Monte Carlo Methods for Pricing and Hedging American Options in High Dimension

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Summary. We numerically compare some recent Monte Carlo algorithms devoted to the pricing and hedging American options in high dimension. In particular, the comparison concerns the quantization method of Barraquand-Martineau and an algorithm based on Malliavin calculus. The (pure) Malliavin calculus algorithm improves the precision of the computation of the delta but, merely for pricing purposes, is uncompetitive with respect to other Monte Carlo methods in terms of computing time. Here, we propose to suitably combine the Malliavin calculus approach with the Barraquand-Martineau algorithm, using a variance reduction technique based on control variables. Numerical tests for pricing and hedging American options in high dimension are given in order to compare the different methodologies.

Keywords: option pricing; hedging; American options; Monte Carlo methods; Malliavin Calculus

1 Introduction

Pricing and hedging American options in high dimension is one of the most interesting and open problems on the practical side and on the theoretical one in the field of computational finance. From a practical point of view, to find an efficient numerical solution for the price and the delta of an American option written on d assets is really a challenge when d is high (say, $d > 3$). The straightforward application of standard numerical schemes (e.g. finite difference, finite element or lattice methods) fails due to the so-called "curse of dimension": the computational cost and the memory requirement increase exponentially as the dimension of the problem increases. Therefore, variational inequalities for parabolic problems in dimension d larger than 3 cannot be solved in practice by conventional, deterministic methods and thus, Monte Carlo methods appear as the the only practicable way.

In the last years, several new ideas appeared in this field. Roughly speaking they may be divided in three main families. In the first one, a tree is built up in order to obtain a discretization of the underlying diffusion on a grid; this family includes, for example, the Broadie and Glasserman algorithm [3], the quantization algorithm developed by Bally, Pagés and Printems [2] or the Barraquand and Martineau method [4]. The second idea is to use regression on a truncated basis of L^2-spaces in order to compute the conditional expectations, as it has been done by Longstaff and Schwartz [12] and by Tsisiklis and Van Roy [14]. Finally, in the papers [5], [7], [8] and [11], the authors develop representation formulas for the conditional expectation using Malliavin calculus and then employ them in order to perform a Monte Carlo algorithm. The peculiarity of this last approach is that it appears as a pure Monte Carlo method despite the nonlinearity.
Another problem of interest is to compute the sensitivity of the solution with respect to some parameter (hedging and Greeks). It seems that Malliavin calculus is an especially promising tool for solving such a problem. It has been used for example by Lions and Reigner [11] and Bouchard and Touzi [5], who follow the third method, as well as by Bally, Pagés and Printems [2], where the quantization algorithm is employed. In [1], Bally, Caramellino and Zanette give a simplified presentation of this topic, including some reduction of variance techniques and with a special interest to the practical implementation and performance of the method.
As a conclusion, it seems that all methods give quite good results for the numerical pricing, but the Malliavin calculus approach behaves in the best way when the numerical hedging is considered. Since the (pure) Malliavin calculus method turns out to be also much slower than the others, the idea is to mix it with some other faster algorithm. Therefore, in this paper we we study some combinations between the Malliavin calculus techniques and the pricing quantization algorithm by Barraquand-Martineau, chosen for the simplicity of the implementation, the accuracy of the pricing results and because it runs quickly.
The outline of the paper is as follows: Section 2 is devoted to the framework of Monte Carlo methods for pricig/hedging American options; in Section 3 and 4, the Malliavin calculus method and the Barraquand-Martineau algorithm are briefly described; Section 5 develops the variance reduction technique here used; finally, numerical tests and results are given in Section 6.

2 Pricing/hedging American options: the Dynamic Programming principle

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where a d-dimensional Browinan motion W is defined and set $\mathcal{F}_t = \sigma(W_s \,:\, s \leq t)$. Let $X = (X_t)_{t\in[0,T]}$ denote the process (on $\mathbb{R}^d$) of the asset prices, which as usual evolves as a diffusion process. An American option with maturity T underlying the asset price process

X, is an option whose holder can exercise his right of option in any time up to T. If $\Phi(X_s)$ denotes the associated cash-flow, the price at time t of such an option is then

$$P(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{t,x}\Big(e^{-r(\tau-t)}\Phi(X_\tau)\Big) \qquad (1)$$

where, here and in the following, $\mathbb{E}_{t,x}$ denotes the expectation conditional to $X_t = x$ and $\mathcal{T}_{t,T}$ stands for the $\mathcal{F}_t$-stopping times taking values in $[t,T]$. It is well known that the price function $P(t,x)$ solves a PDE problem with obstacle allowing to set up a Dynamic Programming Principle, which in turn gives the following analytical approximation for the function P:

Theorem 1. *Given $\Delta t = T/n \in (0,1)$, let $(\bar{X}_{k\Delta t})_{k=0,1,\dots,n}$ a discretization for $(X_t)_{t\in[0,T]}$. Define $\bar{P}_{n\Delta t}(\bar{X}_{n\Delta t}) = \Phi(\bar{X}_{n\Delta t})$ and for any $k = n-1,\dots,1,0$,*

$$\bar{P}_{k\Delta t}(\bar{X}_{k\Delta t}) = \max\Big(\Phi(\bar{X}_{k\Delta t})\,,\, e^{-r\Delta t}\,\mathbb{E}\Big(\bar{P}_{(k+1)\Delta t}(\bar{X}_{(k+1)\Delta t})\,\Big|\,\bar{X}_{k\Delta t}\Big)\Big).$$

Then $\bar{P}_{k\Delta t}(\bar{X}_{k\Delta t}) \simeq P(k\Delta t, X_{k\Delta t})$.

Let us remark that the above statement is heuristic and a rigorous formulation supposes to precise the hypothesis on the diffusion coefficients and on the regularity of the obstacle Φ (see Bally, Pagés and Printems [2]).
Monte Carlo algorithms are based on the dynamic programming principle above. They are obviously backward and, roughly speaking, can be summarized as follows. Take $t_k = k\,\Delta t$, as $k = 0,1,\dots,n$ and, for example, consider X following the Black-Scholes model (geometric Brownian motion).

`Step` n: `simulation of` $(W^q_{t_n})_{q=1,\dots,M}$
$\hookrightarrow$ `simulation of` $(X^q_{t_n})_{q=1,\dots,M}$
$\hookrightarrow$ `computation of` $(\bar{P}_{t_n}(X^q_{t_n}))_{q=1,\dots,M}$: $\bar{P}_{t_n}(X^q_{t_n}) = \Phi(X^q_{t_n})$.

`Step` $n-1$: `given` $(W^q_{t_n})_{q=1,\dots,M}$
$\hookrightarrow$ `simulation of` $(W^q_{t_{n-1}})_{q=1,\dots,M}$ `(use Brownian bridge)`
$\hookrightarrow$ `simulation of` $(X^q_{t_{n-1}})_{q=1,\dots,M}$
$\hookrightarrow$ `computation of` $(\bar{P}_{t_{n-1}}(X^q_{t_{n-1}}))_{q=1,\dots,M}$:

$$\bar{P}_{t_{n-1}}(X^q_{t_{n-1}}) = \max\Big(\Phi(\alpha)\,,\, e^{-r\,\Delta T}\underbrace{\mathbb{E}(\bar{P}_{t_n}(X_{t_n})\,|\,X_{t_{n-1}}=\alpha)}_{\clubsuit}\Big)\Big|_{\alpha=X^q_{t_{n-1}}}$$

$\vdots \leftarrow$ `Step` $k = n-2,\dots,1$ [`replace` $n-1$ `with` k`; at the end, the samples` $(\bar{P}_{t_1}(X^q_{t_1}))_{q=1,\dots,M}$ `are available`]

`Step` 0: `computation of the price:`

$$\bar{P}_0(x) = \max\Big(\Phi(x), e^{-r\,\Delta t}\frac{1}{M}\bar{P}_{t_1}(X^q_{t_1})\Big)$$

In practice, this procedure gives rise to the problem of the computation of a large number of conditional expectations (terms ♣). Now, in order to handle this problem, one can

(a1) approximate by means of a stratification of the path space ([3], [2], [4]);
(a2) approximate by means of regression methods ([12], [14]);
(a3) use representation formulas involving non conditioned expectations ([1], [6], [8], [11]).

Now: what about the delta? Recall that the delta Greek is given by $\Delta(t,x) = \partial_x P(t,x)$. The two main tools are given by:

(b1) the finite difference method, that is

$$\Delta(0,x) \simeq \bar{\Delta}_0(x) = \frac{1}{2\delta}\Big(\bar{P}_0(x+\delta) - \bar{P}_0(x-\delta)\Big) \tag{2}$$

where δ is chosen "small enough";
(b2) representation formulas for the derivative of conditional expectations involving non conditioned expectations ([1], [6], [8], [11]), which will be better clarified in the sequel.

It is well known that the finite difference method does work poorly, especially whenever the function Φ is not regular, so that it would be very important to perform a careful study of the second method.
Now, in the following sections we briefly recall the main procedures we will take into account, that is:

- the (pure) Malliavin calculus algorithm, using in some sense **(a3)** and **(b2)**;
- the (standard) Barraquand-Martineau method, considering **(a1)** and **(b1)**;
- an algorithm following **(a1)** and **(b3)**: it gives the price by the Barraquand-Martineau algorithm and evaluates the delta through Malliavin calculus.

Our framework for the evolution of the underlying asset price process will be the multidimensional Black and Scholes model: under the risk neutral measure,

$$dX_t^i = (r-\eta_i)\, X_t^i\, dt + \sum_{k=1}^d \sigma_{ik}\, X_t^i\, dW_t^k, \quad t \in (0,T],\ i = 1,\dots,d, \tag{3}$$

with $X_0 = x \in \mathbb{R}_+^d$, r being the (constant) spot rate, $\eta_1,\dots,\eta_d$ the continuous dividends and σ the volatility matrix.

3 The (pure) Malliavin calculus algorithm

The (pure) Malliavin calculus algorithm is based on representation formulas of the conditional expectation in terms of non conditional ones, as developed in [1], [5], [7], [8] and [11], and follows the dynamic programming principle described before. It makes use of the following results in order to evaluate the conditional expectations and their derivatives involved in the simulation programme (for notations, proofs and comments we refer to [1]).

3.1 Price

Without loss of generality, one can suppose that the volatility matrix σ is a sub-triangular matrix, that is $\sigma_{ij} = 0$ whenever $i < j$. Thus, any component of X_t can be written as

$$X_t^i = x_i \exp\Big(h_i t + \sum_{j=1}^{i} \sigma_{ij} W_t^j\Big), \qquad i = 1, \dots, d \tag{4}$$

where $h_i = r_i - \eta_i - \frac{1}{2}\sum_{j=1}^{i}\sigma_{ij}^2$, $i = 1, \dots, d$. To our purposes, let us set $\widetilde{X}_t^i = x_i \exp\Big(h_i\, t + \sigma_{ii} W_t^i\Big)$, $i = 1, \dots, d$. The main facts for the introduction of $\widetilde{X}$ are: $\widetilde{X}$ has independent components and for any $t \geq 0$, there exists an invertible transformation $F_t(\cdot) : \mathbb{R}_+^d \to \mathbb{R}_+^d$ such that $X_t = F_t(\widetilde{X}_t)$ and $\widetilde{X}_t = F_t^{-1}(X_t)$: $F_t^1(y) = y_1$ and as $i = 1, \dots, d$,

$$F_t^i(y) = y_i \prod_{j=1}^{i-1}\Big(\frac{y_j}{x_j}\, e^{-h_j t}\Big)^{\widetilde{\sigma}_{ij}}, \quad \text{where} \quad \widetilde{\sigma}_{ij} = \frac{\sigma_{ij}}{\sigma_{jj}} \tag{5}$$

(so, F_t allows to handle $\widetilde{X}$ in place of the original process X). Then, one has

Theorem 2. *For any $0 < s < t$, Φ with polynomial growth and $\alpha \in \mathbb{R}_+^d$, the following localized representation formula holds:*

$$\mathbb{E}(\Phi(X_t) \,|\, X_s = \alpha) = \frac{\mathbb{E}(\Phi(X_t)\Theta_{s,t}^{\psi}(\alpha))}{\mathbb{E}(\Theta_{s,t}^{\psi}(\alpha))} \tag{6}$$

where either $\psi = 0$ (i.e. no localization is considered) or $\psi(x) = \prod_{i=1}^{d}\psi_i(x_i)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, with $\psi_i \geq 0$ and $\int_{\mathbb{R}}\psi_i(\xi)d\xi = 1$, and the weight $\Theta_{s,t}^{\psi}$ is given by:

$$\Theta_{s,t}^{\psi}(\alpha) = \prod_{i=1}^{d}\Big[\psi_i(X_s^i - \alpha^i) + \frac{H(\widetilde{X}_s^i - \widetilde{\alpha}_s^i) - \Psi_i(\widetilde{X}_s^i - \widetilde{\alpha}_s^i)}{\sigma_{ii} s(t-s)\widetilde{X}_s^i}\,\Delta W_{s,t}^i\Big]$$

being: $H(\xi) = \mathbf{1}_{\{\xi>0\}}$, $\Psi_i = 0$ if $\psi = 0$, otherwise $\Psi_i(y) = \int_{-\infty}^{y}\psi_i(\xi)d\xi$; $\widetilde{\alpha}_s = F_s^{-1}(\alpha)$, where F_s^{-1} is the inverse function of F_s, defined in (5); $\Delta W_{s,t}^i = (t-s)(W_s^i + \sigma_{ii}s) - s(W_t^i - W_s^i)$.

Let us spend some words about the localizing function ψ. The non localized representations, that is $\psi = 0$, do not work in practice, mainly because the involved Heaviside function $H(\xi) = \mathbf{1}_{\{\xi>0\}}$ provides high dispersion. Therefore, the use of ψ turns out to be essential. Now, one could think to choose ψ such that $H - \Psi$ is small, but not too much otherwise ψ becomes "close" to the Dirac mass δ_0. In [1], an optimization method for the choice of the "right" function ψ is studied (involving a suitable "integrated variance", following Kohatsu-Higa and Petterson [10]). The result is that a good choice (either from a theoretical and practical point of view when $t - s$ is small, and this is the case) is given by a multivariate Laplace probability density function:

$$\psi^*(x) = \prod_{j=1}^{d} \phi^*(x_j), \quad x = (x_1, \ldots, x_d), \text{ where } \quad \phi^*(\xi) \propto e^{-|\xi|/\sqrt{t-s}}. \tag{7}$$

3.2 Delta

For the computation of the delta $\Delta(t,x) = \partial_x P(t,x)$, one uses the following

Proposition 1. *For any $\Delta t = T/n \in (0,1)$, set*

$$\Gamma_{\Delta t} = \{\alpha \in \mathbb{R}^d \,;\, \bar P_{\Delta t}(\alpha) = \Phi(\alpha)\},$$

where $\bar P_{\Delta t}(\alpha)$ is the approximation of the price from the dynamic programming principle, that is $\bar P_{\Delta t}(\alpha) = \max\Big(\Phi(\alpha)\,,\, e^{-r\Delta t}\,\mathbb{E}\Big(\bar P_{2\Delta t}(\bar X_{2\Delta t})\,\Big|\, \bar X_{\Delta t} = \alpha\Big)\Big)$. Then, by setting

$$\bar\Delta(\alpha) = \partial_\alpha\, \Phi(\alpha)\, \mathbf{1}_{\Gamma_{\Delta t}} + e^{-r\Delta t}\partial_\alpha\, \mathbb{E}\Big(\bar P_{2\Delta t}(\bar X_{2\Delta t})\,\Big|\, \bar X_{\Delta t} = \alpha\Big)\, \mathbf{1}_{\Gamma^c_{\Delta t}} \quad \textit{and}$$
$$\bar\Delta_0(x) = \mathbb{E}_x\Big(\bar\Delta(\bar X_{\Delta t})\Big)$$

where ∂_α denotes the gradient, one has $\Delta(0,x) \simeq \bar\Delta_0(x)$

Again, such an assertion is heuristic and a rigorous statement, including error bound, turns out to be a more difficult problem (in [2] a bound is given in a weak sense).
Now, in order to compute the derivative of the conditional expectation, one makes use of the following representation formula, giving such a derivative in terms of non conditioned expectations:

Theorem 3. *For any $0 < s < t$, Φ with polynomial growth and $\alpha \in \mathbb{R}^d_+$, the following localized representation formula holds:*

$$\partial_\alpha \mathbb{E}(\Phi(X_t)\,|\, X_s = \alpha) = \frac{\mathbb{E}(\Phi(X_t)\Upsilon^{\psi}_{s,t}(\alpha))\mathbb{E}(\Theta^{\psi}_{s,t}(\alpha)) - \mathbb{E}(\Theta^{\psi}_{s,t}(\alpha))\mathbb{E}(\Upsilon^{\psi}_{s,t}(\alpha))}{(\mathbb{E}(\Theta^{\psi}_{s,t}(\alpha)))^2}, \tag{8}$$

where the localizing function ψ enjoys the same properties as in Theorem 2 and the weight $\Upsilon^{\psi}_{s,t} = (\Upsilon^{\psi}_{s,t;1}, \ldots, \Upsilon^{\psi}_{s,t;d})$ is given by:

$$\begin{aligned}\Upsilon^{\psi}_{s,t;j}(\alpha) = -\sum_{k=1}^{j} \widehat{\sigma}_{kj} \frac{\widetilde{\alpha}^k_s}{\alpha_j} \times \Big[\psi_k(\widetilde{X}^k_s - \widetilde{\alpha}^k_s) \frac{\Delta W^k_{s,t}}{\sigma_{kk} s(t-s)\widetilde{X}^k_s} + \\ + \frac{H(\widetilde{X}^k_s - \widetilde{\alpha}^k_s) - \Psi_k(\widetilde{X}^k_s - \widetilde{\alpha}^k_s)}{\sigma_{kk} s(t-s)(\widetilde{X}^k_s)^2} \Big(\frac{(\Delta W^k_{s,t})^2}{\sigma_{kk} s(t-s)} + \Delta W^k_{s,t} - \frac{t}{\sigma_{kk}}\Big)\Big] \times \\ \times \prod_{i=1, i\neq k}^{d} \Big[\psi_i(\widetilde{X}^i_s - \widetilde{\alpha}_i) + \frac{H(\widetilde{X}^i_s - \widetilde{\alpha}^i_s) - \Psi_i(\widetilde{X}^i_s - \widetilde{\alpha}^i_s)}{\sigma_{ii} s(t-s)\widetilde{X}^i_s} \Delta W^i_{s,t}\Big]\end{aligned}$$

Ψ_i, $\widetilde{\alpha}^i_s$, $\Delta W^i_{s,t}$ being defined in Theorem 2 and $\widehat{\sigma} = \widetilde{\sigma}^{-1}$, with $\widetilde{\sigma}$ as in (5).

The previous remarks on the importance of ψ hold as well and again, numerical and theoretical evidence shows that ψ can be taken as in formula (7).

4 The Barraquand-Martineau algorithm

The Barraquand-Martineau algorithm (see [4]) allows to numerically compute the price of an American option by making use of a quantization method acting on a one dimensional dynamical programming algorithm. Let us briefly describe it.

4.1 Price

To overcome the "curse of dimension" problem, Barraquand and Martineau propose to approximate the optimal stopping strategy by the following sub-optimal one. Assume that the option holder knows at time t the payoff values $\{\Phi(X_u); u \le t\}$ but not the stock values $\{X_u; u \le t\}$. Then, the option holder can only exercise according to a strategy optimizing

$$\sup_{\tau \in \mathcal{S}_{0,n}} \mathbb{E}\Big(e^{-r\,\tau}\Phi(X_\tau)\Big) \tag{9}$$

where $\mathcal{S}_{0,n}$ is the set of the $\mathcal{G}_t$-stopping times taking values in $\{t_0, t_1, \ldots, t_n\}$, with $t_k = k\,\Delta t$, and $(\mathcal{G}_t)_t$ being the filtration generated by the payoff process: $\mathcal{G}_t = \sigma(\Phi(X_s)\ :\ s \le t)$. To compute (9), a dynamic programming principle can be set up:

$$\begin{cases} Q_n := \Phi(X_{t_n}) \\ Q_{j-1} := \max\Big(\Phi\big(X_{t_{j-1}}\big), e^{-r\Delta t}\,\mathbb{E}(Q_j \,|\, \mathcal{G}_{t_{j-1}})\Big), \quad j = n, \ldots, 1. \end{cases} \tag{10}$$

Since the process $(\Phi(X_t))_{t\ge 0}$ is (in general) not Markov with respect to $(\mathcal{G}_t)_t$, a second approximation is done and (10) is replaced by

$$\begin{cases} \hat{P}_n := \Phi\left(X_{t_n}\right) \\ \hat{P}_{j-1} := \max\Big(\Phi\left(X_{t_{j-1}}\right), e^{-r\Delta t}\,\mathbb{E}(\hat{P}_j \,|\, \Phi(X_{t_{j-1}}))\Big), \quad j = n, \dots, 1. \end{cases} \tag{11}$$

Barraquand and Martineau then use (11) in order to numerically compute the price of an American option with payoff function Φ and propose to compute the involved conditional expectations through a quantization technique (in few words, this means to use suitable weights built from an approximation of the process $(\Phi(X_t))_t$; for details see [4], where the method is developed). Notice that the algorithm becomes now one dimensional.

4.2 Delta

In the paper by Barraquand and Martineau, no suggestions are given for the computation of the delta, so that a standard implementation of the algorithm would use the standard finite difference method (see (2)).
However, one could also follow the same procedure used in the (pure) Malliavin algorithm, described in Section 3.2, or also something of simpler. Let us explain the simple way (the procedure as in Section 3.2 is an immediate generalization).
Let us come back to the dynamic programming principle described in Section 2. Then, the delta can be numerically computed as

$$\bar{\Delta}_0 = \begin{cases} \partial_x \Phi(x) & \text{if } \Phi(x) > e^{-r\Delta t}\,\mathbb{E}_{0,x}\Big(\bar{P}_{t_1}(X_{t_1})\Big) \\ e^{-r\Delta t}\partial_x\,\mathbb{E}_{0,x}\Big(\bar{P}_{t_1}(X_{t_1})\Big) & \text{if } \Phi(x) < e^{-r\Delta t}\,\mathbb{E}_{0,x}\Big(\bar{P}_{t_1}(X_{t_1})\Big). \end{cases}$$

As for the computation of the derivative of the conditional expectation, one can use the typical formula coming from Malliavin calculus technique (see e.g. [7]):

$$\partial_x \mathbb{E}_{0,x}\Big(\bar{P}_{t_1}(X_{t_1})\Big) = \mathbb{E}_{0,x}\Big(\bar{P}_{t_1}(X_{t_1})\,\pi^{\Delta}\Big)$$

where $\pi^{\Delta} = (\pi^{\Delta}_1, \dots, \pi^{\Delta}_d)$ is a suitable weight which, in the case of the Black and Scholes model, enjoys the formula

$$\pi^{\Delta}_i = \frac{W^i_{t_1}}{X^i_{t_1}\,\sigma_{ii}\,t_1}.$$

Therefore, in practice one can follow the Barraquand and Martineau algorithm, which gives many independent simulations $(\hat{P}^q_1)_{q=1,\dots,M}$ of the random variable $\hat{P}_1$. Since $\hat{P}_1$ is an approximation for $\bar{P}_{t_1}(X_{t_1})$, by using the law of large numbers one can numerically approximate the delta by

$$\hat{\Delta}_0 = \begin{cases} \partial_x \Phi(x) & \text{if } \Phi(x) > e^{-r\Delta t}\,\dfrac{1}{M}\displaystyle\sum_{q=1}^{M} \hat{P}^q_1 \\ e^{-r\Delta t}\,\dfrac{1}{M}\displaystyle\sum_{q=1}^{M} \hat{P}^q_1\,\pi^{\Delta,q} & \text{if } \Phi(x) < e^{-r\Delta t}\,\dfrac{1}{M}\displaystyle\sum_{q=1}^{M} \hat{P}^q_1, \end{cases}$$

where $\pi^{\Delta,q}$ denotes the qth sample of the weight π^{Δ}.

5 Control variate

When one works with Monte Carlo, one typically looks for some ways to speed up the algorithm, that is to reduce the variance. For example, we have already observed that the (pure) Malliavin calculus algorithm does not run without the localizing function. A further technique allowing to reduce the variance is the introduction of a control variate. Unfortunately, there is not a standard way to proceed in this direction. Nevertheless, for pricing an American option it is quite natural to take into account the price of the associated European option as the control variate. The motivation is the following.
For a fixed initial time t and underlying asset price x, let us set $P^{\mathrm{am}}(t,x)$ and $P^{\mathrm{eu}}(t,x)$ as the price of an American and European option respectively, with the same payoff Φ and maturity T. We define

$$P(t,x) = P^{\mathrm{am}}(t,x) - P^{\mathrm{eu}}(t,x).$$

Then it is easy to see that

$$P(t,X_t) = \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\Big(e^{-r(\tau-t)}\widehat{\Phi}(\tau,X_\tau)\,\Big|\,\mathcal{F}_t\Big)$$

where $\mathcal{T}_{t,T}$ stands for the set of all the stopping times taking values on $[t,T]$ and $\widehat{\Phi}$ is defined by

$$\widehat{\Phi}(t,x) = \Phi(x) - P^{\mathrm{eu}}(t,x)$$

(notice that $\widehat{\Phi}(T,x)=0$ and that the obstacle $\widehat{\Phi}$ is now dependent on the time variable also). Thus, for the numerical evaluation of $P(0,x)$, one can set up a dynamic programming principle in point of fact identical to the one in Section 2, with $\widehat{\Phi}$ in place of Φ. Given the estimated "price" $\bar{P}_0(x)$ and "delta" $\bar{\Delta}_0(x)$ (according to $\widehat{\Phi}$), the price and delta of the American option are then approximated by

$$\bar{P}_0^{\mathrm{am}}(x) = \bar{P}_0(x) + P^{\mathrm{eu}}(0,x) \quad \text{and} \quad \bar{\Delta}_0^{\mathrm{am}}(x) = \bar{\Delta}_0(x) + \Delta^{\mathrm{eu}}(0,x)$$

respectively. Since $\widehat{\Phi}$ has to be evaluated at each time step, in order to set up this program one should compute the price/delta of an European option on Φ. For some options, European prices and deltas are known in closed form but in the general case, one can compute them by simulation.

6 Numerical tests and results

Our numerical experiences for pricing/hedging American options concern both a regular and a singular payoff. We present here several tests on the American put on the minimum on 2 and 5 assets (Section 6.1) and on the American relative digital option on 2 assets (Section 6.2). The study will be different according to the type of option because, as we will see, different behaviors will be observed. On these options, we numerically illustrate the behavior of the following methods:

PM method: the pure Malliavin calculus one (as in Section 3.1 and 3.2), where as a control variate we consider the twin European price given by its closed form solution;

BM method: the Barraquand-Martineau algorithm (as in Section 4.1), with the standard implementation (no control variate and deltas numerically computed by the finite difference approximation);

BM-M method: the Barraquand-Martineau algorithm with a Malliavin correction, in which the control variate is given by the twin European price computed with the Barraquand-Martineau algorithm itself and for the deltas, we use the Malliavin approach described in Section 4.2.

Let us briefly explain why, in order to evaluate the control variable, the closed form formula has been used only in the PM method. The associated European price and deltas are available in a closed form solution and entail the computation of the cumulative multivariate normal distribution function (see e.g. Johnson [9]). The computation of such a function is an expensive operation and moreover, it has to be done a number of times proportional to the number of Monte Carlo iterations. Now, as we will see, when using the PM algorithm one is forced (by computational costs) to use few simulations, while in the BM method one has to consider many trials in order to achieve good results, so that the use of the closed formula becomes unfeasible. Moreover, as for the BM method, let us stress that the numerical results using finite differences are all obtained with the classical implementation, that is with $\delta = x \times 10^{-3}$ in formula (2) and, as usual, by using common trials.
Finally, from the computer point of view, all the computations have been performed in double precision on a PC Pentium IV 1.8 GHz with 256 Mb of RAM.

6.1 American put on the minimum

Here, the numerically tested option is a put on the minimum of d assets[3]: its payoff function is given by

$$\Phi(x) = \Big(K - \min(x_1, x_2, .., x_d)\Big)_+$$

As for the dimension, one considers first the case $d = 2$ in order to assess the numerical behavior of the algorithms and then, the case $d = 5$. With a

[3]Such an option has been provided by Gilles Pagés, as the organizer of the session "From the pricing American options on baskets to RBSDE discretization" in the Juan Le Pins MC2QMC 2004 conference.

view to have comparable results at our disposal, a symmetric case is taken into account, in which the deltas are all equal. Thus, both the initial values and the volatilities are assumed to be equal, with null continuous dividend rates and no correlations among the assets. The parameters are: initial values $x_1 = \cdots = x_d = 100$; volatilities $\sigma_1 = \cdots = \sigma_d = 0.2$; exercise price $K = 100$; risk-free interest rate $r = 0.05$.
For the sake of comparison, we use the following "true" reference American put on the minimum price/deltas:

$d = 2$: price and deltas are compared with the ones issued from the Villeneuve-Zanette finite difference algorithm developed in [15] (VZ-P and VZ-Δ in the next tables), with 500 time space steps;

$d = 5$: the price is compared with the result from the Longstaff-Schwarz algorithm corrected with the importance sampling variance reduction technique given by Moreni [13], using 50 time steps and 500.000 Monte Carlo trials; no "true" reference deltas are available.

Put on the minimum of 2 assets

Table 1 shows prices and deltas obtained with the PM method (pure Malliavin, with control variable equal to the European price evaluated in closed form), with varying time periods $n = 10, 20, 50$ and Monte Carlo trials $N_{\mathrm{mc}} = 500, 1000, 5000, 10000, 20000$. The use of the control variate and, obviously, the localization techniques give prices with low dispersion and very stable values for the deltas. Notice that even though the computational costs are devastating when n and N_{mc} increase, nevertheless one obtains good results for few time periods and number of simulations ($n = 10$ and $N_{\mathrm{mc}} = 500$ respectively; a discussion concerning this case is postponed at next page 12).

	N_{mc}	PM-$\bar{P}$	VZ-P	PM-$\bar{\Delta}_1$	PM-$\bar{\Delta}_2$	VZ-Δ	CPU
	500	10.237458		-0.293331	-0.298902		9
10 time periods	1000	10.222146		-0.294641	-0.293187		34
	5000	10.209749		-0.293881	-0.297122		836
	10000	10.215610		-0.296033	-0.296534		3422
	20000	10.200799		-0.295824	-0.294665		13425
	500	10.331336		-0.301507	-0.293753		15
20 time periods	1000	10.312649		-0.291905	-0.291274		61
	5000	10.290837	10.246619	-0.294500	-0.296029	-0.295244	1511
	10000	10.289001		-0.295440	-0.296039		6040
	20000	10.266047		-0.295398	-0.295012		24220
	500	10.456991		-0.305999	-0.285968		36
50 time periods	1000	10.387566		-0.287292	-0.288892		140
	5000	10.359696		-0.300333	-0.298844		3533
	10000	10.367262		-0.296894	-0.298983		14356
	20000	10.345456		-0.297391	-0.296504		56219

Table 1. American put on the minimum of 2 assets: price/deltas by PM method.

Table 2 gives the performance of the BM method (standard Barraquand-Martineau: no control variate and finite differences for the deltas), for varying time periods $n = 50, 70, 90$ and Monte Carlo simulations $N_{\mathrm{mc}} = 20000, 50000, 100000, 200000, 500000$. Notice that the computing time cost is very much lower in spite of the increasing of both n and N_{mc}. While the price seems to be more reliable than the one by the PM method, this procedure produces values for the deltas which are not so stable.

	N_{mc}	BM-$\bar{P}$	VZ-P	FD-$\bar{\Delta}_1$	FD-$\bar{\Delta}_2$	VZ-Δ	CPU
	20000	10.190763		-0.296578	-0.292227		2
50 time	50000	10.127491		-0.293258	-0.293982		6
periods	100000	10.076552		-0.292935	-0.293469		11
	200000	10.062527		-0.292859	-0.292464		23
	500000	10.052767		-0.293358	-0.292993		57
	20000	10.232724		-0.292136	-0.294390		3
70 time	50000	10.154290		-0.292351	-0.291339		8
periods	100000	10.126706	10.244882	-0.292155	-0.293554	-0.295244	16
	200000	10.104338		-0.292689	-0.292538		32
	500000	10.095747		-0.292732	-0.293068		80
	20000	10.259548		-0.297909	-0.291471		4
90 time	50000	10.167042		-0.294732	-0.291964		10
periods	100000	10.144407		-0.290924	-0.291355		20
	200000	10.123695		-0.292693	-0.291131		41
	500000	10.117590		-0.292396	-0.292225		101

Table 2. American put on the minimum of 2 assets: price/deltas by BM method.

In Table 3, prices and deltas are from the BM-M algorithm (Barraquand-Martineau plus Malliavin correction, with control variable equal to the European price evaluated by BM itself), with the number n of time periods and N_{mc} of Monte Carlo trials as in Table 2. The numerical results show good efficacy of the procedure: the use of the control variate gives prices with low dispersion and the Malliavin correction provides very stable deltas. The results show then the precision of the method both for pricing and hedging purposes. Moreover, the computational cost is satisfactory.
As a comparison, we report that the associated European put on the minimum price and deltas are equal to 9.665145 and -0.278340 respectively.

Let us finally give some further remarks concerning the behavior of the pure Malliavin method. We can assert that, here and also in the next 5-dimensional case, the deltas are quite stable but, on the contrary, the price turns out to be satisfactory only with few both time discretization steps and Monte Carlo trials (see the left corner of Table 1 and also of next Table 5 concerning the 5-dimensional case). One may deduce that, as the dimension increases, the convergence (in terms of time iterations) is slower when the number of time periods is high. This could be explained if one knew the theoretical error, for which at the moment there are no results. However, the procedure developed by Bouchard and Touzi in [5], which is not so far from the one

	N_{mc}	BM-$\bar{P}$	VZ-P	EMall-$\bar{\Delta}_1$	EMall-$\bar{\Delta}_2$	VZ-Δ	CPU
	20000	10.190763		-0.296578	-0.292227		2
50 time	50000	10.127491		-0.293258	-0.293982		6
periods	100000	10.076552		-0.292935	-0.293469		11
	200000	10.062527		-0.292859	-0.292464		23
	500000	10.052767		-0.293358	-0.292993		57
	20000	10.232724		-0.292136	-0.294390		3
70 time	50000	10.154290		-0.292351	-0.291339		8
periods	100000	10.126706	10.244882	-0.292155	-0.293554	-0.295244	16
	200000	10.104338		-0.292689	-0.292538		32
	500000	10.095747		-0.292732	-0.293068		80
	20000	10.259548		-0.297909	-0.291471		4
90 time	50000	10.167042		-0.294732	-0.291964		10
periods	100000	10.144407		-0.290924	-0.291355		20
	200000	10.123695		-0.292693	-0.291131		41
	500000	10.117590		-0.292396	-0.292225		101

Table 3. American put on the minimum of 2 assets: price/deltas by BM-M method.

here presented, gives the following theoretical error (Theorem 6.3 in [5]): the maximum of all the L^p distances between the true conditional expectations and the associated regression estimators is of order $n^{d/(4p)} N_{\mathrm{mc}}^{-1/(2p)}$ as $n \to \infty$. Such a result would suggest that as the dimension increases, one should increase a lot the number of Monte Carlo iterations in order to achieve good results. So, in our numerical tests it seems that when a few number of time steps is considered, a good choice of few Monte Carlo trials has been done in order to balance the analytical error and the statistical one. In other words, the underestimate coming from the approximation of the American option with the Bermudean one (giving the analytical error) has been well equilibrated by the "overestimate" coming from the Monte Carlo estimator (giving the statistical error). And let us notice that this empirical evidence is not from a single numerical test but is confirmed by repeating it several times. In fact, as shown in next Table 4, such kind of balancing is empirically proved by the narrow 95% confidence interval for the price and the deltas coming from 100 runs of the PM algorithm.

$\bar{P}$	10.237256
95% CI	$[10.2319346, 10.2425774]$
$\bar{\Delta}_1$	-0.292008
95% CI	$[-0.292843, -0.291173]$
$\bar{\Delta}_2$	-0.292208
95% CI	$[-0.293051, -0.291365]$

Table 4. Price, deltas and their 95% confidence interval from 100 runs of the PM algorithm in dimension $d = 2$, with $n = 10$ time periods and $N_{\mathrm{mc}} = 500$ trials.

Put on the minimum of 5 assets

We consider here as the "true" reference American price the one provided by the Longstaff-Schwartz algorithm with importance sampling: it is equal to 17.225366. Moreover, the closed form solution formula for the associated European price and deltas gives 17.010422 and -0.145269 respectively.
Table 5 reports prices and deltas from the PM method (pure Malliavin plus European price as the control variable, evaluated by the closed form solution). The deltas are quite stable: it seems to be reliable to put them all equal to 0.14. On the contrary, the price turns out to be satisfactory only with few time discretization steps, a phenomenon similar to the one yet observed and discussed in dimension 2 (see page 12).

	N_{mc}	PM-$\bar{P}$	PM-$\bar{\Delta}_1$	PM-$\bar{\Delta}_2$	PM-$\bar{\Delta}_3$	PM-$\bar{\Delta}_4$	PM-$\bar{\Delta}_5$	CPU
3 time periods	500	17.228082	-0.144003	-0.142825	-0.148317	-0.144328	-0.144129	1335
	1000	17.203429	-0.146310	-0.145184	-0.144466	-0.141576	-0.148651	5344
	5000	17.225699	-0.146578	-0.144713	-0.146462	-0.141829	-0.151819	130350
5 time periods	500	17.306792	-0.147045	-0.146015	-0.145506	-0.144541	-0.144248	2539
	1000	17.313300	-0.143961	-0.145753	-0.144254	-0.140675	-0.144558	20315
	5000	17.323421	-0.145170	-0.144910	-0.147152	-0.145496	-0.144370	508016
10 time periods	500	17.420290	-0.139797	-0.144316	-0.140149	-0.142493	-0.138571	4377
	1000	17.407128	-0.142647	-0.141150	-0.143271	-0.145136	-0.141313	35110
	5000	17.422493	-0.142752	-0.143105	-0.144719	-0.143523	-0.144160	850453

Table 5. American put on the minimum of 5 assets: price/deltas by PM method. "True" reference price: 17.225366.

Table 6 gives the performance of the BM method (standard Barraquand-Martineau: no control variate and finite differences for the deltas) for time periods $n = 50, 70, 90$ and Monte Carlo simulations $N_{\mathrm{mc}} = 20000, 50000, 100000, 200000, 500000$. As for $d = 2$, this method works good enough for the pricing but it gives unsatisfactory values for the delta.
Finally, we show in Table 7 prices and deltas produced by the BM-M method (BM algorithm with control variable evaluated by BM itself plus Malliavin correction). Numerical results confirm the efficacy of the procedure: the dispersion of the prices is small and the deltas are extremely stable. Moreover, the BM-M algorithm with 90 time discretization steps and 500.000 Monte Carlo trials runs in a reasonable time (the CPU time is 236.95 sec.), giving the price equal to 17.21464 and the deltas ranging between -0.146917 and -0.147889. Table 7 then confirms the precision of the method both for pricing and hedging purposes.
The conclusion we can derive from our numerical tests is that the BM-M method gives a dispersion which is much lower than the one get from the BM and PM methods. As an example, next tables show the 95% confidence intervals for the deltas arising from our experiments in dimension $d = 5$.

	N_{mc}	BM-$\bar{P}$	FD-$\bar{\Delta}_1$	FD-$\bar{\Delta}_2$	FD-$\bar{\Delta}_3$	FD-$\bar{\Delta}_4$	FD-$\bar{\Delta}_5$	CPU
	20000	17.460207	-0.159696	-0.226632	-0.158597	-0.149758	-0.228941	5
50 time	50000	17.369041	-0.142873	-0.145968	-0.104369	-0.139240	-0.124434	13
periods	100000	17.339862	-0.146710	-0.143540	-0.138880	-0.143685	-0.145569	26
	200000	17.298433	-0.146871	-0.158427	-0.155457	-0.143626	-0.148808	53
	500000	17.260427	-0.144222	-0.144012	-0.147445	-0.149417	-0.152117	131
	20000	17.489552	-0.103888	-0.156574	-0.171656	-0.108386	-0.129727	7
70 time	50000	17.386159	-0.110242	-0.130498	-0.138644	-0.132104	-0.134197	18
periods	100000	17.356226	-0.145139	-0.135330	-0.148800	-0.140391	-0.136315	36
	200000	17.318434	-0.153479	-0.154239	-0.154302	-0.158471	-0.144490	72
	500000	17.271725	-0.143741	-0.151927	-0.143667	-0.155565	-0.145461	179
	20000	17.494507	-0.122205	-0.116665	-0.139285	-0.157534	-0.133938	10
90 time	50000	17.400365	-0.122296	-0.145769	-0.128461	-0.130128	-0.146100	23
periods	100000	17.370721	-0.157246	-0.156728	-0.140088	-0.128637	-0.147778	47
	200000	17.328868	-0.146533	-0.144423	-0.150003	-0.151428	-0.150963	92
	500000	17.284830	-0.150170	-0.147086	-0.149053	-0.140551	-0.144029	230

Table 6. American put on the minimum of 5 assets: price/deltas by BM method. "True" reference price: 17.225366.

	N_{mc}	BM-$\bar{P}$	EMall-$\bar{\Delta}_1$	EMall-$\bar{\Delta}_2$	EMall-$\bar{\Delta}_3$	EMall-$\bar{\Delta}_4$	EMall-$\bar{\Delta}_5$	CPU
	20000	17.211546	-0.147306	-0.148566	-0.148212	-0.147856	-0.148170	6
50 time	50000	17.163299	-0.148192	-0.147771	-0.148391	-0.148597	-0.148087	13
periods	100000	17.141235	-0.148440	-0.148012	-0.148331	-0.148281	-0.147619	26
	200000	17.137925	-0.148454	-0.148048	-0.148370	-0.147960	-0.147591	52
	500000	17.138058	-0.148390	-0.148135	-0.148365	-0.148194	-0.147606	130
	20000	17.277483	-0.148763	-0.147155	-0.147888	-0.147014	-0.147206	8
70 time	50000	17.214609	-0.147278	-0.148236	-0.147995	-0.148457	-0.147413	18
periods	100000	17.193022	-0.148118	-0.147849	-0.148023	-0.147812	-0.147700	37
	200000	17.192147	-0.148143	-0.147818	-0.147840	-0.147997	-0.147291	72
	500000	17.182889	-0.148200	-0.147724	-0.148082	-0.147909	-0.147140	180
	20000	17.299891	-0.149521	-0.148774	-0.148431	-0.147822	-0.147367	10
90 time	50000	17.246921	-0.148203	-0.147783	-0.147231	-0.148884	-0.147334	24
periods	100000	17.227334	-0.147817	-0.147333	-0.147173	-0.147890	-0.147076	48
	200000	17.221100	-0.148092	-0.147787	-0.147871	-0.147587	-0.147059	95
	500000	17.214648	-0.147889	-0.147623	-0.147795	-0.147670	-0.146917	237

Table 7. American put on the minimum of 5 assets: price/deltas by BM method. "True" reference price: 17.225366.

Table 8 reports the deltas and the 95% confidence interval computed by launching 1000 times both the BM-M and the BM algorithm. A comparison between the confidence intervals confirms that the use of the Malliavin approach improves the results.

Concerning the error from the PM method, the slowness of the algorithm does not allow to run it several times. Therefore, in Table 9 we report the 95% confidence interval evaluated on one launch. In terms of the statistical error (which is here only roughly computed), it seems that PM method behaves better than the BM one, but again the BM-M method gives the best performances.

	BM-M method	BM method
$\bar{\Delta}_1$	-0.147413	-0.149163
95% CI	$[-0.147439, -0.147386]$	$[-0.184138, -0.114189]$
$\bar{\Delta}_2$	-0.147406	-0.150679
95% CI	$[-0.147432, -0.147379]$	$[-0.185974, -0.115384]$
$\bar{\Delta}_3$	-0.147422	-0.152518
95% CI	$[-0.147449, -0.147395]$	$[-0.188166, -0.116870]$
$\bar{\Delta}_4$	-0.147432	-0.128041
95% CI	$[-0.147459, -0.147406]$	$[-0.163129, -0.092953]$
$\bar{\Delta}_5$	-0.147439	-0.127282
95% CI	$[-0.147562, -0.147315]$	$[-0.162377, -0.092188]$

Table 8. Deltas and their 95% confidence interval from 1000 runs of the BM-M and BM algorithm in dimension $d = 5$, with $n = 70$ time periods and $N_{\mathrm{mc}} = 50000$ trials.

	PM method
$\bar{\Delta}_1$	-0.146578
95% CI	$[-0.148009, -0.145147]$
$\bar{\Delta}_2$	-0.144713
95% CI	$[-0.146126, -0.143210]$
$\bar{\Delta}_3$	-0.146462
95% CI	$[-0.147932, -0.144992]$
$\bar{\Delta}_4$	-0.141829
95% CI	$[-0, 143859, -0.139799]$
$\bar{\Delta}_5$	-0.151819
95% CI	$[-0.154455, -0.149183]$

Table 9. Price, deltas and their 95% confidence interval from the PM algorithm in dimension $d = 5$, with $n = 3$ time periods and $N_{\mathrm{mc}} = 5000$ trials.

6.2 American relative digital option on 2 assets

In this section, we present the numerical results of the different proposed algorithms in a case of singular payoff. This is done in order to empirically study the sensitivity of the methods with respect to the "regularity" of the considered payoff function. In fact, it is known that in the competition between the Malliavin Monte Carlo method and the finite differences one, the results becomes dramatically better for the Malliavin Monte Carlo one if a singular payoff is taken into account. Therefore, we test here whether the Malliavin correction in the Barraquand and Martineau algorithm brings to good results even in the case of a singular payoff.

In particular, we consider a relative digital option on two stocks. This option pays one unit if the price of stock 1 at maturity is higher than the price of stock 2. Then, its payoff function is given by

$$\Phi(x_1, x_2) = \begin{cases} 1 \text{ if } \quad x_1 \geq x_2, \\ 0 \text{ otherwise} \end{cases}$$

Explicit solutions are immediate to write down in the European case. In the American one, the low dimensionality makes the deterministic methods very accurate, giving results which can be taken as good benchmarks. Here, the "true" reference American relative digital price and deltas are the ones issued from the Villeneuve-Zanette finite difference algorithm [15].
In our experiments, we consider different initial values $x_1 = 100$ and $x_2 = 110$, the other parameters being unchanged (that is, null continuous dividend rates, no correlation, volatilities $\sigma_1 = \sigma_2 = 0.2$ and risk-free interest rate $r = 0.05$). As a comparison, let us report that the associated European relative digital option price is equal to 0.350118 and the deltas are 0.0126763 and -0.0115239 respectively.
Table 10 shows prices and deltas obtained with the PM method (pure Malliavin, with control variable equal to the European price evaluated in closed form), with varying time periods $n = 10, 20, 50$ and Monte Carlo trials $N_{\mathrm{mc}} = 500, 1000, 5000, 10000, 20000$. Notice that the use of the control variate and, obviously, the localization techniques give prices and deltas numerically converging to the "true" reference ones, even if slower and with more dispersion than in the previous case of a regular payoff.

	N_{mc}	PM-$\bar{P}$	VZ-P	PM-$\bar{\Delta}_1$	PM-$\bar{\Delta}_2$	VZ-$\Delta_{1,2}$	CPU
	500	0.594854		0.018308	-0.015955		5
10 time	1000	0.594938		0.021187	-0.015651		19
periods	5000	0.599335		0.019076	-0.016894		484
	10000	0.599038		0.019368	-0.017024		1914
	20000	0.597728		0.019409	-0.017372		7890
	500	0.641910		0.025070	-0.020798		9
20 time	1000	0.638146		0.025047	-0.021920		35
periods	5000	0.639151	0.711646	0.023675	-0.021207	0.026859	138
	10000	0.638669		0.023228	-0.020714	-0.024417	555
	20000	0.635722		0.022973	-0.020185		2215
	500	0.674533		0.030608	-0.017411		20
50 time	1000	0.672565		0.027051	-0.023245		82
periods	5000	0.672784		0.026926	-0.021965		330
	10000	0.672986		0.026567	-0.022705		1328
	20000	0.669231		0.026119	-0.023095		5322

Table 10. American relative digital option on 2 assets: price/deltas by PM method.

Table 11 gives the performance of the BM method (standard Barraquand-Martineau: no control variate and finite differences for the deltas), for varying time periods $n = 10, 20, 50$ and Monte Carlo simulations $N_{\mathrm{mc}} = 20000, 50000, 100000, 200000, 500000$. This procedure produces values for the prices and deltas which are unsatisfactory, a fact suggesting that the Barraquand-Martineau procedure is much more sensible than the PM algorithm to the singularity of the payoff.
In Table 12, prices and deltas are from the BM-M algorithm (Barraquand-Martineau plus Malliavin correction, with control variable equal to the European price evaluated by BM itself), with a number n of time periods and

	N_{mc}	BM-$\bar{P}$	VZ-P	FD-$\bar{\Delta}_1$	FD-$\bar{\Delta}_2$	VZ-$\Delta_{1,2}$	CPU
	20000	0.678769		0.022744	-0.020716		0.28
10 time	50000	0.676988		0.024251	-0.022144		0.68
periods	100000	0.673486		0.023393	-0.020104		1.36
	200000	0.673883		0.024292	-0.022857		2.75
	500000	0.674076		0.024429	-0.021171		6.86
	20000	0.773942		0.027558	-0.019171		0.54
20 time	50000	0.775553		0.018872	-0.021744		1.37
periods	100000	0.770021	0.711646	0.022620	-0.021751	0.026859	2.75
	200000	0.772028		0.023883	-0.021757	-0.024417	5.59
	500000	0.771356		0.023362	-0.020675		13.89
	20000	0.886927		0.016470	-0.013306		1.38
50 time	50000	0.885405		0.017462	-0.013811		3.48
periods	100000	0.882437		0.017189	-0.016284		6.99
	200000	0.883786		0.016070	-0.015490		14.07
	500000	0.882814		0.017018	-0.015744		34.79

Table 11. American relative digital option on 2 assets: price/deltas by BM method.

N_{mc} of Monte Carlo trials as in Table 11. The numerical results confirms the instability of the Barraquand and Martineau method if a singular payoff is taken into account, even if a control variate technique is used.

	N_{mc}	BM-$\bar{P}$	VZ-P	EMall-$\bar{\Delta}_1$	EMall-$\bar{\Delta}_2$	VZ-$\Delta_{1,2}$	CPU
	20000	0.648217		0.021482	-0.020020		0.21
10 time	50000	0.647615		0.022020	-0.020421		0.56
periods	100000	0.647834		0.021832	-0.019955		1.10
	200000	0.648149		0.021893	-0.020103		2.20
	500000	0.648228		0.021825	-0.019969		5.52
	20000	0.757731		0.016761	-0.017507		0.43
20 time	50000	0.760290		0.017234	-0.016148		1.11
periods	100000	0.758649	0.711646	0.017199	-0.016028	0.026859	2.18
	200000	0.760315		0.017744	-0.016152	-0.024417	4.32
	500000	0.759523		0.017648	-0.016039		10.89
	20000	0.878817		0.010585	-0.010919		1.08
50 time	50000	0.878403		0.014053	-0.011941		2.73
periods	100000	0.879533		0.013480	-0.012766		5.43
	200000	0.880806		0.013338	-0.011834		10.85
	500000	0.879467		0.013271	-0.011778		27.03

Table 12. American relative digital option on 2 assets: price/deltas by BM-M method.

As a surprisingly conclusion, the Barraquand and Martineau algorithms do not seem to converge, so that the pure Malliavin method turns out to be the appropriate one in the case of a relative digital option. But, in addition to the singularity of the payoff, there is another aspect which can explain the poor behavior of the approaches inspired by Barraquand and Martineau. In fact, in these cases the exercise date is assumed to optimize with respect to the knowledge of the payoff values $\{\Phi(X_u); u \leq t\}$ (and not the stock values $\{X_u; u \leq t\}$). When dealing with a digital option, this procedure may bring to loose a lot of knowledge.

7 Conclusions

In this paper we give new insight into two Monte Carlo numerical methods (PM and BM-M) to compute price and deltas of American options, and we manage numerical experience on their speed precision efficiency, in the context of a regular payoff (multidimensional put on the minimum) and in the case of a singular payoff (relative digital option on two assets). In the framework of a regular payoff, the pure Malliavin approach has high precision and efficacy in pricing and hedging American options in "low dimension", while the standard Barraquand-Martineau methods works better in high dimension for the pricing, although it gives unstable deltas. Conversely, the Barraquand-Martineau algorithm seems to be satisfactory for pricing and hedging if a control variable and the Malliavin correction are taken into account. In the case of a singular payoff, the pricing stability of Barraquand-Martineau algorithm dramatically decays, involving the Malliavin correction for the delta as well, while the pure Malliavin approach turns out to have good robustness.

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