Pricing American options in the Heston model: an hybrid tree-finite difference approach.

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Abstract

We propose an hybrid tree-finite difference method which permits to obtain efficient and accurate European and American option prices in the Heston model. The theoretical convergence to the Heston model is proved and numerical results are provided, showing the reliability and the efficiency of the algorithm.

Keywords: Heston model; stochastic volatility; American options; tree methods; finite difference.

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1 Introduction

Black-Scholes model was the most popular model for derivative pricing and hedging, although it has shown several problems with capturing dramatic moves in financial markets. In fact, the assumption of a constant volatility in the Black-Scholes model over the lifetime of the derivative is not realistic. As an alternative to the Black-Scholes model, stochastic volatility models emerged. The Heston model [17] is perhaps the most popular stochastic volatility model, allowing one to obtain closed-formulae in the European case using Fourier transform. In the American case the main algorithms are tree methods, Fourier-cosine methods and finite difference methods. Approximating trees for the Heston model have been considered in different papers, see e.g. [22], [12], [13], [18], [15]. The tree approach of Vellekoop and Nieuwenhuis [33] actually provides at our knowledge the best tree procedure in the literature. They use an approach which is based on a modification of an explicitly defined stock price tree where the number of nodes grows quadratically in the number of time steps. Fang and Oosterlee [10] use a Fourier-cosine series expansion approach for pricing Bermudan options under the Heston model. The finite difference methods for solving the parabolic partial differential equation

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associated to the option pricing problems can be based on implicit, explicit or alternating
direction implicit schemes. The implicit scheme requires to solve a sparse system at each time
a linear complementarity problem (LCP) and use an implicit finite difference scheme combined
with a multigrid procedure, whereas Forsyth, Vetzal and Zvan [14] use a penalty method. The
explicit scheme is a quick approach although it requires small time steps to retain the stability.
This request brings to a large number of time steps and is not economic in computation. The
ADI schemes are good alternative methods. For example, Hout and Foulon [20] investigate four
splitting schemes of the ADI type for solving the PDE Heston equation: the Douglas scheme, the
Craigh-Sneyd scheme, the Modified Craigh-Sneyd scheme and the Hundsdorfer-Verwer scheme.
Ikonen and Tovainen [19] propose a componentwise splitting method for pricing American
options in the Heston model. The linear complementarity problem associated to the American
option problem is decomposed into a sequence of five one-dimensional LCP’s problems at each
time step. The advantage is that LCP’s need the use of tridiagonal matrices. In Haentjens,
Hout and Foulon [20], the splitting method of Ikonen and Tovainen is combined with ADI
schemes in order to obtain more efficient numerical results.
In this paper we propose a new approach based both on tree and finite difference methods.
Roughly speaking, our method approximates the CIR type volatility process through a tree
approach already studied in Appolloni, Caramellino and Zanette [3], which works very good.
And at each step, we make use of a suitable transformation of the asset price process allowing
one to take care of a new diffusion process with null correlation w.r.t. the volatility process.
Then, by taking into account the conditional behavior with respect to the evolution of the
volatility process, we consider a finite difference method to deal with the evolution of the
(transformed) underlying asset price process.
The paper is organized as follows. In Section 2, we introduce the model, we study in details
the partial differential equation associated to the pricing problem (Section 2.1) and then we
describe the hybrid-finite difference scheme (Section 2.2). The convergence of the approximating
algorithm is studied in Section 3. Section 4 is devoted to numerical results and comparisons.

2 Construction of the method

The Heston model [17] concerns with cases where the volatility $V$ is assumed to be stochastic.
The dynamics under the risk neutral measure of the share price $S$ and the volatility process $V$
are governed by the stochastic differential equation system

$$\frac{dS(t)}{S(t)} = (r - q)dt + \sqrt{V(t)}dZ_S(t),$$
$$dV(t) = \kappa(\theta - V(t))dt + \sigma \sqrt{V(t)}dZ_V(t),$$

with $S(0) = S_0 > 0$ and $V(0) = V_0 > 0$, where $Z_S$ and $Z_V$ are Brownian motions with
correlation coefficient $\rho$: $d\langle Z_S, Z_V \rangle(t) = \rho dt$. Here $r$ is the risk free rate of interest and $q$
the continuous dividend rate. We recall that the dynamics of $V$ follows a CIR process with
mean reversion rate $\kappa$ and long run variance $\theta$. The parameter $\sigma$ is called the volatility of the
volatility.
From now on we set
\[ \rho = \sqrt{1 - \rho^2} \quad \text{and} \quad Z_V = W, \quad Z_S = \rho W + \rho Z, \]
in which \((W, Z)\) denotes a standard 2-dimensional Brownian motion. So, the dynamics can be written as
\[
\begin{align*}
\frac{dS(t)}{S(t)} &= (r - q)dt + \sqrt{V(t)}(\rho dW(t) + \rho dZ(t)), \\
\frac{dV(t)}{V(t)} &= \kappa(\theta - V(t))dt + \sigma \sqrt{V(t)}dW(t).
\end{align*}
\]
\quad (2.1) \quad (2.2)

We consider the diffusion pair \((Y, V)\), where
\[ Y_t = \log S_t - \rho \sigma V_t. \]
\quad (2.3)

One has
\[
\begin{align*}
dY(t) &= \left(r - q - \frac{1}{2} \kappa(\theta - V_t) - \rho \sigma \kappa(\theta - V_t)\right)dt + \rho \sqrt{V(t)}dZ(t), \\
dV(t) &= \kappa(\theta - V(t))dt + \sigma \sqrt{V(t)}dW(t).
\end{align*}
\]
\quad (2.4) \quad (2.5)

(recall that \(W\) and \(Z\) are independent Brownian motions), with
\[ Y_0 = \exp \left(S_0 - \frac{\rho}{\sigma} V_0\right). \]
\quad (2.6)

In the following, we define \(\mu_Y\) and \(\mu_V\) to be the drift coefficient of \(Y_t\) and \(V_t\) respectively, i.e.
\[ \mu_Y(v) = r - q - \frac{1}{2} \kappa(\theta - v) - \frac{\rho}{\sigma} \kappa(\theta - v) \quad \text{and} \quad \mu_V(v) = \kappa(\theta - v). \]
\quad (2.7)

This means that any functional of the pair \((S_t, V_t)\) can be written as a suitable functional of the pair \((Y_t, V_t)\) by using the transformation (2.3), so \((Y_t, V_t)\) will be our underlying process of interest.

2.1 The associated pricing PDE

Let \(f = f(y, v)\) be a function of the time and the space-variable pair \((y, v)\). For \(h\) small, we need to compute (an estimate for) the quantity \(u(t, y, v)\) defined through
\[ u(t, y, v) = \mathbb{E}(f(Y_{t+h}^{t,y,v}, V_{t+h}^{t,v})) \quad \text{for} \quad t = t_n = nh, \]
in which \((Y_{t+h}^{t,y,v}, V_{t+h}^{t,v})\) denotes the solution to (2.4) and (2.5) with starting conditions \((Y_t, V_t) = (y, v)\). So, we first notice that
\[
\mathbb{E}(f(Y_{t+h}^{t,y,v}, V_{t+h}^{t,v})) = \mathbb{E}\left(\mathbb{E}(f(Y_{t+h}^{t,y,v}, V_{t+h}^{t,v}) | \mathcal{F}_{t+h}^W)\right)
\]
where $F_{t+h}^W = \sigma(W_u : u \leq t + h)$. But conditionally to $F_{t+h}^W$, the volatility process $V$ can be considered deterministic and the process $Y$ turns out to have constant coefficients. More precisely, for $g \in L^2([t, t + h])$ with $g \geq 0$ a.e. and $g_t = v$, set
\[
U_{t+h}^{(g), t, y} = y + \int_t^{t+h} \mu_Y(g_s)d\sigma + \bar{\rho} \int_t^{t+h} \sqrt{g_s}dZ(s).
\] (2.8)

Then
\[
\mathbb{E}(f(Y_{t+h}^{t, y, v}, V_{t+h}^{t, v}) | F_{t+h}^W) = \mathbb{E}(f(U_{t+h}^{(g), t, y}, g_{t+h}) | g = v^{t, v}).
\]

We define now
\[
\bar{u}(t, y; g) = \mathbb{E}(f(U_{t+h}^{(g), t, y}, g_{t+h})),
\]
so that
\[
\mathbb{E}(f(Y_{t+h}^{t, y, v}, V_{t+h}^{t, v}) | F_{t+h}^W) = \bar{u}(t, y; V^{t, v})
\]
and therefore,
\[
u(t, y, v) = \mathbb{E}(\bar{u}(t, y; V^{t, v})).
\] (2.9)

Let us now discuss the quantity $\bar{u}(t, y; g)$, for $g$ fixed as required above, that is
\[
g : [t, t + h] \to \mathbb{R}_+ \text{ and } g_t = v.
\] (2.10)

Set $U_{t+h}^{(g), s, y}$ as the solution $U^{(g)}$ at time $t + h$, with starting condition $U_{t}^{(g)} = y$ of the following stochastic differential equation with deterministic (although path dependent) coefficients:
\[
dU_{t}^{(g)} = \mu_Y(g_u)du + \bar{\rho} \sqrt{g_u}dZ_u.
\] (2.11)

Recall that the associated infinitesimal generator is given by
\[
L_{u}^{(g)} = \mu_Y(g_u)\partial_y + \frac{1}{2} \bar{\rho}^2 g_u \partial_{yy}^2.
\] (2.12)

So, we get
\[
\bar{u}(t, y; g) = \bar{u}(s, y; g)|_{s=t}, \quad \text{with} \quad \bar{u}(s, y; g) = \mathbb{E}(f(U_{t+h}^{(g), s, y}, g_{t+h})).
\]

Now, from the Feynman-Kac formula, the function $(s, y) \mapsto \bar{u}(s, y; g)$ solves the parabolic PDE Cauchy problem
\[
\begin{align*}
\partial_s \bar{u}(s, y; g) + L_{u}^{(g)} \bar{u}(s, y; g) &= 0 \quad y \in \mathbb{R}, s \in (t, t + h), \\
\bar{u}(t + h, y; g) &= f(y, g_{t+h}) \quad y \in \mathbb{R}.
\end{align*}
\] (2.13)

Once the problem (2.13) is solved, we can proceed to compute $u(t, y, v)$ by using (2.9). We stress that the fixed path $g$ plays the role of a parameter and the solution to (2.13) depends in general on the whole trajectory of $g$.

We consider now the case $h \approx 0$, so that, by (2.10), $g_s \approx g_t = v$ and $\mu_Y(g_s) \approx \mu_Y(g_t) = \mu_Y(v)$. This numerically brings to replace (2.13) with a PDE problem with constant coefficients. More precisely, we consider the approximation $\hat{u}^h(s, y; v, g_{t+h})$ for $\bar{u}(t, y; g)$ given by the solution to
\[
\begin{align*}
\partial_s \hat{u}^h(s, y; v, g_{t+h}) + L_{u}^{(v)} \hat{u}^h(s, y; v, g_{t+h}) &= 0 \quad y \in \mathbb{R}, s \in (t, t + h), \\
\hat{u}^h(t + h, y; v, g_{t+h}) &= f(y, g_{t+h}) \quad y \in \mathbb{R},
\end{align*}
\] (2.14)
with
\[ L^{(v)} = \mu_Y(v)\partial_y + \frac{1}{2}\rho^2 v\partial_{yy}. \]

Let us remark that the solution to the problem (2.14) actually depends on \( g \) only through \( v = g_t \) (appearing in the coefficients of the second order operator) and \( g_{t+h} \) (appearing in the Cauchy condition), that is why we used the notation \( \hat{u}^h(s, y; v, g_{t+h}) \). In contrast, the function solving (2.13) depends in principle on the whole trajectory \( g \) over the time interval \([t, t+h]\). Now, problem (2.14) can be easily solved by using a finite differences numerical method. Numerical reasonings suggest the use of an implicit approximation (in time) if \( u \) is “far enough” from zero, otherwise we consider an explicit method - details are given in Section 3.1.1 and 3.1.2. This means that we fix a grid \( \mathcal{Y} = \{y_j\}_{j \in \mathbb{Z}} \) splitting the real line and we approximate the solution \( \hat{u}^h(s, y; v, g_{t+h}) \) to (2.14) on the grid \( \mathcal{Y} \) by means of a linear operator (infinite dimensional matrix) \( \Pi^h(v) = (\Pi^h(v)_{i,j})_{i,j \in \mathbb{Z}} \). In other words, we get
\[ \hat{u}^h(s, y_i; v, g_{t+h}) \simeq \sum_{j \in \mathbb{Z}} \Pi^h(v)_{i,j} f(y_j, g_{t+h}), \quad i \in \mathbb{Z}. \]

Now, recalling (2.9) and the fact that \( \hat{u}^h \simeq \check{u}^h \), on the grid \( \mathcal{Y} \) we approximate the function \( u \) through
\[ u(t, y_i, v) \simeq \mathbb{E}(\hat{u}^h(t, y_i; v, V^l_{t+h})) \simeq \sum_{j \in \mathbb{Z}} \Pi^h(v)_{i,j} \mathbb{E}(f(y_j, V^l_{t+h})), \quad i \in \mathbb{Z}. \quad (2.15) \]

We stress that the expectation on the r.h.s. above is now written in terms of the process \( V \) only, and this is the key point of our story because we can now use the tree method in [3]. But we will examine in depth this point in a moment.

In practice, one cannot solve the PDE problem over the whole real line. So, one takes \( l_h \to +\infty \) as \( s \to 0 \) and considers a discretization of the (space) interval \([-l_h + Y_0, Y_0 + l_h]\) in \( 2M_h + 1 \) equally spaced points \( y_j \), \( j \in J_{M_h} = \{-M_h, \ldots, M_h\} \). Then, the grid \( \mathcal{Y}^h = \{y_j\}_{j \in J_{M_h}} \) is finite and the approximation of \( \hat{u}^h_n(nh, y; v, g_{t+h}) \) is done by adding to (2.14) suitable boundary conditions. By calling again \( \Pi^h(v) \) the matrix (now, finite dimensional) giving the solution from the finite differences approach, we still obtain
\[ \mathbb{E}(f(Y^l_{t+h}, V^l_{t+h})) \big|_{y=y_i} \simeq \sum_{j \in J_{M_h}} \Pi^h(v)_{i,j} \mathbb{E}(f(y_j, V^l_{t+h})), \quad i \in J_{M_h}. \quad (2.16) \]

### 2.2 The hybrid tree-finite difference approach

We describe our approximating algorithm by means of an example. Consider an American option with maturity \( T \) and payoff function \( \Phi(S_t) \) \( t \in [0, T] \). First of all, by using (2.3) we replace the pair \((S, V)\) with the pair \((Y, V)\), so the obstacle will be given by
\[ \Psi(Y_t, V_t) = \Phi(e^{Y_t - \frac{\delta}{2} V_t}), \quad t \in [0, T]. \]

The price at time 0 of such an option is then approximated by a backward dynamic programming algorithm, working as follows. First, consider a discretization of the time interval \([0, T]\) into \( N \) subintervals of length \( h = T/N \): \([0, T] = \cup_{n=0}^{N-1}[nh, (n+1)h] \). Then the price \( P(0, Y_0, V_0) \)
such an American option is numerically approximated through the quantity $P_h(0, Y_0, V_0)$ which is iteratively defined as follows: for $(y, v) \in \mathbb{R} \times \mathbb{R}_+$,

\[
\begin{align*}
&P_h(T, y, v) = \Psi(y, v) \quad \text{and as } n = N - 1, \ldots, 0, \\
&P_h(nh, y, v) = \max \left\{ \Psi(y, v), e^{-rh} \mathbb{E} \left( P_h((n+1)h, Y_{(n+1)h}, V_{(n+1)h}) \right) \right\}.
\end{align*}
\]

From the financial point of view, this means to allow the exercise at the fixed times $nh$, $n = 0, \ldots, N$. Now, what we are going to set up is a contamination of a tree method for the process $V$ with a finite difference method to handle the noise in $Y$ (which is independent of the noise driving $V$). In fact, the expectations appearing in the backward induction can be written as expectations of functions of the process $V$ only, such functions being solution to parabolic PDE's. So, we proceed as described in the previous section: we fix a grid $\mathcal{G}$, thus here it appears the binomial tree method in [3], that we briefly recall now.

By resuming, the price $P(0, y, V_0)$ in $y \in \mathcal{Y}^h$ can be numerically computed from the function $\hat{P}_h(0, y, V_0)$ defined on the grid $\mathcal{Y}^h$ as

\[
\begin{align*}
\hat{P}_h(T, y_i, v) &= \Psi(y_i, v), \quad i \in \mathbb{Z} \quad \text{and as } n = N - 1, \ldots, 0, \\
\hat{P}_h(nh, y_i, v) &= \max \left\{ \Psi(y_i, v), e^{-rh} \sum_{j \in J_{Mh}} \Pi^h(v)_{i,j} \mathbb{E} \left( \hat{P}_h((n+1)h, y_j, V_{(n+1)h}) \right) \right\}, \quad i \in \mathcal{J}_{Mh}.
\end{align*}
\]

We stress that the backward induction (2.18) is now written in terms of the process $V$ only, thus here it appears the binomial tree method in [3], that we briefly recall now.

For $n = 0, 1, \ldots, N$, consider the lattice

\[
\mathcal{Y}^h_n = \{v_{n,k}\}_{k=0,1,\ldots,n} \quad \text{with} \quad v_{n,k} = \left( \sqrt{V_0^h + \frac{\sigma^2}{2}(2n - 1)\sqrt{h}} \right)^2 \mathbf{1}_{\sqrt{V_0^h + \frac{\sigma^2}{2}(2n - 1)\sqrt{h}} > 0} \quad (2.19)
\]

(notice that $v_{0,0} = V_0$) and for each fixed $v_{n,k} \in \mathcal{Y}^h_n$, we define

\[
\begin{align*}
&k_d(n, k) = \max\{k^* : 0 \leq k^* \leq k \text{ and } v_{n,k} + \mu_V(v_{n,k})h \geq v_{n+1,k^*}\}, \\
&k_u(n, k) = \min\{k^* : k + 1 \leq k^* \leq n + 1 \text{ and } v_{n,k} + \mu_V(v_{n,k})h \leq v_{n+1,k^*}\}
\end{align*}
\]

(2.20) and (2.21) with the understanding $k_d(n, k) = 0$ if $\{k^* : 0 \leq k^* \leq k \text{ and } v_{n,k} + \mu_V(v_{n,k})h \geq v_{n+1,k^*}\} = \emptyset$ and $k_u(n, k) = n + 1$ if $\{k^* : k + 1 \leq k^* \leq n + 1 \text{ and } v_{n,k} + \mu_V(v_{n,k})h \leq v_{n+1,k^*}\} = \emptyset$. The transition probabilities are defined as follows: starting from the node $(n, k)$ the probability that the process jumps to $k_u(n, k)$ at time-step $n + 1$ is set as

\[
\begin{align*}
\hat{p}^h_{k_u(n, k)} = 0 &\lor \frac{\mu_V(v_{n,k})h + v_{n,k} - v_{n+1,k_u(n, k)}h}{v_{n+1,k_u(n, k)} - v_{n+1,k_u(n, k)}h} \land 1.
\end{align*}
\]
And of course, the jump to \( (n + 1, k_0'(n, k)) \) happens with probability \( p_{k_0'(n, k)}^h = 1 - p_{k_0''(n, k)}^h \).

This gives rise to a Markov chain \( (\tilde{V}^h_n)_{n=0,\ldots,N} \) that weakly converges, as \( h \to 0 \), to the diffusion process \( (V_t)_{t\in[0,T]} \) and turns out to be a robust tree approximation for the CIR process \( V \). This means that we can approximate the expectation of a functional of the diffusion \( V \) with the same expectation but on the Markov chain \( \tilde{V}^h \). In particular, for a function \( g \) we can write

\[
\mathbb{E}(g(\tilde{V}_{(n+1)h}^h)) \simeq \mathbb{E}(g(\tilde{V}_{n}^h | \tilde{V}^h_{n+1} = v_{n,k})) = g(v_{n+1,k_0'(n, k)}p_{k_0'(n, k)}^h + g(v_{n+1,k_0''(n, k)})p_{k_0''(n, k)}^h).
\]

So, at step \( n \), we can numerically compute the expectation in the backward induction (2.18) on the lattice \( Y^h_n \) as

\[
\mathbb{E}(\hat{P}_h((n+1)h, y_j, V_{(n+1)h}^h)) |_{y=v_{n,k}} \simeq \sum_{k^* \in \{k_0'(n, k), k_0''(n, k)\}} \hat{P}_h((n+1)h, y_j, v_{n+1,k^*})p_{k^*}^h.
\]

We can finally write the backward induction giving our approximating algorithm: for \( n = 0, 1, \ldots, N \), we define \( \hat{P}_h(nh, y, v) \) for \( (y, v) \in Y^h \times V^h_n \) by

\[
\begin{align*}
\hat{P}_h(T, y_i, V_{N,k}) &= \Psi(y_i, V_{N,k}) \quad i \in J_{M_h} \text{ and } v_{N,k} \in V^h_n, \text{ and as } n = N - 1, \ldots, 0 \\
\hat{P}_h(nh, y_i, v_{n,k}) &= \max \left\{ \Psi(y_i, v), e^{-rh} \sum_{k^*, j} \Pi^h(v_{n,k})_{i,j} \hat{P}_h((n+1)h, y_j, v_{n+1,k^*})p_{k^*}^h \right\}, \quad i \in J_{M_h} \text{ and } v_{n,k} \in V^h_n.
\end{align*}
\tag{2.23}
\]

where the sum above is done for \( k^* \in \{k_0'(n, k), k_0''(n, k)\} \) and \( j \in J_{M_h} \). Notice that, at time step \( n \), for every fixed \( i \in J_{M_h} \) and \( k = 0, \ldots, n \) the sum in (2.23) can be seen as an integral w.r.t. the measure

\[
\mu^h(y_i, v_{n,k}; A \times B) = \sum_{k^* \in \{k_0'(n, k), k_0''(n, k)\}} \sum_{j \in J_{M_h}} \Pi^h(v_{n,k})_{i,j}p_{k^*}^h \delta_{\{y_j\}}(A) \delta_{\{v_{n+1,k^*}\}}(B),
\tag{2.24}
\]

\( \delta_{\{a\}} \) denoting the Dirac mass in \( a \). So, \( \mu^h(y_i, v_{n,k}; \cdot) \) is a discrete measure on \( Y^h \times V^h_{n+1} \). Now, if we are able to prove that, at least for small values of \( h \), \( \Pi^h(v) \) is a stochastic matrix, as we will be, we can assert that the expectations are actually done w.r.t. a discrete probability measure giving rise to a Markov chain \( (Y^h_n, V^h_n)_{n=0,1,\ldots,N} \), whose state-space, at time \( n \), is given by \( Y^h \times V^h_n \). In Section 3 we prove that the approximating Markov chain \( (Y^h, V^h) \) really exists (equivalently, the transition kernel \( \Pi^h(v) \) is a stochastic matrix, for every \( v \in \cup_{n=0}^N V^h_n \)) and that it converges to the diffusion process \( (Y, V) \). This gives the convergence of the algorithm approximating the Heston model under appropriate conditions on the divergence to \( \infty \) of \( l_h \) and \( M_h \) as \( h \to 0 \) (see next (3.18) and (3.19)).

### 3 The convergence results

We first set up the finite differences method we take into account. Then, in Section 3.2, we formally define the approximating Markov chain and we prove the convergence to the Heston model.
3.1 The finite difference scheme for the PDE problem (2.14)

As described in Section 2.2, at each time step \( n \) we need to numerically solve problem (2.14). So, we briefly describe the finite differences method we apply to problem (2.14), outlining some key properties of the associated operator allowing us to prove the convergence. For further information on finite difference methods for partial differential equations we refer for instance to [28].

Let \( t = nh, v \) and \( v^* = g_{nh+h} \) be fixed and let us set \( u^n_j = \tilde{u}^h(nh, y_j) \) the discrete solution of (2.14) at time \( nh \) on the point \( y_j \) of the grid \( Y^h \) - for simplicity of notations, we do not stress on \( u^n_j \) the dependence on \( v \) (from the coefficients of the PDE), \( v^* \) (from the Cauchy problem) and \( h \).

It is well known that the behavior of the solution of problem (2.14) changes with respect the magnitude of the rate between the diffusion coefficient \((\rho^2 v/2)\) and the advection term \((\mu_Y(v))\). To deal with these cases, we fix a small real threshold \( \epsilon > 0 \) and in the following we shall describe how to solve both the case \( v < \epsilon \) and \( v > \epsilon \) by applying an explicit in time and an implicit in time approximation respectively.

It is well known that for a big enough diffusion coefficient, to avoid over-restrictive conditions on the grid steps, it is suggested to apply implicit finite differences to problem (2.14). In this case, the discrete solution \( \{u^n_j\}_{j \in \mathbb{Z}} \) at time \( nh \), will then be computed by solving the following discrete problem,

\[
\frac{u^{n+1}_j - u^n_j}{h} + \mu_Y(v) \frac{u^{n+1}_{j+1} - u^{n+1}_{j-1}}{2\Delta y} + \frac{1}{2}\rho^2 v \frac{u^{n+1}_{j+1} - 2u^n_j + u^{n+1}_{j-1}}{\Delta y^2} = 0, \quad j \in \mathbb{Z} \tag{3.1}
\]

where \( \Delta y = y_j - y_{j-1}, \forall j \in \mathbb{Z} \).

On the other hand, when the diffusion coefficient is small compared with the reaction one, it is suggested to apply an explicit in time approximation coupled with a forward or backward finite differences for the first order term \( u_x \) depending on the sign of the reaction coefficient. Specifically, for \( v < \epsilon \) we solve the problem by the following approximation: when \( \mu_Y(v) > 0 \),

\[
\frac{u^{n+1}_j - u^n_j}{h} + \mu_Y(v) \frac{u^{n+1}_{j+1} - u^{n+1}_{j-1}}{\Delta y} + \frac{1}{2}\rho^2 v \frac{u^{n+1}_{j+1} - 2u^n_j + u^{n+1}_{j-1}}{\Delta y^2} = 0, \quad j \in \mathbb{Z}, \tag{3.2}
\]

while, when \( \mu_Y(v) < 0 \),

\[
\frac{u^{n+1}_j - u^n_j}{h} + \mu_Y(v) \frac{u^{n+1}_{j+1} - u^{n+1}_{j-1}}{\Delta y} + \frac{1}{2}\rho^2 v \frac{u^{n+1}_{j+1} - 2u^n_j + u^{n+1}_{j-1}}{\Delta y^2} = 0, \quad j \in \mathbb{Z}. \tag{3.3}
\]

But in this special case, for \( v \) close to zero then \( \mu_Y(v) \simeq \mu_Y(0) = \kappa \theta > 0 \), so here we use (3.2).

As previously mentioned at the end of the Section 2.1, for the numerical tests one has to deal with a finite grid \( Y^h = \{y_j\}_{j \in \mathbb{J}_M} \) (for simplicity, here we simply write \( M \) instead of \( M_h \)) and problems (3.1) and (3.3)-(3.2) have to be coupled with suitable numerical boundary conditions. Here, we shall assume that at each time step \( n \), the two-off the domain values \( u^n_{-M-1} \) and \( u^n_{M+1} \) are defined by the following relations,

\[
u^n_{-M-1} = u^n_{-M+1}, \quad u^n_{M+1} = u^n_{M-1}. \tag{3.4}\]
Other conditions can surely be selected, for example the two boundary values \( u^n_{-M_h} \) and \( u^n_{M_h} \) may be a priori fixed by a known constant (this typically appears in financial problems). All arguments that follow apply.

Hereafter, we shall set
\[
\alpha = \frac{h}{2\Delta y} \mu_Y(v) \quad \text{and} \quad \beta = \frac{h}{2\Delta y^2} \rho^2 v \tag{3.5}
\]
and, for easy of notation we shall omit the dependence on \( h \) on \( M, \ Y \) and \( J_M \).

### 3.1.1 The case \( v > \epsilon \)

By applying implicit finite differences (3.1) coupled with boundary conditions (3.4), we get the solution \( U^n = (u^n_{-M}, \ldots, u^n_{M})^T \) by solving the following linear system:
\[
A U^n = U^{n+1}, \tag{3.6}
\]
where \( A \) is an \((2M+1) \times (2M+1)\) real matrix. \( A \) has the following tridiagonal form
\[
A = \begin{pmatrix}
1 + 2\beta & -2\beta & & & \\
\alpha - \beta & 1 + 2\beta & -\alpha - \beta & & \\
& \ddots & \ddots & \ddots & \\
& & \alpha - \beta & 1 + 2\beta & -\alpha - \beta \\
& & & -2\beta & 1 + 2\beta
\end{pmatrix}. \tag{3.7}
\]

**Proposition 3.1** Assume that \( \beta > |\alpha| \). Then \( A \) is invertible and \( A^{-1} \) is a stochastic matrix, that is all entries are non negative and, for \( 1 = (1, \ldots, 1)^T \), \( A1 = 1 \).

**Proof.** The matrix \( A = (a_{ij})_{i,j \in J_M} \) satisfies

(P1) \( A1 = 1 \), i.e. \( \sum_{j=-M}^M a_{ij} = 1 \) for \( i \in J_M \)

and for \( \beta > |\alpha| \), one has also

(P2) \( a_{ii} > 0 \) for \( i \in J_M \) and for \( j \in J_M, j \neq i, a_{ij} \leq 0 \),

(P3) \( A \) is strict or irreducibly diagonally dominant, i.e. \( \sum_{j=-M, j \neq i}^M |a_{ij}| < a_{ii} \) for \( i \in J_M \).

(P2)-(P3) give that \( A \) is an invertible \( M \)-matrix (see for instance [27]), so that \( A^{-1} \) is non-negative (i.e. \( a^{-1}_{ij} \geq 0, i, j = -M, \ldots, M \)). Moreover, by (P1), \( 1 = A^{-1}1 \). \( \square \)

For each \( l \in \mathbb{N} \), we now consider the polynomial \((y - y_i)^l\) and we call \( \psi_i^l(y) \in \mathbb{R}^{2M+1} \) the associated (vector) function of \( y \in \mathcal{Y}_M \):
\[
(\psi_i^l(y))_k = (y_k - y_i)^l = \Delta y^l (k - i)^l, \quad k = -M, \ldots, M. \tag{3.8}
\]

In next Section 3.2 we need to deal with \( A^{-1}\psi_i^l(y) \) terms for \( l \leq 4 \) and \( i \in J_M \). So, we study such objects. By Proposition 3.1, for \( \beta > |\alpha| \) one has that \( A^{-1} \) is invertible and we may then compute \( A^{-1}\psi_i^l(y) \). We also notice that \( \psi_0^0(y) = 1 \), so that \( A^{-1}\psi_0^0(y) = A^{-1}1 = 1 \).

In the following, the symbol \([\cdot]\) will stand for the floor function and we use the understanding \( \sum_{k=1}^0 (\cdot)_k := 0 \). Moreover, we let \( e_i \) denote the standard orthonormal basis: for \( i, k \in J_M \), \( (e_i)_k = 0 \) for \( k \neq i \) and \( (e_i)_k = 1 \) if \( k = i \).
Lemma 3.2 Let $\psi_i^l(y)$ be defined in (3.8). Then for every $l \in \mathbb{N}$ and $i \in J_M$ one has

$$A\psi_i^l(y) = \psi_i^l(y) - \sum_{j=0}^{l-1} \binom{l}{j} a_{l-j} \Delta y^{l-j} \psi_j^l(y) + b_{l,i}^-M e_{-M} + b_{l,i}^M e_M,$$

where

$$a_n = (\beta - \alpha)(-1)^n + (\beta + \alpha), \quad n \in \mathbb{N},$$

that is $a_n = 2\beta$ if $n$ is even and $a_n = 2\alpha$ if $n$ is odd, and the coefficients $b_{l,i}^M(y)$ are given by

$$b_{l,i}^M(y) = \pm 2 \sum_{j=0, l-j \text{ odd}}^{l-1} \binom{l}{j} (\beta \pm \alpha) \Delta y^{l-j}(y_{\pm M} - y_i)^j. \quad (3.11)$$

Moreover, $b_{l,i}^M(y)$ can be bounded as follows:

$$|b_{l,i}^M(y)| \leq 2(\beta \pm \alpha)(\Delta y + |y_{\pm M} - y_i|)^l \quad (3.12)$$

Proof. One has for $k = -M + 1, \ldots, M - 1$

$$(A\psi_i^l(y))_k = -(\beta - \alpha)(\psi_i^l(y))_{k-1} + (1 + 2\beta)(\psi_i^l(y))_k - (\beta + \alpha)(\psi_i^l(y))_{k+1}$$

$$= -(\beta - \alpha)(y_k - y_i - \Delta y)^l + (1 + 2\beta)(y_k - y_i)^l - (\beta + \alpha)(y_k - y_i + \Delta y)^l$$

$$= (1 + 2\beta)(y_k - y_i)^l - \sum_{n=0}^{l} \binom{l}{n} (y_k - y_i)^{n-l}((\beta - \alpha)(-\Delta y)^n + (\beta + \alpha)\Delta y^n)$$

$$= (1 + 2\beta)(y_k - y_i)^l - \sum_{n=0}^{l} \binom{l}{n} (y_k - y_i)^{n-l}((\beta - \alpha)(-1)^n + (\beta + \alpha))\Delta y^n$$

$$= (y_k - y_i)^l - \sum_{n=1}^{l} \binom{l}{n} (y_k - y_i)^{n-l}((\beta - \alpha)(-1)^n + (\beta + \alpha))\Delta y^n$$

$$= (y_k - y_i)^l - \sum_{n=1}^{l-1} \binom{l}{n} (y_k - y_i)^{n-l}a_n \Delta y^n. \quad (3.11)$$

On the other hand, following the same reasoning for $k = -M$ we obtain

$$(A\psi_i^l(y))_1 = (y_1 - y_i)^l - 2\beta \sum_{j=0}^{l-1} \binom{l}{j} \Delta y^{l-j}(y_1 - y_i)^j,$$

and for $k = M,$

$$(A\psi_i^l(y))_M = (y_M - y_i)^l - 2\beta \sum_{j=0}^{l-1} \binom{l}{j} (-\Delta y)^{l-j}(y_M - y_i)^j.$$ 

So, by (3.11) we get the full form (3.9). Finally, the estimate in (3.12) follows by using Newton’s binomial formula. \(\Box\)

We are now ready to characterize the inverse of any polynomial as in (3.8).
Proposition 3.3 Suppose that $\beta > |\alpha|$ and for $l \geq 1$ let $\gamma_{l,k}$, $k = 0,1,\ldots,l-1$, be iteratively (backwardly) defined as follows:

$$
\gamma_{l,k} = \left( \begin{array}{c} l \\ k \end{array} \right) a_{l-k} \Delta y^{l-k} + \sum_{j=k+1}^{l-1} \gamma_{l,j} \left( \begin{array}{c} j \\ k \end{array} \right) a_{j-k} \Delta y^{j-k}, \quad k = l - 1, \ldots, 0,
$$

where the coefficients $a_n$ are given in (3.10). Then

$$
A^{-1} \psi_i^l(y) = \psi_i^l(y) + \sum_{j=0}^{l-1} \gamma_{l,j} \psi_j^l(y) + \phi_{l,M}^i(y),
$$

in which

$$
\phi_{l,M}^i(y) = T_{l,i,M}^- (y) A^{-1} e_{-M} + T_{l,i,M}^+(y) A^{-1} e_M \quad \text{with} \quad T_{l,i,M}^\pm (y) = -b_{+,M}^i(y) - \sum_{j=0}^{l-1} \gamma_{l,j} b_{+,M}^j(y),
$$

the $b_{+,M}^i$ being given in (3.11). Moreover, if $M \Delta y > 1$ and $l(\beta + |\alpha|) \Delta y (1 + \Delta y)^l < 1$, the following estimate holds for $T_{l,i,M}^\pm (y)$: for every $i \in J_M$,

$$
|T_{l,i,M}^\pm (y)| \leq \frac{2(\beta \pm \alpha) \Delta y^l (1 + 2M)^l}{1 - l(\beta + |\alpha|) \Delta y (1 + \Delta y)^l}.
$$

Proof. It is clear that $A^{-1} \psi_i^l(y) = \psi_i^l(y) + \sum_{j=1}^{l-1} \gamma_{l,j} \psi_j^l(y) + \phi_i^l(y)$ if and only if

$$
\psi_i^l(y) = A \psi_i^l(y) + \sum_{j=0}^{l-1} \gamma_{l,j} A \psi_j^l(y) + A \phi_{l,i,M}^l(y).
$$

We call (*) the r.h.s. above. By using Lemma 3.2 and setting

$$
\theta_i^l(y) = b_{-,M}^i(y) e_{-M} + b_{+,M}^i(y) e_M,
$$

one has

$$
(*) = \psi_i^l(y) - \sum_{k=0}^{l-1} \left( \begin{array}{c} l \\ k \end{array} \right) a_{l-k} \Delta y^{l-k} \psi_k^i(y) + \theta_i^l(y) +
$$

$$
+ \sum_{j=0}^{l-1} \gamma_{l,j} \psi_j^l(y) - \sum_{k=0}^{j-1} \left( \begin{array}{c} j \\ k \end{array} \right) a_{j-k} \Delta y^{j-k} \psi_k^i(y) + \theta_j^l(y) + \phi_{l,i,M}^l(y)
$$

$$
= \psi_i^l(y) - \sum_{k=0}^{l-1} \left( \begin{array}{c} l \\ k \end{array} \right) a_{l-k} \Delta y^{l-k} \psi_k^i(y) + \sum_{j=0}^{l-1} \gamma_{l,j} \psi_j^l(y)
$$

$$
- \sum_{j=0}^{l-1} \gamma_{l,j} \sum_{k=0}^{j-1} \left( \begin{array}{c} j \\ k \end{array} \right) a_{j-k} \Delta y^{j-k} \psi_k^i(y) + \theta_j^l(y) + \sum_{j=0}^{l-1} \gamma_{l,j} \theta_j^l(y) + A \phi_{l,i,M}^l(y)
$$

$$
= \psi_i^l(y) + \sum_{k=0}^{l-1} \left( - \left( \begin{array}{c} l \\ k \end{array} \right) a_{l-k} \Delta y^{l-k} + \gamma_{l,k} - \sum_{j=k+1}^{l-1} \gamma_{l,j} \left( \begin{array}{c} j \\ k \end{array} \right) a_{j-k} \Delta y^{j-k} \right) \psi_k^i(y) +
$$

$$
+ \theta_i^l(y) + \sum_{j=0}^{l-1} \gamma_{l,j} \theta_j^l(y) + A \phi_{l,i,M}^l(y).
$$
By the definition of the $\gamma_{j,k}$’s and $\phi_{i,M}^j(y)$’s, each coefficients in the above (first) sum is null and the sum of the last three terms is zero, so that $(\ast) = \psi_i^j(y)$. Let us discuss (3.14). By using (3.12) and the fact that $|y_{\pm,M} - y_i| \leq 2M \Delta y$, we have

$$|T_{l,i,M}^\pm(y)| \leq 2(\beta \pm \alpha)\Delta y^l(1 + 2M)^l \left(1 + \sum_{j=0}^{l-1} |\gamma_{l,j}|\right).$$

Now, from (3.10) it follows that $|a_n| \leq \beta + |\alpha|$ for every $n$ and by inserting in the definition of the $\gamma_{l,j}$’s we obtain

$$\sum_{k=0}^{l-1} |\gamma_{l,k}| \leq (\beta + |\alpha|) \sum_{k=0}^{l-1} \left(\binom{l}{k} \Delta y^{l-k} + \sum_{j=k+1}^{l-1} |\gamma_{l,j}| \binom{j}{k} \Delta y^{j-k}\right)$$

$$= (\beta + |\alpha|) \sum_{k=0}^{l-1} \binom{l}{k} \Delta y^{l-k} + \sum_{j=1}^{l} |\gamma_{l,j}| \sum_{k=0}^{j-1} \binom{j}{k} \Delta y^{j-k}. $$

For $j \leq l$, we use the simple estimate $\sum_{k=0}^{j-1} \binom{j}{k} \Delta y^{j-k} \leq l \Delta y(1 + \Delta y)^l$, and we get

$$\sum_{k=0}^{l-1} |\gamma_{l,k}| \leq (\beta + |\alpha|) \times l \Delta y(1 + \Delta y)^l \left(1 + \sum_{j=1}^{l-1} |\gamma_{l,j}|\right).$$

By assuming that $(\beta + |\alpha|) \times l \Delta y(1 + \Delta y)^l < 1$, we get

$$\sum_{k=0}^{l-1} |\gamma_{l,k}| \leq \frac{(\beta + |\alpha|) \times l \Delta y(1 + \Delta y)^l}{1 - (\beta + |\alpha|) \times l \Delta y(1 + \Delta y)^l}.$$ 

Therefore,

$$|T_{l,i,M}^\pm(y)| \leq 2(\beta \pm \alpha)\Delta y^l(1 + 2M)^l \left(1 + \sum_{j=0}^{l-1} |\gamma_{l,j}|\right) \leq \frac{2(\beta \pm \alpha)\Delta y^l(1 + 2M)^l}{1 - (\beta + |\alpha|) \times l \Delta y(1 + \Delta y)^l}.$$

$\Box$

**Remark 3.4** For further use, we write down explicitly the vector $A^{-1}\psi_i^j(y)$ for $l = 1, 2, 4$. Straightforward computations give the following:

$$A^{-1}\psi_1^i(y) = \psi_1^i(y) + 2\alpha \Delta y \mathbf{1} + \phi_{1,M}^i(y),$$

$$A^{-1}\psi_2^i(y) = \psi_2^i(y) + 4\alpha \Delta y \psi_1^i(y) + 2(\beta + 2\alpha)\Delta y^2 \mathbf{1} + \phi_{2,M}^i(y),$$

$$A^{-1}\psi_3^i(y) = \psi_3^i(y) + 8\alpha \Delta y \psi_2^i(y) + 12(\beta + 4\alpha^2)\Delta y^2 \psi_1^i(y) + 8(\alpha + 12\alpha^2 + 18\alpha \beta)\Delta y^3 \psi_1^i(y) + 2(\beta + 16\alpha^2 + 96\alpha^3 + 12\beta^2 + 192\alpha^3 \beta)\Delta y^4 \mathbf{1} + \phi_{4,M}^i(y).$$
In particular, since \((\psi_i^l(y))_i = 0\) for every \(l \geq 1\), the \(i\)th entry of the above sequences are given by

\[
(A^{-1}\psi_1^l(y))_i = 2\alpha \Delta y + (\phi_{1,M}^l(y))_i, \\
(A^{-1}\psi_2^l(y))_i = 2(\beta + 2\alpha)\Delta y^2 + (\phi_{2,M}^l(y))_i, \\
(A^{-1}\psi_3^l(y))_i = 2(\beta + 16\alpha^2 + 96\alpha^3 + 12\beta^2 + 192\alpha^3\beta)\Delta y^4 + (\phi_{4,M}^l(y))_i.
\]

By replacing the \(\alpha\) and \(\beta\) expressions (3.5), we get the formulas

\[
(A^{-1}\psi_1^l(y))_i = h\mu_Y(v) + (\phi_{1,M}^l(y))_i, \\
(A^{-1}\psi_2^l(y))_i = h\bar{\rho}^2 v + 2h\Delta y\mu_Y(v) + (\phi_{2,M}^l(y))_i, \\
(A^{-1}\psi_3^l(y))_i = h\Delta y^2\bar{\rho}^2 v + 8h^2\Delta y^2\mu_Y(v)^2 + 24h^3\mu_Y(v)^3 + 6h^2\bar{\rho}^4 v^2 + \\
+ 24h^4\rho^4 v \mu_Y(v)^3 + (\phi_{4,M}^l(y))_i.
\] (3.15)

Furthermore, to deal with the numerical boundary conditions, as those given in (3.4), we need to study the behavior of the \(i\)th component of the boundary term \(\phi_{l,M}^i(y)\) in (3.13) as \(i\) is “far from the boundary” and \(l = 1, 2, 4\). Here, we use a quite general result (allowing one to set up different boundary conditions) whose proof is postponed in Appendix A.

**Proposition 3.5** Suppose that \(\beta > |\alpha|\). Let \(l \in \mathbb{N}, i \in \mathcal{J}_M\) and let \(\phi_{l,M}^i(y)\) denote the boundary term in (3.13). Assume that \(M \Delta y > 1\) and \(l(\beta + |\alpha|)(1 + \Delta y)^l < 1\). Then one has

\[
|\phi_{l,M}^i(y)| \leq \frac{4 \Delta y^l(1 + 2M)^l}{1 - l(\beta + |\alpha|)(1 + \Delta y)^l} \times \left(1 - \frac{1}{1 + \beta + |\alpha|}\right)^{M - i + 1}.
\]

**Proof.** Since \(\beta > |\alpha|\), \(A\) is an \(M\)-matrix and we can apply what developed in Appendix A for the matrix \(A\) in (3.7). So, by using Proposition A.1, we get

\[
|(A^{-1}e_M)_i| \leq \frac{1}{\gamma^*} \left(\frac{\beta + \alpha}{\gamma^*}\right)^{M - i},
\]

where \(\gamma^* = \min(\gamma_2, z_+)^*\) with

\[
\gamma_2 = 1 + 2\beta - \frac{2\beta(\beta \alpha)}{1 + 2\beta} \quad \text{and} \quad z_+ = \frac{1 + 2\beta}{2} + \frac{1}{2} \sqrt{(1 + 2\beta)^2 - 4(\beta^2 - \alpha^2)}.
\]

Straightforward computations give \(\gamma^* \geq 1 + \beta + \alpha\), and we get

\[
|(A^{-1}e_M)_i| \leq \frac{1}{1 + \beta + \alpha} \left(\frac{\beta + \alpha}{1 + \beta + \alpha}\right)^{M - i}.
\]

Similarly, we obtain

\[
|(A^{-1}e_{-M})_i| \leq \frac{1}{1 + \beta - \alpha} \left(\frac{\beta - \alpha}{1 + \beta - \alpha}\right)^{M - i}.
\]

We insert now the above estimates and estimate (3.14) in (3.13), and we get

\[
|\phi_{l,M}^i(y)| \leq \left[\left(\frac{\beta - \alpha}{1 + \beta - \alpha}\right)^{M - i + 1} + \left(\frac{\beta + \alpha}{1 + \beta + \alpha}\right)^{M - i + 1}\right] \times \frac{2 \Delta y^l(1 + 2M)^l}{1 - l(\beta + |\alpha|)(1 + \Delta y)^l}.
\]

The statement now follows by observing that

\[
\frac{\beta \pm \alpha}{1 + \beta \pm \alpha} = 1 - \frac{1}{1 + \beta \pm \alpha} \leq 1 - \frac{1}{1 + \beta + |\alpha|}.
\]

\[\square\]
3.1.2 The case $v < \epsilon$

Here we would go through the procedure done in the previous section for the explicit in time approximation. Specifically, we assume that $\mu_Y(v) > 0$ (the same argument can be applied to the case with $\mu_Y(v) < 0$) and we analyze the discrete problem (3.2). We shall then obtain the solution $U^n$ by

$$U^n = C U^{n+1},$$

where

$$C = \begin{pmatrix}
1 - \alpha - 2\beta & \alpha + 2\beta & \beta & 1 - 2\beta - 2\alpha & 2\alpha + \beta \\
\beta & 1 - 2\beta - 2\alpha & 2\alpha + \beta & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
\beta & 1 - 2\beta - 2\alpha & 2\alpha + \beta & \alpha + 2\beta & 1 - \alpha - 2\beta
\end{pmatrix}, \quad (3.16)$$

$\alpha$ and $\beta$ being given in (3.5). The matrix $C$ is a stochastic matrix under the condition

$$1 - 2\beta - 2\alpha > 0,$$

or equivalently

$$\frac{h}{\Delta y} \rho^2 v + \frac{h}{\Delta y} \mu_Y(v) < 1.$$ 

Since $v < \epsilon$, the condition is not too restrictive.

In next Section 3.2 we need to deal with $C \psi_l^i(y)$ for $l \leq 4$ and $i \in \mathbb{Z}$, where the function $\psi_l^i(y) \in \mathbb{R}^Z$ are defined in (3.8). So, for further use, we write down explicitly the vector $C \psi_l^i(y)$ for $l = 1, 2, 4$. Straightforward computations give

$$C \psi_l^i(y) = \psi_l^i(y) + \sum_{j=0}^{l-1} \binom{l}{j} ((-1)^{l-j} \beta + \beta + 2\alpha) \Delta y^{l-j} \psi_j^i(y) + c_i^{-M} e_1 + c_i^M e_M,$$

where

$$c_i^{-M} = -\sum_{j=0}^{l-1} \binom{l}{j} (\beta - \alpha - (-1)^{l-j} \beta) \Delta y^{l-j} (y_M - y_j),$$

$$c_i^M = -\sum_{j=0}^{l-1} \binom{l}{j} ((-1)^{l-j} (\beta + \alpha) - \beta - 2\alpha) \Delta y^{l-j} (y_M - y_j).$$

Then, the following equalities hold

$$C \psi_1^i(y) = \psi_1^i(y) + 2\alpha \Delta y 1 + c_1^{-M} e_1 + c_1^M e_M,$$

$$C \psi_2^i(y) = \psi_2^i(y) + 4\alpha \Delta y \psi_1^i(y) + 2(\alpha + \beta) \Delta y^2 1 + c_2^{-M} e_1 + c_2^M e_M,$$

$$C \psi_4^i(y) = \psi_4^i(y) + 8\alpha \Delta y \psi_3^i(y) + 12(\alpha + \beta) \Delta y^2 \psi_2^i(y) + 8\alpha \Delta y^3 \psi_1^i(y) + 2(\alpha + \beta) \Delta y^4 1 +$$

$$+ c_4^{-M} e_1 + c_4^M e_M.$$

In particular, since $(\psi_l^i(y))_i = 0$ for every $l \geq 1$, assuming $|i| < M$, the $i$th entry of the above sequences are given by

$$(C \psi_1^i(y))_i = h \mu_Y(v),$$

$$(C \psi_2^i(y))_i = h \rho^2 v + h \Delta y \mu_Y(v),$$

$$(C \psi_4^i(y))_i = h \Delta y^2 \rho^2 v + h \Delta y^3 \mu_Y(v).$$

(3.17)
3.2 The associated Markov chain and the convergence of the hybrid algorithm

We denote, as in Section 2.2, \((V_n^h)_{n=0,1,...,N}\), with \(Nh = T\), the Markov chain approximating the volatility process \(V\) introduced in [3]. We recall that the state-space at time-step \(n\) is given by \(\mathcal{V}_n^h\) defined in (2.19). In particular, we observe that there exists \(C_* > 0\) such that

\[
0 \leq v \leq C_* h^{-1} \quad \text{for every} \quad v \in \bigcup_{n=0}^{N} \mathcal{V}_n^h.
\]

We define now the \(Y\)-component of our Markov chain.

First, we set up the dependence on \(h\) for the space-step \(\Delta y\), the number \(M\) giving the points of the grid \(\mathcal{Y}_M\) and the threshold \(\epsilon\) that allows us to use the explicit or the implicit finite differences method: we assume that

\[
\Delta y \equiv \Delta y_h = c_y h^p, \quad M \equiv M_h = c_M h^{-q}, \quad \epsilon \equiv \epsilon_h = c_e h^p
\]

where \(c_y, c_M\) are positive constants and the quantities \(c_e, p, q > 0\) are chosen as follows

\[
p < 1, \quad q > p, \quad c_e > \frac{2 c_y \kappa \theta}{\rho^2}.
\]

First, from (3.18) and (3.19) we immediately have \(l_h = M_h \Delta y_h = O(h^{-q+p}) \to \infty\) as \(h \to 0\), so that \(\mathcal{Y}_M \uparrow \mathbb{R}\) as \(h \to 0\). Moreover, one has

**Proposition 3.6** Let \(\beta = \beta_h\) and \(\alpha = \alpha_h\) be given in (3.5) with the constraints (3.18) and (3.19). Then there exists \(h_0 > 0\) such that for every \(h < h_0\) one has

i) if \(v > \epsilon_h\), then \(\beta_h > |\alpha_h|\);

ii) if \(v \leq \epsilon_h\), then \(1 - 2\beta_h - 2\alpha_h < 1\).

**Proof.** i) One has \(\beta_h = \frac{h_1 - 2p}{2c_y \kappa p} \rho^2 v\) and \(|\alpha_h| \leq \frac{h_1 - 2p}{2c_y \kappa (\theta + v)}\), so \(\beta_h > |\alpha_h|\) if \(\frac{h_1 - 2p}{2c_y \kappa (\theta + v)} > \frac{h_1 - 2p}{2c_y p}\).

But this holds if \(v(\frac{\rho^2}{2c_y} - \frac{h_1 p}{2c_y}) > \frac{h_1 p}{2c_y}\), which in turn is implied by \(v > \epsilon_h\) when \(h\) is small enough.

ii) One has \(2\beta_h + 2\alpha_h \leq \frac{h_1 - 2p}{c_y} \rho^2 c_e h^p + \frac{h_1 - 2p}{c_y} \kappa \theta = h_1^{-p}(\frac{1}{c_y} \rho^2 c_e + \frac{1}{c_y} \kappa \theta) < 1\) for every \(h\) small enough. \(\square\)

Now, Proposition 3.6 ensures that there exists \(h_0 > 0\) such that for every \(h < h_0\) and for every \(v \in \bigcup_{n=0}^{N} \mathcal{V}_n^h\), the matrix \(A^{-1}\) discussed in Section 3.1.1 and the matrix \(C\) discussed in Section 3.1.2 are both well defined and stochastic matrices. So, for \(h\) small \((h < h_0\) as discussed above), we define \(\Pi^h(v)\) as follows:

- if \(v > \epsilon_h\), \(\Pi^h(v)\) is the inverse of the matrix \(A\) in (3.7),

- if \(v \leq \epsilon_h\), \(\Pi^h(v)\) is the matrix \(C\) in (3.16).

As a consequence, we can assert that for every \(v \in \bigcup_{n=0}^{N} \mathcal{V}_n^h\), \(\Pi^h(v) = (\Pi^h(v)_{ij})_{i,j \in \mathcal{J}_M^h}\) is a stochastic matrix. We now define the following transition probability law: at time-step \(n \in \{0, 1, \ldots, N\}\), for \((y_i, v_{n,k}) \in \mathcal{Y} \times \mathcal{V}_n\) we set \(\mu^h(y_i, v_{n,k}; \cdot)\) the law on \(\mathbb{R}^2\) as in (2.24), that is

\[
\mu^h(y_i, v_{n,k}; A \times B) = \sum_{k^* \in \{k^*_1(n,k), k^*_2(n,k)\}} \sum_{y_j \in \mathcal{Y}} \Pi^h(v_{n,k})_{i,j} p^h_{k^*}(y_j)(A)\delta_{\{v_{n+1,k^*}\}}(B).
\]
So, we call $X^h = (X^h_n)_{n=0,1,\ldots,N}$ the 2-dimensional Markov chain having (3.20) as its transition probability law at time-step $n \in \{0, 1, \ldots, N\}$, that is for $(y_i, v_{n,k}) \in Y^h \times V^h_n$ and $(y_j, v_{n+1,k^*}) \in Y^h \times V^h_{n+1}$

$$
\mathbb{P}(X^h_{n+1} = (y_j, v_{n+1,k^*}) \mid X^h_n = (y_i, v_{n,k})) = \begin{cases} 
\Pi(v_{n,k})_{ij} p^h_{k^*_h(n,k)} & \text{if } k^* = k^*_h(n,k) \\
\Pi(v_{n,k})_{ij} p^h_{k^*_2(n,k)} & \text{if } k^* = k^*_d(n,k) \\
0 & \text{otherwise.}
\end{cases}
$$

Since $\sum_j \Pi(v)_{ij} = 1$, one gets that the second component of $X^h$ is a Markov chain itself and it equals, in law, to $V^h$. So, we write $X^h = (Y^h, V^h)$ and for every function $f : \mathbb{R}^2 \to \mathbb{R}$ we have

$$
\mathbb{E}(f(Y^h_{n+1}, V^h_{n+1}) \mid (Y^h_n, V^h_n) = (y_i, v_{n,k})) = \sum_{k^* \in \{k^*_h(n,k), k^*_2(n,k)\}} p^h_{k^*} \sum_{y_j \in Y^h} \Pi^h(v_{n,k})_{ij} f(y_j, v_{n+1,k^*}).
$$

(3.21)

As already discussed in Section 2.2, (3.21) gives that our algorithm is actually given by approximating in law the diffusion pair $X = (Y, V)$ with the Markov chain $X^h = (Y^h, V^h)$. And this means that the convergence of the method relies to the convergence in law of the family of Markov chains $(X^h)_h$ as $h \to 0$. So, we set $X^h = (Y^h, V^h)$ as the first order continuous interpolation of $X^h$, that is

$$
X^h_t = X^h_n + \frac{t - nh}{h} (X^h_{n+1} - X^h_n), \quad t \in [nh, (n+1)h], \quad n = 0, 1, \ldots, N - 1
$$

(3.22)

and we state our result:

**Theorem 3.7** Suppose that (3.18) and (3.19) hold. Then as $h \to 0$, the process $X^h = (Y^h, V^h)$ converges in law on the space of the continuous paths in the time interval $[0,T]$ to the diffusion process $X = (Y, V)$ solution of (2.4)-(2.5).

**Proof.** The idea of the proof is standard, see e.g. Nelson and Ramaswamy [24] or also classical books such as Billingsley [4], Ethier and Kurtz [9] or Stroock and Varadhan [29].

To simplify the notations, let us set

$$
\mathcal{M}^Y_{n,i,k}(h;l) = \mathbb{E}((Y^h_{n+1} - y_i)^l \mid (Y^h_n, V^h_n) = (y_i, v_{n,k})), \quad l = 1, 2, 4,
$$

$$
\mathcal{M}^V_{n,i,k}(h;l) = \mathbb{E}((V^h_{n+1} - v_{n,k})^l \mid (Y^h_n, V^h_n) = (y_i, v_{n,k})), \quad l = 1, 2, 4,
$$

$$
\mathcal{M}^{Y,V}_{n,i,k}(h) = \mathbb{E}((Y^h_{n+1} - y_i)(V^h_{n+1} - v_{n,k}) \mid (Y^h_n, V^h_n) = (y_i, v_{n,k})).
$$

It is clear that $\mathcal{M}^{Y}_{n,i,k}(h;l)$ is the local moment of order $l$ at time $nh$ related to $Y$, as well as $\mathcal{M}^V_{n,i,k}(h;l)$ is related to the component $V$, and $\mathcal{M}^{Y,V}_{n,i,k}(h)$ is the local cross-moment of the two components at the generic time step $n$. So, the proof of the theorem relies in checking that, for fixed $r_*, v_* > 0$ and setting $\Lambda_* = \{(n,i,k) : v_{n,k} \leq v_*, |y_i| \leq r_*\}$, then the following properties i), ii) and iii) hold:
i) (convergence of the local drift)
\[
\lim_{h \to 0} \sup_{(n,i,k) \in \Lambda_*} \frac{1}{h} |M_{n,i,k}^Y(h; 1) - (\mu_Y)_{n,k} h| = 0,
\]
\[
\lim_{h \to 0} \sup_{(n,i,k) \in \Lambda_*} \frac{1}{h} |M_{n,i,k}^V(h; 1) - (\mu_V)_{n,k} h| = 0;
\]
where we have set \((\mu_Y)_{n,k} = \mu_Y(v_{n,k})\) and \((\mu_V)_{n,k} = \mu_V(v_{n,k})\);

ii) (convergence of the local diffusion coefficient)
\[
\lim_{h \to 0} \sup_{(n,i,k) \in \Lambda_*} \frac{1}{h} |M_{n,i,k}^Y(h; 2) - \bar{\rho}^2 v_{n,k} h| = 0,
\]
\[
\lim_{h \to 0} \sup_{(n,i,k) \in \Lambda_*} \frac{1}{h} |M_{n,i,k}^V(h; 2) - \sigma^2 v_{n,k} h| = 0
\]
\[
\lim_{h \to 0} \sup_{(n,i,k) \in \Lambda_*} \frac{1}{h} |M_{n,i,k}^{Y,V}(h)| = 0;
\]

iii) (fast convergence to 0 of the fourth order local moments)
\[
\lim_{h \to 0} \sup_{(n,i,k) \in \Lambda_*} \frac{1}{h} M_{n,i,k}^Y(h; 4) = 0,
\]
\[
\lim_{h \to 0} \sup_{(n,i,k) \in \Lambda_*} \frac{1}{h} M_{n,i,k}^V(h; 4) = 0.
\]

We recall that the \(V\)-component of the 2-dimensional Markov chain is a Markov chain itself and we have
\[
M_{n,i,k}^V(h; l) \equiv M_{n,k}^V(h; l) = \mathbb{E}((V_{n+1}^h - v_{n,k})^l | V_n^h = v_{n,k}), \quad l = 1, 2, 4.
\]
The limits in i), ii) and iii) containing \(M_{n,k}^V(h; l)\) for \(l = 1, 2, 4\), have been already proved in [3] (see Theorem 7 therein), so we prove the validity of the remaining limits.
We set \(\phi^i_j(y)\) the vector in \(\mathbb{R}^{2M_h + 1}\) given by \((\phi^i_j(y))_j = (y_j - y_i)^j, \ j \in J_{M_h}\). From (3.21) we get
\[
M_{n,i,k}^Y(h; l) = \sum_{y_j \in \mathcal{V}_h} \Pi^h(v_{n,k})_{i,j} \psi^i_j(y_j) = (\Pi^h(v_{n,k}) \phi^i_j(y))_i
\]
and we notice that the above quantity has been already discussed in the previous sections. We set \(\Lambda_* = \Lambda_{*,1,h} \cup \Lambda_{*,2,h}\) with
\[
\Lambda_{*,1,h} = \Lambda_* = \{(n, i, k) : \epsilon_h < v_{n,k} \leq v_*, |y_i| \leq \rho_1\},
\]
\[
\Lambda_{*,2,h} = \Lambda_* = \{(n, i, k) : v_{n,k} \leq \epsilon_h, |y_i| \leq \rho_1\}.
\]
For \((n, i, k) \in \Lambda_{*,1,h}\), \(\Pi^h(v_{n,k})\) is the inverse of the matrix in (3.7). So, by using (3.15), we have
\[
(A^{-1} \phi^i_j(y))_i = h(\mu_Y)_{n,k} + (\phi^i_{1,M})_i,
\]
\[
(A^{-1} \phi^i_2(y))_i = h\rho^2 v_{n,k} + 2h \Delta y_i (\mu_Y)_{n,k} + (\phi^i_{2,M})_i,
\]
\[
(A^{-1} \phi^i_3(y))_i = h\Delta y_i^2 \rho^2 v_{n,k} + 8h^2 \Delta y_i^2 (\mu_Y)_{n,k}^2 + 24h^3 (\mu_Y)_{n,k}^3 + 6h^2 \rho^4 v_{n,k}^2 + 24h^4 \rho^6 v_{n,k} (\mu_Y)_{n,k}^3 + (\phi^i_{3,M})_i,
\]
\[
(A^{-1} \phi^i_4(y))_i = 0.
\]
\(\phi_{i,Mh}^l(y)\) being given in (3.13). In Lemma 3.8 below, we prove that for \(l \leq 4\),

\[
\sup_{(n,i) \in \Lambda_{*,1,h}} \frac{1}{h} |(\phi_{i,Mh}^l(y))_i| \to 0 \quad \text{as} \quad h \to 0.
\]

And since \((\mu_Y)_{n,k}\) is bounded on \(\Lambda_*\), the limits in \(i)\), \(ii)\) and \(iii)\) associated to \(M_{n,i,k}^Y(h;l)\), \(l = 1, 2, 4\), hold uniformly in \(\Lambda_{*,1,h}\). We prove the same on the set \(\Lambda_{*,2,h}\). For \((n,i,k) \in \Lambda_{*,2,h}\), the matrix \(\Pi^h(v_{n,k})\) to be taken into account is given by (3.16). Moreover, for \(h\) small enough, we have that if \((n,i,k) \in \Lambda_{*,2,h}\) then \(|i| < M_h\). Therefore, by (3.17) we obtain

\[
\begin{align*}
M_{n,i,k}^Y(h;1) &= h(\mu_Y)_{n,k}, \\
M_{n,i,k}^Y(h;2) &= h\beta^2 v_{n,k} + h\Delta y_h(\mu_Y)_{n,k}, \\
M_{n,i,k}^Y(h;4) &= h\Delta y_h^2 v_{n,k} + h\Delta y_h^3(\mu_Y)_{n,k}
\end{align*}
\]

and again the limits in \(i)\), \(ii)\) and \(iii)\) concerning \(M_{n,i,k}^Y(h;l), l = 1, 2, 4,\) hold uniformly in \(\Lambda_{*,2,h}\).

It remains to study the cross-moment. By using (3.21), it is given by

\[
M_{n,i,k}^{Y,V}(h) = M_{n,i,k}^Y(h;1)M_{n,i,k}^V(h;1)
\]

and the convergence as in \(ii)\) immediately follows from the already proved limits in \(i)\).

In order to conclude, we only need to prove next

**Lemma 3.8** Assume that (3.18) and (3.19) both hold. Let \(v_*, r_* > 0\) and set

\[
\Lambda_{*,1,h} = \Lambda_* = \{(n,i,k) : \epsilon_h < v_{n,k} \leq v_* , |y_i| \leq r_* \}
\]

Then one has

\[
\lim_{h \to 0} \sup_{(n,i) \in \Lambda_{*,1,h}} \frac{1}{h} |(\phi_{i,Mh}^l(y))_i| = 0,
\]

for every \(l \leq 4\), where \(\phi_{i,Mh}^l(y)\) is defined in (3.13) with \(M = M_h\).

**Proof.** We use Proposition 3.5. Under (3.18) and (3.19), for \((n,i,k) \in \Lambda_{*,1,h}\) we have already observed that \(\beta > |\alpha|\), \(\alpha, \beta\) being given in (3.5) and the constraint \(M\Delta y > 1\) is trivially satisfied for \(h\) small. Moreover, on the set \(\Lambda_{*,1,h}\), for \(h\) small enough, there exists \(C > 0\) such that

\[
l(\beta + |\alpha|)\Delta y(1 + \Delta y)^l \leq Ch^{1-p} \leq \frac{1}{2} < 1,
\]

for every \(l \leq 4\). Moreover, \(1 + \beta + |\alpha| \leq Ch^{1-2p}\) gives

\[
1 - \frac{1}{1 + \beta + |\alpha|} \leq 1 - \frac{1}{Ch^{1-2p}}.
\]

Then, by applying Proposition 3.5, there exists \(C > 0\) such that definitely for \(h\) close to 0

\[
\sup_{(n,i,k) \in \Lambda_{*,1,h}} \frac{1}{h} |(\phi_{i,Mh}^l(y))_i| \leq Ch^{-(q-p)l-1} \left(1 - \frac{1}{Ch^{1-2p}}\right)^{2c_Mh^{-q}} \quad \text{for} \quad l \leq 4,
\]

\(c_M\) being the constant in (3.18). Now, passing to the log in the above r.h.s. we have for \(h \approx 0\)

\[
\log \left(h^{-(q-p)l-1} \left(1 - \frac{1}{Ch^{1-2p}}\right)^{2c_Mh^{-q}}\right) \leq -(q-p)l + 1 - \frac{2c_Mh^{-q}}{C} h^{-(q+1-2p)}.
\]

But (3.19) gives \(q > p\), so \(q + 1 - 2p > 1 - p > 0\) and the limit, as \(h \to 0\), of the above r.h.s. is \(-\infty\). And this completes the proof.  \(\Box\)
4 Numerical results

In this section we provide numerical results in order to assess the efficiency and the robustness of our hybrid tree-finite difference method in the case of plain vanilla options and barrier options.

4.1 European and American options

We compare here the performance of the hybrid tree-finite difference algorithm (called HTFD) introduced in Section 2.2 with the tree method of Vellekoop Nieuwenhuis (called VN) in [33] for the computation of European and American options in the Heston model. All the numerical results (except for our method) are obtained by using the Premia software [26]. In the European and American option contracts we are dealing with, we consider the following set of parameters:

Initial price $S_0 = 100$, strike price $K = 100$, maturity $T = 1$, interest rate $r = \log(1.1)$, dividend rate $q = 0$, initial volatility $V_0 = 0.1$, long-mean $\theta = 0.1$, speed of mean-reversion $\kappa = 2$, correlation $\rho = -0.5$. In order to study the numerical robustness of the algorithms, we choose three different values for $\sigma$: we set $\sigma = 0.04, 0.5, 1$. We first consider the case $\sigma = 0.04$, that is $\sigma$ close to zero (which implies that the Heston PDE is convection-dominated in the $V$-direction). Moreover, for $\sigma = 1$, we stress that the Novikov condition $2\kappa\theta \geq \sigma^2$ is not satisfied.

In the (pure) tree method VN, we fix the number of points in the $V$ coordinate as $N_V = 50$, with varying number of time and space steps: $N_t = N_S = 50, 100, 200, 400$.

The numerical study of the hybrid tree-finite difference method HTSD is split in two cases: HTFD1 refers to the (fixed) number of time steps $N_t = 100$ and varying number of space steps $N_S = 50, 100, 200, 400$; we add the situation HTFD2 where the number of time steps is equal to the number of space steps $N_t = N_S = 50, 100, 200, 400$.

Table 1 reports European put option prices. Comparisons are given with a benchmark value obtained using the Heston closed formula $\text{CF}$ in [17].

In Table 2 we provide results for American put option prices. In this case we use a benchmark from the Monte Carlo Longstaff-Schwartz algorithm, called MC-LS, as in [23], with a huge number of Monte Carlo simulation (1 million iterations) which are done by means of the accurate Alfonsi [1] discretization scheme for the CIR process with $M = 100$ discretization time steps and bermudan exercise dates. We recall that the Alfonsi method provides a Monte Carlo weak second-order scheme for the CIR process, without any restriction on its parameters.

Table 3 refers to the computational time cost (in seconds) of the different algorithms for $\sigma = 0.5$ in the European case.

The numerical results show that the hybrid tree-finite difference method is very accurate, reliable and efficient.

4.2 Barrier options

We study here the continuously monitored barrier options case and we compare our hybrid tree-finite difference algorithm with the numerical results of the method of lines provided in Chiarella et al. [6]. So, we consider European and American up-and-out call options with the following set of parameters: $K = 100$, $T = 0.5$, $r = 0.03$, $q = 0.05$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$. The up barrier is $H = 130$. We choose different values for $S_0$: $S_0 = 80, 100, 120$. 


### Table 1: Prices of European put options.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$N_S$</th>
<th>VN</th>
<th>HTFD1</th>
<th>HTFD2</th>
<th>CF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>50</td>
<td>8.040982</td>
<td>7.934492</td>
<td>7.911034</td>
<td></td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>8.021780</td>
<td>7.970437</td>
<td>7.970437</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>8.003938</td>
<td>7.978890</td>
<td>7.983188</td>
<td>7.994716</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>7.984248</td>
<td>7.980984</td>
<td>7.990825</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>50</td>
<td>8.148234</td>
<td>7.758954</td>
<td>7.746533</td>
<td></td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>7.727191</td>
<td>7.804520</td>
<td>7.804520</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>7.813599</td>
<td>7.816749</td>
<td>7.821404</td>
<td>7.8318540</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>7.910909</td>
<td>7.81596</td>
<td>7.827805</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>50</td>
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<td></td>
<td>100</td>
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<tr>
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<td>7.964052</td>
<td>7.22835</td>
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<td></td>
<td>400</td>
<td>6.639931</td>
<td>7.224356</td>
<td>7.233742</td>
<td></td>
</tr>
</tbody>
</table>

We also compare with a benchmark value obtained by using the method of lines, called MOL, with mesh parameters 100, 200, 6400 (see Chiarella et al. [6]).

Table 4 and Table 5 report European and American Up-and-Out option prices respectively.

Table 6 refers to the computational time cost (in seconds) of the various algorithms for the European barrier case.

**Acknowledgment:** The authors want to thank Professor Alexander Ern for his advice.
Table 3: Computational times (in seconds) for European put options for $\sigma = 0.5$.

<table>
<thead>
<tr>
<th>$N_S$</th>
<th>VN</th>
<th>HTFD1</th>
<th>HTDF2</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.11</td>
<td>0.02</td>
<td>0.007</td>
</tr>
<tr>
<td>100</td>
<td>0.42</td>
<td>0.04</td>
<td>0.040</td>
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<tr>
<td>200</td>
<td>1.73</td>
<td>0.08</td>
<td>0.380</td>
</tr>
<tr>
<td>400</td>
<td>7.06</td>
<td>0.16</td>
<td>3.040</td>
</tr>
</tbody>
</table>

Table 4: Prices of European call up-and-out options. Up barrier is $H = 130$. $K = 100$, $T = 0.5$, $r = 0.03$, $q = 0.05$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$.

<table>
<thead>
<tr>
<th>$N_S$</th>
<th>HTFD1</th>
<th>HTFD2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0 = 80$</td>
<td>50</td>
<td>0.913861</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.893484</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>0.895127</td>
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<tr>
<td></td>
<td>400</td>
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<tr>
<td>$S_0 = 100$</td>
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<tr>
<td></td>
<td>100</td>
<td>2.606249</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>2.597363</td>
</tr>
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<td></td>
<td>400</td>
<td>2.603679</td>
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<tr>
<td>$S_0 = 120$</td>
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<tr>
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<tr>
<td></td>
<td>400</td>
<td>1.504755</td>
</tr>
</tbody>
</table>

Table 5: Prices of American call up-and-out options. Up barrier is $H = 130$. $K = 100$, $T = 0.5$, $r = 0.03$, $q = 0.05$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$.

<table>
<thead>
<tr>
<th>$N_S$</th>
<th>HTFD1</th>
<th>HTFD2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0 = 80$</td>
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</tr>
<tr>
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<td>1.369914</td>
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</tr>
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<tr>
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<td>8.274116</td>
</tr>
<tr>
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<td>8.286667</td>
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<tr>
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<td>8.283815</td>
</tr>
<tr>
<td>$S_0 = 120$</td>
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<td>21.943742</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>21.820015</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>21.779648</td>
</tr>
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Table 6: Computational times (in seconds) for European Barrier options.

<table>
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<th>$N_S$</th>
<th>HTFD1</th>
<th>HTDF2</th>
</tr>
</thead>
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<tr>
<td>50</td>
<td>0.007</td>
<td>0.017</td>
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<tr>
<td>100</td>
<td>0.132</td>
<td>0.132</td>
</tr>
<tr>
<td>200</td>
<td>0.284</td>
<td>1.079</td>
</tr>
<tr>
<td>400</td>
<td>0.535</td>
<td>8.901</td>
</tr>
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</table>
We study here the behavior of the solution \( x = (x_1, \ldots, x_N)^T \) of the two following linear systems

\[ Ax = v_1, \quad (A.1) \]

and

\[ Ax = v_N, \quad (A.2) \]

where \( (v_i)_k = 0 \) for \( k \neq i \) and \( (v_i)_i = 1, i = 1, \ldots, N \) and, more in general, \( A \) has the following tridiagonal form

\[
A = \begin{pmatrix}
a_1 & c_1 & & \\
b & a & c & \\
& \ddots & \ddots & \ddots \\
& b & a & c \\
b_N & & a_N \\
\end{pmatrix}. \quad (A.3)
\]

We assume that \( A = (a_{ij})_{1 \leq i, j \leq N} \) satisfies the hypothesis (P2)-(P3) in the proof of Proposition 3.1, that is

(P2) \( a_{ii} > 0 \) for \( i \in \{1, \ldots, N\} \) and for \( i, j \in \{1, \ldots, N\} \) with \( j \neq i \), \( a_{ij} \leq 0 \),

(P3) \( A \) is strict or irreducibly diagonally dominant, i.e. \( \sum_{j=1, j \neq i}^N |a_{ij}| < a_{ii} \) for \( i \in \{1, \ldots, N\} \).

We recall that (P2)-(P3) give that \( A \) is an invertible \( M \)-matrix (see for instance [27]).

**Proposition A.1** Suppose that the matrix \( A \) in (A.3) satisfies (P2)-(P3), i.e.

\[
a, a_1, a_N > 0, \quad b, c \leq 0, \quad c_1, b_N \leq 0, \quad a > |b| + |c|, \quad a_1 > |c_1|, \quad a_N > |b_N|. \quad (A.4)
\]

Assume moreover the following stability conditions on the “boundary” values \( a_1, a_N, c_1 \) and \( b_N \):

\[
\frac{|bc_1|}{a_1} < z_+, \quad \frac{|b_N|}{a_N} < z_+, \quad z_+ = \frac{a + \sqrt{a^2 - 4|bc|}}{2}. \quad (A.5)
\]

Then the solution \( x \) of (A.1) is defined by a sequence \( \{x_k\}_{k=1, \ldots, N} \) of positive terms and there exists a positive value \( \tilde{\gamma}^* > |b| \) such that, for \( k = 2, \ldots, N - 1 \)

\[
x_{N+1-k} \leq x_1 \left( \frac{|b|}{\tilde{\gamma}^*} \right)^{N-k} \quad \text{and} \quad x_N \leq x_1 \frac{|b_N|}{a_N} \left( \frac{|b|}{\tilde{\gamma}^*} \right)^{N-2}. \quad (A.6)
\]

Similarly, for the solution \( x \) of (A.2) it holds \( x_k > 0 \), for all \( k = 1, \ldots, N \) and there exists a positive value \( \gamma^* > |c| \) such that for \( k = 2, \ldots, N - 1 \),

\[
x_k \leq x_N \left( \frac{|c|}{\gamma^*} \right)^{N-k} \quad \text{and} \quad x_1 \leq x_N \frac{|c_1|}{a_1} \left( \frac{|c|}{\gamma^*} \right)^{N-2}. \quad (A.7)
\]
Proof. Let us start by estimating first the solution $x$ of system (A.2). By applying the Thomas algorithm, also known as tridiagonal matrix algorithm \[30\], we may obtain the solution by back substitution,

$$x_N = \frac{1}{\gamma_N}, \quad x_k = \frac{|c|}{\gamma_k} x_{k+1} \text{ for } k = N - 1, \ldots, 2, \quad x_1 = \frac{|c_1|}{\gamma_1} x_2,$$

where the coefficients $\gamma_k$ are also recursively defined by

$$\gamma_1 = a_1, \quad \gamma_2 = a - \frac{|bc_1|}{\gamma_1}, \quad \gamma_k = a - \frac{|bc|}{\gamma_{k-1}}, \quad k = 3, \ldots, N - 1, \quad \gamma_N = a_N - \frac{|b_N c|}{\gamma_{N-1}}.$$

We first show that for $k = 3, \ldots, N - 1$ the sequence $\{\gamma_k\}$ has two strictly positive fixed points. The generating function of the sequence is

$$f(z) = a - \frac{|bc|}{z},$$

and the two fixed points are $z_\pm = (a \pm \sqrt{a^2 - 4|bc|})/2$, which are well defined and strictly positive by condition $|b| + |c| < a$. Moreover, for $z > 0$ $f$ is a concave function and it is easy to check that $z_-$ is an unstable fixed point while $z_+$ is stable. According to (A.5) $z_+ > |bc_1|/a_1$.

Since $\gamma_2 = a - |bc_1|/a_1$ we have that

$$\gamma_2 > z_-. $$

So, starting from $\gamma_2$ the sequence converges to $z_+$ and we have that for $\gamma^* = \min \{\gamma_2, z_+\}$

$$\gamma_k \geq \gamma^*, \quad k = 2, \ldots, N - 1. \quad \text{(A.8)}$$

Furthermore, for the strictly diagonal dominant condition on $A$, we have that $\gamma_N > 0$. Specifically $a > |c| + |b| > |c| + |b| |c_1|/a_1$ implies that $\gamma_2 > |c|$ and $\gamma_2 a_N > |b_N c|$, therefore

$$\gamma_N = a_N - \frac{|b_N c|}{\gamma_{N-1}} > a_N - \frac{|b_N c|}{\gamma_2} > 0.$$

Going back to sequence $\{x_k\}_{i=1,\ldots,N}$, we first notice that since $x_N > 0$ then $x_k > 0$ for all $k = 1, \ldots, N$. Moreover, for (A.8) we obtain (A.7). In fact, for $k = 2, \ldots, N - 1$ we have

$$x_k = \frac{|c|}{\gamma_k} x_{k+1} \leq \frac{|c|}{\gamma^*} x_{k+1} \leq \cdots \leq \left(\frac{|c|}{\gamma^*}\right)^{N-k} x_N$$

and thus

$$x_1 = \frac{|c_1|}{\gamma_1} x_2 \leq \frac{|c_1|}{a_1} \left(\frac{|c|}{\gamma^*}\right)^{N-2} x_N.$$

It remains to verify that $\gamma^* > |c|$. This is a consequence of the strictly diagonal dominant condition on $A$. We have already seen that $\gamma_2 > |c|$; so, it remains to verify the case $\gamma^* = z_+$, i.e.

$$a + \sqrt{a^2 - 4|bc|} > 2|c|.$$
By hypothesis, we have that \( a + \sqrt{a^2 - 4|bc|} > |c| + |b| + \sqrt{a^2 - 4|bc|} \), it is then sufficient to show that \( \sqrt{a^2 - 4|bc|} > |c| - |b| \). If \(|c| - |b| < 0\), it is trivially true. While if \(|c| - |b| > 0\) we get \( a^2 - 4|bc| > c^2 + b^2 - 2|bc| \Rightarrow a^2 > (|c| + |b|)^2 \) that is nothing else than the strictly diagonal dominant condition.

To obtain the estimate (A.6) we introduce the \( N \times N \) matrix \( U \) satisfying

\[
U = \begin{pmatrix}
0 & 0 & 1 \\
& & \\
& & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

Since \( UV_N = v_1 \) and \( UU = I \) (i.e. \( U^{-1} = U \)), to compute (A.1) we use that

\[
Ax = v_1 \iff \tilde{A} \tilde{x} = v_N,
\]

where

\[
\tilde{A} = UAU = \begin{pmatrix}
a_N & b_N \\
c & a & b \\
& & \ddots & \ddots & \ddots \\
& & & c & a & b \\
& & & & c_1 & a_1
\end{pmatrix}.
\]

and

\[
\tilde{x} = Ux = (x_N, x_{N-1}, \ldots, x_1)^T.
\]

So, following the same reasoning as above, we get \( \tilde{\gamma}_2 = a - \frac{|b|c}{a_N} \) and \( \tilde{\gamma}^* = \min(\tilde{\gamma}_2, z^+) > |b| \) such that

\[
\tilde{x}_k \leq \tilde{x}_N \left( \frac{|b|}{\tilde{\gamma}^*} \right)^{N-k}, \quad k = 2, \ldots, N - 1
\]

i.e.,

\[
x_{N+1-k} \leq x_1 \left( \frac{|b|}{\tilde{\gamma}^*} \right)^{N-k}, \quad k = 2, \ldots, N - 1.
\]

□

This results tells us that after a single time step, for \( N \) large enough the value at \( k = N/2 \) for instance, is influenced by the right side numerical boundary condition by a “very small” term

\[
x_{N/2} \leq \frac{1}{a} \left( \frac{|c|}{a} \right)^{N/2},
\]

i.e. if for example \(|c|/a = 1/2\) and \( M = 100 \) then

\[
x_{M/2} < 0.25 \times 10^{-50}.
\]
References


