. 130. $z = 1 + i$. $\left[\sqrt{2},\ \dfrac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}\right]$

Ex. 131. $z = 1 - \sqrt{3}i$. $\left[2, -\dfrac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z}\right]$

Ex. 132. $z = \dfrac{\sqrt{3}}{3} + \dfrac{i}{3}$. $\left[\dfrac{2}{3},\ \dfrac{\pi}{6} + 2k\pi,\ k \in \mathbb{Z}\right]$

Ex. 133. $z = i(1 - i)$. $\left[\sqrt{2},\ \dfrac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}\right]$

Ex. 134. $z = \dfrac{2}{1 + \sqrt{3}i}$. $\left[1,\ -\dfrac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z}\right]$

Ex. 135. $z = \dfrac{3}{\sqrt{3}+i}$. $\left[\dfrac{3}{2}, -\dfrac{\pi}{6} + 2k\pi, k \in \mathbb{Z}\right]$

Ex. 136. $z = (1-i)(\sqrt{3}+i)$. $\left[2\sqrt{2}, -\dfrac{\pi}{12} + 2k\pi, k \in \mathbb{Z}\right]$

Ex. 137. $z = \dfrac{1-i}{\sqrt{3}+i}$. $\left[\dfrac{1}{\sqrt{2}}, -\dfrac{5}{12}\pi + 2k\pi, k \in \mathbb{Z}\right]$

Ex. 138. $z = \dfrac{\sqrt{3}-i}{1+\sqrt{3}i}$. $\left[1, -\dfrac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\right]$

Ex. 139. $z = (1-i)^{12}$. $\left[64, \pi + 2k\pi, k \in \mathbb{Z}\right]$

Ex. 140. $z = \left(1 + \dfrac{i}{\sqrt{3}}\right)^{14}$. $\left[\dfrac{2^{14}}{3^7}, \dfrac{\pi}{3} + 2k\pi, k \in \mathbb{Z}\right]$

Ex. 141. $z = \dfrac{i^{324} - i^{261}}{i^{145} + i^{492}}$. $\left[1, -\dfrac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\right]$

Ex. 142. $z = \dfrac{1 - i^{1039}}{i^{2048} - i^{1457}}$. $\left[1, \dfrac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\right]$

Ex. 143. $z = \dfrac{\cos 2\theta - i \sin 2\theta}{\sin\theta + i\cos\theta}$. $\left[1, -\theta - \dfrac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\right]$

Ex. 144. $z = \dfrac{\frac{1}{2}\sin 2\theta + i\cos\theta}{1 - i\sin\theta}$.
$\left[|\cos\theta|, \dfrac{\pi}{2} \text{ se } \cos\theta > 0, -\dfrac{\pi}{2} \text{ se } \cos\theta < 0, \text{ not determinate if } \cos\theta = 0\right]$

2.2 Roots of complex numbers

Compute the following roots of the complex numbers:

Ex. 145. $\sqrt{1+\sqrt{3}i}$. $\left[\pm\dfrac{\sqrt{3}+i}{\sqrt{2}}\right]$

Ex. 146. $\sqrt[3]{1+\sqrt{3}i}$. $\left[\sqrt[3]{2}\left(\cos\dfrac{\pi}{9} + i\sin\dfrac{\pi}{9}\right), \sqrt[3]{2}\left(\cos\dfrac{7\pi}{9} + i\sin\dfrac{7\pi}{9}\right),\right.$
$\left.\sqrt[3]{2}\left(\cos\dfrac{5\pi}{9} - i\sin\dfrac{5\pi}{9}\right)\right]$

Ex. 147. $\sqrt{\dfrac{1+i}{1-i}}$. $\left[\pm\dfrac{1+i}{\sqrt{2}}\right]$

Ex. 148. $\sqrt[4]{(1-i)^3 + (1+i)^3}$. $\left[1+i, 1-i, -1+i, -1-i\right]$

Ex. 149. $\sqrt{2\dfrac{(1-i)^2}{(1+i)^3}}$. $\left[\pm\sqrt[4]{2}\left(\cos\dfrac{3\pi}{8} + i\sin\dfrac{3\pi}{8}\right) = \pm\dfrac{\sqrt{\sqrt{2}-1} + i\sqrt{\sqrt{2}+1}}{\sqrt{2}}\right]$

Compute the following expressions containing complex numbers:

Ex. 150. $\sqrt{\dfrac{\left((1+i)^2-(1-i)^3\right)-2}{(3-i)^2+6i}}$. $\left[\pm\dfrac{1+i}{2}\right]$

Ex. 151. $\sqrt[3]{1+\dfrac{(2-3i)^2+7i}{5}}$. $\left[i,\ \pm\dfrac{\sqrt{3}}{2}-\dfrac{i}{2}\right]$

Ex. 152. $\sqrt[4]{\dfrac{(1+2i)(3-i)}{5}-\dfrac{(1-i)(1+3i)}{2}}$. $\left[\dfrac{1+i}{\sqrt{2}},\ \dfrac{1-i}{\sqrt{2}},\ \dfrac{-1+i}{\sqrt{2}},\ \dfrac{-1-i}{\sqrt{2}}\right]$

Ex. 153. $\sqrt[3]{\dfrac{(2-i)^2-3}{(1+2i)^3+11}}$. $\left[\sqrt[3]{2},\ \dfrac{-1\pm\sqrt{3}}{\sqrt[3]{4}}\right]$

Ex. 154. $\sqrt{-1}+\sqrt[3]{i}$. $\left[0,\ -2i,\ \dfrac{3i}{2}\pm\dfrac{\sqrt{3}}{2},\ -\dfrac{i}{2}\pm\dfrac{\sqrt{3}}{2}\right]$

2.3 Complex equations

2.3.1 Algebraic complex equations

Determine the solutions of the following algebraic equations:

Ex. 155. $z^2+2z+2=0$. $\left[z=-1+i,\ -1-i\right]$

Ex. 156. $z^2-6z+13=0$. $\left[z=3+2i,\ 3-2i\right]$

Ex. 157. $4z^2-4z+17=0$. $\left[z=\dfrac{1}{2}+2i,\ \dfrac{1}{2}-2i\right]$

Ex. 158. $z^3+3z^2+z-5=0$. $\left[z=1,\ -2+i,\ -2-i\right]$

Ex. 159. $z^4+4=0$. $\left[z=1+i,\ 1-i,\ -1+i,\ -1-i\right]$

Ex. 160. $z^2-iz+6=0$. $\left[z=-2i,\ 3i\right]$

Ex. 161. $4z^2-2(1-i)z-i=0$. $\left[z=\dfrac{1}{2},\ -\dfrac{i}{2}\right]$

Ex. 162. $2z^2+iz+3=0$. $\left[z=i,\ -\dfrac{3i}{2}\right]$

Ex. 163. $2z^2-5iz-2=0$. $\left[z=2i,\ \dfrac{i}{2}\right]$

Ex. 164. $6z^2-(3+2i)z+i=0$. $\left[z=\dfrac{1}{2},\ \dfrac{i}{3}\right]$

Ex. 165. $8z^2-2(16+i)z+5(5+2i)=0$. $\left[z=1+\dfrac{i}{2},\ 3-\dfrac{i}{4}\right]$

Ex. 166. $2z^3-(2-i)z^2+(1-i)z-1=0$. $\left[z=1,\ \dfrac{i}{2},\ -i\right]$

Ex. 167. $2z^3+(2+5i)z^2+(3+5i)z+3=0$. $\left[z=-1,\ -3i,\ \dfrac{i}{2}\right]$

2.3.2 Non-algebraic complex equations

Ex. 168. $z|z^3| + 16 = 0.$ $\left[z = -2\right]$

Ex. 169. $z^2|z^2| + 16 = 0.$ $\left[z = 2i,\ z = -2i\right]$

Ex. 170. $z^3|z| + 16 = 0.$ $\left[z = -2,\ z = 1 + \sqrt{3}i,\ z = 1 - \sqrt{3}i\right]$

Ex. 171. $z^2(1 + |z^2|) = -20.$ $\left[z = 2i,\ z = -2i\right]$

Ex. 172. $z^2(1 - |z^2|) = -20.$ $\left[z = \sqrt{5},\ z = -\sqrt{5}\right]$

Ex. 173. $z^2(4 - |z^2|) = 5.$ $\left[z = \sqrt{5}i,\ z = -\sqrt{5}i\right]$

Ex. 174. $z^2(4 - |z^2|) = 4.$ $\left[z = \sqrt{2}i,\ z = -\sqrt{2}i\ \sqrt{2 + 2\sqrt{2}}i,\ -\sqrt{2 + 2\sqrt{2}}i\right]$

Ex. 175. $z^2(4 - |z^2|) = 3.$ $\left[z = 1,\ z = -1,\ z = \sqrt{3},\ z = -\sqrt{3},\ z = \sqrt{2 + \sqrt{7}}i,\ -\sqrt{2 + \sqrt{7}}i\right]$

Ex. 176. $\dfrac{z^2}{1 + |z^2|} = -\dfrac{1}{2}.$ $\left[z = i,\ z = -i\right]$

Ex. 177. $\dfrac{z^2}{1 + |z^2|} = -2.$ $\left[\text{Nessuna soluzione}\right]$

Ex. 178. $\dfrac{z^2}{1 - |z^2|} = -\dfrac{1}{2}.$ $\left[z = \sqrt{\dfrac{2}{3}}i,\ z = -\sqrt{\dfrac{2}{3}}i\right]$

Ex. 179. $\dfrac{z^2}{1 - |z^2|} = -2.$ $\left[z = \dfrac{i}{\sqrt{3}},\ z = -\dfrac{i}{\sqrt{3}}\right]$

Ex. 180. $\dfrac{z^4}{|z^2|} == -8.$ $\left[z = 2 + 2i,\ z = 2 - 2i,\ z = -2 + 2i,\ z = -2 - 2i\right]$

Ex. 181. $\dfrac{z^2}{|z^4|} == -\dfrac{1}{8}.$ $\left[z = 2\sqrt{2}i,\ z = -2\sqrt{2}i\right]$

Ex. 182. $\dfrac{z^4}{|z^6|} == 8.$ $\left[z = \dfrac{1}{2\sqrt{2}},\ z = -\dfrac{1}{2\sqrt{2}},\ z = \dfrac{i}{2\sqrt{2}},\ z = -\dfrac{i}{2\sqrt{2}}\right]$

Ex. 183. $\dfrac{z^2 - |z^2|}{4 + |z^2|} + 1 = 0.$ $\left[\text{No solution}\right]$

Ex. 184. $\dfrac{z^2 - |z^2|}{4 + |z^2|} + \dfrac{1}{2} = 0.$ $\left[z = \sqrt{2}i,\ z = -\sqrt{2}i\right]$

Ex. 185. $\dfrac{z^2 - |z^2|}{4 + |z^2|} + \dfrac{1}{4} = 0.$ $\left[z = (1 + \sqrt{3})i,\ z = (1 - \sqrt{3})i,\ z = -(1 + \sqrt{3})i,\ z = -(1 - \sqrt{3})i\right]$

Ex. 186. $\dfrac{z^2 - |z^2|}{4 + |z^2|} = 0.$ $\left[\operatorname{Im}(z) = 0\right]$

Ex. 187. $z(2 + |z^2|) = \dfrac{3}{\bar{z}}.$ $\left[|z| = 1\right]$

3 Limits of one real variable funtions

3.1 Check, using the definition, the following limits

Verify the definition of limit in the following cases:

Ex. 188. $\lim_{x\to 1} x = 1.$

Ex. 189. $\lim_{x\to +\infty} \frac{1}{x} = 0.$

Ex. 190. $\lim_{x\to 3}(2x + 1) = 7.$

Ex. 191. $\lim_{x\to 2} x^2 = 4.$

Ex. 192. $\lim_{x\to 0} \frac{1}{x^2} = +\infty.$

Ex. 193. $\lim_{x\to 0} \frac{1}{x^3} \quad \nexists.$

Ex. 194. $\lim_{x\to 1} 3^x = 3.$

Ex. 195. $\lim_{x\to \pi/2} \sin x = 1.$

Ex. 196. $\lim_{x\to (\pi/2)^-} \tan x = +\infty.$

Ex. 197. $\lim_{x\to 1^+} x - [x] = 0.$

Ex. 198. $\lim_{x\to 1^-} x - [x] = 1.$

Ex. 199. $\lim_{x\to 0^+} \log_{1/2} x = +\infty.$

Ex. 200. $\lim_{x\to +\infty} \frac{x+2}{2x+2} = \frac{1}{2}.$

Ex. 201. $\lim_{x\to +\infty} \sin\frac{1}{x} = 0.$

3.2 Computation of limits

Calculate, if they exist real or infinite,the following limits:

Ex. 202. $\lim_{x\to 2}\left(x^2 + \frac{1}{x}\right).$ $\left[\frac{9}{2}\right]$

Ex. 203. $\lim_{x\to+\infty} \dfrac{x+x^2}{x^3+1}$. $\big[0\big]$

Ex. 204. $\lim_{x\to 0} \dfrac{x\sin x}{1-\cos x}$. $\big[2\big]$

Ex. 205. $\lim_{x\to+\infty} \left(\sqrt{x^2+4}-x\right)$. $\big[0\big]$

Ex. 206. $\lim_{x\to+\infty} \left(\sqrt{2x+x^2}-x\right)$. $\big[1\big]$

Ex. 207. $\lim_{x\to+\infty} \dfrac{\log_2(x+x^2)}{\log_3 x - 1}$. $\big[2\log_2 3\big]$

Ex. 208. $\lim_{x\to 0} \dfrac{\sin x - x}{x^{9/10}}$. $\big[0\big]$

Ex. 209. $\lim_{x\to 4} \dfrac{\sqrt{x^2+1}-\sqrt{17}}{x-4}$. $\Big[\dfrac{4}{\sqrt{17}}\Big]$

Ex. 210. $\lim_{x\to 0^+} 4^{1/x}$. $\big[+\infty\big]$

Ex. 211. $\lim_{x\to 0^-} 4^{1/x}$. $\big[0\big]$

Ex. 212. $\lim_{x\to 0} \dfrac{\sin x - \sqrt{x}}{1-\cos\sqrt[4]{x}}$. $\big[-2\big]$

Ex. 213. $\lim_{x\to 0} \dfrac{\sin x - x^2}{\sqrt{1-\cos x^2}}$. $\big[+\infty\big]$

Ex. 214. $\lim_{x\to \pi/2} 2^{(\sin x - 1)/x^4}$. $\big[1\big]$

Ex. 215. $\lim_{x\to 0} \dfrac{\dfrac{\sin^2 x}{2}+\cos x - 1}{x^2}$. $\big[0\big]$

Ex. 216. $\lim_{x\to+\infty} \log_4\left(\dfrac{x+1}{x-1}\right)$. $\big[0\big]$

Ex. 217. $\lim_{x\to 0} \dfrac{\log_3(x+1)}{x}$. $\big[\log_3 e\big]$

Ex. 218. $\lim_{x\to 0} \dfrac{\log_{1/2}\cos x}{x^2}$. $\big[\log_4 e\big]$

Ex. 219. $\lim_{x\to 1^+} (\sin x)^{1/\log_2 x}$. $\big[0\big]$

Ex. 220. $\lim_{x\to+\infty} \dfrac{x^3}{2^x}$. $\big[0\big]$

Ex. 221. $\lim_{x\to+\infty} \dfrac{\log_3 x}{x}$. $\big[0\big]$

Ex. 222. $\lim_{x\to+\infty} \dfrac{x^3}{2^{\log_3(\log_2 x)}}$. $\Big[+\infty\Big]$

Determine domain and image of the following functions, indicating if they are periodic and even or odd.

Ex. 223. $f(x) = \sqrt{2\sin^2 x + \cos x - 1}$.

Ex. 224. $f(x) = \log_3(\sin^3 x - \cos^3 x)$.

Ex. 225. $f(x) = \log_{1/2}(|\sin 2x| + \cos x)$.

Ex. 226. $f(x) = 4^{(\sin x+\cos x)/(\sin x-\cos x)}$.

Ex. 227. $f(x) = \dfrac{1}{2^{\sin x} - 3^{\cos x}}$.

Ex. 228. $f(x) = |x|^\alpha \sin \dfrac{1}{x^3}$, al variare di $\alpha \in \mathbb{R}$.

Ex. 229. $f(x) = \arcsin\left(\dfrac{2+e^x}{e^{2x}-3}\right)$.

Ex. 230. $f(x) = \sqrt[4]{\tan^2(x^2+1) - \tan(x^2+1) - 6}$.

Ex. 231. $f(x) = \dfrac{5^x + 5^{-x}}{2}$.

Ex. 232. $f(x) = \arctan \dfrac{5^x - 5^{-x}}{2}$.

Draw a qualitative graph of the functions studied in the exercises 223, 226, 228, 231 and 232 above.

Calculate, if they exist real or infinite,the following limits:

Ex. 233. $\lim_{x\to+\infty} (x+5)\sqrt{\dfrac{x+1}{x-1}} - x$. $\Big[6\Big]$

Ex. 234. $\lim_{x\to+\infty} x(\log(x+1) - \log x)$. $\Big[1\Big]$

Ex. 235. $\lim_{x\to+\infty} \left(\dfrac{x^3-2x+1}{x^2+x^3}\right)^{(2x^2+1)/(x-3)}$. $\Big[e^{-2}\Big]$

Ex. 236. $\lim_{x\to 0} \dfrac{\log\cos x}{\sin 2x^2}$. $\Big[-\frac{1}{4}\Big]$

Ex. 237. $\lim_{x\to 0^+} (\sin x)^{x^2+3x\log x}$. $\Big[1\Big]$

Ex. 238. $\lim_{x\to 0} \dfrac{\log(1+\sin x)}{\sin 2x + x^2 \log x}$. $\Big[\frac{1}{2}\Big]$

Ex. 239. $\lim_{x\to 0} \dfrac{e^{2x-3}-e^{-3}}{\sin x}$. $\left[2e^{-3}\right]$

Ex. 240. $\lim_{x\to 0} \dfrac{\sin\left(\sqrt{1+x^2}-1\right)}{x}$. $\left[0\right]$

Ex. 241. $\lim_{x\to 0} \left(\dfrac{\log(1+x)+\sin x+x}{x+x^2}\right)^2$. $\left[9\right]$

Ex. 242. $\lim_{x\to 0} \dfrac{e^{-1/x^2}+\log\left(1+x^{1/5}-\sin\sqrt[3]{x}\right)}{\sqrt[3]{x}-2\sqrt[5]{x}}$. $\left[-\frac{1}{2}\right]$

Ex. 243. $\lim_{x\to +\infty} \dfrac{\sin(x^5/3^x)}{x^4 2^{-x}}$. $\left[0\right]$

Calculate, if they exist real or infinite,the following limits:

Ex. 244. $\lim_{x\to 1} \dfrac{\sqrt{x}-\cos(x-1)}{\log x}$. $\left[\frac{1}{2}\right]$

Ex. 245. $\lim_{x\to 2} \left(\sin\dfrac{\pi x}{4}\right)^{1/\log(3-x)}$. $\left[1\right]$

Ex. 246. $\lim_{x\to 1} |x-1|^{x-1}$. $\left[1\right]$

Ex. 247. $\lim_{x\to 0^+} x^{1/\log x}$. $\left[e\right]$

Ex. 248. $\lim_{x\to +\infty} \left(\dfrac{\cos(1/x)}{\cos(2/x)}\right)^{(x^2+1)/x}$. $\left[1\right]$

Ex. 249. $\lim_{x\to 0^+} \dfrac{(2x^x-1)^{1/\sqrt{x}}-1}{\sqrt{x}\log x}$. $\left[2\right]$

Ex. 250. $\lim_{x\to +\infty} x^2((e^{1/x}+1)^{1/2}-\cos(1/x))$. $\left[+\infty\right]$

Ex. 251. $\lim_{x\to +\infty} x^2((2e^{1/x^2}-1)^{1/2}-\cos(1/x))$. $\left[\frac{3}{2}\right]$

Ex. 252. $\lim_{x\to 3} \dfrac{e^{-1/(3-x)^2}+e(4-3\cos(x-3))^{1/5}-e^{\sqrt{4-x}}}{\sqrt{1-\cos(x-3)}}$. $\left[-\frac{e}{\sqrt{2}}\right]$

Ex. 253. $\lim_{x\to +\infty} \sin(1/x)\cdot\log(x^2+e^{1/x}+2^{x^2/(x+1)})$. $\left[\log 2\right]$

Ex. 254. $\lim_{x\to +\infty} \dfrac{1}{\log^{10}(x^2+x+1)}\left(\sin\dfrac{1}{\frac{x}{x+1}\log^{10}(x^3+x+1)}\right)^{-1}$. $\left[(\frac{3}{2})^{10}\right]$

Ex. 255. $\lim_{x\to 0} \dfrac{\arcsin\sqrt{x}}{\sqrt{\cos\sqrt[4]{x}-1}}$. $\left[-4\right]$

Ex. 256. $\lim_{x\to 0}(1+\sin x)^{1/\arctan x}$. $\left[e\right]$

Calculate, if they exist real or infinite,the following limits:

Ex. 257. $\lim_{x\to+\infty}\dfrac{x^2+\sin x}{x+\log\left(x+e^{2x^2}\right)}$. $\left[\frac{1}{2}\right]$

Ex. 258. $\lim_{x\to+\infty}\left(x^4e^{-x}+\sin(1/x^2)+1\right)^{\sqrt{1+2x^4}}$. $\left[e^{\sqrt{2}}\right]$

Ex. 259. $\lim_{x\to+\infty}\dfrac{\left(\sqrt{x+x^3}-x\right)\log\left(\dfrac{\sqrt{4x+1}}{2\sqrt{x}+3}\right)}{x\arctan x}$. $\left[-\frac{3}{\pi}\right]$

Ex. 260. $\lim_{x\to+\infty}\dfrac{x^{\arctan x}-x^{\pi/2}}{(1+x)^{\pi/2+1/\sqrt{\log x}}}$. $\left[0\right]$

Ex. 261. $\lim_{x\to+\infty}\dfrac{x^{\arctan x}-x^{\pi/2}}{(1+x)^{\pi/2-1}}$. $\left[-\infty\right]$

Ex. 262. $\lim_{x\to 0^+}\dfrac{e^{-1/x}+x^2+\dfrac{1}{\log^2 x}+x\log\left(e^{-1/x}+e^{-2/x}\right)+1}{e^x-1}$. $\left[+\infty\right]$

Ex. 263. $\lim_{x\to 0^+}\dfrac{x\sin x-\cos x+e^{x^2/2}}{\sqrt{1-\cos x}\cdot\arcsin x}$. $\left[\frac{4}{\sqrt{2}}\right]$

Ex. 264. $\lim_{x\to 1}\dfrac{(x^2-2x+1)\tan(x-1)-\sin^3(x-1)}{\sqrt{\cos(x-1)}-1}$. $\left[0\right]$

Ex. 265. $\lim_{x\to 0^+}\dfrac{x\left(\cos\sqrt{x^3}-1\right)+\sin^2 x^{3/4}}{x^3e^{-1/\sqrt{x}}+\sqrt{x}\left(e^{x^2}-1\right)}$. $\left[+\infty\right]$

Ex. 266. $\lim_{x\to 0^+}\dfrac{x\left(\cos\sqrt{x^3}-1\right)+\sin^2 x^{3/4}}{x^3e^{-1/\sqrt{x}}+\left(e^{x^2}-1\right)/\sqrt{x}}$. $\left[1\right]$

Ex. 267. $\lim_{x\to 0^+}\dfrac{\log|\log x|+\log x}{\log\left(1+x^{\log x}\right)}$. $\left[0\right]$

Ex. 268. $\lim_{x\to 1}\dfrac{e^{3x-x^2}-e^2\cos(x-1)-x+1}{\log(\sin \pi x/2)}$. $\left[(\frac{2e}{\pi})^2\right]$

Calculate, if they exist real or infinite,the following limits:

Ex. 269. $\lim_{x\to 0^+}x^x;\ \lim_{x\to 0^+}x^{x^x};\ \lim_{x\to 0^+}x^{x^{x^x}}$. $\left[1;\ 0;\ 1\right]$

Ex. 270. $\lim_{x\to 0^+}\overbrace{x^{x^{\cdot^{\cdot^{x}}}}}^{n\text{ times}}$. $\left[1 \text{ if } n \text{ is even, } 0 \text{ if } n \text{ is odd}\right]$

Ex. 271. $\lim_{x\to 1}\left(\frac{(x-1)^2}{\sin(\pi x)(e-e^x)}\right)^{\log x}$. $\left[1\right]$

Ex. 272. $\lim_{x\to+\infty}\frac{\cos(1/x)-e^{-1/x^2}}{\left(\sqrt{x^4-x^2}-x^2\right)\log\sqrt{\frac{x^2+2}{x^2+1}}}$. $\left[-2\right]$

Ex. 273. $\lim_{x\to 0^+}\frac{\sin(e^{x^2}-\cos x+2\sin x^2\sqrt{1+2\sin^2 x})}{2\sin^2 x}$. $\left[\frac{7}{4}\right]$

Ex. 274. $\lim_{x\to 0^+}\frac{\sqrt{e^{x^2}-\cos x+2\sin x^3\sqrt{1+2\sin x^3}}}{2\sin^3 x}$. $\left[+\infty\right]$

Ex. 275. $\lim_{x\to 0^+}\frac{\sqrt{1+x\sin x}-\sqrt{\cos 2x}}{\tan^2(x/2)}$. $\left[6\right]$

Ex. 276. $\lim_{x\to 0}\frac{\log(2-\cos x)(2-\cos x)^{1/x^2}\sin x^2}{\sin^2 x^2}$. $\left[\frac{\sqrt{e}}{2}\right]$

Ex. 277. $\lim_{x\to 0}\frac{\log(\cos^2 x)(x-\sqrt{x^2+3x+1})}{1+e^{-1/x}-\cos x}$. $\left[2\right]$

Ex. 278. $\lim_{x\to 0}\left((\sin x+2)^2\log(\sin x+1)\right)^{(\sqrt{1+3x^2}-1)/x^2}$. $\left[0\right]$

Ex. 279. $\lim_{x\to+\infty}\frac{\log\left(\frac{e^{-1/x}}{x^4}+1\right)+\sin^3(1/x)}{\log\left(\frac{2+x^3}{x^3}\right)}$. $\left[\frac{1}{2}\right]$

Arrange in growing order of infinity (infinitesimal) the following functions and sequences, after having determined the order of infinite (infinitesimal), if it exits as a real number.

Ex. 280. For $x\to+\infty$: a) $\frac{e^x}{x^2}$, b) $x\log x$, c) $\frac{x^2}{\log x}$, d) $\frac{1}{\sin(1/x)}$. [d, b, c, a. ord d=1]

Ex. 281. For $n\to+\infty$: a) 2^n, b) $n!$, c) n^n, d) $\left(\frac{3}{2}\right)^{n^2}$. [a, d, b, c]

Ex. 282. For $x\to+\infty$: a) x^x, b) $x\log^2 x$, c) $x^{2\log x}$, d) $\frac{x^5+x^3+2}{x^2+1}\log\frac{x+1}{x}$.
[b, d, c, a. ord d=2]

Ex. 283. For $x\to 0^+$: a) $\frac{1}{\log x}$, b) x^2, c) $\frac{\sqrt[3]{1-\cos x}}{\sqrt{\arcsin x}}$, d) $\log x\cdot\arcsin x$.
[a, c, d, b. ord b=2, c=$\frac{1}{6}$]

Ex. 284. For $x\to 0^+$: a) $\log x$, b) $\log|\log x|$, c) $\frac{1}{x\log x}$, d) $\frac{1}{\log(1+x)}$.
[b, a, c, d. ord d=1]

Ex. 285. For $x \to 1^+$: a) $e^{-1/(x-1)^2}$, b) $\sqrt[10]{x} - \cos(x-1)$, c) $\sin^3 \sqrt[3]{x^2 - x}$, d) $\dfrac{x-1}{\log^{20}(x-1)}$.
$\left[\text{c, d, b, a. ord b=1, c=}\frac{1}{3}\right]$

Ex. 286. For $x \to 2^+$: a) $\dfrac{1}{(x-2)^{3/2}}$, b) $\dfrac{1}{(x-2)^{3/4}\log(x-2)}$, c) $e^{\sqrt{x-2}/\sin(x-2)}$,
d) $(x-2)^{1/(2-x)}$. $\left[\text{a, b, c, d. ord a=}\frac{3}{2}\right]$

Ex. 287. For $x \to 0^+$: a) $x \arctan x$, b) $\dfrac{1-\cos x}{\log x}$, c) $x^x - 1$, d) $\sin^3 \sqrt[4]{x}$.
$\left[\text{d, c, b, a. ord a=2, d=}\frac{3}{4}\right]$

Arrange in growing order of infinity (infinitesimal) the following functions and sequences, after having determined the order of infinite (infinitesimal), if it exits as a real number.

Ex. 288. For $x \to +\infty$: a) x^2, b) $\log(1 + x^3 + e^{x^3})$, c) $\dfrac{x^2}{x+1}$, d) $\left(\dfrac{x^2}{x+1}\right)^{1+1/\sqrt{\log x}}$.
$\left[\text{c, d, a, b. ord a=2, b=3, c=1}\right]$

Ex. 289. For $n \to +\infty$: a) $\dfrac{\sqrt{n}}{n^2+1}$, b) $\dfrac{1}{n \log n}$, c) $\dfrac{\log^2 n}{n}$, d) $\dfrac{n!}{(n+1)! - (n-1)!}$.
$\left[\text{c, d, b, a. ord a=}\frac{3}{2}\text{, d=1}\right]$

Ex. 290. For $n \to +\infty$: a) $(\sqrt[n]{n} - 1)^{-1}$, b) $n(\sqrt{3+n^2} - n)$, c) $(\cos(1/n) - 1) \cdot 2^{n^3/(n+1)}$, d) n^n.
$\left[\text{a, b, c, d. ord b=2}\right]$

Ex. 291. For $x \to 0^+$: a) $\dfrac{x^2(1-\cos x)^2}{\log(1+\sin^4 x)}$, b) $\log(x+1)$, c) $x \log x$, d) $\sin(x \log(1+x)) \cdot \log x$.
$\left[\text{c, b, d, a. ord a=2, b=1}\right]$

Ex. 292. For $x \to +\infty$: a) $\dfrac{x^2 \log(2 - \cos(1/x))}{\sin^2(1/x)}$, b) $\dfrac{x^{\sqrt{x}}}{x^{100}}$, c) $x^2 \log\left(\dfrac{x^2+1}{x}\right)$,
d) $x \log^{100}(1+x)$. $\left[\text{d, a, c, b. ord a=2}\right]$

Ex. 293. For $x \to 3^+$: a) $(e^{\frac{(x-3)^2}{(3-x)(x+1)^3}} - 1)\sin(x-3)^{9/4}$, b) $\sin^3(x-3)$, c) $(x-3)^3 \log(x-2)$,
d) $(x-3)^3 \log^{10}(x-3)$. $\left[\text{d, b, a, c. ord a=}\frac{13}{4}\text{, b=3, c=4}\right]$

Ex. 294. For $x \to 0^+$: a) $x \log(1+x^2)$, b) $x^{2-x/(x^2+1)}$, c) $\left(\dfrac{\sqrt[3]{x^2+x}}{\sqrt[4]{x^2+2x}}\right)^{25}$, d) $x^3 \log^{10} x$.
$\left[\text{b, c, d, a. ord a=3, b=2, c=4, c=}\frac{25}{12}\right]$

Ex. 295. For $x \to 0^+$: a) $x \arctan \sqrt{x}$, b) $\dfrac{(1-\cos x)^2 \sqrt{x+1}}{\sqrt{x^4+1}\,\log(1+x^2)}$, c) $x^2 \log\left(\dfrac{x^2+1}{x}\right) e^{\sqrt{x}}$,
d) $\sin(x^3 \log x)$. $\left[\text{a, c, b, d. ord a=}\frac{3}{2}\text{, b=2}\right]$

Calculate the limit of the following sequences:

Ex. 296. $\lim_{n\to+\infty} \dfrac{e^{n^2}}{n^n}$. $\Big[+\infty\Big]$

Ex. 297. $\lim_{n\to+\infty} \dfrac{e^{n^{3/2}}}{n^{n^2}+e^n}$. $\Big[0\Big]$

Ex. 298. $\lim_{n\to+\infty} \dfrac{\sqrt{n^2+n^3}-n+\sin n}{\sqrt[4]{1+n^5+2n^6}}$. $\Big[\dfrac{\sqrt[4]{2}}{2}\Big]$

Ex. 299. $\lim_{n\to+\infty} \dfrac{2^{(1+\log^{1/2} n)}}{n^{1/2}}$. $\Big[0\Big]$

Ex. 300. $\lim_{n\to+\infty} (\log(n^2+1)-\log n-\log(n+1))\sqrt{1+n^2}$. $\Big[-1\Big]$

Ex. 301. $\lim_{n\to+\infty} \dfrac{2^n-3^n}{4^n}$. $\Big[0\Big]$

Calculate the limit of the following sequences:

Ex. 302. $\lim_{n\to+\infty} \sqrt[n]{(n^2+1)\sin(1/n)}$. $\Big[1\Big]$

Ex. 303. $\lim_{n\to+\infty} \dfrac{n^n}{(n!)!}$. $\Big[0\Big]$

Ex. 304. $\lim_{n\to+\infty} \sqrt[n]{n!}$. $\Big[+\infty\Big]$

Ex. 305. $\lim_{n\to+\infty} \left|\log\dfrac{n}{n+1}\right|^{\frac{1-2\sqrt{n+1}}{n+\sqrt{n}}}$. $\Big[1\Big]$

Ex. 306. $\lim_{n\to+\infty} \left(1+\dfrac{n!}{n^n}\right)^{\frac{(n-1)^n n}{(n+1)!}}$. $\Big[\sqrt[e]{e}\Big]$

Ex. 307. $\lim_{n\to+\infty} \sqrt{n^2+1}\arcsin\left(e^{-n}+\dfrac{1}{n^2+n}\right)$. $\Big[0\Big]$

Ex. 308. $\lim_{n\to+\infty} (1+\cos(1/n)-\cos(2/n))^{-(\arcsin(1/n))^{n^2}}$. $\Big[1\Big]$

Ex. 309. $\lim_{n\to+\infty} (\dfrac{n^2+n^3+3}{\sqrt{n}+n+n^3-1}e^{-1/n})^n$. $\Big[1\Big]$

Ex. 310. $\lim_{n\to+\infty} \dfrac{n^6+e^{n\log n}+2^{n^4}\arcsin(1/n)}{n^n-n!+e^{n^3}}$. $\Big[1\Big]$

Ex. 311. $\lim_{n\to+\infty} \sqrt[n^2]{n^3+1+e^{n^2}}$. $\Big[e\Big]$

Ex. 312. $\lim_{n\to+\infty} \sqrt[n]{e^n+\sin(\pi n/2)}$. $\Big[e\Big]$

Ex. 313. $\lim_{n\to+\infty} \sqrt[n]{2+\sin n}$. $\Big[\nexists\Big]$

***Ex. 314.** Let $\{a_n\}$ be a positive terms sequence such that

$$\lim_{n\to+\infty} \log \frac{a_n}{a_{n+1}} \geq 0.$$

Give at least two counterexamples showing that from this relation is not possible deduce that

$$\lim_{n\to+\infty} a_n = +\infty$$

Moreover, say under which further hypothesis the result would be true.

Ex. 315. Using the comparison theorem, show that

$$\lim_{n\to+\infty} \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n} = 0.$$

***Ex. 316.** Let $\{a_n\}$ be a positive terms sequence. Show that

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = r \geq 0 \quad \Rightarrow \quad \lim_{n\to+\infty} \sqrt[n]{a_n} = r$$

Use the sequence $a_n = e^{1/n} + \sin(\pi n/2) + 1$ to show that in general the converse is not true.

***Ex. 317.** Show, exhibiting a counterexample, that if $\{a_n\}$ is a non-negative terms sequence, then

$$\lim_{n\to+\infty} a_n^{1/n} = l \quad \not\Rightarrow \quad \lim_{n\to+\infty} \frac{a_n}{l^n} = 1$$

Moreover, show that if $\lim\limits_{n\to\infty} a_n^{1/n} = l > 1$ then $a_n \to +\infty$ for $n \to +\infty$.

4 Study of functions of one real variable

4.1 Asymptotes

Determine the possible asymptotes (vertical, horizontal, oblique) for the following functions, after having indicated their domain. Moreover, calculate the limit of the functions to the boundary points of their domain.

Ex. 318. $f(x) = \dfrac{x+1}{3-2x}$.

Ex. 319. $f(x) = \dfrac{1}{x(x-2)}$.

Ex. 320. $f(x) = \dfrac{\sqrt{x^4+1}}{x-2}$.

Ex. 321. $f(x) = x\log(1+x)$.

Ex. 322. $f(x) = \dfrac{x}{x^2+1}$.

Ex. 323. $f(x) = \dfrac{x}{x^2-1}$.

Ex. 324. $f(x) = x\arcsin\dfrac{1}{x+1}$.

Ex. 325. $f(x) = e^{\log^2(x/(x-1))+\log(3x-3)+2}$.

Ex. 326. $f(x) = \log(1-3e^x+2e^{2x})$.

Ex. 327. $f(x) = x\,e^{x/(x^2-1)}$.

Ex. 328. $f(x) = x\sqrt{\cos\dfrac{x}{x^2+1}}$.

Ex. 329. $f(x) = \dfrac{\log|x|}{3+\log|x|} + \sqrt{x^2+2x}$.

Ex. 330. $f(x) = x\arctan x$. (Use the formula $\arctan x + \arctan\dfrac{1}{x} = \dfrac{\pi}{2}$, $x > 0$).

Ex. 331. $f(x) = x^{1+1/\log x}$.

Ex. 332. $f(x) = x^{1+\log x/\sqrt{1+\log^2 x}}$.

Ex. 333. $f(x) = \dfrac{x^2}{x^4-1}e^{-1/x^2}$.

4.2 Continuity and derivability

Determine the domain and the set of continuity of the following functions.

Ex. 334.

$$f(x) = \begin{cases} x - [x] - 1, & x \leq 2 \\ x - [x], & x > 2. \end{cases}$$

Ex. 335. $f(x) = [x] + \sqrt{x - [x]}$.

Ex. 336. $f(x) = 4^{1/\sin x}$.

Ex. 337. $f(x) = \dfrac{\sin(\log x)}{\log x}$.

Ex. 338.

$$f(x) = \begin{cases} \sin(\cot x), & x \neq k\pi,\ k \in \mathbb{Z} \\ 0, & x = k\pi,\ k \in \mathbb{Z}. \end{cases}$$

Ex. 339. Determine $a \in \mathbb{R}$ such that the following function result to be continuous

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x + 1}, & x \neq -1 \\ a, & x = -1. \end{cases}$$

Ex. 340. Say if it is possible to apply the Weierstass theorem about the existence of the extremes to the following function

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 1 - x, & 1 \leq x \leq 3. \end{cases}$$

Determine the set of continuity and the set of derivability of the following functions and calculate their derivative.

Ex. 341. $f(x) = \tan 2x$.

Ex. 342. $f(x) = e^{2x} - e^{-2x}$.

Ex. 343. $f(x) = 3^{2x}$.

Ex. 344. $f(x) = x^{x^2+1}$.

Ex. 345. $f(x) = \dfrac{2x + 3}{x - 4}$.

Ex. 346. $f(x) = \sqrt{\dfrac{2x + 3}{x - 4}}$.

Ex. 347. $f(x) = \dfrac{\sqrt{x}}{x^2 + 1}$.

Ex. 348. $f(x) = (\arcsin x)^3$.

Ex. 349. $f(x) = e^{\sin x}$.

Ex. 350. $f(x) = \arctan\left(\dfrac{x}{1-x^2}\right)$.

Ex. 351. $f(x) = \log \tan x$.

Ex. 352. $f(x) = \arcsin\left(\dfrac{1}{1+\sqrt{x}}\right)$.

Ex. 353. $f(x) = \arcsin\left(\dfrac{x^2}{x^2-1}\right)$.

Ex. 354. $f(x) = x\, e^{1/(1-x)}$.

Ex. 355. $f(x) = 2^{\arccos 3x}$.

Ex. 356. $f(x) = \log 2|x|$.

Ex. 357. $f(x) = \dfrac{\log x}{3 - 2\log(2x)}$.

Ex. 358. $f(x) = |x|x + e^x$.

Ex. 359. $f(x) = \sqrt{x^2 + x^4} \arctan x$.

Ex. 360. $f(x) = \sqrt{1 - \cos x}$.

Ex. 361. $f(x) = \sqrt{\log\left(\dfrac{x^2}{x^2-1}\right)}$.

The same work (determination of continuity, derivability and calculation of the derivative) is recommended also for the functions in the exercises in paragraphs 1.8 and 4.1.

4.3 Invertibility and derivative of the inverse function

Verify the invertibility of the following functions and determine the domain of derivability of the respective inverse functions.

Ex. 362. $f(x) = 2x + \log x$.

Ex. 363. $f(x) = -x + e^{-2x}$.

Ex. 364. $f(x) = x|x| + \log(1 + x)$.

Ex. 365. $f(x) = x + \sin x$.

Ex. 366. $f(x) = x\sqrt{|x|} + \arctan x$.

Ex. 367. $f(x) = \sqrt[5]{1 - x - \cos x}$.

Ex. 368. For any of the following function $f(x)$, determine:
$f'(2)$, $f'(1)$, $f'(1 + \log 2)$, $f'(\frac{\pi}{2} + 1)$, $f'(1 + \frac{\pi}{4})$, $f'(0)$.
Moreover, write the equation of the tangent line passing for the point indicated.

Ex. 369. Use the mean value theorem to show that

$$|\sin x - \sin y| \leq |x - y|, \quad x, y \in \mathbb{R}.$$

4.4 Critical points

Determine the possible critical points for the following functions.

Ex. 370. $f(x) = \dfrac{x}{x^2 + 1}$.

Ex. 371. $f(x) = \dfrac{x}{x^2 - 1}$.

Ex. 372. $f(x) = \dfrac{\log x}{x}$.

Ex. 373. $f(x) = xe^{-1/x}$.

Ex. 374. $f(x) = \sqrt{x}\left|1 + \dfrac{1}{\log x}\right|$.

Ex. 375. $f(x) = x \log x$.

Ex. 376. $f(x) = x^3 + x^2 - x$.

Ex. 377. $f(x) = \sqrt{-x(x + 1)}$.

Ex. 378. $f(x) = e^x\left(\dfrac{3}{2}|x| + \dfrac{1}{2}(3x - 8)\right)$.

Ex. 379. $f(x) = ((2 - x)^6)^{\log|x-2|}$.

4.5 Derivability and Monotony

Determine the intervals of monotony for the functions in the paragraph 4.4.

4.6 Taylor and Mac Laurin Polynomials

Determine the Mac Laurin polynomial of the following functions to the indicated order.

Ex. 380. $f(x) = \sin(x^2)$, to the order 4.

Ex. 381. $f(x) = \sqrt{1+2x}$, to the order 3.

Ex. 382. $f(x) = \log(1+x^3)$, to the order 8.

Ex. 383. $f(x) = \sin^2 x$, to the order 4.

Ex. 384. $f(x) = e^{x+1}$, to the order 5.

Determine the Taylor polynomial, centered in x_0 and to the indicated order, for the following functions.

Ex. 385. $f(x) = e^x$, $x_0 = 2$, to the order 3.

Ex. 386. $f(x) = \cos x$, $x_0 = 3$, to the order 4.

Ex. 387. $f(x) = \log(1+x)$, $x_0 = 2$, to the order 3.

Ex. 388. Determine the Mac Laurin polynomial of order 4, for the function $f(x) = \log(1 + x\sin x)$.

Determine the Mac Laurin polynomial of order 5, for the following functions.

Ex. 389. $f(x) = (1+x)e^x$.

Ex. 390. $f(x) = x\sin x + \cos x$.

Ex. 391. $f(x) = \sin x \cdot \log(1+x)$.

4.7 Using Taylor polynomials for the calculation of limits

Calculate the following limits.

Ex. 392. $\displaystyle\lim_{x\to+\infty} \frac{x^3}{x+1}\left(e^{1/(x+1)} - 1\right) - x.$ $\left[-\frac{3}{2}\right]$

Ex. 393. $\displaystyle\lim_{x\to 0^+} \frac{\frac{1}{2}x\sin x + \cos x - e^{x^4}}{x^2\log(1+x^2)}.$ $\left[-\frac{25}{24}\right]$

Ex. 394. $\displaystyle\lim_{x\to 1^+} \frac{e^{-1/(x+1)} + \sqrt{x^x} - x\log x}{(x\log(x\cos(x-1)))^2}.$ $\left[+\infty\right]$

Ex. 395. $\displaystyle\lim_{x\to+\infty} \frac{x^5 + x^2\log x}{x^3 + x^6\log\left(\frac{2\arctan x}{\pi}\right) - \frac{2}{\pi}x^5}$ $\left[\frac{\pi}{4}\right]$

4.8 Uniform Continuity

Ex. 396. Verify, using the definition, that $f(x) = x^2$ is not an uniform continuous function over $X = [1, +\infty)$.

Ex. 397. Establish if $f(x) = \dfrac{\arctan x}{x}$ is a uniform continuous function over the following domains:
$D_1 = (0, +\infty)$; $D_2 = (1, +\infty)$; $D_3 = [1, +\infty)$; $D_4 = (-\infty, -1) \cup (2, +\infty)$.

Ex. 398. Verify if $f(x) = x - \log x$ results to be a Lipschitz function over the domain $D = [1, +\infty)$.

Verify if the following functions result to be uniformly continuous over their domain:

Ex. 399.

$$f(x) = \begin{cases} xe^{-1/|x|}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Ex. 400.

$$f(x) = \begin{cases} 2\sin x + 1, & \text{if } x < 0, \\ \log(e(2x+1)), & \text{if } x \geq 0. \end{cases}$$

Ex. 401. $f(x) = \sin(e^{\sin x})$

For any of the following functions determine $a \in \mathbb{R}$ such that they result to be continuous. Then check if, for such an a, the functions result to be also uniformly continuous throughout their domain of definition.

Ex. 402.

$$f(x) = \begin{cases} a(e^x - 1), & \text{if } x < 1, \\ e^{-x}, & \text{if } x \geq 1. \end{cases}$$

Ex. 403.

$$f(x) = \begin{cases} \dfrac{\log x}{x} + e^{1-x}, & \text{if } x > 1, \\ a, & \text{if } x = 1, \\ \sqrt{2-x} + \dfrac{\pi}{4} - \arctan x, & \text{if } x < 1. \end{cases}$$

Ex. 404.

$$f(x) = \begin{cases} \sqrt{x^2 - 2x + 2}, & \text{if } x \leq 0, \\ \dfrac{a\log(x+1)}{x}, & \text{if } x > 0. \end{cases}$$

Ex. 405.
$$f(x) = \begin{cases} \dfrac{2\sin x}{2} - \dfrac{1}{\sqrt{\log(1+x)+1}}, & \text{if } x > 0, \\ a, & \text{if } x = 0, \\ x(e^x+1)-1, & \text{if } x < 0. \end{cases}$$

5 Integrals of one-variable functions and numerical series

5.1 Immediate indefinite integrals (primitives)

Calculate the following indefinite integrals (primitives).

Ex. 406. $\displaystyle\int \frac{1}{\sqrt[4]{x^3}}\,\mathrm{d}x.$ $\left[4x^{1/4}+c\right]$

Ex. 407. $\displaystyle\int \sqrt{3qx}\,\mathrm{d}x, \quad q\in\mathbb{R}^+.$ $\left[\frac{2}{3}(3q)^{1/2}x^{3/2}+c\right]$

Ex. 408. $\displaystyle\int (a^{2/3}-x^{2/3})^3\,\mathrm{d}x, \quad a\in\mathbb{R}.$ $\left[a^2-\frac{9}{5}a^{4/3}x^{5/3}+\frac{9}{7}a^{2/3}x^{7/3}-\frac{1}{3}x^3+c\right]$

Ex. 409. $\displaystyle\int P_n(x)\,\mathrm{d}x, \quad P_n(x)=\sum_{k=0}^{n}a_kx^k,\ a_k\in\mathbb{R}.$ $\left[\sum_{k=0}^{n}\frac{a_k}{k+1}x^{k+1}+c\right]$

Ex. 410. $\displaystyle\int \sum_{k=0}^{n}\alpha_ke^{\beta_kx}\,\mathrm{d}x, \quad \alpha_k,\beta_k\in\mathbb{R},\ \beta_k\neq 0.$ $\left[\sum_{k=0}^{n}\frac{\alpha_k}{\beta_k}e^{\beta_kx}+c\right]$

Ex. 411. $\displaystyle\int \sum_{k=0}^{n}\alpha_k\sin\beta_kx\,\mathrm{d}x, \quad \alpha_k,\beta_k\in\mathbb{R},\ \beta_k\neq 0.$ $\left[-\sum_{k=0}^{n}\frac{\alpha_k}{\beta_k}\cos\beta_kx+c\right]$

Ex. 412. $\displaystyle\int \frac{x^2-3x+1}{x}\,\mathrm{d}x.$ $\left[\frac{1}{2}x^2-3x+\log|x|+c\right]$

Ex. 413. $\displaystyle\int \frac{3+\sqrt{x}}{\sqrt[5]{x^2}}\,\mathrm{d}x.$ $\left[5\sqrt[5]{x^3}+\frac{10}{11}\sqrt[10]{x^{11}}+c\right]$

Ex. 414. $\displaystyle\int \frac{a+\sqrt{1-x^2}}{\sqrt{1-x^2}}\,\mathrm{d}x, \quad a\in\mathbb{R}.$ $\left[a\arcsin x+x+c\right]$

Ex. 415. $\displaystyle\int \frac{x^2}{1+x^2}\,\mathrm{d}x.$ $\left[x-\arctan x+c\right]$

Ex. 416. $\displaystyle\int \tan^2 x\,\mathrm{d}x.$ $\left[\tan x-x+c\right]$

Ex. 417. $\displaystyle\int \cot^2 x\,\mathrm{d}x.$ $\left[-\cot x-x+c\right]$

Ex. 418. $\displaystyle\int \frac{1+2x^2}{x^2(1+x^2)}\,\mathrm{d}x.$ $\left[-\frac{1}{x}+\arctan x+c\right]$

Ex. 419. $\displaystyle\int \frac{\sin 2x}{\cos x}\,dx.$ $\Big[-2\cos x + c\Big]$

Ex. 420. $\displaystyle\int \frac{x^5+1}{x+1}\,dx.$ $\Big[\frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x + c\Big]$

Ex. 421. $\displaystyle\int \frac{x^n - a^n}{x-a}\,dx, \quad a \in \mathbb{R}.$ $\Big[\frac{x^n}{n} + a\frac{x^{n-1}}{n-1} + a^2\frac{x^{n-2}}{n-2} + \cdots + a^{n-1}x + c\Big]$

Ex. 422. $\displaystyle\int \frac{dx}{\sin^2 x\cos^2 x}.$ $\Big[\tan x - \cot x + c\Big]$

Ex. 423. $\displaystyle\int \frac{\cos 2x}{\sin x + \cos x}\,dx.$ $\Big[\cos x + \sin x + c\Big]$

Ex. 424. $\displaystyle\int \sin^2\frac{x}{2}\,dx.$ $\Big[\frac{1}{2}(x - \sin x) + c\Big]$

Ex. 425. $\displaystyle\int \cos^2\frac{x}{3}\,dx.$ $\Big[\frac{1}{2}x + \frac{3}{4}\sin\frac{2x}{3} + c\Big]$

Ex. 426. $\displaystyle\int \frac{1}{\sin^2\frac{x}{2}\,\cos^2\frac{x}{2}}\,dx.$ $\Big[2\tan\frac{x}{2} - 2\cot\frac{x}{2} + c\Big]$

5.2 Indefinite integrals by substitution

Calculate the following indefinite integrals using, for instance, the method of substitution of variable.

Ex. 427. $\displaystyle\int \sqrt{\sin x}\cos x\,dx.$ $\Big[\frac{2}{3}\sin^{3/2} x + c\Big]$

Ex. 428. $\displaystyle\int \frac{x}{1-x^2}\,dx.$ $\Big[-\frac{1}{2}\log\left|1-x^2\right| + c\Big]$

Ex. 429. $\displaystyle\int \frac{1}{a^2+x^2}\,dx, \quad a \in \mathbb{R}^+.$ $\Big[\frac{1}{a}\arctan\frac{x}{a} + c\Big]$

Ex. 430. $\displaystyle\int \frac{1}{a^2-x^2}\,dx, \quad a \in \mathbb{R}^+.$ $\Big[-\frac{1}{2a}\log\left|\frac{a-x}{a+x}\right| + c\Big]$

Ex. 431. $\displaystyle\int \frac{\sqrt{a-x}}{\sqrt{a+x}}\,dx, \quad a \in \mathbb{R}^+.$ $\Big[a\arcsin\frac{x}{a} + \frac{1}{2}\sqrt{a^2-x^2} + c\Big]$

Ex. 432. $\displaystyle\int \frac{1+e^{-x}}{1+xe^{-x}}\,dx.$ $\Big[\log|x+e^x| + c\Big]$

Ex. 433. $\displaystyle\int \frac{1}{\sqrt{a-bx^2}}\,dx, \quad a, b \in \mathbb{R}^+$ $\Big[\frac{1}{\sqrt{b}}\arcsin\sqrt{\frac{b}{a}}x + c\Big]$

Ex. 434. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx.$ $\Big[-\sqrt{1-x^2} + c\Big]$

Ex. 435. $\displaystyle\int \frac{1}{x\sqrt{5x-7}}\,dx.$ $\left[\frac{2}{\sqrt{7}}\arctan\sqrt{\frac{5x+7}{7}}+c\right]$

Ex. 436. $\displaystyle\int \sin^\alpha x\cos x\,dx,\quad \alpha\neq -1.$ $\left[\frac{1}{\alpha+1}\sin^{\alpha+1}x+c\right]$

Ex. 437. $\displaystyle\int \frac{1}{e^{-x}+e^x}\,dx.$ $\left[\arctan e^x+c\right]$

Ex. 438. $\displaystyle\int \frac{\cos(\log x)}{x}\,dx.$ $\left[\sin(\log x)+c\right]$

Ex. 439. $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx.$ $\left[2(\sqrt{x}-\arctan\sqrt{x})+c\right]$

Ex. 440. $\displaystyle\int \frac{x^2}{(x-1)^3}\,dx.$ $\left[\log|x-1|-\frac{2}{x-1}-\frac{1}{2(x-1)^2}+c\right]$

Ex. 441. $\displaystyle\int \frac{x}{\sqrt{a^4-x^4}}\,dx,\quad a\neq 0.$ $\left[\frac{1}{2}\arcsin\frac{x^2}{a^2}+c\right]$

Ex. 442. $\displaystyle\int \frac{\cot x}{\sin^\alpha x}\,dx,\quad \alpha\in\mathbb{R}^+.$ $\left[-\frac{1}{\alpha\sin^\alpha x}+c\right]$

Ex. 443. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx,\quad a\neq 0.$ $\left[-\frac{1}{a}\arctan\frac{a}{x}+c\right]$

Ex. 444. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x}\,dx,\quad a\in\mathbb{R}.$ $\left[\sqrt{x^2-a^2}-a\arccos\frac{a}{x}+c\right]$

Ex. 445. $\displaystyle\int \frac{ax+b}{cx+d}\,dx,\quad a,b,c,d\in\mathbb{R}, c\neq 0.$ $\left[\frac{1}{c^2}\left(a(cx+d)+(bc-ad)\log|cx+d|\right)+cons.\right]$

Ex. 446. $\displaystyle\int \frac{e^{2x}}{\sqrt{e^x-1}}\,dx.$ $\left[\frac{2}{3}\sqrt{(e^x-1)^3}+2\sqrt{e^x+1}+c\right]$

Ex. 447. $\displaystyle\int \frac{\tan x}{\log(\cos x)}\,dx.$ $\left[-\log\left|\log(\cos x)\right|+c\right]$

Ex. 448. $\displaystyle\int \frac{1}{(a+x)(a^2-x^2)^{1/2}}\,dx,\quad a\neq 0.$ $\left[-\frac{1}{a}\sqrt{\frac{a-x}{a+x}}+c\right]$

Ex. 449. $\displaystyle\int \frac{1}{\sin x\cos x}\,dx.$ $\left[\log|\tan x|+c\right]$

Ex. 450. $\displaystyle\int \frac{1}{\sin x}\,dx.$ $\left[\log\left|\frac{\sin x}{1+\cos x}\right|+c\right]$

Ex. 451. $\displaystyle\int \frac{1}{\cos x}\,dx.$ $\left[\log\left|\frac{1+\sin x}{\cos x}\right|+c\right]$

Ex. 452. $\displaystyle\int \frac{1}{\sqrt{x}(a+x)}\,dx,\quad a\in\mathbb{R}^+$ $\left[\frac{2}{\sqrt{a}}\arctan\sqrt{\frac{x}{a}}+c\right]$

Ex. 453. $\displaystyle\int \frac{1}{\sqrt{a^2+x^2}}\,dx, \quad a \in \mathbb{R}.$ $\Big[\log\Big|\sqrt{a^2+x^2}+x\Big|+c\Big]$

Ex. 454. $\displaystyle\int \frac{1}{(a^2+x^2)^{3/2}}\,dx, \quad a \neq 0.$ $\Big[\frac{1}{a^2}\frac{x}{\sqrt{a^2+x^2}}+c\Big]$

Ex. 455. $\displaystyle\int \frac{1}{(a^2+x^2)^{5/2}}\,dx, \quad a \neq 0.$ $\Big[\frac{1}{a^4}\Big(\frac{x}{\sqrt{a^2+x^2}}-\frac{1}{3}\frac{x^3}{\sqrt{(a^2+x^2)^3}}\Big)+c\Big]$

Ex. 456. $\displaystyle\int \frac{1}{(a^2+x^2)^{7/2}}\,dx, \quad a \neq 0.$ $\Big[\frac{1}{a^6}\Big(\frac{x}{\sqrt{a^2+x^2}}-\frac{2}{3}\frac{x^3}{\sqrt{(a^2+x^2)^3}}+\frac{1}{5}\frac{x^5}{\sqrt{(a^2+x^2)^5}}\Big)+c\Big]$

***Ex. 457.** $\displaystyle\int \frac{1}{(a^2+x^2)^{(2n+1)/2}}\,dx, \quad a \neq 0, n \in \mathbb{N}.$ [use the results of the previous exercises]

5.3 Indefinite integrals by parts

Calculate the following indefinite integrals using, for instance, the method of integration by parts.

Ex. 458. $\displaystyle\int \frac{\log x}{x^3}\,dx.$ $\Big[-\frac{1}{2x^2}\Big(\log x+\frac{1}{2}\Big)+c\Big]$

Ex. 459. $\displaystyle\int \sin^3 x\,dx.$ $\Big[-\frac{1}{3}\Big(\sin^2 x\cos x+2\cos x\Big)+c\Big]$

Ex. 460. $\displaystyle\int \sin^4 x\,dx.$ $\Big[\frac{3}{8}x-\frac{1}{4}\sin^2 x+\frac{1}{32}\sin 4x+c\Big]$

Ex. 461. $\displaystyle\int \sin^5 x\,dx.$ $\Big[-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+c\Big]$

Ex. 462. $\displaystyle\int \frac{\sin x}{e^x}\,dx.$ $\Big[-\frac{1}{2}(\sin x e^{-x}+\cos x e^{-x})+c\Big]$

Ex. 463. $\displaystyle\int x^3\arctan x\,dx.$ $\Big[\frac{x^4}{4}\arctan x+\frac{1}{2}\Big(x-\frac{x^3}{3}-\arctan x\Big)+c\Big]$

Ex. 464. $\Big[\frac{\log^{\alpha+1}x}{\alpha+1}+c \text{ if } \alpha \neq 0,1;\ \log|x|+c, \text{ if } \alpha=0;\ \log|\log x|+c \text{ if } \alpha=-1\Big]$

Ex. 465. $\displaystyle\int xe^x\,dx.$ $\Big[xe^x-e^x+c\Big]$

Ex. 466. $\displaystyle\int x^2e^x\,dx.$ $\Big[e^x(x^2-2x+2)+c\Big]$

Ex. 467. $\displaystyle\int x^ne^x\,dx, \quad n \in \mathbb{N}.$ $\Big[e^x(x^n-nx^{n-1}+n(n-1)x^{n-2}-\cdots+(-1)^n n!)+c\Big]$

Ex. 468. $\displaystyle\int x\sin x\,dx.$ $\Big[-x\cos x+\sin x+c\Big]$

Ex. 469. $\displaystyle\int x\cos x\,\mathrm{d}x.$ $\Big[x\sin x+\cos x+c\Big]$

Ex. 470. $\displaystyle\int x^2\sin x\,\mathrm{d}x.$ $\Big[-x^2\cos x+2x\sin x+2\cos x+c\Big]$

Ex. 471. $\displaystyle\int x^2\cos x\,\mathrm{d}x.$ $\Big[-x^2\cos x+2x\sin x+2\cos x+c\Big]$

Ex. 472. $\displaystyle I_n=\int x^n\sin x\,\mathrm{d}x,\quad n\in\mathbb{N},n>1.$ $\displaystyle\Big[\text{Let } I_1=\int x\cos x\,\mathrm{d}x,\ I_n=-x^n\cos x+nI_{n-1}=$
$\displaystyle=-x^n\cos x+n(-x^{n-1}\cos x+(n-1)(-x^{n-2}\cos x+\ldots+2\int x\cos x\,\mathrm{d}x)\ldots)\Big]$

Ex. 473. $\displaystyle I_n=\int x^n\cos x\,\mathrm{d}x,\quad n\in\mathbb{N},n>1.$ $\displaystyle\Big[\text{Let } I_1=\int x\sin x\,\mathrm{d}x,\ I_n=x^n\sin x-nI_{n-1}=$
$\displaystyle=x^n\sin x-n(x^{n-1}\sin x-(n-1)(x^{n-2}\sin x-\ldots-2\int x\sin x\,\mathrm{d}x)\ldots)\Big]$

Ex. 474. $\displaystyle\int x\sin^2 x\,\mathrm{d}x.$ $\displaystyle\Big[\frac{1}{2}\left(-x\sin x\cos x+\frac{x^2}{2}+\frac{\sin^2 x}{2}\right)+c\Big]$

Ex. 475. $\displaystyle\int\sqrt{1-x^2}\,\mathrm{d}x.$ $\displaystyle\Big[\frac{1}{2}\left(x\sqrt{1-x^2}+\arcsin x\right)+c\Big]$

Ex. 476. $\displaystyle\int x\arcsin x\,\mathrm{d}x.$ $\displaystyle\Big[\frac{x^2}{2}\arcsin x+\frac{1}{4}\left(x\sqrt{1-x^2}-\arcsin x\right)+c\Big]$

Ex. 477. $\displaystyle\int\frac{x}{\cos^2 x}\,\mathrm{d}x.$ $\Big[x\tan x+\log|\cos x|+c\Big]$

Ex. 478. $\displaystyle\int\arcsin^2 x\,\mathrm{d}x.$ $\Big[x\arcsin^2 x+2\sqrt{1-x^2}\arcsin x-2x+c\Big]$

Ex. 479. $\displaystyle\int e^{\arcsin x}\,\mathrm{d}x.$ $\displaystyle\Big[\frac{1}{2}e^{\arcsin x}\left(x+\sqrt{1-x^2}\right)+c\Big]$

Ex. 480. $\displaystyle\int\sin px\cos qx\,\mathrm{d}x,\quad p,q\in\mathbb{R},p\neq q.$ $\displaystyle\Big[\frac{q}{q^2-p^2}\sin px\sin qx+\frac{p}{q^2-p^2}\cos px\cos qx+c\Big]$

Ex. 481. $\displaystyle\int\sqrt{x^2+a}\,\mathrm{d}x,\quad a\in\mathbb{R}.$ $\displaystyle\Big[\frac{1}{2}(x\sqrt{x^2+a}+a\log\left|\sqrt{x^2+a}+x\right|+c\Big]$

Ex. 482. $\displaystyle\int e^x\sin x\,\mathrm{d}x.$ $\displaystyle\Big[\frac{1}{2}e^x(\sin x-\cos x)+c\Big]$

Ex. 483. $\displaystyle\int e^x\cos x\,\mathrm{d}x.$ $\displaystyle\Big[\frac{1}{2}e^x(\sin x+\cos x)+c\Big]$

Ex. 484. $\displaystyle\int e^{\alpha x}\sin x\,\mathrm{d}x,\quad \alpha\in\mathbb{R}.$ $\displaystyle\Big[\frac{1}{\alpha^2+1}e^{\alpha x}(\alpha\sin x-\cos x)+c\Big]$

Ex. 485. $\displaystyle\int e^{\alpha x}\cos x\,\mathrm{d}x,\quad \alpha\in\mathbb{R}.$ $\displaystyle\Big[\frac{1}{\alpha^2+1}e^{\alpha x}(\sin x+\alpha\cos x)+c\Big]$

Ex. 486. $\displaystyle\int e^{\alpha x}\sin\beta x\,\mathrm{d}x,\quad (\alpha,\beta)\in\mathbb{R}^2, (\alpha,\beta)\neq(0,0).$ $\quad\left[\dfrac{1}{\alpha^2+\beta^2}e^{\alpha x}(\alpha\sin\beta x-\beta\cos\beta x)+c\right]$

Ex. 487. $\displaystyle\int e^{\alpha x}\cos\beta x\,\mathrm{d}x,\quad (\alpha,\beta)\in\mathbb{R}^2, (\alpha,\beta)\neq(0,0).$ $\quad\left[\dfrac{1}{\alpha^2+\beta^2}e^{\alpha x}(\beta\sin\beta x+\alpha\cos\beta x)+c\right]$

***Ex. 488.** $\displaystyle\int e^{x}\cos^n x\,\mathrm{d}x,\quad n\in\mathbb{N}.$
(Use the formula:

$$\cos^n x=\begin{cases}\dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\cos(n-2k)x, & n\text{ odd}\\[2ex] \dfrac{1}{2^n}\dbinom{n}{n/2}+\dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\lfloor n/2\rfloor-1}\binom{n}{k}\cos(n-2k)x, & n\text{ even}\end{cases}$$

and the result of the exercise 487.)

***Ex. 489.** $\displaystyle\int e^{x}\sin^n x\,\mathrm{d}x,\quad n\in\mathbb{N}.$
(Use the formula:

$$\sin^n x=\begin{cases}\dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^{\lfloor n/2-k\rfloor}\binom{n}{k}\sin(n-2k)x, & n\text{ odd}\\[2ex] \dfrac{1}{2^n}\dbinom{n}{n/2}+\dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{\lfloor n/2\rfloor-1}(-1)^{\lfloor n/2-k\rfloor}\binom{n}{k}\sin(n-2k)x, & n\text{ even}\end{cases}$$

and the result of the exercise 486.)

***Ex. 490.** $\displaystyle I_{m,n}=\int\sin^m x\cos^n x\,\mathrm{d}x,\quad m,n\in\mathbb{Z}.$
(One obtains the following equivalent reduction formulas:

$$\begin{aligned}I_{m,n}&=-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}I_{m-2,n+2}=\frac{\sin^{m+1}x\cos^{n-1}x}{m+1}+\frac{n-1}{m+1}I_{m+2,n-2}=\\&=\frac{\sin^{m+1}x\cos^{n+1}x}{m+1}+\frac{m+n+2}{m+1}I_{m+2,n}=-\frac{\sin^{m+1}x\cos^{n+1}x}{n+1}+\frac{m+n+2}{n+1}I_{m,n+2})\end{aligned}$$

***Ex. 491.** $\displaystyle\int e^{\alpha x}\sin^m\beta x\,\mathrm{d}x,\quad \alpha,\beta\in\mathbb{R},\ m\in\mathbb{Z}.$
(Use the results of the previous exercises)

***Ex. 492.** $\displaystyle\int e^{\alpha x}\cos^n\beta x\,\mathrm{d}x,\quad \alpha,\beta\in\mathbb{R},\ n\in\mathbb{Z}.$
(Use the results of the previous exercises)

5.4 Determine the following indefinite integrals (primitives).

Ex. 493. $\displaystyle\int\frac{x^2+2}{(x-3)^2(x+2)}\,\mathrm{d}x.$ $\quad\left[\dfrac{19}{25}\log|x-3|-\dfrac{11}{5(x-3)}+\dfrac{6}{25}\log|x+2|+c\right]$

Ex. 494. $\displaystyle\int \frac{4x-3}{(x-1)(x-2)^3}\,dx.$ $\displaystyle\Big[-\log|x-1|+\log|x-2|+\frac{2}{(x-2)}-\frac{5}{2(x-2)}+c\Big]$

Ex. 495. $\displaystyle\int \frac{x^5+x^4-8}{x^3-4x}\,dx.$ $\displaystyle\Big[\frac{x^3}{3}+\frac{x^2}{2}+4x+2\log|x|+5\log|x-2|-3\log|x+2|+c\Big]$

Ex. 496. $\displaystyle\int \frac{x}{(x^2+1)(x-1)}\,dx.$ $\displaystyle\Big[\frac{1}{2}\Big(-\frac{1}{2}\log(x^2+1)-\frac{1}{2}\arctan x+\log|x-1|\Big)+c\Big]$

Ex. 497. $\displaystyle\int \frac{x+1}{x^2+1}\,dx.$ $\displaystyle\Big[\frac{1}{2}\log(x^2+1)+\arctan x+c\Big]$

Ex. 498. $\displaystyle\int \frac{x^3-6}{x^4+6x^2+8}\,dx.$ $\displaystyle\Big[-\frac{5}{2(x-2)^2}+\frac{1}{x-2}+\log|x-2|-\log|x-1|+c\Big]$

Ex. 499. $\displaystyle\int \frac{x^3-2x^2+5}{x^4+3x^3+3x^2-3x-4}\,dx.$
$$\Big[\frac{1}{4}\log|x-1|-\frac{1}{2}\log|x+1|+\frac{5}{8}\log|x^2+3x+4|-\frac{31\sqrt{7}}{8}\arctan\frac{2x+3}{\sqrt{7}}+c\Big]$$

Ex. 500. $\displaystyle\int \frac{2x^3-3x+3}{(x-1)(x^2-2x+5)}\,dx.$
$$\Big[\frac{11}{4}\log\left(|x^2-2x+5|\right)+\frac{1}{2}\log\left(|x-1|\right)-\frac{5}{2}\arctan\Big(\frac{1}{4}(2x-2)\Big)+2x+c\Big]$$

Ex. 501. $\displaystyle\int \frac{x^2+x+1/2}{x^2+1}\,dx.$ $\displaystyle\Big[\frac{1}{2}\log(x^2+1)-\frac{1}{2}\arctan(x)+x+c\Big]$

Ex. 502. $\displaystyle\int \frac{3x^2-6x+7}{(x-2)^2(x+5)}\,dx.$ $\displaystyle\Big[\frac{16}{7}\log\left(|x+5|\right)+\frac{5}{7}\log\left(|x-2|\right)-\frac{1}{x-2}+c\Big]$

Ex. 503. $\displaystyle\int \frac{2x^2+x}{(x^2+1)(x^2+2x+2)}\,dx.$
$$\Big[-\frac{1}{2}\log\left(|x^2+2x+2|\right)+\frac{1}{2}\log(x^2+1)+\arctan\Big(\frac{1}{2}(2x+2)\Big)+c\Big]$$

Ex. 504. $\displaystyle\int \frac{x^3+x-1}{(x^2+2)^2}\,dx.$ $\displaystyle\Big[\frac{1}{2}\log(x^2+2)-\frac{1}{2^{\frac{5}{2}}}\arctan(\frac{x}{\sqrt{2}})-\frac{x-2}{4x^2+8}+c\Big]$

Ex. 505. $\displaystyle\int \frac{1}{(x^3+1)^2}\,dx.$
$$\Big[\frac{1}{9}\log\left(|x^2-x+1|\right)+\frac{2}{9}\log\left(|x+1|\right)+\frac{2}{3^{\frac{3}{2}}}\arctan(2x-\frac{1}{\sqrt{3}})+\frac{x}{3x^3+3}+c\Big]$$

Ex. 506. $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx.$ $\displaystyle\Big[\frac{1}{2}\arctan x+\frac{x}{2x^2+2}+c\Big]$

Ex. 507. $\displaystyle\int \frac{4}{x^4+1}\,dx.$
$$\Big[4\Big(\frac{1}{2^{\frac{5}{2}}}\log\left(|x^2+\sqrt{2}x+1|\right)-\frac{1}{2^{\frac{5}{2}}}\log\left(|x^2-\sqrt{2}x+1|\right)+\frac{1}{2^{\frac{3}{2}}}\arctan\Big(\frac{2x+\sqrt{2}}{\sqrt{2}}\Big)+\frac{1}{2^{\frac{3}{2}}}\arctan\Big(\frac{2x-\sqrt{2}}{\sqrt{2}}\Big)\Big)+c\Big]$$

Ex. 508. $\displaystyle\int \frac{\tan^2 x}{\tan^3 x + 1}\,\mathrm{d}x.$

$$\left[\frac{1}{6}\log\left(|\tan^2 x - \tan x + 1|\right) + \frac{1}{6}\log\left(|\tan x + 1|\right) - \frac{1}{4}\log\left(\tan^2 x + 1\right) + \frac{\arctan\left(\frac{2\tan x - 1}{\sqrt{3}}\right)}{\sqrt{3}} - \frac{1}{2}x + c\right]$$

Ex. 509. $\displaystyle\int \frac{\sin^2 x}{\cos^2 x + 2\sin^2 x}\,\mathrm{d}x.$ $\quad \left[x - \frac{\arctan\left(\sqrt{2}\tan x\right)}{\sqrt{2}} + c\right]$

Ex. 510. $\displaystyle\int \frac{1}{\sin^m x \cos^n x}\,\mathrm{d}x, \quad m, n \in \mathbb{N}.$ $\quad []$

Ex. 511. $\displaystyle\int \cos mx \sin nx\,\mathrm{d}x, \quad m, n \in \mathbb{N}.$ $\quad []$

Ex. 512. $\displaystyle\int \frac{\sqrt{x}}{\sqrt[4]{x} + 1}\,\mathrm{d}x.$

$$\left[20\left(x^{\frac{1}{4}} + 1\right) + \frac{4}{5}\left(x^{\frac{1}{4}} + 1\right)^5 - 5\left(x^{\frac{1}{4}} + 1\right)^4 + \frac{40}{3}\left(x^{\frac{1}{4}} + 1\right)^3 - 20\left(x^{\frac{1}{4}} + 1\right)^2 - 4\log\left(x^{\frac{1}{4}} + 1\right) + c\right]$$

Ex. 513. $\displaystyle\int \frac{\sqrt[3]{x}}{\sqrt{x} + x^2}\,\mathrm{d}x.$

$[]$

Ex. 514. $\displaystyle\int \frac{1 + \tan x}{1 - \tan x}\,\mathrm{d}x.$ $\quad \left[\frac{1}{2}\log\left(\tan^2 x + 1\right) - \log\left(|\tan x - 1|\right) + c\right]$

Ex. 515. $\displaystyle\int \frac{1}{3 + 5\cos x}\,\mathrm{d}x.$ $\quad \left[2\left(\frac{1}{8}\log\left(\left|\frac{\sin x}{\cos x + 1} + 2\right|\right) - \frac{1}{8}\log\left(\left|\frac{\sin x}{\cos x + 1} - 2\right|\right)\right) + c\right]$

5.5 Definite Integrals

Determine the following definite integrals:

Ex. 516. $\displaystyle\int_{-2}^{3} \frac{x}{x^2+1}\,\mathrm{d}x.$ $\left[\dfrac{\log 2}{2}\right]$

Ex. 517. $\displaystyle\int_{-3}^{3} \frac{x}{x^2+1}\,\mathrm{d}x.$ $\left[0\right]$

Ex. 518. $\displaystyle\int_{-3}^{3} \frac{x^2}{x^2+1}\,\mathrm{d}x.$ $\left[6-2\arctan 3\right]$

Ex. 519. $\displaystyle\int_{-3}^{3} \sin^3 x\cos x\,\mathrm{d}x.$ $\left[0\right]$

Ex. 520. $\displaystyle\int_{0}^{2\pi} \sin^3 \cos 2x\,\mathrm{d}x.$ $\left[0\right]$

Ex. 521. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{x}{\sin^2 x}\,\mathrm{d}x.$ $\left[\dfrac{\pi}{4}+\log\sqrt{2}\right]$

Ex. 522. $\displaystyle\int_{-\pi/2}^{\pi/2} x\sin x\cos x\,\mathrm{d}x.$ $\left[\dfrac{\pi}{4}\right]$

Ex. 523. $\displaystyle\int_{0}^{\pi} x\sin^2 x\,\mathrm{d}x.$ $\left[\frac{\pi^2}{4}\right]$

Ex. 524. $\displaystyle\int_{\frac{1}{e}}^{e} x|\log x|\,\mathrm{d}x.$ $\left[\dfrac{e^2}{4}+\dfrac{1}{2}-\dfrac{3}{4e^2}\right]$

Ex. 525. $\displaystyle\int_{2}^{5} \frac{e^{2x}}{\sqrt{e^x-1}}\,\mathrm{d}x.$ $\left[\dfrac{2}{3}\left((2+e^5)\sqrt{e^5-1}-(2+e^2)\sqrt{e^2-1}\right)\right]$

Ex. 526. Calculate $\displaystyle\int_{-10}^{10} f(x)\,\mathrm{d}x$, where:

$$f(x)=\begin{cases} x^2+2\,, & x\le -2,\\ \dfrac{\sqrt{x^2-4}}{x}\,, & -2<x<2,\\ \sqrt{x}\,, & x\ge 2.\end{cases}$$

Ex. 527. Calculate $\displaystyle\int_{-10}^{10} f(x)\,\mathrm{d}x$,where:

$$f(x)=\begin{cases} \dfrac{1}{\sqrt{x^2+4}}\,, & x\le 0,\\ \dfrac{x^2}{x^2+1}\,, & x>0.\end{cases}$$

Ex. 528. Calcolate $\int_{-3}^{5} f(x)\,\mathrm{d}x$,where:

$$f(x) = \begin{cases} \sin\dfrac{x}{2}, & x > 0, \\ \cos\dfrac{x}{2}, & x < 0. \end{cases}$$

Ex. 529. Calculate the area of the surface between the graph of the curves of equation $y = x^3$, $y = 2 - x^2$ under the condition $x < 0$. Say if it exists (finite) the area of the surface between the graph of the two curves, without the condition $x < 0$.

Ex. 530. Calculate the area of the surface between the graph of the curves of equation $y = -x^2 + x + 2$ ed $y = x^2 - 1$.

5.6 Improper Integrals

5.6.1 Determine the convergence or divergence for the following improper integrals

Ex. 531. $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$ Calculate, if it exist, the value of the integral. $\big[\pi\big]$

Ex. 532. $\displaystyle\int_{0}^{\log 3} \frac{1}{e^x-3}\,\mathrm{d}x.$ $\big[\text{Divergent}\big]$

Ex. 533. $\displaystyle I = \int_{2}^{\infty} \frac{1}{x\log^\alpha x}\,\mathrm{d}x, \quad \alpha \in \mathbb{R}.$

$\left[\text{If } \alpha > 1\,,\ I = \dfrac{1}{(\alpha-1)\log^{\alpha-1} 2};\ \text{if } \alpha \leq 1\,,\ \text{then } I \text{ is divergent.}\right]$

Ex. 534. $\displaystyle\int_{4}^{6} \frac{1}{(x-4)-\log(x-3)}\,\mathrm{d}x.$ $\big[\text{Divergent}\big]$

Ex. 535. $\displaystyle\int_{2}^{4} \frac{1}{|\cos \pi x/2|^{3/5}}\,\mathrm{d}x.$ $\big[\text{Convergent}\big]$

Ex. 536. $\displaystyle\int_{1}^{+\infty} \frac{1}{((\log x)(x^5+x-2))^{1/5}}\,\mathrm{d}x.$ $\big[\text{Divergent}\big]$

Ex. 537. $\displaystyle\int_{0}^{+\infty} \frac{\sin x \log x}{(x+1)^{3/2}-1}\,\mathrm{d}x.$ $\big[\text{Convergent}\big]$

Ex. 538. $\displaystyle\int_{1}^{+\infty} \frac{e^{1/x^2}-e^{1/x}}{\sqrt{x}}\,\mathrm{d}x.$ $\big[\text{Convergent}\big]$

Ex. 539. $\displaystyle\int_{0}^{\pi/2} \frac{e^{-1/x}}{\sqrt{\sin x}}\,\mathrm{d}x.$ $\big[\text{Convergent}\big]$

Ex. 540. $\displaystyle\int_{0}^{+\infty} \frac{1}{mx+e^x}\,\mathrm{d}x, \quad m \in \mathbb{R}^+.$ $\big[\text{Convergent}\big]$

Ex. 541. $\displaystyle\int_{2}^{+\infty} \frac{1}{\sqrt{(\log x)^2(x^3+x)}}\,\mathrm{d}x.$ $\big[\text{Convergent}\big]$

5.6.2 Discuss the integrability in an improper sense of the following integrals.

Ex. 542. $\displaystyle\int_{1}^{+\infty} \frac{\log(t+1)}{t^3+2t+1}\,dt.$ $\big[\text{Convergent}\big]$

Ex. 543. $\displaystyle\int_{0}^{1} \frac{\log t}{(1-t)^{5/4}t^{1/2}}\,dt.$ $\big[\text{Convergent}\big]$

Ex. 544. $\displaystyle\int_{0}^{+\infty} \frac{1}{\sqrt{t}(t^2+1)\log(1+\sqrt{t})}\,dt.$ $\big[\text{Divergent}\big]$

Ex. 545. $\displaystyle\int_0^{+\infty} \frac{\sin\frac{1}{\sqrt{y}}}{(y-1)^{1/2}}\,dy.$ [Divergent]

Ex. 546. $\displaystyle\int_1^{+\infty} \frac{\log(2+x^2)}{\sqrt{x}\arctan x^2}\,\mathrm{d}x.$ [Divergent]

Ex. 547. $\displaystyle\int_0^{+\infty} \frac{e^{-y^2/2}}{\sqrt{2y}+\arctan(y^{1/4})}\,dy.$ [Convergent]

Ex. 548. $\displaystyle\int_{1/2}^{+\infty} \frac{e^{-x}}{(x-3)^{1/3}(x-1/2)^{1/2}}\,\mathrm{d}x.$ [Convergent]

Ex. 549. $\displaystyle\int_{+\infty}^{-1} \frac{e^{-x}}{(x-4)^2(x+1/2)^{1/3}}\,\mathrm{d}x.$ [Divergent]

Ex. 550. $\displaystyle\int_{1/2}^{+\infty} \frac{1}{(y-3)^{1/3}(y-1/2)^{1/2}}\,dy.$ [Divergent]

Ex. 551. $\displaystyle\int_{1/2}^{+\infty} \frac{1}{|x-3|^{3/4}(x-1/2)^{1/2}}\,\mathrm{d}x.$ [Convergent]

Ex. 552. $\displaystyle\int_3^{+\infty} \frac{\log(3+x^{-1/4})}{(x-3)^{3/4}(x-1/2)^{1/2}}\,\mathrm{d}x.$ [Convergent]

Ex. 553. $\displaystyle\int_0^1 \frac{\log x^2}{(1-x)^{9/4}x^{1/2}}\,\mathrm{d}x.$ [Divergent]

Ex. 554. If $\displaystyle I_a = \int_a^{+\infty} \frac{e^{-x}}{(x-3)^2(x-1/2)^{1/2}}\,\mathrm{d}x,$ find $a \in \mathbb{R}$ such that $I_a < +\infty$. [$a > 3$]

Ex. 555. If $\displaystyle I_a = \int_1^{+\infty} \frac{dy}{(1+y)^2(y+2)^a}\,dy,$ find $a \in \mathbb{R}$ such that $I_a < +\infty$.
Moreover, calculate I_1. [$a > -1.\ I_1 = \frac{1}{2} + \log\frac{2}{3}$]

5.6.3 Determine the values of $\alpha \in \mathbb{R}$ s.t. the following improper integrals result to be convergent.

Ex. 556. $\displaystyle\int_0^1 \frac{(\tan x)^\alpha}{\log(1+\sin x)}\,\mathrm{d}x.$ [$\alpha > 0$]

Ex. 557. $\displaystyle\int_0^{+\infty} \frac{\arctan(1/x^\alpha)}{\sqrt{x}+2}\,\mathrm{d}x.$ [$\alpha > \frac{1}{2}$]

Ex. 558. $\displaystyle\int_0^1 \frac{\cos x+3}{x^\alpha+\sqrt{x}}\,\mathrm{d}x.$ [$\alpha < 1$]

Ex. 559. $\displaystyle\int_2^{+\infty} \frac{\arctan(x+7)}{x\log^\alpha(x-2)}\,\mathrm{d}x.$ [$\alpha > 1$]

Ex. 560. $\displaystyle\int_2^{+\infty} \frac{\log^\alpha(1+1/x)}{\sqrt{x}+1}\,\mathrm{d}x.$ $\left[\alpha > \frac{1}{2}\right]$

Ex. 561. $\displaystyle\int_1^{+\infty} \frac{|\sin(1/x)-1/x|^{\alpha^2/2}}{\sqrt[3]{x}}\,\mathrm{d}x.$ $\left[|\alpha| > \frac{2}{3}\right]$

Ex. 562. $\displaystyle\int_1^{+\infty} \left(1-\cos\frac{1}{x^3}\right)^\alpha x^{\alpha/2}\,\mathrm{d}x.$ $\left[\alpha > \frac{2}{11}\right]$

Ex. 563. $\displaystyle\int_0^{+\infty} (\arctan x)^\alpha(\sqrt{x}+3)^{2\alpha}\,\mathrm{d}x.$ [Divergent for any α]

Ex. 564. $\displaystyle\int_0^{+\infty} (e^{-x}+\frac{x^{2\alpha}+1}{\sqrt{x}})\,\mathrm{d}x.$ [Divergent for any α]

Ex. 565. $\displaystyle\int_{-1}^{+\infty} \frac{\arctan(x^2+3)}{(x+1)^\alpha(x+2)}\,\mathrm{d}x.$ $\left[0 < \alpha < 1\right]$

Ex. 566. $\displaystyle\int_0^{+\infty} \arctan(1/x)^\alpha(x^2+3)^{2\alpha}\,\mathrm{d}x.$ $\left[-\frac{1}{4} < \alpha < 0\right]$

Ex. 567. $\displaystyle\int_3^{+\infty} \frac{e^{-t}}{(t-3)^\alpha\sqrt{t}}\,dt.$ $\left[\alpha < 1\right]$

Ex. 568. $\displaystyle\int_0^{+\infty} \frac{\sin^\alpha(1/\sqrt{t})}{\sqrt{t}\log^\alpha(t+1)}\,dt.$ [Divergent for any α]

Ex. 569. $\displaystyle\int_{-1}^{2} \frac{(e^{x+3}+7\sin^2 x)}{x^\alpha(e^x+1)}\,\mathrm{d}x.$ $\left[\alpha < 1\right]$

Ex. 570. $\displaystyle\int_{-\infty}^{+\infty} e^{-\alpha x^2/2}\,\mathrm{d}x.$ $\left[\alpha > 0\right]$

Ex. 571. $\displaystyle\int_1^{+\infty} (e^{1/x}-1)^\alpha\frac{\log(2+x)}{x^2}\,\mathrm{d}x.$ $\left[\alpha > -1\right]$

Ex. 572. $\displaystyle\int_4^{+\infty} \frac{\log^{\alpha+1}(x-3)}{\sqrt{e^{x-4}-1}}\,\mathrm{d}x.$ Moreover, calculate its value for $\alpha = -1$. $\left[\alpha > -\frac{3}{2}.\,\pi\right]$

Ex. 573. $\displaystyle\int_0^{+\infty} \frac{\sin\left(\frac{x}{x^2+1}\right)}{(x^2-\sin x^2)^\alpha}\,\mathrm{d}x.$ $\left[0 < \alpha < \frac{1}{3}\right]$

Ex. 574. $\displaystyle\int_0^{+\infty} \frac{3+2\sin x}{(x-1)^{1/3}(x+2)^{4\alpha}}\,\mathrm{d}x.$ $\left[\alpha > \frac{1}{6}\right]$

Ex. 575. $\displaystyle\int_0^{+\infty} \frac{\log(1+x^\alpha)}{x^3}\,\mathrm{d}x.$ $\left[\alpha > 2\right]$

Ex. 576. $\displaystyle\int_0^{1} \frac{1}{x(-\log x)^\alpha+x^2(1-x^2)^{1/3}}\,\mathrm{d}x.$ $\left[\alpha > 1\right]$

5.6.4 Determine the values of α and β such that the following improper integrals converge

Ex. 577. $\displaystyle\int_0^1 \frac{|\log x|^\alpha}{|\sin \pi x|^\beta}\,\mathrm{d}x.$ $\left[\beta < 1,\ \beta - \alpha < 1\right]$

Ex. 578. $\displaystyle\int_0^{+\infty} \frac{e^{\alpha x + \beta/x}}{x+1}\,\mathrm{d}x.$ $\left[\alpha < 0,\ \beta \le 0\right]$

Ex. 579. $\displaystyle\int_0^{+\infty} \frac{(\arctan x)^\alpha}{x^\beta(2+\cos x)}\,\mathrm{d}x.$ $\left[\beta > 1,\ \beta - \alpha < 1\right]$

5.7 Numerical Series

5.7.1 Determine the nature of the following numerical series

Ex. 580. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k + \sqrt{k}}.$ [Divergent]

Ex. 581. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k + \log k}.$ [Divergent]

Ex. 582. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\log k}}.$ [Convergent]

Ex. 583. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{\log(\log k)}{\log k}\right)^k.$ [Convergent]

Ex. 584. $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}.$ [Convergent]

Ex. 585. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-\sqrt{k}}.$ [Convergent]

Ex. 586. $\displaystyle\sum_{k=1}^{\infty} \left(\sqrt{k^2+1} - k\right)\log\left(1 + \frac{1}{k}\right).$ [Convergent]

Ex. 587. $\displaystyle\sum_{k=1}^{\infty} \left(\sqrt{k+1} - \sqrt{k}\right)^2.$ [Divergent]

Ex. 588. $\displaystyle\sum_{k=1}^{\infty} \left(\sqrt{1 + \sin\frac{3}{k}} - 1\right)\left(1 - e^{-1/k}\right).$ [Convergent]

Ex. 589. $\displaystyle\sum_{k=1}^{\infty} (e^{1/k^2} - 2\cos\frac{1}{k} + 1).$ [Convergent]

Ex. 590. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3+e^{\alpha k}}, \quad \alpha \in \mathbb{R}.$ $\Big[$Convergent if $\alpha > 0,$ divergent otherwise$\Big]$

Ex. 591. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{4+e^{\alpha k}}, \quad \in \mathbb{R}.$ $\Big[$Convergent if $\alpha > 0,$ divergent otherwise$\Big]$

Ex. 592. $\displaystyle\sum_{k=1}^{\infty} \left(1-\sqrt{e}\left(\cos\frac{1}{k}\right)^{k^2}\right).$ $\Big[$Convergent$\Big]$

Ex. 593. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{5}{9-2\cos k}\right)^{k}.$ $\Big[$Convergent$\Big]$

Ex. 594. $\displaystyle\sum_{k=1}^{\infty} \left(\log(1+\frac{3}{\sqrt[3]{k^2}})-\frac{\alpha}{\sqrt[3]{k^2}}\right).$ $\Big[$Convergent if $\alpha = 3,$ divergent otherwise$\Big]$

Ex. 595. $\displaystyle\sum_{k=1}^{\infty} \left(k\sin\left(\frac{1}{k}\right)\right)^{k^3}.$ $\Big[$Convergent$\Big]$

5.7.2 Determine the nature of the following series

Ex. 596. $\displaystyle\sum_{k=1}^{\infty} \left(1-\frac{1}{k^{1/3}}\right)^{k^2}.$ $\Big[$Convergent$\Big]$

Ex. 597. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3}{5+\cos^2 k}\right)^{k}.$ $\Big[$Convergent$\Big]$

Ex. 598. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{6}{3^n}+\frac{(-1)^{n+1}}{4^n}\right).$ Moreover, if it exists, calculate the sum. [Converges to $\frac{23}{3}$]

Ex. 599. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{\sqrt{n}}}{e^{n^2}}.$ $\Big[$Convergent$\Big]$

Ex. 600. $\displaystyle\sum_{k=1}^{\infty} \left(\left(\frac{3x^2-3}{x^2+1}\right)^{2n}+\frac{n+1}{n^2(\log n)^x+2}\right), \qquad x \in \mathbb{R}.$

$\Big[$Converges if$1 < x < \sqrt{2},$ diverges otherwise$\Big]$

5.7.3 Determine the nature of the following series for any $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}$

Ex. 601. $\displaystyle\sum_{k=4}^{\infty} \frac{1}{k^2}\left(1-\frac{1}{k}\right)^{k^\alpha}.$ $\Big[$Converges for any $\alpha \in \mathbb{R}\Big]$

Ex. 602. $\displaystyle\sum_{n=1}^{\infty} \frac{n^\alpha (x+1)^{2n}}{(2n)!}, \quad x \in \mathbb{R}.$ $\Big[$Converges for any $\alpha, x \in \mathbb{R}\Big]$

Ex. 603. $\displaystyle\sum_{n=1}^{\infty} n\left(1-\left(1+\frac{1}{n^{2\alpha}}\right)^{1/4}\right).$ $\left[\text{Converges if } \alpha > 1, \text{ divergent otherwise}\right]$

Ex. 604. $\displaystyle\sum_{n=1}^{\infty} \frac{n^8}{(n-\log n)^{10}-n^{\alpha}}.$ $\left[\text{Converges if } \alpha \neq 10, \text{ divergent otherwise}\right]$

Ex. 605. $\displaystyle\sum_{n=1}^{\infty} n^{\alpha}\left(\left(n^4-5n^2\right)^{1/4}-\left(n^3-3n\right)^{1/3}\right).$ $\left[\text{Convergent if } \alpha < 0, \text{ divergent otherwise}\right]$

Ex. 606. Determine the values of $\alpha \in \mathbb{R}$ such that the following two series have the same nature: $\displaystyle\sum_{n=1}^{\infty}\left(e^{(n^{\alpha}+1/n)}-1\right), \quad \sum_{n=1}^{\infty}\log(1+n^{\alpha}).$ $\left[\alpha \geq -1\right]$

Ex. 607. Study the nature of the series $\displaystyle\sum_{n=1}^{\infty} 3^{(-1)^n} 3^{\alpha n}$ for any $\alpha \in \mathbb{R}$. Moreover, calculate its sum once calculated the one of the two series $\displaystyle\sum_{n=1}^{\infty} 3\cdot 3^{2\alpha n}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3}\cdot 3^{(2n+1)\alpha}$.

$\left[\text{Converges for } \alpha < 0 \text{ to the value } \dfrac{9+3^{\alpha}}{3(1-3^{2\alpha})}\right]$

5.7.4 Discuss the simply and absolute convergence of the following series

Ex. 608. $\displaystyle\sum_{n=0}^{\infty}\arctan\frac{1}{n+1}.$ $\left[\text{Simply and absolutely convergent}\right]$

Ex. 609. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{\alpha}{2\alpha+3}\right)^{n}\frac{1}{\log n}.$

$\left[\text{Simply convergent if } \alpha < -3, \text{absolutely convergent if } \alpha < -3, \alpha \geq -1\right]$

Ex. 610. $\displaystyle\sum_{k=1}^{\infty}(-1)^k\left(e^{1/k^{1/4}}-1\right)^{\alpha}.$ $\left[\text{Simply convergent if } \alpha > 0, \text{absolutely convergent if } \alpha > 4\right]$