General Freidlin-Wentzell Large Deviations and positive Diffusions

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Abstract

We prove Freidlin-Wentzell Large Deviation estimates under rather minimal assumptions. This allows to derive Wentzell-Freidlin Large Deviation estimates for diffusions on the positive half line with coefficients that are neither bounded nor Lipschitz continuous, in particular for the CIR and the CEV models.

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1 Introduction

The classical Freidlin-Wentzell Large Deviation estimates are concerned with the asymptotics of a family of Stochastic Differential Equations of the type

$$\begin{aligned} dY_t^\varepsilon &= b_\varepsilon(Y_t^\varepsilon)dt + \varepsilon\sigma_\varepsilon(Y_t^\varepsilon)dB_t \\ Y_0^\varepsilon &= x \end{aligned} \tag{1.1}$$

as $\varepsilon \to 0+$. Usually (see [3] e.g.) they are stated for coefficients $b_\varepsilon \equiv b$ and $\sigma_\varepsilon \equiv \sigma$ non depending of ε and under the requirement for b and σ to be bounded and (globally) Lipschitz continuous.

Recently applications to finance have attracted the attention to the study of models that are based on diffusion processes whose state space is the positive half line. Instances of these situations are e.g. the CEV model

$$dr_t = \alpha(b - r_t)\,dt + \rho\, r_t^\gamma\, dB_t, \quad r_0 > 0 \tag{1.2}$$

where $\frac{1}{2} \le \gamma \le 1$ and in particular the CIR model that corresponds to the case $\gamma = \frac{1}{2}$. The coefficients here are neither bounded nor Lipschitz continuous.

The problem of deriving Freidlin-Wentzell Large Deviations for diffusions of the form (1.2) has been studied in [4], where the case of a diffusion coefficient of the form $\sigma(x) = \rho\sqrt{x}$ and a general drift is taken into account.

In this paper we give two results. First we prove Freidlin-Wentzell estimates for coefficients b_ε σ_ε possibly depending on ε and under assumptions of local boundedness and local Lipschitz continuity. These results are derived using an extension of the classical transfer technique and are not entirely new, as they are based substantially on the work of Priouret [5] and the refinements of [2]. Then, using these extended estimates we obtain Freidlin-Wentzell estimates for positive diffusions including, among others, those of the type (1.2).

To summarize our results for the case of positive diffusions, let X^ε the solution of the SDE with values in $\mathbb{R}^+$

$$dX_t^\varepsilon = b(X_t^\varepsilon)\,dt + \varepsilon\sigma(X_t^\varepsilon)\,dB_t, \quad X_0^\varepsilon = x > 0 \tag{1.3}$$

We make the following assumption:

(A1.1) *a) The diffusion coefficient* $\sigma : [0, +\infty[\to \mathbb{R}^+$ *is Hölder continuous with exponent* $\gamma \geq \frac{1}{2}$*, is locally Lipschitz continuous on* $]0, +\infty[$*, vanishes at* 0 *and has a sublinear growth at* ∞*.*

b) The drift $b : [0, +\infty[\to \mathbb{R}$ *is locally Lipschitz continuous, has a sublinear growth at* ∞ *and* $b(0) > 0$*.*

Remark that Assumption (A1.1) implies the existence of $\delta > 0, \beta > 0$ such that $\sigma(x) \leq \delta\sqrt{x}$ and $b(x) \geq \beta$ respectively in a right neighborhood of 0. (A1.1) ensures that (1.3) has a unique pathwise solution (see Theorem 3.5, chap.XIX [6]). Our main result is

Theorem 1.2 *Let* X^ε *be the solution of (1.3) in the time interval* $[0, T]$ *with* $x > 0$*. Then under (A1.1) it holds*

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}(X^\varepsilon \in F) \leq -\inf_{\psi\in F} J(\psi) \tag{1.4}$$

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}(X^\varepsilon \in G) \geq -\inf_{\psi\in G} J(\psi) \tag{1.5}$$

for every closed set $F \subset \mathscr{C}_x([0, T], \mathbb{R}^+)$ *and open set* $G \subset \mathscr{C}_x([0, T], \mathbb{R}^+)$*, where*

$$J(\psi) = \frac{1}{2}\int_0^T \frac{(\dot{\psi}_t - b(\psi_t))^2}{\sigma(\psi_t)^2}\,dt \tag{1.6}$$

(with the understanding $J(\psi) = +\infty$ *if* ψ *is not differentiable).*

In order to make a comparison with the above mentioned result of [4], we are able to consider a more general class of diffusion coefficients but we need to assume that the starting point x is strictly positive, whereas in [4] the case $x = 0$ was taken into account. Also we need to assume $b(0) > 0$, whereas in [4] the weaker assumption $b(0) \geq 0$ was required.

Remark finally that the technique developed here can possibly be applicable in more general situations (multidimensional diffusions on a wedge e.g.).

The plan of the paper is as follows. In §2 we prove general Freidlin-Wentzell estimates for coefficients that are not necessarily Lipschitz continuous and bounded. They are also allowed to depend on ε. In §3 we apply the results of §2 and prove Freidlin-Wentzell estimates for diffusions on the half-line as stated in Theorem 1.2. Finally, in order to be self-contained, we put some proofs in §4.

2 Large Deviation estimates

In this section we give a proof of the Freidlin-Wentzell Large Deviation estimates with the aim of making the assumptions as weak as possible. The idea which was originally of Azencott [1] reduces to the remark that Ito's mapping, associating the Brownian path to the corresponding path of the solution of a Stochastic Differential Equation, is not, in general, continuous but is regular enough to develop a kind of contraction procedure.

2.1 The main result

For $T > 0$, let $\mathscr{C}^m = \mathscr{C}([0,T], \mathbb{R}^m)$ denote the space of continuous paths on $[0,T]$ taking values in $\mathbb{R}^m$, endowed with the topology of uniform convergence. We set $\mathscr{C}^m_x$ as the closed hyperplane of the paths starting at $x \in \mathbb{R}^m$. Let $\mathscr{H}^m = \mathscr{H}([0,T], \mathbb{R}^m)$ be the subspace of $\mathscr{C}^m_0$ of paths that are absolutely continuous and whose derivative is square integrable on $[0,T]$ and endowed with the Hilbert norm $|\ |_1$, that is

$$|h|_1^2 = \|\dot h_s\|_{L^2}^2 = \int_0^T |\dot h_s|^2\, ds.$$

Let us set, for $h \in \mathscr{C}^m_x$ For $h \in \mathscr{H}^k$ we set

$$I(h) = \begin{cases} \frac12\, |h|_1^2 & \text{if } h \in \mathscr{H}^k \\ +\infty & \text{otherwise} \end{cases}$$

The classical Schilder's theorem [7] states that I is the rate function of the Large Deviation Principle on $\mathscr{C}^k_0$ satisfied by a Brownian motion with a small parameter. Let, for $f \in \mathscr{C}^m$,

$$\|f\|_t = \sup_{0\le s\le t} |f_s|, \quad \|f\| = \|f\|_T, \quad B(f,\rho) = \{g, \|g-f\| \le \rho\}$$

For $\varepsilon > 0$, let $b_\varepsilon : \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma_\varepsilon : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^k$ families of vector and matrix fields respectively. Let B a k-dimensional Brownian motion on some probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_t, \mathrm{P})$ and Y^ε the solution of the Stochastic Differential Equation (SDE) (1.1). Let us consider the following assumption.

(A2.3) *There exist a vector field $b : \mathbb{R}^m \to \mathbb{R}^m$ and a matrix field $\sigma : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^k$ such that*

a) for every $h \in \mathscr{H}^k$ and $x \in \mathbb{R}^m$ the ordinary differential equation

$$\begin{aligned} \dot g_t &= b(g_t) + \sigma(g_t)\dot h_t \\ g_0 &= x \end{aligned} \tag{2.7}$$

has a unique solution on $[0,T]$.

b) Let $S_x(h)$ denote the solution of (2.7). Therefore $S_x : \mathscr{H}^k \to \mathscr{C}^m_x$. For any $a > 0$, the restriction of S_x to the compact set $K_a = \{|h|_1 \le a\}$ is continuous with respect to the uniform norm: for any $\{h_n\}_n \subset K_a$ such that $\|h_n - h\| \to_{n\to\infty} 0$ with $h \in K_a$ then $\|S_x(h_n) - S_x(h)\| \to_{n\to\infty} 0$.

c) (The quasi-continuity property) For every $R > 0, \rho > 0, a > 0, c > 0$ there exist $\varepsilon_0 > 0, \alpha > 0$ such that, if $\varepsilon < \varepsilon_0$,

$$\mathrm{P}\big(\|Y^\varepsilon - g\| > \rho, \|\varepsilon B - h\| \le \alpha\big) \le \mathrm{e}^{-R/\varepsilon^2} \tag{2.8}$$

uniformly for $|h|_1 \le a$ and $|x| \le c$, where $g = S_x(h)$.

Assumption (A2.3) c) means that if the Brownian path is such that $\|\varepsilon B - h\| \le \alpha$, then the corresponding path of the diffusion Y^ε is near the path $g = S_x(h)$ with a probability converging to 1 as $\varepsilon \to 0$ at a high exponential rate. If $b_\varepsilon, \sigma_\varepsilon$ do not depend of ε then (A2.3) c) would be trivially true if Ito's mapping, associating the Brownian path to the corresponding path of the solution of a SDE were continuous, which is the case in some situations, mostly in dimension 1. It can be viewed as a weak continuity property of Ito's mapping. For this property to be true it will be necessary that the coefficients $b_\varepsilon, \sigma_\varepsilon$ converge suitably to b, σ respectively. Then

Theorem 2.4 *Suppose that $b_\varepsilon, \sigma_\varepsilon$ are locally Lipschitz continuous and the SDE (1.1) has a strong solution for every $\varepsilon > 0$. Then if (A2.3) holds, the family $\{Y^\varepsilon\}_\varepsilon$ satisfies a Large Deviation Principle on $\mathscr{C}_x^m$ with inverse speed ε^2 and (good) rate function*

$$\lambda(g) = \inf \big\{I(h); S_x(h) = g\big\},$$

with the understanding $\lambda(g) = +\infty$ if $\big\{I(h); S_x(h) = g\big\}$ is empty. This means that

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}(X^\varepsilon \in F) \le -\inf_{\psi\in F} \lambda(\psi) \tag{2.9}$$

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}(X^\varepsilon \in G) \ge -\inf_{\psi\in G} \lambda(\psi) \tag{2.10}$$

for every closed set $F \subset \mathscr{C}_x([0,T], \mathbb{R}^m)$ and open set $G \subset \mathscr{C}_x([0,T], \mathbb{R}^m)$ and that the level sets of λ are compact.

The idea of exploiting the quasi-continuity properties of Ito's mapping goes back to Azencott [1]. For a proof of Theorem 2.4 at this level of generality one can refer to Priouret [5] or to Baldi and Chaleyat-Maurel [2]. In order to be self contained we give a sketch of the proof in §4.

In the next section we give exlicit conditions on the coefficients $b_\varepsilon, \sigma_\varepsilon$ that ensure that Assumption (A2.3) is satisfied.

2.2 The Large Deviation transfer

Let us first give conditions ensuring that (A2.3) a) and b) hold.

Lemma 2.5 *If b and σ are locally Lipschitz continuous and have a sublinear growth at infinity, then (A2.3) a) and (A2.3) b) hold. Moreover, for every compact set $K \subset \mathbb{R}^m$ and $a > 0$ there exists $H > 0$ such that*

$$\sup_{x\in K} \sup_{h\,:\,|h|_1\le a} \|S_x(h)\| \le H. \tag{2.11}$$

Proof. Existence and uniqueness of the solutions of (2.7) are standard facts. Let us prove (2.11), which will follow from a standard application of Gronwall lemma. Let $C_0 \ge 0$ be such that $|b(x)| \le C_0(1 + |x|)$, $|\sigma(x)| \le C_0(1 + |x|)$. Setting $g = S_x(h)$, we have by the Cauchy-Schwarz inequality

$$|g_t| \le |x| + C_0 \int_0^t (1 + |g_s|)\, ds + C_0 \int_0^t (1 + |g_s|)|\dot h_s|\, ds \le$$
$$\le |x| + C_0\sqrt{T}\Big(\int_0^t (1 + |g_s|)^2\, ds\Big)^{1/2} + C_0 a\Big(\int_0^t (1 + |g_s|)^2\, ds\Big)^{1/2}.$$

Taking the square and denoting by R the radius of a sphere centered at the origin and containing the compact set K,

$$|g_t|^2 \le 2|x|^2 + 2C_0^2(\sqrt{T} + a)^2 \int_0^t (1 + |g_s|)^2\, ds \le$$
$$\le 2R^2 + 4C_0^2 T(\sqrt{T} + a)^2 + 4C_0^2(\sqrt{T} + a)^2 \int_0^t |g_s|^2\, ds$$

so that Gronwall lemma gives the bound

$$|g_t|^2 \le \big(2R^2 + 4C_0^2 T(\sqrt{T}+a)^2\big)\exp\Big(4C_0^2(\sqrt{T}+a)^2 T\Big) := H$$

for every $t \in [0,T]$.

Let us prove (A2.3) b). Let $h_1, h_2 \in K_a = \{|h|_1 \le a\}$ and $g_i = S_x(h_i)$, $i = 1,2$. From (2.11) we have $\|g_i\| \le H$. Recall also that in the ball of radius H and centered at the origin b and σ are bounded (constant M) and Lipschitz continuous (constant L). Then,

$$\begin{gathered} g_1(t) - g_2(t) = \\ = \int_0^t (b(g_1(s)) - b(g_2(s)))\, ds + \int_0^t (\sigma(g_1(s)) - \sigma(g_2(s)))\dot{h}_1(s)\, ds + \\ + \int_0^t \sigma(g_2(s))(\dot{h}_1(s) - \dot{h}_2(s))\, ds \end{gathered} \tag{2.12}$$

By the next Lemma 2.6 if $|h_1|_1 \le a, |h_2|_1 \le a$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|h_1 - h_2\| < \delta$

$$\Big| \int_0^t \sigma(g_2(s))(\dot{h}_1(s) - \dot{h}_2(s))\, ds \Big| < \varepsilon$$

Therefore

$$\begin{gathered} |g_1(t) - g_2(t)| \le \\ \le \varepsilon + L\int_0^t |g_1(s) - g_2(s)|\, ds + L\int_0^t |g_1(s) - g_2(s)| \cdot |\dot{h}_1(s)|\, ds \le \\ \le \varepsilon + L\int_0^t |g_1(s) - g_2(s)|\, ds + L\Big(\int_0^t |g_1(s) - g_2(s)|\, ds\Big)^{1/2}\Big(\int_0^t |\dot{h}_1(s)|\, ds\Big)^{1/2} \le \\ \le \varepsilon + L(\sqrt{T}+a)\Big(\int_0^t |g_1(s) - g_2(s)|^2\, ds\Big)^{1/2} \end{gathered} \tag{2.13}$$

Therefore, if $\|h_1 - h_2\| < \delta$,

$$|g_1(t) - g_2(t)|^2 \le 2\varepsilon^2 + 2L^2(\sqrt{T}+a)^2 \int_0^t |g_1(s) - g_2(s)|^2\, ds \tag{2.14}$$

and by Gronwall lemma

$$|g_1(t) - g_2(t)|^2 \le 2\varepsilon^2 \mathrm{e}^{2L^2(\sqrt{T}+a)^2 t}$$

which allows to conclude.

Lemma 2.6 *Let Ψ be a bounded (constant M) Lipschitz function (constant L) and $h_1, h_2 \in \mathscr{H}^k$, $g_2 = S_x(h_2)$ with $|h_1|_1 \le a$, $|h_2|_1 \le a$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|h_1 - h_2\| < \delta$*

$$\Big| \int_0^t \Psi(g_2(s))(\dot{h}_1(s) - \dot{h}_2(s))\, ds \Big| < \varepsilon$$

Proof. Let us assume first that Ψ is differentiable. As $|\Psi'(x)| \le L$, Ψ being Lipschitz continuous, we have integrating by parts,

$$\Big|\int_0^t \Psi(g_2(s))(\dot h_1(s) - \dot h_2(s))\,ds\Big| =$$

$$= \Big|\Psi(g_2(t))(h_1(t) - h_2(t)) - \int_0^t \frac{d}{ds}\Psi(g_2(s))(h_1(s) - h_2(s))\,ds\Big| \le$$

$$\le M\|h_1 - h_2\| + L\|h_1 - h_2\| \int_0^t |\dot g_2(s)|\,ds \le$$

$$\le \|h_1 - h_2\|\Big(M + L\int_0^t \big|b(g_2(s)) + \sigma(g_2(s))\dot h_2(s)\big|\,ds\Big) \le$$

$$\le \|h_1 - h_2\|\Big(M + LMT + LM\int_0^t |\dot h_2(s)|\,ds\Big) \le$$

$$\le M\|h_1 - h_2\|\big(1 + LT + L\sqrt{T}\,|h_2|_1\big) \le M\|h_1 - h_2\|\big(1 + L(T + a\sqrt{T})\big)$$

and the statement is proved. In general Ψ is not differentiable but it is easy to approximate it with a regular function. Let $\phi \in C^\infty$ such that $\int_{\mathbb{R}^m} \phi = 1$, $\phi(x) = 0$ if $|x| > 1$ and $0 \le \phi \le 1$. For $\eta > 0$, set $\phi_\eta(x) = \frac{1}{\eta^m}\phi(\frac{x}{\eta})$. Then ϕ_η is a mollifier and if

$$\Psi_\eta(x) := \Psi * \phi_\eta(x) = \int_{\mathbb{R}^m} \Psi(z)\phi_\eta(x - z)\,dy = \int_{\mathbb{R}^m} \Psi(x - z)\phi_\eta(z)\,dz\ ,$$

then Ψ_η is differentiable (C^∞, actually). Ψ_η is still Lipschitz continuous with Lipschitz constant equal to L, as

$$|\Psi_\eta(x) - \Psi_\eta(y)| \le \int_{\mathbb{R}^m} |\Psi(x - z) - \Psi(y - z)|\phi_\eta(z)\,dz \le L|x - y| \int_{\mathbb{R}^m} \phi_\eta(z)\,dz = L|x - y|$$

and also bounded with the same bound M as Ψ, so that by the first part of the proof

$$\Big|\int_0^t \Psi_\eta(g_2(s))(\dot h_1(s) - \dot h_2(s))\,ds\Big| \le M\|h_1 - h_2\|\big(1 + L(T + a\sqrt{T})\big)\ . \tag{2.15}$$

Remark that the bound in the right hand side only depends on L and M (and not on η). It is straightforward that $|\Psi_\eta(x) - \Psi(x)| \le L\eta$ so that

$$\Big|\int_0^t (\Psi(g_2(s))(\dot h_1(s) - \dot h_2(s))\,ds - \int_0^t \Psi_\eta(g_2(s))(\dot h_1(s) - \dot h_2(s))\,ds\Big| \le$$

$$\le \int_0^t |\Psi(g_2(s)) - \Psi_\eta(g_2(s))| \cdot |\dot h_1(s) - \dot h_2(s)|\,ds \le L\sqrt{T}\,\eta|h_1 - h_2|_1 \le$$

$$\le 2\eta La\sqrt{T}$$

therefore, by the first part of the proof,

$$\Big|\int_0^t \Psi(g_2(s))(\dot h_1(s) - \dot h_2(s))\,ds\Big| \le M\|h_1 - h_2\|\big(1 + L(T + a\sqrt{T})\big) + 2\eta La\sqrt{T}$$

and η being arbitrary the proof is complete.

□

We now tackle the problem of giving reasonable conditions under which Assumption (A2.3) c) holds. A natural set of hypotheses (we dot claim that it is the only one) is

(A2.7) *b and σ are locally Lipschitz continuous, have a sublinear growth at infinity and are such that*

$$\begin{aligned}\lim_{\varepsilon\to 0+} |b_\varepsilon(y) - b(y)| = 0\\ \lim_{\varepsilon\to 0+} |\sigma_\varepsilon(y) - \sigma(y)| = 0\end{aligned} \tag{2.16}$$

uniformly on compact sets.

We prove now that Assumption (A2.7) implies Assumption (A2.3) c). In order to do this we recall the following result.

Lemma 2.8 *Let $c, c_\varepsilon : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ be vector fields such that*

$$|c_\varepsilon(s,x)| + |c(s,x)| \le \phi(s), \quad 0 \le s \le T \tag{2.17}$$

$$|c(s,y) - c(s,z)| \le \psi(s)|y-z|, \quad 0 \le s \le T \tag{2.18}$$

for some function $\phi \in L^2([0,T])$ and $\psi \in L^1([0,T])$ respectively and such that

$$\lim_{\varepsilon\to 0} \int_0^T \sup_y |c_\varepsilon(s,y) - c(s,y)|\, ds = 0 \tag{2.19}$$

Let σ_ε, σ be $k \times m$ matrix fields such that σ is Lipschitz continuous, bounded (bound M) and such that (2.16) holds uniformly in $y \in \mathbb{R}^m$. Let X^ε, γ the solutions of

$$\begin{aligned}X_t^\varepsilon &= x + \int_0^t c_\varepsilon(s, X_s^\varepsilon)\, ds + \varepsilon \int_0^t \sigma_\varepsilon(s, X_s^\varepsilon)\, dB_s\\ \gamma_t &= x + \int_0^t c(s,\gamma_s)\, ds\end{aligned} \tag{2.20}$$

respectively. Then for every $R > 0$, $\rho > 0$, $a_1 > 0$ there exists $\varepsilon_0 > 0, \alpha > 0$ such that for every $x \in \mathbb{R}^m$ and ϕ, ψ such that

$$\|\phi\|_{L^2} \le a_1, \quad \|\psi\|_{L^1} \le a_1,$$

we have

$$\mathrm{P}\big(\|X^\varepsilon - \gamma\| > \rho, \|\varepsilon B\| \le \alpha\big) \le \mathrm{e}^{-R/\varepsilon^2}$$

for every $\varepsilon < \varepsilon_0$.

Lemma 2.8 is actually Proposition 1.2 in Baldi and Chaleyat-Maurel [2]. In order to be self contained we sketch its proof in §4.

We can now state the following theorem, which is an extension of Theorem III 2.4 of Azencott [1] (see also Priouret [5] or Baldi and Chaleyat-Maurel [2]).

Theorem 2.9 *Under (A2.7), for every $R > 0, \rho > 0, a > 0, c > 0$ there exist $\varepsilon_0 > 0, \alpha > 0$ such that, if $\varepsilon < \varepsilon_0$,*

$$\mathrm{P}\big(\|Y^\varepsilon - g\| > \rho, \|\varepsilon B - h\| \le \alpha\big) \le \mathrm{e}^{-R/\varepsilon^2} \tag{2.21}$$

where $h \in \mathscr{H}^k$ and $g = S_x(h)$, uniformly for $|h|_1 \le a$ and $|x| \le c$. Moreover if b and σ are bounded and the convergence in (2.16) is uniform in y, then (2.8) is uniform in (the starting point) x.

Proof. Let

$$L_\varepsilon = \exp\Big(\frac{1}{\varepsilon}\int_0^T \dot h_s\, dB_s - \frac{1}{2\varepsilon^2}\int_0^T |\dot h_s|^2\, ds\Big)$$

and P^ε the probability on $(\Omega, \mathscr{F}_T)$ having density L_ε with respect to P. By Girsanov's theorem, under P^ε the process $B^\varepsilon_t = B_t - \frac{1}{\varepsilon}\, h_t$ is a Brownian motion for $0 \le t \le T$ and Y^ε solves

$$\begin{aligned} dY^\varepsilon_t &= c_\varepsilon(t, Y^\varepsilon_t)\, dt + \varepsilon\sigma_\varepsilon(Y^\varepsilon_t) dB^\varepsilon_t \\ Y^\varepsilon_0 &= x \end{aligned}$$

where $c_\varepsilon(t,y) = b_\varepsilon(y) + \sigma_\varepsilon(y)\dot h_t$. We set $c(t,y) = b(y) + \sigma(y)\dot h_t$ and suppose, at first, that σ and b are bounded (constant M), Lipschitz continuous (constant L) and that the convergence in (2.16) is uniform. Then

$$\begin{aligned} |c_\varepsilon(s,x)| + |c(s,x)| &\le 2M(1 + |\dot h_s|) \\ |c(s,y) - c(s,z)| &\le L(1 + |\dot h_s|)|y - z| \end{aligned}$$

and

$$\sup_y |c_\varepsilon(s,y) - c(s,y)| \le (1 + |\dot h_s|) \sup_y \{|b_\varepsilon(y) - b(y)| + |\sigma_\varepsilon(y) - \sigma(y)|\}$$

Then the hypotheses of Lemma 2.8 are verified and for every $R' > 0$ there exists $\varepsilon_0 > 0, \alpha > 0$ such that if $\varepsilon < \varepsilon_0$ and

$$A_\varepsilon = \{\|Y^\varepsilon - g\| > \rho\varepsilon, \|\varepsilon B^\varepsilon\| \le \alpha\}$$

then

$$\mathrm{P}^\varepsilon(A_\varepsilon) < \exp\Big(-\frac{R'}{\varepsilon^2}\Big)\ . \tag{2.22}$$

As

$$\begin{aligned} \frac{d\mathrm{P}}{d\mathrm{P}^\varepsilon} = L_\varepsilon^{-1} &= \exp\Big(-\frac{1}{\varepsilon}\int_0^T \dot h_s\, dB_s + \frac{1}{2\varepsilon^2}\int_0^T |\dot h_s|^2\, ds\Big) = \\ &= \exp\Big(-\frac{1}{\varepsilon}\int_0^T \dot h_s\, dB^\varepsilon_s - \frac{1}{2\varepsilon^2}\int_0^T |\dot h_s|^2\, ds\Big) \end{aligned}$$

from Cauchy-Schwarz's inequality

$$\mathrm{P}(A_\varepsilon) = \int_{A_\varepsilon} L_\varepsilon^{-1}\, d\mathrm{P}^\varepsilon \le \mathrm{P}^\varepsilon(A_\varepsilon)^{1/2}\mathrm{E}^\varepsilon[(L_\varepsilon^{-1})^2]^{1/2}, \tag{2.23}$$

E^ε being the expectation under P^ε. Now

$$\begin{aligned} \mathrm{E}^\varepsilon[(L_\varepsilon^{-1})^2] &= \mathrm{E}^\varepsilon\Big[\exp\Big(-\frac{2}{\varepsilon}\int_0^T \dot h_s\, dB_s + \frac{1}{\varepsilon^2}\int_0^T |\dot h_s|^2\, ds\Big)\Big] = \\ &= \underbrace{\mathrm{E}^\varepsilon\Big[\exp\Big(-\frac{2}{\varepsilon}\int_0^T \dot h_s\, dB_s + \frac{2}{\varepsilon^2}\int_0^T |\dot h_s|^2\, ds\Big)\Big]}_{=1} \times \exp\big(\tfrac{1}{\varepsilon}|h|_1^2\big) = \\ &= \exp\big(\tfrac{1}{\varepsilon}|h|_1^2\big) \end{aligned} \tag{2.24}$$

Therefore, for every h with $|h|_1 \le a$,

$$\mathrm{P}(A_\varepsilon) < \exp\Big(-\frac{R' - a^2}{2}\,\frac{1}{\varepsilon^2}\Big)$$

which actually gives (2.8).

It remains to drop the hypotheses of boundedness and global Lipschitz continuity for b and σ and of the uniformity of the convergence in (2.16). This can be done easily using the following localization argument. The idea is very simple: the event $\{\|Y^\varepsilon - g\| > \rho\}$ only depends on the value of the coefficients in a neighborhood of the path g, therefore in a bounded set, where they are Lipschitz continuous and bounded.

Indeed recall that by Lemma 2.5 the set of paths g that solve (2.7) as h varies in $\{|h|_1 \le a\}$ and x in a compact set $K \subset \mathbb{R}^m$, remains inside an open ball of radius H and centered at the origin in $\mathbb{R}^m$. Now let

$$\tilde{b}_\varepsilon(y) = \begin{cases} b_\varepsilon(y) & \text{if } |y| < H + 2\rho \\ b_\varepsilon(\frac{y}{|y|}H) & \text{if } |y| \ge H + 2\rho \end{cases}$$

and in a similar way $\tilde{b}$, $\tilde{\sigma}_\varepsilon$ and $\tilde{\sigma}$. The new coefficients $\tilde{b}$, $\tilde{b}_\varepsilon$ $\tilde{\sigma}_\varepsilon$ and $\tilde{\sigma}$ are obviously bounded, Lipschitz continuous and of course

$$\lim_{\varepsilon\to 0+} |\tilde{b}_\varepsilon(y) - \tilde{b}(y)| = \lim_{\varepsilon\to 0+} |\tilde{\sigma}_\varepsilon(y) - \tilde{\sigma}(y)| = 0$$

uniformly in $y \in \mathbb{R}^m$. Moreover, let $\tilde{Y}^\varepsilon$ and $\tilde{g}$ denote the solutions of

$$\begin{aligned} \tilde{Y}^\varepsilon_t &= x + \int_0^t \tilde{b}_\varepsilon(\tilde{Y}^\varepsilon_s)ds + \int_0^t \varepsilon\tilde{\sigma}_\varepsilon(\tilde{Y}^\varepsilon_s)dB_s \\ \tilde{g}_t &= x + \int_0^t \tilde{b}(\tilde{g}_s)ds + \int_0^t \tilde{\sigma}(\tilde{g}_s)\dot{h}_s ds \end{aligned}$$

respectively. Then of course $\tilde{g} \equiv g$ and, as $\tilde{b}_\varepsilon \equiv b_\varepsilon$ and $\tilde{\sigma}_\varepsilon \equiv \sigma_\varepsilon$ in the ball of radius $H + 2\rho$, Y^ε and $\tilde{Y}^\varepsilon$ coincide up the exit from this ball and

$$\{\|Y^\varepsilon - g\| > \rho\} = \{\|\tilde{Y}^\varepsilon - g\| > \rho\}$$

Therefore,

$$\{\|Y^\varepsilon - g\| > \rho, \|\varepsilon B - h\| \le \delta\} = \{\|\tilde{Y}^\varepsilon - \tilde{g}\| > \rho, \|\varepsilon B - h\| \le \delta\}$$

which concludes the proof.

It should be stressed that in this section we proved Freidlin-Wentzell estimates under rather minimal assumptions. In particular

• it is not necessary for the coefficients to be bounded or globally Lipschitz continuous. Only local Lipschitz continuity is needed. Of course in this case some additional assumption is needed in order to ensure the existence of global solutions. The assumption of sublinear growth for b can be removed in presence of assumptions (of the type of contractivity) ensuring that there is no explosion and that the solutions of the system (2.7) remain inside a compact set as h remains in a bounded set of $\mathscr{H}^k$ and x in a compact set of $\mathbb{R}^m$. Large Deviation estimates can also been obtained in the case of explosion, (see [1] e.g.), but this is beyond the scope of these notes.

• Both coefficients of Y^ε can depend on ε and it is not required for them to enjoy uniform (in ε) properties of regularity or of boundedness or of sublinear growth. They only need to be locally Lipschitz continuous and to ensure that the SDE (1.1) has a unique strong solution for every ε.

3 Pathwise Large Deviations for positive diffusions

In this section we prove Theorem 1.2. Under Assumption (A1.1) it is clear that the rate function

$$\lambda(g) = \inf\{\tfrac{1}{2}\,|h|_1, S_x(h) = g\}$$

is given by (1.6), as $S_x(h) = g$ is equivalent to $\dot h_t = (\dot g_t - b(g_t))/\sigma(g_t)$. Theorem 1.2 is therefore a consequence of Theorem 2.4, as soon as we show that (A2.3) is satisfied.

Proposition 3.10 *Let X^ε the solution of (1.3) and assume that Assumption (A1.1) is satisfied. Then Assumption (A2.3) holds.*

Proof. Suppose that we can prove that there exists a compact set $K \subset]0,+\infty[$ such that for every $h \in H^1([0,T])$ with $|h|_1 \le a$ and for every x in some compact set of $]0,+\infty[$ the corresponding solutions g of (2.7) remain in K. Let $\delta > 0$ such that the neighborhood of radius δ, K^δ, of K is such that $\overline{K^\delta} \subset]0,+\infty[$, then the same localization argument as at the end of the proof of Theorem 2.9 would allow to conclude. We already know (Lemma 2.5) that these solutions remain in a bounded set. We must therefore only prove that there exists $\eta > 0$ such that $g_t \ge \eta$ for every $t \le T$. This is proved in the next Proposition 3.11.

Proposition 3.11 *Under (A1.1) the equation*

$$\begin{aligned} \dot\psi_t &= b(\psi_t) + \sigma(\psi_t)\dot h_t \\ \psi_0 &= x_0 \end{aligned} \tag{3.25}$$

for $h \in H^1[0,T]$, $x_0 > 0$, admits a unique solution for $t \in [0,T]$ for every $T > 0$. Moreover for every compact set $K \subset \mathbb{R}^+$ and $a > 0$ there exists $\eta > 0$ such that $\psi_t \ge \eta$ for every $x_0 \in K$, and $|h|_1 \le a$.

The proof of this proposition follows from the following elementary computations of calculus of variations.

Lemma 3.12 *For any absolutely continuous paths ψ with a square integrable derivative, let*

$$J_T(\psi) = \int_0^T \mathscr{L}(\psi_t, \dot\psi_t)\,dt, \quad \mathscr{L}(\psi, \dot\psi) = \frac{(\dot\psi - b(\psi))^2}{2\sigma(\psi)^2}$$

where b and σ are strictly positive continuous functions on $]0,+\infty[$ ($J_T = +\infty$ possibly). Let

$$V(x) = \inf_{T>0} \inf_{\substack{\psi_0 = x_0 \\ \psi_T = x}} J_T(\psi) \tag{3.26}$$

the associated free time cost function, where $x_0 > 0$ is a fixed starting point. Then for $0 < x < x_0$ it holds

$$V(x) = -2\int_{x_0}^x \frac{b(z)}{\sigma(z)^2}\,dz \tag{3.27}$$

Proof. Let

$$H(x,p) = \sup_{v\in\mathbb{R}}\big(vp - \mathscr{L}(x,v)\big) = \sup_{v\in\mathbb{R}}\Big(vp - \frac{(v-b(x))^2}{2\sigma(x)^2}\Big) \tag{3.28}$$

the associated Hamiltonian. A straightforward computation yields

$$H(x,p) = b(x)p + \frac{1}{2}\,\sigma(x)^2 p^2$$

the supremum in the right hand side of (3.28) being attained at $v^* = \sigma(x)^2 p + b(x)$. This implies that

$$H(x,p) + \mathscr{L}(x,v) \ge vp, \quad \text{for every } v, p \text{ and } x > 0 \tag{3.29}$$

and

$$H(x,p) + \mathscr{L}(x,v) = vp, \quad \text{if } v = v^* = \sigma(x)^2 p + b(x) \tag{3.30}$$

Let us denote by V_1 the right hand side of (3.27). Then it is immediate that, for $x \le x_0$, V_1 is the largest solution of the Hamilton-Jacobi equation

$$\begin{cases} H(x,w') = 0 \\ w(x_0) = 0 \end{cases} \tag{3.31}$$

that is

$$\begin{cases} w'(x)\big(b(x) + \frac{1}{2}\,\sigma(x)^2 w'(x)\big) = 0 \\ w(x_0) = 0 \end{cases} \tag{3.32}$$

Let $0 < x < x_0$ and let us first prove that $V_1(x) \le V(x)$. Let ψ an absolutely continuous path such that $\psi_0 = x_0$, $\psi_T = x$. We can assume that $\psi_t \ne x_0$ for every $t > 0$. Otherwise we replace ψ by the path $\widetilde{\psi}$ defined by $\widetilde{\psi}_t = \psi_{t+t^*}$, t^* being $t^* = \max\{t, \psi_t = x_0\}$. Then the new path $\widetilde{\psi}$ satisfies $\widetilde{\psi}_0 = \psi_{t^*} = x_0$, $\widetilde{\psi}_{T-t^*} = x$, $\widetilde{\psi}_t \ne x_0$ for $t > 0$ and $J_{T-t^*}(\widetilde{\psi}) \le J_T(\psi)$, as the integrand $\mathscr{L}$ is positive. By the same argument we can also assume that $\psi_t \ne x$ for every $t < T$.

As ψ takes its values in the interval $[x, x_0]$ and V_1 is differentiable with bounded derivative in this interval, the function $t \to V_1(\psi_t)$ is absolutely continuous for $t \in [0,T]$ and $\frac{d}{dt}\,V_1(\psi_t) = V_1'(\psi_t)\dot{\psi}_t$. Thanks to (3.29) applied to $x = \psi_s$, $p = V_1'(\psi_s)$, $v = \dot{\psi}_s$,

$$\begin{aligned} V_1(x) &= \int_0^T \frac{d}{ds} V_1(\psi_s)\,ds = \int_0^T V_1'(\psi_s)\dot{\psi}_s\,ds \le \\ &\le \int_0^T \Big(\underbrace{H(\psi_s, V_1'(\psi_s)}_{=0} + \mathscr{L}(\psi_s, \dot{\psi}_s)\Big)\,ds = J_T(\psi) \end{aligned} \tag{3.33}$$

This being true for every absolutely continuous path ψ with a square integrable derivative and such that $\psi_0 = x_0$, $\psi_T = x$ and for every $T > 0$, this proves that

$$V_1(x) \le \inf_{T>0} \inf_{\substack{\psi_0 = x_0 \\ \psi_T = x}} J_T(\psi) = V(x)\ .$$

In order to obtain the opposite inequality, let ξ the solution of

$$\begin{cases} \dot{\xi}_t = -b(\xi_t) \\ \xi_0 = x_0 \end{cases}$$

As b is assumed to be strictly positive, there exists $T > 0$ such that $\xi_T = x$ (recall that $x < x_0$). It is immediate that

$$\sigma(\xi_s)^2 V_1'(\xi_s) + b(\xi_s) = -b(\xi_s) = \dot{\xi}_s$$

so that, thanks to (3.30),

$$V_1'(\xi_s)\dot{\xi}_s = H(\xi_s, V_1'(\xi_s)) + \mathscr{L}(\xi_s, \dot{\xi}_s)\ .$$

The same argument as in (3.33) gives now an equality, that is

$$V_1(x) = J_T(\xi)\ .$$

Therefore $V_1(x) \geq V(x)$ which completes the proof of (3.27).

$\square$

In particular, if $b(x) \equiv \beta > 0$ and $\sigma(x) = \rho\sqrt{x}$, we find

$$V(x) = -2\frac{\beta}{\rho^2}\log\frac{x}{x_0}$$

which implies that for every absolutely continuous path ψ joining x_0 to $x < x_0$ in the time interval $[0,T]$, we have

$$\frac{1}{2}\int_0^T \frac{(\dot\psi_t - \beta)^2}{\rho^2\psi_t}\,dt \geq -2\frac{\beta}{\rho^2}\log\frac{x}{x_0}\ . \tag{3.34}$$

Remark that this quantity diverges as $x \to 0^+$.

Proof of Proposition 3.11. We must prove that the solutions of (3.25) for $|h|_1 \leq a$ stay away from 0. This will be a consequence of Lemma 3.12 (and in particular of (3.34)) and of a comparison argument.

Let $\bar x > 0$ such that $b(x) \geq \beta > 0$ and $\sigma(x) \leq \delta\sqrt{x}$ for some $\beta > 0, \delta > 0$ and for $x \in [0,\bar x]$ (recall Assumption (A1.1)). For a compact set $K \subset]0,+\infty[$, possibly taking a smaller value for $\bar x$, we can assume also that $\bar x < x_0$ for every $x_0 \in K$.

Let ξ, $0 < \xi < \bar x$, such that $-2\frac{\beta}{\delta^2}\log\frac{\bar x}{\xi} > a^2$ and let us prove that if $|h|_1 \leq a$ then $\psi_t > \xi$ for every $t \leq T$. Actually, otherwise, there would exist two times $t_1 < t_2$ such that $\psi_{t_1} = \bar x$ for some $t_1 \leq T$, $\psi_{t_2} = \xi$ and $\psi_t \leq \bar x$ for $t_1 \leq t \leq t_2$ (recall that $\xi \leq \bar x < x_0$). Then

$$|h|_1^2 = \frac{1}{2}\int_0^T \frac{(\dot\psi_t - b(\psi_t))^2}{\sigma(\psi_t)^2}\,dt \geq \frac{1}{2}\int_{t_1}^{t_2} \frac{(\dot\psi_t - b(\psi_t))^2}{\sigma(\psi_t)^2}\,dt\ .$$

As $b(x) \geq \beta > 0$ and $\sigma(x) \leq \delta\sqrt{x}$ for $x \leq \bar x$, by Lemma 3.12,

$$\frac{1}{2}\int_{t_1}^{t_2} \frac{(\dot\psi_t - b(\psi_t))^2}{\sigma(\psi_t)^2}\,dt \geq -2\int_{\bar x}^{\xi} \frac{b(y)}{\sigma(y)^2}\,dy \geq 2\int_{\xi}^{\bar x} \frac{\beta}{\delta^2 y}\,dy = -2\frac{\beta}{\delta^2}\log\frac{\bar x}{\xi} > a^2$$

which is in contradiction with the assumption $|h|_1 \leq a$. This proves also that the solution ψ stays away from 0, so that it has a unique solution, as the coefficients b and σ are assumed to be Lipschitz continuous on $]0,+\infty)$.

4 Appendix: proofs of Theorem 2.4 and Lemma 2.8

Proof of Theorem 2.4. Thanks to Assumption (A2.3) b) in the definition

$$\lambda(g) = \inf\left\{\tfrac{1}{2}\,|h\|_1; S_x(h) = g\right\},$$

the infimum is attained unless $\lambda(g) = +\infty$. Actually if $\lambda(g) = a$, then we have also

$$\lambda(g) = \inf\left\{\tfrac{1}{2}\,|h\|_1; S_x(h) = g, \tfrac{1}{2}\,|h\|_1 \leq a+1\right\},$$

and it suffices now to remark that in the uniform norm the set $\{S_x(h) = g\}$ is closed thanks to (A2.3) b), $\{|h\|_1 \le a+1\}$ is compact and $h \to \frac{1}{2}\,|h\|_1$ is lower semi continuous.

The same argument proves that λ is lower semi-continuous with compact level sets, as $\{\lambda \le a\}$ turns out to be the image of $C_a = \{\frac{1}{2}\,|h\|_1 \le a\}$, which is compact in the uniform norm, through the transformation S_x, whose restriction to C_a is continuous in the uniform norm.

We must prove the lower and upper bounds: if, for every Borel set $A \subset \mathscr{C}_x([0,T],\mathbb{R}^+)$,

$$\Lambda(A) := \inf_{g\in A} \lambda(g)$$

then we must prove that

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}(Y^\varepsilon \in F) \le -\Lambda(F) \tag{4.35}$$

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}(Y^\varepsilon \in G) \ge -\Lambda(G) \tag{4.36}$$

for every closed set $F \subset \mathscr{C}_x([0,T],\mathbb{R}^+)$ and open set $G \subset \mathscr{C}_x([0,T],\mathbb{R}^+)$

Lower bound. Let $\delta > 0$ and $g \in G$ such that $\lambda(g) \le \Lambda(G) + \delta$ and $h \in \mathscr{H}^k$ such that $S_x(h) = g$ and $\frac{1}{2}\,|h|_1^2 = \lambda(g)$. Thus if $\rho > 0$ is such that the neighborhood of radius ρ of g in $\mathscr{C}_x^m$ is contained in G, then for every $\alpha > 0$

$$\begin{aligned}\mathrm{P}(Y^\varepsilon \in G) \ge \mathrm{P}\big(\|Y^\varepsilon - g\| < \rho\big) \ge \mathrm{P}\big(\|Y^\varepsilon - g\| < \rho, \|\varepsilon B - h\| < \alpha\big) = \\ = \mathrm{P}\big(\|\varepsilon B - h\| < \alpha\big) \;-\; \mathrm{P}\big(\|Y^\varepsilon - g\| > \rho, \|\varepsilon B - h\| < \alpha\big)\end{aligned}$$

Now for every $\alpha > 0$, thanks to the classical Schilder estimate [7],

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}\big(\|\varepsilon B - h\| < \alpha\big) \ge -\frac{1}{2}\,|h|_1^2 = -\lambda(g) \ge -\Lambda(G) - \delta\ .$$

By (A2.3) c), for $\alpha > 0$ small enough,

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}\big(\|Y^\varepsilon - g\| > \rho, \|\varepsilon B - h\| < \alpha\big) < -R$$

with $R > \Lambda(G) + 1$, so that

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}(Y^\varepsilon \in G) \ge -\Lambda(G) - \delta$$

which, δ being arbitrary, allows to conclude.

Upper bound. If $\Lambda(F) = 0$ there is nothing to prove. Otherwise let $0 < a < \Lambda(F)$ and consider the compact sets (in $\mathscr{C}_x^m$ and $\mathscr{C}_0^k$ respectively)

$$\begin{aligned}K_a &= \{g \in \mathscr{C}_x^m; \lambda(g) \le a\}\\ C_a &= \{h \in \mathscr{C}_0^k; \tfrac{1}{2}\,|h|_1^2 \le a\}\end{aligned}$$

Then $K_a \cap F = \emptyset$ and, F being closed and K_a compact, for every $g \in K_a$ there exists $\rho = \rho_g$ such that $B(g,\rho) \cap F = \emptyset$. For every $h \in C_a$ the path $g = S_x(h)$ belongs to K_a and, by (A2.3) c), there exists $\alpha = \alpha_h$ such that

$$\mathrm{P}\big(\|Y^\varepsilon - g\| > \rho, \|\varepsilon B - h\| < \alpha\big) \le \mathrm{e}^{-R/\varepsilon^2}$$

for $\varepsilon \le \varepsilon_0 = \varepsilon_{0,h}$. The balls $B(h,\alpha_h)$, $h\in C_a$, form an open cover of C_a which is compact, so that there exist $h_1,\dots,h_r$ such that $B(h_i,\alpha_i)$, $i=1,\dots,r$ is a finite subcover of C_a. Let $A=\bigcup_{i=1}^r B(h_i,\alpha_i)$ and $g_i=S_x(h_i)$. Then

$$\mathrm{P}(Y^\varepsilon\in F)\le \mathrm{P}\big(Y^\varepsilon\in F,\varepsilon B\in A\big)\;+\;\mathrm{P}\big(\varepsilon B\in A^c\big) \tag{4.37}$$

Now, again thanks to Schilder estimates ([7]), as A^c is a closed set such that $\frac12\,|h|_1^2>a$ for every $h\in A$,

$$\mathrm{P}\big(\varepsilon B\in A^c\big)\le \mathrm{e}^{-a/\varepsilon^2}$$

for small ε whereas, if $g_i=S_x(h_i)$,

$$\mathrm{P}\big(Y^\varepsilon\in F,\varepsilon B\in A\big)\le\sum_{i=1}^r \mathrm{P}\big(Y^\varepsilon\in F,\|\varepsilon B-h_i\|<\alpha_i\big)\le$$

$$\le\sum_{i=1}^r \mathrm{P}\big(\|Y^\varepsilon-g_i\|>\rho_i,\|\varepsilon B-h_i\|<\alpha_i\big)$$

so that, again for small ε and a possibly smaller α_i,

$$\mathrm{P}\big(Y^\varepsilon\in A\big)\le r\mathrm{e}^{-R/\varepsilon^2}+\mathrm{e}^{-a/\varepsilon^2}$$

which, for $R>a$ gives

$$\limsup_{\varepsilon\to0}\varepsilon^2\log\mathrm{P}\big(Y^\varepsilon\in A\big)\le -a$$

for every $a<\Lambda(F)$, which allows to conclude.

□

Proof of Lemma 2.8. We have

$$X_t^\varepsilon-\gamma_t=\int_0^t c_\varepsilon(s,X_s^\varepsilon)-c(s,X_s^\varepsilon)\,ds+\int_0^t c(s,X_s^\varepsilon)-c(s,\gamma_s)\,ds+\varepsilon\int_0^t\sigma_\varepsilon(X_s^\varepsilon)\,dB_s$$

For small ε, thanks to (2.19),

$$\Big|\int_0^t c_\varepsilon(s,X_s^\varepsilon)-c(s,X_s^\varepsilon)\,ds\Big|\le\frac{\rho}{2}\,\mathrm{e}^{-a_1T}$$

whereas (2.18) gives

$$\Big|\int_0^t c(s,X_s^\varepsilon)-c(s,\gamma_s)\,ds\Big|\le\int_0^t|\psi(s)||X_s^\varepsilon-\gamma_s|\,ds\le\int_0^t|\psi(s)|\cdot\|X^\varepsilon-\gamma\|_s\,ds$$

(recall that $\|X^\varepsilon-\gamma\|_s=\sup_{u\le s}|X_s^\varepsilon-\gamma_s|$). Therefore, if $U_\varepsilon(t)=\varepsilon\int_0^t\sigma_\varepsilon(X_s)\,dB_s$,

$$\|X^\varepsilon-\gamma\|_t\le\frac{\rho}{2}\,\mathrm{e}^{-a_1T}+\|U_\varepsilon\|+\int_0^t|\psi(s)|\cdot\|X^\varepsilon-\gamma\|_s\,ds$$

and from Gronwall Lemma, for $\|\psi\|_{L^1}\le a_1$,

$$\|X^\varepsilon-\gamma\|\le\frac{\rho}{2}+\|U_\varepsilon\|\,\mathrm{e}^{a_1T}$$

Thus

$$\mathrm{P}\big(\|X^\varepsilon-\gamma\|>\rho,\|\varepsilon B\|\le\alpha\big)\le\mathrm{P}\big(\|U_\varepsilon\|>\tfrac{\rho}{2}\,\mathrm{e}^{-a_1T},\|\varepsilon B\|\le\alpha\big)$$

The conclusion comes now from Lemma 4.13 below.

Lemma 4.13 *(Assumptions and notations of Lemma 2.8) Let* $U_\varepsilon(t) = \varepsilon \int_0^t \sigma_\varepsilon(X_s)\, dB_s$ *as above. Then for every* $R > 0$, $\rho > 0$ *there exist* ε_0, $\alpha > 0$ *such that, if* $\varepsilon < \varepsilon_0$

$$\varepsilon^2 \log \mathrm{P}\big(\|U_\varepsilon\| > \rho, \|\varepsilon B\| \leq \alpha\big) \leq -R$$

Proof. For every $n > 0$ let

$$t_0 = 0, t_1 = \tfrac{T}{n}, \dots, t_k = \tfrac{kT}{n}, \dots t_n = T$$

be a discretization of $[0, T]$ and define the approximations

$$X_t^{\varepsilon,n} = X_{t_k}^\varepsilon \text{ if } t_k \leq t < t_{k+1} \ .$$

We have

$$\big\{\|U_\varepsilon\| > \rho, \|\varepsilon B\| \leq \alpha\big\} \subset A \cup B \cup C$$

where

$$\begin{aligned} A &= \big\{\|X^\varepsilon - X^{\varepsilon,n}\| > \beta\big\} \\ B &= \Big\{\sup_{t\leq T}\Big|\varepsilon \int_0^t \big(\sigma_\varepsilon(X_s^\varepsilon) - \sigma_\varepsilon(X_s^{\varepsilon,n})\big)\, dB_s\Big| > \tfrac{\rho}{2}, \|X^\varepsilon - X^{\varepsilon,n}\| \leq \beta\Big\} \\ C &= \Big\{\sup_{t\leq T}\Big|\varepsilon \int_0^t \sigma_\varepsilon(X_s^{\varepsilon,n})\, dB_s\Big| > \tfrac{\rho}{2}, \|X^\varepsilon - X^{\varepsilon,n}\| \leq \beta, \|\varepsilon B\| \leq \alpha\Big\} \end{aligned}$$

In order to give an upper bound for $\mathrm{P}(B)$ we split $B = B_1 \cup B_2 \cup B_3$ where

$$\begin{aligned} B_1 &= \Big\{\sup_{t\leq T}\Big|\varepsilon \int_0^t \big(\sigma_\varepsilon(X_s^\varepsilon) - \sigma(X_s^\varepsilon)\big)\, dB_s\Big| > \tfrac{\rho}{6}, \Big\} \\ B_2 &= \Big\{\sup_{t\leq T}\Big|\varepsilon \int_0^t \big(\sigma(X_s^\varepsilon) - \sigma(X_s^{\varepsilon,n})\big)\, dB_s\Big| > \tfrac{\rho}{6}, \|X^\varepsilon - X^{\varepsilon,n}\| \leq \beta\Big\} \\ B_3 &= \Big\{\sup_{t\leq T}\Big|\varepsilon \int_0^t \big(\sigma_\varepsilon(X_s^{\varepsilon,n}) - \sigma(X_s^{\varepsilon,n})\big)\, dB_s\Big| > \tfrac{\rho}{6}, \Big\} \end{aligned}$$

As, thanks to (2.16), for every $\eta > 0$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$

$$\sup_y \big|\sigma_\varepsilon(y) - \sigma(y)\big| \leq \eta \ ,$$

the exponential inequality gives for small η

$$\begin{aligned} \mathrm{P}(B_1) &\leq 2m \exp\Big(-\frac{\rho^2}{72T\eta^2}\,\frac{1}{\varepsilon^2}\Big) < \mathrm{e}^{-R/\varepsilon^2} \\ \mathrm{P}(B_3) &\leq 2m \exp\Big(-\frac{\rho^2}{72T\eta^2}\,\frac{1}{\varepsilon^2}\Big) < \mathrm{e}^{-R/\varepsilon^2} \end{aligned}$$

Then, as σ is supposed to be Lipschitz continuous (constant K_L) on B_2 it holds $|\sigma(X_s^\varepsilon) - \sigma(X_s^{\varepsilon,n})| \leq K_L\beta$ and again the exponential inequality gives, for small β,

$$\mathrm{P}(B_2) \leq 2m \exp\Big(-\frac{\rho^2}{72TK_L^2\beta^2}\,\frac{1}{\varepsilon^2}\Big) < \mathrm{e}^{-R/\varepsilon^2}$$

Thus

$$\mathrm{P}(B) \leq 3\mathrm{e}^{-R/\varepsilon^2} \tag{4.38}$$

for $\varepsilon < \varepsilon_0$ and small β, independently of n. As for C, on the set $\{\|\varepsilon B\| \leq \alpha\}$ it holds

$$\Big|\varepsilon \int_0^t \sigma_\varepsilon(X_s^{\varepsilon,n})\, dB_s\Big| = \Big|\varepsilon \sum_{k=0}^{n-1} \sigma(X_{t_k}^\varepsilon)(B_{t_{k+1}\wedge t} - B_{t_k\wedge t})\Big| \leq 2Mn\alpha$$

which gives $C = \emptyset$ if $\alpha < \frac{\rho}{4Mn_0}$. As for A

$$\mathrm{P}(\|X^\varepsilon - X^{\varepsilon,n}\| > \beta) = \mathrm{P}\Big(\bigcup_{k=0}^{n-1}\{\sup_{t_k\leq t<t_{k+1}} |X_t^\varepsilon - X_{t_k}^\varepsilon| > \beta\}\Big) \leq$$

$$\leq \sum_{k=0}^{n-1}\Big\{\mathrm{P}\Big(\sup_{t_k\leq t<t_{k+1}} \Big|\int_{t_k}^t c_\varepsilon(s, X_s^\varepsilon)\, ds\Big| + \sup_{t_k\leq t<t_{k+1}} \varepsilon\Big|\int_{t_k}^t \sigma_\varepsilon(X_s^\varepsilon)\, dB_s\Big| > \beta\Big\} .$$

By Cauchy-Schwarz inequality

$$\Big|\int_{t_k}^t c_\varepsilon(s, X_s^\varepsilon)\, ds\Big| \leq \sqrt{\frac{T}{n}}\Big(\int_0^T |\phi(s)|^2\, ds\Big)^{1/2} \leq a_1\sqrt{\frac{T}{n}}$$

so that for $n \geq n_0$ large enough, independently of ε, the events

$$\Big\{\sup_{t_k\leq t<t_{k+1}} \Big|\int_{t_k}^t c_\varepsilon(s, X_s^\varepsilon)\, ds\Big| > \frac{\beta}{2}\Big\}, \quad k = 0, \ldots, n-1$$

are empty. Moreover, from the exponential inequality of martingales

$$\mathrm{P}\Big(\sup_{t_k\leq t<t_{k+1}} \varepsilon\Big|\int_{t_k}^t \sigma_\varepsilon(X_s^\varepsilon)\, dB_s\Big| > \frac{\beta}{2}\Big) \leq 2m\exp\Big(-\frac{n\beta^2}{8M^2T}\,\frac{1}{\varepsilon^2}\Big) .$$

Thus, for $n > n_0$ and $\varepsilon > 0$,

$$\varepsilon^2 \log \mathrm{P}\big(\|X^\varepsilon - X^{\varepsilon,n}\| > \beta\big) \leq \varepsilon^2 \log(2mn) - \frac{n\beta^2}{8M^2T}$$

and the quantity in the right hand side, for a possibly larger value of n_0, is smaller than $-R$ for every $\varepsilon \leq 1$. This together with (4.38) allows to conclude.

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