Notes on
Stochastic Calculus applied to Finance

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Contents

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1 The model

We consider a market model where trading is allowed over the time interval $[0, T^*]$, where $T^* > 0$ is a fixed horizon date. In our market, we consider N risky assets, whose prices are denoted by $X^1, \dots, X^N$, and one risk-free asset (the money market) with price X^0. We suppose that the risk is driven by a d-dimensional standard Brownian motion $B = (B^1, \dots, B^d)$. Thus, we are assuming a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ where the d-dimensional standard Brownian motion B is defined, and let us denote by $(\mathscr{F}_t)_t$ the natural σ-algebra generated by B and augmented by the $\mathbb{P}$-null sets: $\mathscr{F}_t = \sigma(B_s\,;\, s \leq t) \vee \{A \in \mathscr{F} \,:\, \mathbb{P}(A) = 0\}$. We are then considering a continuous time trading market and we suppose that the underlying $(N+1)$-dimensional asset price process $X = (X^0, X^1, \dots, X^N)$ follows the *generalized Black and Scholes model*, that is, X is the solution to the stochastic differential equation (s.d.e.)

$$
\begin{aligned}
&\frac{dX_t^0}{X_t^0} = r(t)\,dt, \\
&\frac{dX_t^i}{X_t^i} = b^i(t, X_t)\,dt + \sum_{k=1}^{d} \sigma_k^i(t, X_t)\,dB^k(t), \quad i = 1, \dots, N,
\end{aligned}
\tag{1}
$$

with the starting conditions $X_0^0 = 1$ and $X_0^i = x^i$, as $i = 1, \dots, N$.
The N-vector field b stands fon an *appreciation rate* and the $(N \times d)$-matrix field σ is called the *volatility* matrix.
Throughout these pages, we will always suppose that the coefficients b and σ and the process r fulfill the following

Assumption.

(*i*) The vector field $b(t, x)$ and the matrix field $\sigma(t, x)$ are both globally bounded and locally Lipschitz continuous in x, uniformly in t: for some positive M,

$$|b(t, x)| + |\sigma(t, x)| \leq M,$$

for any t and x, and for any $K > 0$ there exists a positive constant L_K such that if $|x|, |y| \leq K$ then

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq L_K\,|x - y|,$$

for any t.

(*ii*) The process $r(t)$ is supposed to be non negative, bounded and progressively measurable.

Notice that[1] condition (i) ensures the strong existence and uniqueness of the solution to (1) and, in particular, for any $i = 1, \ldots, N$,

$$\int_0^t \sigma_k^i(s, X_s) X_s^i \, dB^k(s)$$

is a (square integrable) martingale w.r.t. the filtration $\mathscr{F}_t$.
The process $r(t)$ denotes the *short term interest rate* and obviously, X^0 can be explicitly written:

$$X_t^0 = e^{\int_0^t r_s \, ds}.$$

The asset whose price is given by X^0 plays the role of the *benchmark* asset: it represents a primary security and we consider it as a *numeraire*, that is all *relative* prices will be referred with respect to X^0. We will also speak about *discounted* prices when divided by the benchmark price X^0.

A special case is given by assuming $N = d$ and that the diffusion coefficients and the short term interest rate are constant and deterministic, that is

$$\begin{aligned} \frac{dX_t^0}{X_t^0} &= r \, dt, \\ \frac{dX_t^i}{X_t^i} &= b^i \, dt + \sum_{k=1}^d \sigma_k^i \, dB^k(t), \quad i = 1, \ldots, d. \end{aligned} \tag{2}$$

The model given by (2) corresponds to the *standard multidimensional Black and Scholes model.*

In the follows, we will always assume the standard hypothesis for our market model, that is

> *Our market is* ***frictionless****, that is, there are no transaction costs in buying or selling stocks, there are no taxes, as well as there are no penalties to short-selling, i.e. investors who do not own a stock, can buy shares of it and arrange with the buyer at some future date to pay an amount equal to the price at that date. Moreover, it is possible to borrow any fraction of the price of a security, to buy it or to hold it, at the risk-free interest rate.*

2 Trading European options

In a stock exchange, there are plenty of securities or derivative securities which are quoted and traded. A *derivative security* or a *contingent claim*,

[1]This follows from the well-known existence theorem for the solutions to S.D.E. in the localized form, see e.g. Theorem 8.10, p. 169, in Baldi [2].

as opposed to a primary security, is a security whose value depends on the prices of the assets of the market. As an example, consider a *European option* and in particular, a *call* or *put* option: a call (put) option is the right, and not the obligation, to buy (sell) shares of the underlying assets in a contractual pre-specified future date, called *maturity*, at a contractual pre-specified price, called *exercise price*. Just to simplify, consider that the call or put option is written on a single asset whose price process is X_t. Then, setting T as the maturity and K as the exercise price, then the *value* Z of the call and put option at time T is given by

$$\begin{aligned} Z &= g_{\text{call}}(X_T) = (X_T - K)_+ = \max(0, X_T - K) \quad \text{and} \\ Z &= g_{\text{put}}(X_T) = (K - X_T)_+ = \max(0, K - X_T), \end{aligned}$$

respectively.
There are plenty of European options each of them characterized by its maturity and its value at maturity, which is commonly called the *payoff*. One can formalize this as follows

Definition 2.1. *A European option Z with maturity T* is characterized by a pair (Z, T), where T stands for the maturity date and Z for the payoff, modelled as a non negative and $\mathscr{F}_T$-measurable random variable.

There are other kind of options, such as *American* options, which differ from the European ones in the exercise date: they can be exercised in some instant $t \leq T$. In the following, we will assume European contingent claims: the American options will be studied in a section constructed *ad hoc*.
With the options, the problem is to find the fair price (and to this purpose, we will need some suitable integrability properties for the payoff). Indeed, let us come back to the example of the call or put option: it is a right to buy or sell something in a future date at a price fixed at the initial date. Thus, the buyer has to pay this right to the seller, who in turn should use this money in order to deliver the contract. This means that an option should have an *initial price* and we will see that it can be fairly fixed as the initial investment of a portfolio constructed on a fairly strategy.

2.1 Trading strategies and arbitrage

Definition 2.2. *A* trading strategy *(or simply, a strategy) over the trading interval $[0, T]$, with $T \leq T^*$, is an adapted $(N+1)$-dimensional process $H_t = (H_t^0, H_t^1, , \dots, H_t^N)$ whose general component H_t^i stands for the number of units of the i^{th} security held by an investor at time t. The* portfolio *associated to the strategy H is the wealth process corresponding to the trading strategy H:*

$$V_t(H) = \langle H_t, X_t \rangle = \sum_{i=0}^{N} H_t^i X_t^i, \quad t \in [0, T]. \tag{3}$$

The initial value of the portfolio $V_0(H)$ represents the *initial investment* of the strategy H.
In the sequel, with the notation T we mean a date such that $T \leq T^*$ and the interval $[0, T]$ will stand for the trading interval of interest.

Definition 2.3. *Let a trading strategy $H_t = (H_t^0, H_t^1, \ldots, H_t^N)$ be such that*

$$\int_0^T |H_t^0|\, dt + \sum_{i=1}^N \int_0^T |H_t^i|^2\, dt < \infty, \qquad a.s.$$

H is said to be self-financing *over $[0, T]$ if its associated portfolio $V_t(H)$ is an Ito process satisfying*

$$\begin{aligned} dV_t(H) &= \langle H_t, dX_t \rangle = \sum_{i=0}^N H_t^i\, dX_t^i \\ &= H_t^0\, r_t\, X_t^0 dt + \sum_{i=1}^N H_t^i\, X_t^i b^i(t, X_t) dt + \sum_{i=1}^N H_t^i\, X_t^i \sum_{k=1}^d \sigma_k^i(t, X_t)\, dB_t^k. \end{aligned}$$

Notice that the requirements $H^0 \in L^1([0,T])$ a.s. and $H^i \in L^2([0,T])$ a.s. as $i = 1, \ldots, N$, are technical and allow one to write the above integrals[2] w.r.t. dt and dB_t^k, the latter having the usual local martingale property[3].
Intuitively, a strategy is self-financing if the variations of the associated portfolio in a small time period, depend on the asset prices $X^0, \ldots, X^N$ and are independent of the strategies (number of units) $H^0, \ldots, H^N$. In other words, changes in the portfolio are due to capital gains and are not due to increase or decrease of funds.

Let us now introduce the discounted price processes and the discounted portfolio: we set

$$\begin{aligned} \tilde{X}_t^i &= X_t^i / X_t^0 = e^{-\int_0^t r_s\, ds} X_t^i, \text{ as } i = 1, \ldots, N, \\ \text{and} \quad \tilde{V}_t(H) &= V_t(H)/X_t^0 = e^{-\int_0^t r_s\, ds} V_t(H). \end{aligned}$$

Notice that $\tilde{V}_t(H) = H_t^0 + \sum_{i=1}^N H_t^i\, \tilde{X}_t^i$. In the sequel, we will refer to $\tilde{X}_t$ as the N-dimensional discounted price process $\tilde{X}_t = (\tilde{X}_t^1, \ldots, \tilde{X}_t^N)$.
As a first result, one has

Proposition 2.4. *Let the trading strategy $H_t = (H_t^0, H_t^1, \ldots, H_t^N)$ be such that $\int_0^T |H_t^0|\, dt + \sum_{i=1}^N \int_0^T |H_t^i|^2\, dt < \infty$, a.s. Then H is self financing if and only if*

$$\tilde{V}_t(H) = V_0(H) + \int_0^t \sum_{i=1}^N H_t^i\, d\tilde{X}_t^i, \quad t \in [0, T].$$

[2] In fact, since X is a continuous process and r, b and σ are all bounded, then $H_t^0\, r_t\, X_t^0 \in L^1([0,T])$ and $H_t^i\, X_t^i b^i(t, X_t) \in L^1([0,T])$ a.s. for any i, as well as $H_t^i\, X_t^i \sigma_k^i(t, X_t) \in L^2([0,T])$ a.s. for any i and k.

[3] This is an immediate consequence of the fact that $H_t^i X_t^i \sigma_k^i(t, X_t) \in L^2([0,T])$ a.s. for any i and k.

Proof. Suppose first that H is self financing. Then, by recalling (1), by using Ito's lemma one has

$$\begin{aligned} d\tilde{V}_t(H) &= -r_t\,\tilde{V}_t(H)\,dt + e^{-\int_0^t r_s\,ds}\,dV_t(H) \\ &= e^{-\int_0^t r_s\,ds}\Big(-r_t \sum_{i=0}^{N} H_t^i\,X_t^i\,dt + \sum_{i=0}^{N} H_t^i\,dX_t^i \Big) \\ &= \sum_{i=1}^{N} H_t^i\,\Big(-r_t\,\tilde{X}_t^i + e^{-\int_0^t r_s\,ds}\,dX_t^i \Big) = \sum_{i=1}^{N} H_t^i\,d\tilde{X}_t^i. \end{aligned}$$

Conversely, if $\tilde{V}_t(H)$ satisfies the Ito differential above, then by applying again Ito's lemma to $V_t(H) = e^{\int_0^t r_s\,ds}\tilde{V}_t(H)$, one obtains

$$dV_t(H) = \sum_{i=0}^{N} H_t^i\,d\tilde{X}_t^i,$$

that is H is self financing.

□

As a consequence of Proposition 2.4, we will obtain that whenever H is self financing then for any probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that $\tilde{X}_t$ is a $\mathbb{P}^*$-martingale then $\tilde{V}_t(H)$ is a $\mathbb{P}^*$-local martingale. We will discuss this point more deeply in next Section 2.2.
Let us introduce now the following sub-class of self-financing strategies.

Definition 2.5. *A self financing strategy H is said to be* admissible if $V_t(H) \geq 0$ for any t, a.s.

It then follows that the admissible strategies never give a loss in the wealth. We are now ready to introduce the concept of arbitrage.

Definition 2.6. *A self financing trading strategy H over $[0,T]$ is said to be an* arbitrage opportunity *if the associated portfolio $V_t(H)$ satisfies*

$$V_0(H) = 0, \mathbb{P}(V_t(H) \geq 0) = 1 \textit{ for any } t \leq T \textit{ and } \mathbb{P}(V_T(H) > 0) > 0.$$

First, notice that an arbitrage opportunity is in fact an admissible strategy. Moreover, an arbitrage strategy remains an arbitrage one under any probability measure $\mathbb{P}^*$ equivalent to the original one $\mathbb{P}$. Roughly speaking, a strategy gives rise to an arbitrage if even though an investor starts with a null initial investment, then not only he will never suffer a loss (that is, $V_t(H) \geq 0$) but also will have a final gain (that is, $V_T(H) > 0$) with positive probability. Thus, arbitrage means that there are no limits in creating wealth and then it should be forbidden in a well-functioning market. Arbitrage opportunities might happen in practice (for example, by involving transactions in two or more markets), but since they give risk-free profits, they would create forces of demand and supply which would cause the nullification of the arbitrage, so that the arbitrage would disappear quickly. Therefore, we will always consider markets where arbitrage opportunities are not allowed, and we will say that these markets are *arbitrage-free*. We can mathematically describe this concept by the following

Definition 2.7. *Our market model is said to be* arbitrage-free *if any admissible strategy H on $[0,T]$ with $V_0(H)=0$ is such that $\mathbb{P}(V_T(H)>0)=0$.*

As we will see, the arbitrage-free property is strictly connected to the so called *equivalent martingale measures*, which we are now going to introduce.

2.2 Equivalent martingale measures

As we will see, in order to tackle the two main problems in finance, that is the no arbitrage problem and the market completeness, it would be important to have a measure $\mathbb{P}^*$, equivalent to $\mathbb{P}$, under which the discounted price process is a martingale.
Let us first study which are the measures equivalent to $\mathbb{P}$ on $(\Omega, \mathscr{F}_{T^*})$ in our model. To this purpose, let us define

$$\xi_t^\gamma := \exp\Big(\int_0^t \gamma_s\, dB_s - \frac{1}{2}\int_0^t |\gamma_s|^2\, ds\Big), \tag{4}$$

where γ denotes an adapted process belonging to $L^2([0,T^*])$ a.s. Then it is well known that ξ_t^γ follows in general a $\mathbb{P}$-local martingale and since it is bounded from below ($\xi_t^\gamma \geq 0$) then it is also a $\mathbb{P}$-supermartingale, so that $\mathbb{E}(\xi_{T^*}^\gamma) \leq \mathbb{E}(\xi_0^\gamma) = 1$.
Processes like ξ^γ allow to identify the measures which are equivalent to $\mathbb{P}$. In fact, let us define $\mathbb{P}^*$ as the measure on $(\Omega, \mathscr{F}_{T^*})$ whose Radon-Nicodym derivative w.r.t. $\mathbb{P}$ is given by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \xi_{T^*}^\gamma = \exp\Big(\int_0^{T^*} \gamma_s\, dB_s - \frac{1}{2}\int_0^{T^*} |\gamma_s|^2\, ds\Big). \tag{5}$$

First, one has the following well known result:

Theorem 2.8. [Girsanov] *Let ξ^γ, defined in (4), be such that $\mathbb{E}(\xi_{T^*}^\gamma)=1$. Then the measure $\mathbb{P}^*$ defined by (5) is a probability measure and the process*

$$B_t^* = B_t - \int_0^t \gamma_s\, ds$$

is a d-dimensional Brownian motion on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T^*]}, \mathbb{P}^*)$.

For the proof, we refer e.g. to Theorem 5.1 in Karatzas and Shreve [7] or also Teorema 7.22 in Baldi [2].

Remark 2.9. *First, notice that the hypotheses of the Girsanov's Theorem hold if and only if* ξ^γ *is a martingale over* $[0,T^*]$ *under* $\mathbb{P}$*. In fact, if* ξ^γ *is a martingale under* $\mathbb{P}$ *then obviously* $\mathbb{E}(\xi^\gamma_{T^*}) = \mathbb{E}(\xi^\gamma_0) = 1$*. Conversely, since* ξ^γ *is a* $\mathbb{P}$*-supermartingale then for any* $T^* \geq u \geq v \geq 0$ *one has* $\mathbb{E}(\xi^\gamma_{T^*}) \leq \mathbb{E}(\xi^\gamma_u) \leq \mathbb{E}(\xi^\gamma_v) \leq \mathbb{E}(\xi^\gamma_0) = 1$*. Now, if* ξ^γ *is not a* $\mathbb{P}$*-martingale then for some* $t > s$ *one has* $\mathbb{E}(\xi^\gamma_t \,|\, \mathscr{F}_s) < \xi^\gamma_s$ *a.s., so that* $1 = \mathbb{E}(\xi^\gamma_{T^*}) \leq \mathbb{E}(\xi^\gamma_t) < \mathbb{E}(\xi^\gamma_s) \leq \mathbb{E}(\xi^\gamma_0) = 1$*, from which a nonsense follows.*

Let us recall that a sufficient condition in order that ξ^γ *is a martingale over* $[0,T^*]$ *is that*

$$\int_0^{T^*} |\gamma_s|^2 ds \leq K,$$

for some constant $K > 0$*, which in turn guarantees that* ξ^γ *is a martingale bounded in* $L^p(\Omega, \mathscr{F}_{T^*}, \mathbb{P})$*, for any* p *(see Baldi, Proposizione 7.19, p. 145). Another (classical) condition providing that* ξ^γ *is a* $\mathbb{P}$*-martingale over* $[0,T^*]$ *is the Novikov condition (see e.g. Proposition 5.12, p. 198, in Karatzas and Shreve [7]), that is*

$$\mathbb{E}\left(e^{\frac{1}{2}\int_0^{T^*} |\gamma_s|^2 ds}\right) < \infty.$$

A very important fact is that the Girsanov's Theorem 2.8 has also a counterpart, allowing one to give a characterization of all the measures equivalent to $\mathbb{P}$ on $(\Omega, \mathscr{F}_{T^*})$. In fact, one has

Proposition 2.10. *If a probability measure* $\mathbb{P}^*$ *on* $(\Omega, \mathscr{F}_{T^*})$ *is equivalent to* $\mathbb{P}$ *then there exists an adapted* d*-dimensional process* $\gamma \in L^2([0,T^*])$ *a.s. such that (5) holds. In particular, one has that* ξ^γ_t *is a martingale under* $\mathbb{P}$*, and then* $B^*_t = B_t - \int_0^t \gamma_s \, ds$ *is an* $\mathscr{F}_t$*-Brownian motion under* $\mathbb{P}^*$*.*

Proof. The proof uses the representation theorem for Brownian martingales[4] (see e.g. Theorem 4.15 Karatzas and Shreve [7] or Teoremi 7.26 and 7.27 in Baldi [2]).

First, let us notice that, for any $t \leq T^*$, $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ also on $\mathscr{F}_t$, because $\mathscr{F}_t \subset \mathscr{F}_{T^*}$. Let us denote by ξ_t the Radon-Nicodym derivative of $\mathbb{P}^*$ w.r.t. $\mathbb{P}$ on $(\Omega, \mathscr{F}_t)$, that is

$$\xi_t = \frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathscr{F}_t}.$$

[4] **Representation theorem for Brownian martingales.** *Let* $\mathscr{F}_t$ *denote the natural filtration of a Brownian motion* B *augmented with the* $\mathbb{P}$*-null sets. Let* $(Y_t)_{t \leq T^*}$ *be a square integrable* $\mathscr{F}_t$*-martingale [resp.: is an* $\mathscr{F}_t$*-local martingale]. Then there exists a unique constant* c *and a unique adapted process* α *in* $L^2(\Omega \times [0,T^*])$ *[resp.: in* $L^2([0,T^*])$ *a.s.] such that*

$$Y_t = c + \int_0^t \alpha_s \, dB_s, \quad t \in [0,T^*].$$

Then, $(\xi_t)_{t\leq T^*}$ becomes an adapted process which is integrable under $\mathbb{P}$ for any $t \leq T^*$ (in fact, $\xi_t > 0$ a.s. and $\mathbb{E}(\xi_t) = \mathbb{P}^*(\Omega) = 1$). Moreover, it is a martingale under $\mathbb{P}$, because for any $0 < s < t < T^*$ and $A \in \mathscr{F}_s \subset \mathscr{F}_t$,

$$\mathbb{E}(\xi_t 1_A) = \mathbb{P}^*(A) = \mathbb{E}(\xi_s 1_A).$$

Now, by using the Brownian martingale representation theorem, one has

$$\xi_t = 1 + \int_0^t \alpha_s \, dB_s, \quad t \in [0, T^*].$$

for a ("unique") adapted process $\alpha \in L^2([0,T^*])$ a.s. By using the Ito's Lemma, one obtains as $t \in [0,T^*]$,

$$\xi_t = \exp\Big(\int_0^t \gamma_s \, dB_s - \frac{1}{2}\int_0^t |\gamma_s|^2 \, ds\Big), \quad \text{with } \gamma_s = \alpha_s \, \xi_s^{-1}.$$

Since ξ is a.s. positive and continuous as a function of t, it immediately follows that γ is adapted and belongs to $L^2([0,T^*])$ a.s., and the statement follows.

□

Let us come back to our model in finance and propose the following definition:

Definition 2.11. *A measure $\mathbb{P}^*$ on $(\Omega, \mathscr{F}_{T^*})$ is called an* equivalent martingale measure *if $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ on $\mathscr{F}_{T^*}$ and $(\tilde{X}_t)_{t\leq T^*}$ follows a martingale under $\mathbb{P}^*$.*

By using Theorem 2.8 and 2.10, we obtain the following result.

Theorem 2.12. *An equivalent martingale measure $\mathbb{P}^*$ exists if and only if there exists a d-dimensional adapted process $\gamma^* \in L^2([0,T^*])$ a.s. solution to*

$$\sum_{k=1}^{d} \sigma_k^i(t, X_t)\gamma_t^{*k} = r_t - b^i(t, X_t), \quad i = 1, \dots, N, \tag{6}$$

for which $\mathbb{E}(\xi_{T^*}^{\gamma^*}) = 1$ (or equivalently, ξ^{γ^*} is a martingale under $\mathbb{P}$).
In such a case, under $\mathbb{P}^*$ the discounted price process evolves following

$$d\tilde{X}_t^i = \tilde{X}_t^i \sum_{k=1}^{d} \sigma_k^i(t, X_t) \, dB_t^{*k}, \quad \tilde{X}_0^i = x^i, \quad i = 1, \dots, N, \tag{7}$$

and the price process solves

$$dX_t^i = X_t^i r_t \, dt + X_t^i \sum_{k=1}^{d} \sigma_k^i(t, X_t) \, dB_t^{*k}, \quad X_0^i = x^i, \quad i = 1, \dots, N, \tag{8}$$

where $B_t^* = B_t - \int_0^t \gamma_s^* \, ds$ is a Brownian motion.

The process γ^* solution to (6) is called the *market price of risk.* If the market price of risk exists, then an equivalent martingale measure $\mathbb{P}^*$ exists and under $\mathbb{P}^*$, the price process evolve following (8), so that the appreciation rate b plays no role and is replaced by the spot rate r. In other words, under $\mathbb{P}^*$, in mean the risky asset prices evolve as the riskless asset price. That is why an equivalent martingale measure is often called a *risk neutral measure.*

Proof of Theorem 2.12. First, suppose there exists an equivalent martingale measure $\mathbb{P}^*$. Then, by Proposition 2.10, there exists an adapted process $\gamma \in [0,T^*] \in L^2([0,T^*])$ a.s. such that the associated process ξ^γ as in (4) satisfies the hypotheses of the Girsanov's Theorem, so that

$$B_t^* = B_t - \int_0^t \gamma_s \, ds, \quad t \le T^*$$

is a Brownian motion under $\mathbb{P}^*$. Now, by replacing the Brownian motion B^* in the s.d.e. driving $\tilde{X}$, one obtains, for $i = 1, \dots, N$,

$$d\tilde{X}_t^i = X_t^i \Big(b^i(t,X_t) - r_t + \sum_{k=1}^d \sigma_k^i(t,X_t)\gamma_t^k \Big) dt + X_t^i \sum_{k=1}^d \sigma_k^i(t,X_t)\gamma_t^k \, dB_t^{*k}.$$

In order to get the martingale property for $\tilde{X}$, it must be

$$b^i(t,X_t) - r_t + \sum_{k=1}^d \sigma_k^i(t,X_t)\gamma_t^k = 0, \quad i = 1, \dots, N,$$

for any $t \le T^*$, a.s. Therefore, the statement holds and in effect, under $\mathbb{P}^*$ one obtains both (7) and (8).
Viceversa, if there exists an adapted process γ^* satisfying (6) and fulfilling the requirements of the Girsanov's Theorem 2.8, then obviously (7) and (8) hold. Now, by (7) and by using the Ito's Lemma, one has

$$\tilde{X}_t^i = x^i \exp\Big(-\frac{1}{2} \int_0^t |\theta_s^i|^2 ds + \int_0^t \theta_s^i \, dB_s^* \Big)$$

with $\theta_k^i(s) = \sigma_k^i(s, X_s)$, $k = 1, \dots d$. Since θ^i is bounded, the process $\tilde{X}^i$ is just the exponential martingale associated to B^* (see e.g. Remark 2.9), and the statement holds.

□

Finally, the following result gives us a simple sufficient condition ensuring the existence of an equivalent martingale measure.

Proposition 2.13. *Suppose that $d = N$, σ is invertible and the matrix field $a = \sigma\sigma^*$ is uniformly elliptic (that is, there exists $\alpha > 0$ such that*

$\langle a(t,x)\lambda,\lambda\rangle \geq \alpha\,|\lambda|^2$, for any $\lambda\in\mathbb{R}^d$ and $t\in[0,T^*]$). Then, there exists a unique equivalent martingale measure $\mathbb{P}^*$, given by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \xi^{\gamma^*}_{T^*} = \exp\Big(\int_0^{T^*}\gamma^*_s\,dB_s - \frac{1}{2}\int_0^{T^*}|\gamma^*_s|^2\,ds\Big)$$

with

$$\gamma^*_t = \sigma^{-1}(t,X_t)(r_t\,\mathbf{1} - b(t,X_t)),$$

where $\mathbf{1}$ denotes the d-dimensional vector whose entries are all equal to 1.

Proof. By using Theorem 2.11, an equivalent martingale measure exists if and only if equation (6) holds and the resulting process γ is adapted, belongs to $L^2([0,T^*])$ a.s. and makes ξ^γ, defined in (4), a martingale under $\mathbb{P}$. In our context, (6) is satisfied if and only if

$$\gamma_t = \gamma^*_t = \sigma^{-1}(t,X_t)(r_t\mathbf{1} - b(t,X_t)),$$

which makes sense because of our hypothesis (and $\gamma=\gamma^*$ is the unique solution to (6)). The above process is evidently adapted. Moreover, it is bounded. In fact, since a is uniformly elliptic, that is $|\sigma\lambda|^2 = \langle a\lambda,\lambda\rangle \geq \alpha|\lambda|^2$ with $\alpha>0$, then $\|\sigma^{-1}\|^2 \leq 1/\alpha$. By recalling that both r and b are supposed to be bounded, it follows that γ^* is bounded, and this ensures the martingale property for ξ^{γ^*} (see Remark 2.9).

□

2.3 Martingale properties for discounted portfolios

Let us now come back to self-financing strategies. The following proposition states the previously remarked martingale property for portfolios associated to self-financing strategies, that is

Proposition 2.14. *If there exists an equivalent martingale measure $\mathbb{P}^*$, the discounted portfolio associated to any self-financing strategy is a local martingale under $\mathbb{P}^*$. Moreover, if H is an admissible strategy, then $\tilde V(H)$ is a supermartingale under $\mathbb{P}^*$*

Proof. Let H be a self financing strategy. Then by Proposition 2.4

$$d\tilde V_t(H) = \sum_{i=1}^N H^i_t\,d\tilde X^i_t,$$

as $t\leq T\leq T^*$. Now, if $\mathbb{P}^*$ is an equivalent martingale measure then by Theorem 2.12 (in particular, by (7)) one has

$$d\tilde V_t(H) = \sum_{i=1}^N H^i_t\,\tilde X^i_t\sum_{k=1}^d \sigma^i_k(t,X_t)\,dB^{*k}_t.$$

Since $\sigma_k^i(t, X_t)$ is bounded, for any i and k, $\tilde{X}_t = e^{-\int_0^t r_s ds} X_t$ is a continuous process and $H^i \in L^2([0, T^*])$ a.s. for any i (recall that this is inside the definition of self-financing strategies), it follows that $\sum_{i=1}^N H_t^i \tilde{X}_t^i \sigma_k^i(t, X_t) \in L^2([0, T^*])$ a.s., and this proves that $\tilde{V}(H)$ is a local martingale.
Suppose now that H is admissible, that is self-financing and such that $V_t(H) \geq 0$ a.s. under $\mathbb{P}$. Since $\mathbb{P}^*$ is equivalent to $\mathbb{P}$, we obtain that the discounted portfolio $\tilde{V}_t(H)$ is a $\mathbb{P}^*$-local martingale bounded from below ($\tilde{V}_t(H) \geq 0$ a.s. under $\mathbb{P}^*$) so that (see e.g. Proposizione 6.24 in Baldi [2]) it is a supermartingale.

□

We can finally study the link between equivalent martingale measures and the arbitrage-free property of the market model.

Proposition 2.15. *If there exists an equivalent martingale measure $\mathbb{P}^*$, the market model is arbitrage-free.*

Proof. We have to prove that if H denotes an admissible strategy over $[0, T]$ such that $V_0(H) = 0$ a.s. then $V_T(H) = 0$ a.s. By Proposition 2.14, $\tilde{V}_t(H)$ is a non negative supermartingale under $\mathbb{P}^*$, so that

$$0 \leq \mathbb{E}^*(\tilde{V}_T(H)) \leq \mathbb{E}^*(\tilde{V}_0(H)) = 0.$$

This proves that necessarily $\mathbb{P}^*(\tilde{V}_T(H) > 0) = 0$, an then $\mathbb{P}(V_T(H) > 0) = 0$.

□

Often we will need martingale properties for the discounted portfolio, and not only local ones. Thus, we introduce the following definition, which requires the existence of an equivalent martingale measure $\mathbb{P}^*$.

Definition 2.16. *Let $\mathbb{P}^*$ be an equivalent martingale measure. The class $\mathscr{M}_a(\mathbb{P}^*)$ will denote the set of the admissible strategies H on $[0, T]$ such that the associated discounted portfolio $\tilde{V}_t(H)$ is a $\mathbb{P}^*$-martingale. For some equivalent martingale measure $\mathbb{P}^*$, the pair $(X, \mathscr{M}_a(\mathbb{P}^*))$ will be our quoted market model.*

In view of Proposition 2.13, we can assert that, whenever an equivalent martingale measure $\mathbb{P}^*$ exists, the market model $(X, \mathscr{M}_a(\mathbb{P}^*))$ is arbitrage-free.

2.4 Replicating strategies e market completeness

Let (Z, T) denote a European option, that is T is the exercise date (maturity) and Z denotes the payoff (see Definition 2.1).

Definition 2.17. *An admissible trading strategy H is said* to replicate *a European option (Z,T) if $V_T(H) = Z$ $\mathbb{P}$-a.s. If a claim Z admits an arbitrage-free and replicating strategy H, Z is said to be* attainable *in the market and the corresponding portfolio $V_t(H)$ is called the* replicating portfolio.

If an equivalent martingale measure $\mathbb{P}^*$ exists, we know that the market model $(X, \mathscr{M}_a(\mathbb{P}^*))$ is arbitrage-free, so that the attainability of an option can be defined as follows.

Definition 2.18. *If an equivalent martingale measure $\mathbb{P}^*$ exists, an option (Z,T) is said to be attainable in the market model $(X, \mathscr{M}_a(\mathbb{P}^*))$ if Z is integrable under $\mathbb{P}^*$ and there exists a strategy $H \in \mathscr{M}_a(\mathbb{P}^*)$ such that $V_T(H) = Z$.*

Recall that $\mathscr{M}_a(\mathbb{P}^*)$ is the set of the $\mathbb{P}^*$-admissible trading strategies such that the corresponding discounted portfolio $\tilde{V}(H)$ is a martingale (and not only a local one or a supermartingale) under $\mathbb{P}^*$.
Intuitively, it is important to work in an arbitrage-free framework (recall that we need a "fair game" between buyer and seller!) and to have a unique (up to indistinguishability) replicating portfolio. Indeed, this portfolio will be the one allowing the seller to deliver the contract.
From now on, we will use the notation $\mathbb{E}^*$ for the expectation under $\mathbb{P}^*$.
We have the following first important result:

Proposition 2.19. *If an equivalent martingale measure $\mathbb{P}^*$ exists, any European option (Z,T) which is attainable in $(X, \mathscr{M}_a(\mathbb{P}^*))$ admits a unique replicating portfolio under $\mathbb{P}^*$, given by*

$$V_t = \mathbb{E}^*(e^{-\int_t^T r_s\,ds} Z \mid \mathscr{F}_t)$$

Proof. First, let us notice that $e^{-\int_t^T r_s\,ds} Z$ is indeed integrable under $\mathbb{P}^*$, because r in non negative and bounded a.s. and Z is integrable under $\mathbb{P}^*$. Fix a European option Z with maturity T which is attainable in $(X, \mathscr{M}_a(\mathbb{P}^*))$. For any replicating strategy H, $\tilde{V}_t(H)$ is a $\mathbb{P}^*$-martingale: if Z is attainable then

$$\begin{aligned} V_t(H) &= e^{\int_0^t r_s\,ds}\,\tilde{V}_t(H) = e^{\int_0^t r_s\,ds}\,\mathbb{E}^*(\tilde{V}_T(H) \mid \mathscr{F}_t) \\ &= e^{\int_0^t r_s\,ds}\,\mathbb{E}^*(e^{-\int_0^T r_s\,ds} V_T(H) \mid \mathscr{F}_t) \\ &= e^{\int_0^t r_s\,ds}\,\mathbb{E}^*(e^{-\int_0^T r_s\,ds} Z \mid \mathscr{F}_t) \\ &= \mathbb{E}^*(e^{-\int_t^T r_s\,ds} Z \mid \mathscr{F}_t) =: V_t. \end{aligned}$$

□

For $t < T$, the value of V_t is called the *no-arbitrage price* of the European option Z at time t. In fact, whenever an equivalent martingale measure exists, one can take V_t as the price of the option dealt at time t, since an admissible strategy covering the payoff at maturity exists. This is obviously subject to the existence of a martingale measure $\mathbb{P}^*$ for the discounted price process. But, it could not be unique and it would be unpleasant if two different martingale measures gave two different no-arbitrage prices. This is not really a problem, as it follows from the next result.

Proposition 2.20. *Suppose there exist two equivalent martingale measures $\mathbb{P}_1^*$ and $\mathbb{P}_2^*$. Let Z be a European option with maturity T and attainable both in $(X, \mathscr{M}_a(\mathbb{P}_1^*))$ and $(X, \mathscr{M}_a(\mathbb{P}_2^*))$. Then*

$$V_t = \mathbb{E}_1^*(e^{-\int_t^T r_s\,ds} Z \mid \mathscr{F}_t) = \mathbb{E}_2^*(e^{-\int_t^T r_s\,ds} Z \mid \mathscr{F}_t),$$

in which $\mathbb{E}_i^*$ denotes the expectation under $\mathbb{P}_i^*$, $i = 1, 2$. In particular, the no-arbitrage prices under $\mathbb{P}_1^*$ and $\mathbb{P}_2^*$ agree.

Proof. Let H_1 and H_2 be replicating strategies for (Z, T) in $\mathscr{M}_a(\mathbb{P}_1^*)$ and $\mathscr{M}_a(\mathbb{P}_2^*)$ respectively. In particular, they are both admissible. Since $\mathbb{P}_1^*$ and $\mathbb{P}_2^*$ are both martingale measures, by Proposition 2.13 one hat that $\tilde{V}_t(H_1)$ is a $\mathbb{P}_2^*$-supermartingale, as well as $\tilde{V}_t(H_2)$ is a $\mathbb{P}_1^*$-supermartingale. Moreover, Z is attainable by H_1 and H_2, so that $V_T(H_1) = Z = V_T(H_2)$ and thus $\tilde{V}_T(H_1) = e^{-\int_0^T r_s\,ds} Z = \tilde{V}_T(H_2)$. Finally, since $H_1 \in \mathscr{M}_a(\mathbb{P}_1^*)$ and $H_2 \in \mathscr{M}_a(\mathbb{P}_2^*)$, by Proposition 2.19, one has that $\tilde{V}_t(H_1) = \mathbb{E}_1^*(e^{-\int_0^T r_s\,ds} Z \mid \mathscr{F}_t)$ and $\tilde{V}_t(H_2) = \mathbb{E}_2^*(e^{-\int_0^T r_s\,ds} Z \mid \mathscr{F}_t)$. By using such properties, it follows that

$$\begin{aligned} \tilde{V}_t(H_2) = \;& \mathbb{E}_2^*(e^{-\int_0^T r_s\,ds} Z \mid \mathscr{F}_t) = \mathbb{E}_2^*(\tilde{V}_T(H_1) \mid \mathscr{F}_t) \\ \leq \;& \tilde{V}_t(H_1) = \mathbb{E}_1^*(e^{-\int_0^T r_s\,ds} Z \mid \mathscr{F}_t). \end{aligned}$$

By interchanging the role of $\mathbb{P}_1^*$ and $\mathbb{P}_2^*$, one obtains $\tilde{V}_t(H_2) \geq \tilde{V}_t(H_1)$ as well, so that the two expectations agree.

□

By summarizing, the existence of an equivalent martingale measure $\mathbb{P}^*$ (that is, a measure equivalent to $\mathbb{P}$ under which the discounted price process is a martingale) allows to state that the discounted portfolio is a $\mathbb{P}^*$-local martingale. Moreover, the market given by the $\mathbb{P}^*$-admissible strategies (that is, the ones for which the associated portfolio is a non negative $\mathbb{P}^*$-martingale) turns out to be arbitrage free and the notion of no-arbitrage price for attainable options is well defined. Moreover, one might ask if a double link between the existence of $\mathbb{P}^*$ and no arbitrage strategies does hold. The answer is positive: such a kind of result can be stated, and is called *fundamental theorem of asset pricing.* In literature, there are several results in this direction, according to the model chosen for the market: the interested reader can find good references quoted in Musiela and Rutkowski [10].

By resuming, if an equivalent martingale measure exits, the no arbitrage-price is well defined for any attainable option. Therefore, it would be nice that any options (at least with good integrable properties) were attainable. In financial terms, this means that the market is *complete*. Let us now formalize the concept of market completeness.

Definition 2.21. *Let $\mathbb{P}^*$ be an equivalent martingale measure. The model is said to be* complete *if any European option whose payoff Z belongs to $L^p(\Omega, \mathbb{P}^*)$ for some $p > 2$ is attainable in $(X, \mathscr{M}_a(\mathbb{P}^*))$. In the opposite case, the market model is said to be* incomplete.

We have the following fundamental result

Theorem 2.22. *The following statements hold:*

(*i*) *If the model is complete then there exists a unique equivalent martingale measure.*

(*ii*) *If $N = d$, σ is invertible and the matrix field $a = \sigma\sigma^*$ is uniformly elliptic [that is, there exists $\alpha > 0$ such that $\langle a(t,x)\lambda, \lambda\rangle \geq \alpha\,|\lambda|^2$, for any $\lambda \in \mathbb{R}^d$ and $t \leq T^*$], then the model is complete.*

Proof. (*i*) Suppose there exist two martingale measures $\mathbb{P}_1^*$ and $\mathbb{P}_2^*$. Take $A \in \mathscr{F}_{T^*}$ and consider the option (Z, T^*) with $Z = e^{\int_0^{T^*} r_s\,ds}\,1_A$. Notice that Z is $\mathscr{F}_{T^*}$-measurable and $Z \in L^p(\Omega, \mathbb{P}_i^*)$ for any p and $i = 1, 2$. Since the market is complete, Z is attainable both in $(X, \mathscr{M}_a(\mathbb{P}_1^*))$ and $(X, \mathscr{M}_a(\mathbb{P}_2^*))$. Then, by Proposition 2.20,

$$\mathbb{E}_1^*(e^{-\int_0^{T^*} r_s\,ds}\,Z) = \mathbb{E}_2^*(e^{-\int_0^{T^*} r_s\,ds}\,Z)$$

which gives $\mathbb{P}_1^*(A) = \mathbb{P}_2^*(A)$, and this holds for any $A \in \mathscr{F}_{T^*}$. Then, $\mathbb{P}_1^* \equiv \mathbb{P}_2^*$ on $\mathscr{F}_{T^*}$, that is there exists only one equivalent martingale measure.

(*ii*) By Proposition 2.13, an equivalent martingale measure $\mathbb{P}^*$ exists. In view of Proposition 2.19, we have now to show that for any non negative, belonging to $L^p(\Omega, \mathbb{P}^*)$ for some $p > 2$ and $\mathscr{F}_T$-measurable random variable Z, there exists a self financing strategy H such that

$$V_t(H) = \mathbb{E}^*(e^{-\int_t^T r_s ds}\,Z \,|\, \mathscr{F}_t),$$

or equivalently

$$\tilde{V}_t(H) = \mathbb{E}^*(e^{-\int_0^T r_s ds}\,Z \,|\, \mathscr{F}_t).$$

Let us set

$$\tilde{M}_t = \mathbb{E}^*(e^{-\int_0^T r_s ds}\,Z \,|\, \mathscr{F}_t).$$

By Proposition 2.4 and Theorem 2.12, we need to show that there exist d adapted processes $H_t^1, \dots, H_t^d$, all belonging to $L^2([0,T])$ a.s., such that

$$d\tilde{M}_t = \sum_{i=1}^d H_t^i d\tilde{X}_t^i = \sum_{i=1}^d H_t^i \tilde{X}_t^i \sum_{k=1}^d \sigma_k^i(t, X_t)\, dB_t^{*k} \tag{9}$$

where $B_t^* = B_t - \int_0^t \gamma_s^*\, ds$ and γ^* is the (bounded) process given in Proposition 2.13. In fact, if such H_t^i's exists, then taking $H_t^0 = \tilde{M}_t - \sum_{i=1}^d H_t^i \tilde{X}_t^i$, one obtains immediately that $H = (H^0, \dots, H^d)$ is a self financing strategy such that $\tilde{V}_t(H) = \tilde{M}_t$. This in turn shows that $H \in \mathscr{M}_a(\mathbb{P}^*)$, because $\tilde{M}_t$ is a non negative martingale under $\mathbb{P}^*$. Moreover, by construction, one has $V_T(H) = Z$, and the statement follows.

By resuming, we have only to prove that (9) holds, for some adapted processes $H_t^1, \dots, H_t^d \in L^2([0,T])$ a.s. The proof is a consequence of the representation theorem for Brownian martingales. In fact, since $\tilde{M}_t$ is a Brownian and square integrable martingale, one has

$$\tilde{M}_t = c + \int_0^t \tilde{Y}_s\, dB_s^*, \quad t \in [0,T],$$

in which $\tilde{Y}$ is an adapted process in $L^2([0,T])$ a.s. Now, by choosing H as a solution to

$$\sum_{i=1}^d H_t^i \tilde{X}_t^i \sigma_k^i(t, X_t) = \tilde{Y}_s^k, \quad k = 1, \dots, n$$

(which exists because σ is invertible), then the statement follows. Notice that

$$H_t^i = \frac{[(\sigma^*)^{-1}(t, X_t)\tilde{Y}_t]^i}{\tilde{X}_t^i} \quad i = 1, \dots, d,$$

in which σ^* denotes the transpose of σ. Now, since $(\sigma^*)^{-1}$ is bounded (recall that $a = \sigma\sigma^*$ is uniformly elliptic) and $\tilde{X}_t$ is a continuous process, then $H_t^i \in L^2([0,T])$ a.s for any i because $\tilde{Y}_t^k \in L^2([0,T])$ a.s for any k.

Unfortunately, the representation theorem for Brownian martingales cannot be applied in this way: in the procedure developed above, we have done a big mistake! Indeed, the technical point is that, since we are now working under $\mathbb{P}^*$ and then with the Brownian motion B^*, the referring filtration is not $\mathscr{F}_t$ but $\mathscr{F}_t^*$, that is the sigma algebra generated by B^* and completed by the $\mathbb{P}^*$-null sets (and in general, $\mathscr{F}_t^* \subset \mathscr{F}_t$). In other words, the above reasoning would be right if one had to work with $\mathbb{E}^*(e^{-\int_t^T r_s\, ds} Z \,|\, \mathscr{F}_t^*)$, and not with $\mathbb{E}^*(e^{-\int_t^T r_s\, ds} Z \,|\, \mathscr{F}_t)$.

So, in order to overcome this complication, let us put

$$\bar{M}_t = \mathbb{E}(e^{-\int_0^T r_s\, ds} Z\, \xi_T^{\gamma^*} \,|\, \mathscr{F}_t)$$

(let us stress the fact that the expectation is taken under the original measure $\mathbb{P}$). Notice that $\mathbb{E}(e^{-\int_0^T r_s\,ds}\,Z\,\xi_T^{\gamma*}) = \mathbb{E}^*(e^{-\int_0^T r_s\,ds}\,Z)$, so that the random variable $e^{-\int_0^T r_s\,ds}\,Z\,\xi_T^{\gamma*}$ is $\mathbb{P}$-integrable. Therefore, $\bar{M}_t$ is an $\mathscr{F}_t$-martingale under $\mathbb{P}$ and $\mathscr{F}_t$ gives the right filtration: we can assert that there exists an adapted process Y such that $\int_0^T |Y_s|^2\,ds < \infty$ a.s. and

$$d\bar{M}_t = \bar{M}_0 + \int_0^t Y_s\,dB_s.$$

Now, since $\tilde{M}_t = \mathbb{E}^*(e^{-\int_0^T r_s\,ds}\,Z\,|\,\mathscr{F}_t)$, by the Bayes rule one has

$$\tilde{M}_t = \frac{\mathbb{E}(e^{-\int_0^T r_s\,ds}\,Z\,\xi_T^{\gamma*}\,|\,\mathscr{F}_t)}{\mathbb{E}(\xi_T^{\gamma*}\,|\,\mathscr{F}_t)} = \frac{\mathbb{E}(e^{-\int_0^T r_s\,ds}\,Z\,\xi_T^{\gamma*}\,|\,\mathscr{F}_t)}{\xi_t^{\gamma*}} = \frac{\bar{M}_t}{\xi_t^{\gamma*}}.$$

Since $d\bar{M}_t = Y_t\,dB_t$ and $d\xi_t^{\gamma*} = \xi_t^{\gamma*}\,\gamma_t^*\,dB_t$, then by Ito's formula it follows that

$$d\tilde{M}_t = d[(\xi_t^{\gamma*})^{-1}\,\bar{M}_t] = (\xi_t^{\gamma*})^{-1}\,(Y_t - \bar{M}_t\gamma_t^*)\,dB_t^*.$$

We have then found the right representation formula for $\tilde{M}_t$, with $\tilde{Y}_t = (\xi_t^{\gamma*})^{-1}\times(Y_t - \bar{M}_t\gamma_t^*)$. The statement now follows if we show that $\tilde{Y}_t \in L^2([0,T])$ a.s. This is a consequence of the fact that $\bar{M} \in L^2(\Omega\times[0,T], \mathbb{P}\times dt)$ (so that $\bar{M} \in L^2([0,T])$ a.s under $\mathbb{P}$ and then a.s. under $\mathbb{P}^*$): by using first the Jensen inequality (for conditional expectation) and secondly the Hölder inequality, one has

$$\begin{aligned}\mathbb{E}\int_0^T \bar{M}_t^2\,dt &= \int_0^T \mathbb{E}(|\mathbb{E}(e^{-\int_0^T r_s\,ds}\,Z\,\xi_T^{\gamma*}\,|\,\mathscr{F}_t)|^2)\,dt\\ &\leq \int_0^T \mathbb{E}(e^{-2\int_0^T r_s\,ds}\,Z^2\,(\xi_T^{\gamma*})^2)\,dt\\ &\leq const\cdot\mathbb{E}(Z^{2\bar{p}})^{1/\bar{p}}\,\mathbb{E}((\xi_T^{\gamma*})^{2\bar{q}})^{1/\bar{q}},\end{aligned}$$

in which $\bar{p},\bar{q} > 0$ with $1/\bar{p} + 1/\bar{q} = 1$. Now, taking $\bar{p} = p/2$ and recalling that γ^* is bounded, the last r.h.s. is finite, and the proof is now completed.

□

2.5 Pricing and hedging European options. Greeks

In a market completeness framework, the unique equivalent martingale measure $\mathbb{P}^*$ is called *the risk-neutral measure*. We will put ourselves in such a framework: we suppose the uniqueness of the equivalent martingale measure $\mathbb{P}^*$, that is of the market price of risk γ^*. Therefore, our market model is arbitrage free and the no-arbitrage price of any European option Z with maturity T is well defined and given by

$$V_t = \mathbb{E}^*(e^{-\int_t^T r_s\,ds}\,Z\,|\,\mathscr{F}_t). \tag{10}$$

We refer to (10) as the *price* of the option with payoff Z and maturity T, as seen at time $t < T$. Thus, if one sees V_t as t varies, it gives the value of the replicating portfolio; for a fixed time t, V_t gives the price of the associated option contract written at time t.
Now, this solves the problem on the hand of the buyer, who has to pay an amount equal to V_t to the seller. But now we have to give an answer to the problem from the point of view of the seller: which strategy should the seller take into account in order to deliver the contract? In other words, we have here the value of the replicating portfolio but the seller needs the strategy: we should produce a replicating strategy, giving for it some usable representation. This is called the *hedging* problem.
In the proof of Theorem 2.22 we have seen a replicating strategy but only from a theoretical point of view, since it has been built by means of the martingale representation theorem and this is not a constructive result. Let us see another way to hedge options: the *delta hedging* way.

Let us consider a European option, with payoff Z and maturity T. Since Z is $\mathscr{F}_T$-measurable, we can consider the case in which

$$Z = \phi_T(X),$$

that is, a function of the prices process $(X_t)_{t\in[0,T]}$. Since the riskless asset whose price is X^0 plays a separate role, we can now consider X as the vector of the prices of the risk assets, that is $X_t = (X_t^1, \dots, X_t^N)$. For example, in the call option case $\phi_T(X) = g_{\text{call}}(X_T) = (X_T - K)_+$ and in the put option one, $\phi_T(X) = g_{\text{put}}(X_T) = (K - X_T)_+$.
Let us stress that this representation for the payoff is not really restrictive: almost any option has a payoff depending on the prices process X over $[0, T]$. More precisely, whenever ϕ is not a function depending strictly on X_T (as it happens in the case of call or put options) but depends on the path $(X_t)_{t\in[0,T]}$, then the associated option is called *path dependent* (for example, barrier options, Asiatic ones etc., we will see in the sequel some examples).

Whenever $Z = \phi(X_T)$, the replicating portfolio can be written as $V_t = \mathbb{E}^*(e^{-\int_t^T r_s\,ds}\,\phi(X_T)\,|\,\mathscr{F}_t)$. Now, let us assume from now on that the randomness of the spot rate r is driven by the risky asset prices, that is

$$r_t = r(t, X_t).$$

Under $\mathbb{P}^*$, X is now a diffusion and then a Markov process: the conditional expectation giving the option price is done on a random variable depending on a Markov process, and therefore the result is a function of (t, X_t). In other words, thanks to the Markov property we can write

$$V_t = \mathbb{E}^*(e^{-\int_t^T r(s,X_s)\,ds}\,\phi(X_T)\,|\,\mathscr{F}_t) = P(t, X_t),$$

for some suitable function P. By setting $\tilde{P}(t,x) = e^{-\int_0^t r_s\,ds} P(t, x\, e^{\int_0^t r_s\,ds})$, then the replicating discounted portfolio is given by

$$\tilde{V}_t = \tilde{P}(t, \tilde{X}_t).$$

Now, suppose that P is in the class $C^{1,2}$ of the continuous function with first derivative in the first variable and second derivative in the second variable. By (8), one has

$$\frac{d\tilde{X}_t^i}{\tilde{X}_t^i} = \sum_{k=1}^d \sigma_k^i(t, X_t)\, dB^{*k}(t), \quad i = 1, \dots, N,$$

so that by applying Ito's formula one has:

$$d\tilde{V}_t = d\tilde{P}(t, \tilde{X}_t) = \Big(\partial_t \tilde{P}(t, \tilde{X}_t) + \tfrac{1}{2} \sum_{i,j=1}^N \partial^2_{x_i x_j} \tilde{P}(t, \tilde{X}_t)\, a_{ij}(t, X_t)\, \tilde{X}_t^i\, \tilde{X}_t^j \Big) dt$$
$$+ \sum_{i=1}^N \partial_{x_i} \tilde{P}(t, \tilde{X}_t)\, d\tilde{X}_t^i$$

where $a = \sigma\, \sigma^*$. Now, since $\tilde{V}$ is a martingale, it then has to follow that

$$\partial_t \tilde{P}(t, \tilde{X}_t) + \frac{1}{2} \sum_{i,j=1}^N \partial^2_{x_i x_j} \tilde{P}(t, \tilde{X}_t)\, a_{ij}(t, X_t)\, \tilde{X}_t^i\, \tilde{X}_t^j = 0, \quad \text{a.s.} \tag{11}$$

and, by Proposition 2.4, the strategy defined by

$$H_t^i = \partial_{x_i} \tilde{P}(t, \tilde{X}_t),\ i = 1, \dots, N, \text{ and } H_t^0 = \tilde{V}_t - \sum_{i=1}^N H_t^i\, \tilde{X}_t^i \tag{12}$$

is the required replicating one. Such equalities can be written also in terms of the function P giving the (non discounted) portfolio. In such a case, straightforward computations allow to rewrite (11) as

$$\partial_t P(t,x) + \mathscr{L}_t^* P(t,x) - r(t,x)\, P(t,x) = 0, \quad t \in (0,T),\ x \in \mathbb{R}^N \tag{13}$$

where $\mathscr{L}_t^*$ is the generator of X under the risk neutral measure (acting on the space variable only, see (8)), that is

$$\mathscr{L}_t^* g(x) = r_t \sum_i^N \partial_{x_i} g(x)\, x_i + \frac{1}{2} \sum_{i,j=1}^N \partial^2_{x_i x_j} g(x)\, a_{ij}(t,x)\, x_i\, x_j.$$

Equation (13) is sometimes called the *fundamental PDE following from the no-arbitrage approach.*

Concerning (12), it becomes

$$H_t^i = \partial_{x_i} P(t, X_t), \; i = 1, \dots, N, \text{ and } H_t^0 = V_t - \sum_{i=1}^{N} H_t^i \tilde{X}_t^i \qquad (14)$$

The above quantities H_t^i, $i = 1, \dots, N$, are called the *delta* of the option and are usually denoted with the notation Δ:

$$\Delta_i(t, X_t) = \partial_{x_i} P(t, X_t), \; i = 1, \dots, N.$$

The delta gives then the replicating portfolio. A really interesting fact is that the delta gives also the sensitivity of the price with respect to the initial values of the underlying asset prices. This is a special case of *Greek*. The Greeks are indeed quantities giving the sensitivity of the price with respect to several parameters which have a financial meaning. They are taken into special account by practitioners, also because they have themselves special financial interests and meanings. The most used Greeks can be summarized as follows:

- *delta*: sensitivity of the price w.r.t. the initial values:

$$\Delta_i = \partial_{x_i} P;$$

- *gamma*: sensitivity of the deltas w.r.t. the initial values:

$$\Gamma_{ij} = \partial^2_{x_i x_j} P;$$

- *theta*: sensitivity of the price w.r.t. the initial time:

$$\Theta = \partial_t P;$$

- *rho*: sensitivity of the price w.r.t. the spot rate:

$$\mathbf{Rho} = \partial_r P;$$

- *vega*: sensitivity of the price w.r.t. the volatility:

$$\mathbf{Vega} = \partial_\sigma P.$$

Obviously, in the rho and vega cases, the derivatives has to be understood in a suitably functional way whenever r and σ are not constant. These last Greeks gives the behavior of the price and then of the portfolio with respect to purely financial quantities (i.e. the interest rate and the volatility), while the other ones (delta, gamma and theta) give the rate of change of the portfolio with respect to parameters connected to the assets on which the European option is written (the starting instant and prices of the assets). But these other Greeks have also other interesting meanings. For example, Equation (13) can be rewritten in terms of the Greeks Δ, Γ and Θ, so that they are deeply connected. In the one dimensional case, Equation (13) becomes

$$\Theta + \frac{1}{2}\sigma^2\, x^2\Gamma + r\,\Delta\, x = r\,P.$$

If we consider a delta-neutral portfolio, that is $\Delta = 0$, then the equation above becomes $\Theta + \frac{1}{2}\sigma^2\, x^2\Gamma = r\,P$, which shows that Θ and Γ are negatively correlated (if Θ becomes large and positive then Γ has to become large and negative, and vice versa). The theta, gamma and delta Greeks are also widely used to tackle credit risk problems, such as the VaR (Value at Risk) one. The interested reader can find useful discussions (also from a practical point of view), remarks and references on topics regarding Greeks in Hull [6].

Example 2.23. (The Black and Scholes price and Greek formulas for call and put options) Let us evaluate the price of a call and put option in the standard one dimensional Black and Scholes model, as well as the associated Greeks.
Under the risk-neutral measure $\mathbb{P}^*$, the price of the risk asset evolve as

$$\frac{dX_t}{X_t} = r\,dt + \sigma\,dB^*(t) \tag{15}$$

and the price of the call option with maturity T as seen at time t, is given by

$$P_{\text{call}}(t, X_t) = \mathbb{E}^*(e^{-r(T-t)}\,(X_T - K)_+ \,|\, \mathscr{F}_t),$$

where K stands for the strike price. If we use the notation $X^{t,x}$ to denote the solution X of (15) starting at x at time t, by using Ito's lemma it easily follows that

$$X_s^{t,x} = x\, e^{(r-\frac{1}{2}\,\sigma^2)(s-t)+\sigma(B_s^*-B_t^*)}, \quad s \geq t,$$

and we can write

$$\begin{aligned} P_{\text{call}}(t,x) &= \mathbb{E}^*(e^{-r(T-t)}\,(X_T^{t,x} - K)_+) \\ &= e^{-r(T-t)}\,\mathbb{E}^*((x\, e^{(r-\frac{1}{2}\,\sigma^2)(T-t)+\sigma(B_T^*-B_t^*)} - K)_+). \end{aligned}$$

The expectation above is easy to compute: straightforward computations allow to write

$$P_{\text{call}}(t,x) = x\,\mathscr{N}(d_1(T-t,x)) - K\,e^{-r(T-t)}\,\mathscr{N}(d_2(T-t,x)), \tag{16}$$

where $\mathcal{N}$ denotes the standard Gaussian cumulative distribution function and, for $s, x > 0$,

$$d_1(s,x) = \frac{\ln(x/K) + (r + \sigma^2/2)s}{\sigma\sqrt{s}} \quad \text{and} \quad d_2(s,x) = d_1(s,x) - \sigma\sqrt{s}. \tag{17}$$

Concerning the delta, again by straightforward computations one obtains

$$\Delta_{\text{call}}(t,x) = \partial_x P_{\text{call}}(t,x) = \mathcal{N}(d_1(T-t,x)). \tag{18}$$

As for the put option, one could use similar arguments or also the call-put parity property (that is $P_{\text{call}}(t,x) - P_{\text{put}}(t,x) = e^{-r(T-t)}\,\mathbb{E}^*((X_T^{t,x} - K)))$, in order to obtain the associated price and delta following the formulas

$$\begin{aligned} P_{\text{put}}(t,x) &= K\,e^{-r(T-t)}\,\mathcal{N}(-d_2(T-t,x)) - x\,\mathcal{N}(-d_1(T-t,x)), \\ \Delta_{\text{put}}(t,x) &= \partial_x P_{\text{put}}(t,x) = \mathcal{N}(d_1(T-t,x)) - 1. \end{aligned} \tag{19}$$

Also the other Greeks can be explicitly written[5]. They can be summarized as follows:

$$\begin{aligned} \Gamma_{\text{call}}(t,x) &= \frac{\mathcal{N}'(d_1(T-t,x))}{x\,\sigma\sqrt{T-t}} = \Gamma_{\text{put}}(t,x) \\ \Theta_{\text{call}}(t,x) &= -\frac{x\,\mathcal{N}'(d_1(T-t,x))\,\sigma}{2\sqrt{T-t}} - r\,x\,e^{-r(T-t)}\,\mathcal{N}(d_2(T-t,x)) \\ \Theta_{\text{put}}(t,x) &= -\frac{x\,\mathcal{N}'(d_1(T-t,x))\,\sigma}{2\sqrt{T-t}} + r\,x\,e^{-r(T-t)}\,\mathcal{N}(-d_2(T-t,x)) \\ \mathbf{Rho}_{\text{call}}(t,x) &= x\,(T-t)\,e^{-r(T-t)}\,\mathcal{N}(d_2(T-t,x)) \\ \mathbf{Rho}_{\text{put}}(t,x) &= -x\,(T-t)\,e^{-r(T-t)}\,\mathcal{N}(-d_2(T-t,x)) \\ \mathbf{Vega}_{\text{call}}(t,x) &= x\sqrt{T-t}\,\mathcal{N}'(d_1(T-t,x)) = \mathbf{Vega}_{\text{put}}(t,x) \end{aligned} \tag{20}$$

(obviously, the function $\mathcal{N}'$ is the standard Gaussian probability density function).

[5]Since this model assumes constant spot rate r and volatility σ, the rho and vega Greeks - sensitivities w.r.t. spot rate and volatility - do not need any special definition.

3 Pricing American options

As already mentioned, an American option differs from a European one in the fact that it can be exercised in any instant $t \le T$, being T the associated maturity. More precisely, an American contingent claim can be formalized as a pair $Z^a = (T, g)$, consisting of a maturity date T and a *cash flows function* $g : [0,T] \times \mathbb{R}^N_+ \to \mathbb{R}_+$ such that the associated payoff at the exercise date $\tau \in [0,T]$ is given by $g(\tau, X_\tau)$. Now, since exercising the option does depend on the market developments, it is quite natural to require that τ can be random and more precisely a stopping time w.r.t. the filtration $\{\mathscr{F}_t\}_{0\le t\le T}$ giving the market information at any instant up to the maturity date T. Thus, we require that $\tau \in \mathscr{T}_{[0,T]}$, being $T_{[0,T]}$ the set of all stopping times over $[0,T]$, that is the stopping times w.r.t. $\{\mathscr{F}_t\}_{0\le t\le T}$ taking values on $[0,T]$ a.s.

Typical examples are American call and put options, in which

$$g_{\rm call}(t,x) = (x-K)_+ \quad \text{and} \quad g_{\rm put}(t,x) = (K-x)_+.$$

Here, the difference from the European case is in the fact that the option can be exercised at any time before T. We are in fact treating here *standard* American options. Sometimes, the set of the exercise dates is taken as a subset $\mathscr{T}'$ of $\mathscr{T}_{[0,T]}$. For example, one could take $\mathscr{T}' = \{t_1, t_2, \dots, t_n\}$ with $t_1, t_2, \dots, t_n$ fixed and deterministic times in $[0,T]$. In such a case, one is speaking about *Bermudian* options. And obviously, the Bermudian case becomes the European one whenever $\mathscr{T}' = \{T\}$.

Our aim here is to find the fair price and the exercise date for an American security by means of "perfect hedging" as well as "no arbitrage" arguments. For the sake of clearness of notations, we will do this by assuming 0 as the initial referring date.

The definitions above are sufficient to deduce that such a problem is much more sophisticated to handle than the European one. Therefore, since the underlying mathematical theory becomes more and more advanced as difficulties increase, to avoid a too technical machinery let us assume here that:

- the spot rate process r_t is deterministic;
- the market is complete; in particular, the assumptions in (*ii*) of Theorem 2.22 are fulfilled, that is $N = d$, σ is invertible and the matrix field $a = \sigma\sigma^*$ is uniformly elliptic. Recall that in this case the market price of risk γ^* uniquely exists (being also bounded) and the risk neutral measure $\mathbb{P}^*$ given by (5) is well defined.

3.1 Price by perfect-hedging arguments

To our purpose, it is useful to consider more general trading strategies. In fact, differently from the previously used kind of strategies, we consider here also the possibility that some wealth is not reinvested, that is "consumed", and C_t gives the cumulative money which is put aside at time t.

Definition 3.1. *A* trading and consumption strategy *is a pair* (H, C) *given by a classical trading strategy* H *and a consumption process* C*, which is defined as a non-decreasing and adapted process such that* $C_0 = 0$*.* (H, C) *is said to be self-financing if* $H^0 \in L^1([0, T])$ *a.s.,* $H^i \in L^2([0, T])$ *a.s. for any* $i = 1, \dots, N$ *and the associated portfolio, given by*

$$V_t(H, C) = \sum_{i=0}^{N} H_t^i X_t^i, \quad t \in [0, T],$$

satisfies

$$V_t(H, C) = V_0(H, C) + \int_0^t \sum_{i=0}^{N} H_t^i \, dX_t^i - C_t, \quad t \in [0, T].$$

Similarly to what previously proved, one has:

Proposition 3.2. (H, C) *is a self-financing trading and consumption strategy if and only if the discounted portfolio* $\tilde{V}(H, C)$ *satisfies*

$$d\tilde{V}_t(H, C) = \sum_{i=1}^{N} H_t^i \, d\tilde{X}_t^i - e^{-\int_0^t r_s \, ds} dC_t, \quad t < T.$$

If moreover $V_t(H, C) \geq 0$ *a.s. for any* t *then* $\tilde{V}_t(H, C)$ *is a supermartingale under* $\mathbb{P}^*$*.*

Proof. For the first part, it is sufficient to apply Ito's lemma (as in Proposition 2.4). As for the second part, recall that

$$M_t = \int_0^t \sum_{i=1}^{N} H_s^i \, d\tilde{X}_s^i$$

is a continuous local martingale which is also bounded from below, because

$$\tilde{M}_t = \tilde{V}_t(H, C) - V_0 + \int_0^t e^{-\int_0^u r_s \, ds} dC_u \geq -V_0$$

(recall that C is non-decreasing). Therefore, $\tilde{M}_t$ is a supermartingale. Now, as $t > s$,

$$\begin{aligned} \mathbb{E}^*(\tilde{V}_t(H, C) \,|\, \mathscr{F}_s) &= \mathbb{E}^*(V_0 + \tilde{M}_t - \int_0^t e^{-\int_0^u r_v \, dv} dC_u \,|\, \mathscr{F}_s) \\ &\leq V_0 + \tilde{M}_s - \int_0^s e^{-\int_0^u r_v \, dv} dC_u - \mathbb{E}^*(\int_s^t e^{-\int_0^u r_v \, dv} dC_u \,|\, \mathscr{F}_s) \end{aligned}$$

$$= \tilde{V}_s(H, C) - \mathbb{E}^*(\int_s^t e^{-\int_0^u r_v\, dv} dC_u \,|\, \mathscr{F}_s) \leq \tilde{V}_s(H, C)$$

and the statement follows.

□

Definition 3.3. *A self-financing trading and consumption strategy (H, C) is said to hedge the American option $Z^a = (T, g)$ if*

$$\text{for any } t \in [0, T], \quad V_t(H, C) \equiv \sum_{i=0}^{N} H_t^i X_t^i \geq g(t, X_t) \quad a.s.$$

The set $\mathscr{H}(Z^a)$ will denote all the self-financing trading and consumption strategies hedging the American option Z^a.

Now, if the writer of the option follows a strategy $(H, C) \in \mathscr{H}(Z^a)$ then at any instant $t \leq T$ he will have a wealth at least equal to $g(t, X_t)$, an amount which is exactly the payoff of the option if it is exercised at time t. Therefore, it would be useful and important to study the minimal value of a hedging scheme for an American option. To this purpose, let us introduce the concept of "essential supremum": for a given family $\mathscr{H}$ of $\mathscr{G}$-measurable r.v.'s, define

$$Z = \operatorname{ess\,sup}_{Y \in \mathscr{H}} Y$$

as the r.v. Z defined as 0 if $\mathscr{H} = \emptyset$, otherwise it is such that

- Z is $\mathscr{G}$-measurable;
- for any $Y \in \mathscr{H}$, $Z \geq Y$ a.s.;
- if Z' is a $\mathscr{G}$-measurable r.v. such that $Z' \geq Y$ a.s. for all $Y \in \mathscr{H}$, then $Z' \geq Z$.

Let us stress that the essential supremum always exists, being a.s. unique (see e.g. Theorem 2.3.1 of Wong [5]).
For $t \leq T$, let us define $\mathscr{T}_{[t,T]}$ as the set of the $\mathscr{F}_t$-stopping times taking values on $[t, T]$. Then, one has the following important property.

Proposition 3.4. *Let $(Y_t)_{t \leq T}$ be a stochastic process which is $\mathscr{F}_t$-adapted, $Y_t \geq 0$ a.s. for all t and $\mathbb{E}^*(\sup_{t \leq T} Y_t) < \infty$. For $t \leq T$, set*

$$\Gamma_t^T(A) = \sup_{\tau \in \mathscr{T}_{[t,T]}} \mathbb{E}^*(Y_\tau \, 1_A), \quad A \in \mathscr{F}_t.$$

Then, Γ_t^T is a (positive) measure on $(\Omega, \mathscr{F}_t)$ which is absolutely continuous w.r.t. $\mathbb{P}^*$ on $\mathscr{F}_t$ and

$$\frac{d\Gamma_t^T}{d\mathbb{P}^*} = \operatorname{ess\,sup}_{\tau \in \mathscr{T}_{[t,T]}} \mathbb{E}^*(Y_\tau \,|\, \mathscr{F}_t).$$

Proof. Γ_t^T is a positive set function defined on $\mathscr{F}_t$. One has first to prove the σ-additivity property. Take $\{A_n\}_n \subset \mathscr{F}_t$ such that $A_n \cap A_m = \emptyset$ as $n \neq m$. One immediately has that $\Gamma_t^T(\cup_n A_n) \leq \sum_n \Gamma_t^T(A_n)$, so that

$$\sum_n \Gamma_t^T(A_n) = \Gamma_t^T(\cup_n A_n) + \varepsilon$$

with $\varepsilon \geq 0$. We have to show that $\varepsilon = 0$. Then, suppose $\varepsilon > 0$. By the definition of sup, for any n there exists $\tau_n^* \in \mathscr{F}_{[t,T]}$ such that

$$\Gamma_t^T(A_n) \leq \mathbb{E}^*(Y_{\tau_n^*} 1_{A_n}) + \varepsilon/2^{n+1}.$$

Take now $\tau^* = \sum_n \tau_n^* 1_{A_n} + \bar{\tau} 1_{A^c}$, in which $A = \cup_n A_n$ and $\bar{\tau} \in \mathscr{T}_{[t,T]}$. Then $\tau^* \in \mathscr{F}_{[t,T]}$ and

$$\sum_n \Gamma_t^T(A_n) \leq \mathbb{E}^*(Y_{\tau^*} 1_A) + \varepsilon/2.$$

Therefore,

$$\sum_n \Gamma_t^T(A_n) \leq \Gamma_t^T(\cup_n A_n) + \varepsilon/2,$$

which gives a contradiction. Then, $\varepsilon = 0$ and Γ_t^T is in effect a measure on $(\Omega, \mathscr{F}_t)$. It is trivially absolutely continuous w.r.t. $\mathbb{P}^*$, so that the Radon-Nicodym derivative actually exists and is, by definition, $\mathscr{F}_t$-measurable. Now, for any $A \in \mathscr{F}_t$,

$$\mathbb{E}^*\Big(\frac{d\Gamma_t^T}{d\mathbb{P}^*} 1_A\Big) = \Gamma_t^T(A) \geq \mathbb{E}^*(Y_\tau 1_A) = \mathbb{E}^*\Big(\mathbb{E}^*(Y_\tau \,|\, \mathscr{F}_t) 1_A\Big).$$

Therefore, $\frac{d\Gamma_t^T}{d\mathbb{P}^*} \geq \mathbb{E}^*(Y_\tau \,|\, \mathscr{F}_t)$ a.s. for any $\tau \in \mathscr{F}_{[t,T]}$, so that $\frac{d\Gamma_t^T}{d\mathbb{P}^*} \geq \operatorname{ess\,sup}_{\tau\mathscr{F}_{[t,T]}} \mathbb{E}^*(Y_\tau \,|\, \mathscr{F}_t)$ a.s. In order to prove the equality, take a r.v. Z' which is $\mathscr{F}_t$-measurable and $Z' \geq \mathbb{E}^*(Y_\tau \,|\, \mathscr{F}_t)$ a.s. for any $\tau \in \mathscr{F}_{[t,T]}$. Then, for all $A \in \mathscr{F}_t$ one has $\mathbb{E}^*(Z' I_A) \geq \mathbb{E}^*(\mathbb{E}^*(Y_\tau \,|\, \mathscr{F}_t) 1_A) = \mathbb{E}^*(Y_\tau 1_A)$ for all $\tau \in \mathscr{T}_{[t,T]}$, that is

$$\mathbb{E}^*(Z' I_A) \geq \sup_{\tau \in \mathscr{T}_{[t,T]}} \mathbb{E}^*(Y_\tau 1_A) = \Gamma_t^T(A) = \mathbb{E}^*\Big(\frac{d\Gamma_t^T}{d\mathbb{P}^*} 1_A\Big).$$

Therefore, $Z' \geq \frac{d\Gamma_t^T}{d\mathbb{P}^*}$ a.s. and the statement follows.

□

We are now ready to study the perfect hedging strategy, that is a strategy which hedges the American option and gives the minimal portfolio.

Theorem 3.5. *Let $Z^a = (T, g)$ be an American option such that the process $(g(t, X_t))_{0 \leq t \leq T}$ is càdlàg and $\sup_{t\in[0,T]} g(t, X_t) \in L^p(\Omega, \mathbb{P}^*)$ for some $p > 2$. Set*

$$V_t = \operatorname{ess\,sup}_{\tau \in \mathscr{T}_{[t,T]}} \mathbb{E}^*(e^{-\int_t^\tau r_s\, ds} g(\tau, X_\tau) \,|\, \mathscr{F}_t), \quad t \in [0, T], \tag{21}$$

where $\mathscr{T}_{[t,T]}$ stands for the set of the stopping times over $[t,T]$. Then there exists a self financing trading and consumption strategy $(H,C) \in \mathscr{H}(Z^a)$ such that $V_t(H,C) = V_t$. Moreover, for any $(\bar{H},\bar{C}) \in \mathscr{H}(Z^a)$ one has $V_t(\bar{H},\bar{C}) \geq V_t$.

Before to continue with the proof, let us observe that Theorem 3.5 allows one to naturally define V_t as the fair price of the American option Z^a as seen at time $t \in [0,T]$. In fact, Theorem 3.5 tells us that V_t is the minimal value of the portfolios associated to a strategy hedging the option. In other words, there exists a self-financing trading and consumption strategy giving a *perfect hedging* against the American option.
Moreover, let us notice that, since $\tau \equiv T \in \mathscr{T}_{[t,T]}$ for any $t \leq T$, one has

$$\operatorname{ess\,sup}_{\tau\in\mathscr{T}_{[t,T]}} \mathbb{E}^*(e^{-\int_t^\tau r_s\,ds} g(\tau, X_\tau) \,|\, \mathscr{F}_t) \geq \mathbb{E}^*(e^{-\int_t^T r_s\,ds} g(T, X_T) \,|\, \mathscr{F}_t)$$

which tells us that the price of an American option is always greater than the price of the associated European option (that is, the European option with maturity T and payoff $g(T, X_T)$).

Proof of Theorem 3.5. We only give a sketch of the proof, since it involves a very technical machinery.
Let $\tilde{g}(t,X_t) = e^{-\int_0^t r_s\,ds} g(t,X_t)$ denote the discounted cash flows process and let J_t denote the Snell envelope of $\tilde{g}(t,X_t)$, that is the smallest supermartingale over $[0,T]$ greater or equal to $\tilde{g}(t,X_t)$. Let us first show that

$$\begin{aligned} J_t &= \operatorname{ess\,sup}_{\tau\in\mathscr{T}_{[t,T]}} \mathbb{E}^*(\tilde{g}(\tau, X_\tau) \,|\, \mathscr{F}_t) \\ &= \operatorname{ess\,sup}_{\tau\in\mathscr{T}_{[t,T]}} \mathbb{E}^*(e^{-\int_0^\tau r_s\,ds} g(\tau, X_\tau) \,|\, \mathscr{F}_t), \end{aligned}$$

$t \in [0,T]$, and notice that $V_t = e^{\int_0^t r_s ds} J_t$.
Take $s \leq t$ and $A \in \mathscr{F}_s$. Since $\mathscr{T}_{[t,T]} \subset \mathscr{T}_{[s,T]}$, one has

$$\sup_{\tau\in\mathscr{F}_{[t,T]}} \mathbb{E}^*(\tilde{g}(\tau, X_\tau)\, 1_A) \leq \sup_{\tau\in\mathscr{F}_{[s,T]}} \mathbb{E}^*(\tilde{g}(\tau, X_\tau)\, 1_A)$$

If Γ_t^T and Γ_s^T denote the measures defined as in Proposition 3.4 with $Y_t = \tilde{g}(t,X_t)$, we can say that

$$\Gamma_t^T(A) \leq \Gamma_s^T(A)$$

for all $A \in \mathscr{F}_s \subset \mathscr{F}_t$. Therefore,

$$\mathbb{E}^*\Big(\frac{d\Gamma_t^T}{d\mathbb{P}^*}\, 1_A\Big) \leq \mathbb{E}^*\Big(\frac{d\Gamma_s^T}{d\mathbb{P}^*}\, 1_A\Big)$$

for al $A \in \mathscr{F}_s$. By Proposition 3.4, $\frac{d\Gamma_t^T}{d\mathbb{P}^*} = J_t$ and $\frac{d\Gamma_s^T}{d\mathbb{P}^*} = J_s$, so that $\mathbb{E}^*(J_t 1_A) \leq \mathbb{E}^*(J_s 1_A)$, and then $(J_t)_{t\leq T}$ is actually a supermartingale.

Obviously, $J_t \geq \tilde{g}(t, X_t)$ because $t \in \mathscr{T}_{[t,T]}$. Now, let U_t be a supermartingale such that $U_t \geq \tilde{g}(t, X_t)$ a.s. for any $t \leq T$. Then, by using the Optional Sampling Theorem, for any $\tau \in \mathscr{T}_{[t,T]}$ one has

$$U_t \geq \mathbb{E}^*(U_\tau \,|\, \mathscr{F}_t) \geq \mathbb{E}^*(\tilde{g}(\tau, X_\tau) \,|\, \mathscr{F}_t)$$

so that

$$U_t \geq \operatorname{ess\,sup}_{\tau \in \mathscr{T}_{[t,T]}} \tilde{\mathbb{E}}^*(g(\tau, X_\tau) \,|\, \mathscr{F}_t) = J_t.$$

Therefore, J_t is the Snell envelope of $\tilde{g}(t, X_t)$.

Let us now prove that there exists $(H, C) \in \mathscr{H}(Z^a)$ such that $V_t(H, C) = V_t$. It can be shown that J_t is a càdlàg and regular[6] process. Then, since all the technical assumptions are verified, one can use the Doob-Meyer decomposition Theorem (see Karatzas and Shreve [7], Theorem 4.10 and Theorem 4.14): $J_t = M_t - A_t$, where M is a (square integrable under $\mathbb{P}^*$) martingale and A is a continuous non-decreasing process with $A_0 = 0$. Moreover, by using (in the right way) the representation theorem for Brownian martingales (see the discussion in the proof of Theorem 2.22), one has $dM_t = Y_t\, dB_t^*$, for some adapted process Y such that $\mathbb{E}^*(\int_0^T |Y_t|^2\, dt) < \infty$. Then,

$$d(e^{-\int_0^t r_s\, ds} V_t) = dJ_t = Y_t\, dB_t^* - dA_t.$$

Take now H such that $\sum_{i=1}^N H_s^i X_s^i\, \sigma_k^i(s, X_s) = Y_s^k$ and C such that $C_0 = 0$ and $dC_t = e^{\int_0^t r_s\, ds} dA_t$ (and notice that C is non decreasing). Then one has

$$d(e^{-\int_0^t r_s\, ds} V_t) = \sum_{i=1}^N H_t^i\, d\tilde{X}_t^i - e^{-\int_0^t r_s\, ds}\, dC_t,$$

so that $V_t = V_t(H, C)$ and also the self financing property of (H, C) holds by Proposition 3.2. Since we have already seen that $V_t \geq g(t, X_t)$, we have shown the existence of $(H, C) \in \mathscr{H}(Z^a)$ such that $V_t = V_t(H, C)$.

Finally, take $(\bar{H}, \bar{C}) \in \mathscr{H}(Z^a)$. Since $V_t(\bar{H}, \bar{C}) \geq g(t, X_t) \geq 0$, by Proposition 3.2 one has that $\tilde{V}_t(\bar{H}, \bar{C})$ is a supermartingale such that $\tilde{V}_t(\bar{H}, \bar{C}) \geq \tilde{g}(t, X_t)$. Then, $\tilde{V}_t(\bar{H}, \bar{C}) \geq J_t$, so that $V_t(\bar{H}, \bar{C}) \geq e^{\int_0^t r_s\, ds} J_t = V_t$.

□

Let us add some remarks to the proof of Theorem 3.5, involving the Optimal Stopping Theory, so that we give no proofs and refer to Wong [5] or also Karatzas and Shreve [8].

First, one has that $J_t \geq e^{-\int_0^t r_s\, ds} g(t, X_t)$, for any t (in fact, $\tau = t \in \mathscr{T}_{[t,T]}$). Then, the Optimal Stopping Theory allows to find the stopping time τ_0 giving J_0: setting

$$\tau_0 = \inf\{t \in [0, T] \,:\, J_t = e^{-\int_0^t r_s\, ds} g(t, X_t)\},$$

[6] Y_t is called a *regular* process if for any $\alpha > 0$, $\mathbb{E}(Y_{\tau_n}) \to \mathbb{E}(Y_\tau)$ as $n \to \infty$ for any $\{\tau_n\} \in \mathscr{T}_{[0,\alpha]}$ such that $\tau_n \uparrow \tau \in \mathscr{T}_{[0,\alpha]}$.

where $\tau_0 = T$ if $J_t > e^{-\int_0^t r_s\,ds} g(t, X_t)$ for any $t \in [0, T]$, then

$$J_0 = \mathbb{E}^*(e^{-\int_0^{\tau_0} r_s\,ds} g(\tau_0, X_{\tau_0})). \tag{22}$$

Since $V_t = e^{-\int_0^t r_s\,ds} J_t$, everything can be rewritten in terms of the portfolio V_t defined in (21). One has $V_t \geq g(t, X_t)$ for any $t \in [0, T]$ and

$$\tau_0 = \inf\{t \in [0, T] \,:\, V_t = g(t, X_t)\}, \tag{23}$$

and therefore

$$V_{\tau_0} = g(\tau_0, X_{\tau_0}). \tag{24}$$

Furthermore, it follows that $C_{t\wedge\tau_0} = 0$, that is, there is no consumption up to the exercise date τ_0. In fact, $J_{t\wedge\tau_0}$ can be shown to be a martingale, so that the process A given by the Boob-Meyer decomposition satisfies $A_{t\wedge\tau_0} = 0$. Since $C_t = \int_0^t e^{\int_0^s r_u du} dA_s$, one immediately obtains that

$$C_{t\wedge\tau_0} = 0.$$

Roughly speaking, the self-financing trading and consumption strategy giving the perfect hedging behaves as a standard self-financing strategy up to the exercise date.
Obviously, everything can be translated to the time interval $[t, T]$: the stopping time

$$\tau_t = \inf\{s \in [t, T] \,:\, V_s = g(s, X_s)\},$$

can be shown to be optimal, that is

$$V_t = \mathbb{E}^*(e^{-\int_t^T r_s ds} g(\tau_t, X_{\tau_t}) \,|\, \mathscr{F}_t),$$

and he consumption process is null in between t and τ_t.

Example 3.6. (Price of an American call option) Suppose that $d = 1$ and consider an American call option, whose price as seen at time 0 is then given by

$$P_0^{\mathrm{am}} = \operatorname{ess\,sup}_{\tau\in\mathscr{T}_{[0,T]}} \mathbb{E}^*(e^{-\int_0^\tau r_s ds}(X_\tau - K)_+)$$

Setting P_0^{eu} as the price of the European call option, one obviously has $P_0^{\mathrm{am}} \geq P_0^{\mathrm{eu}}$, but in this special case the converse inequality holds as well[7], so that for a call option the American and European prices agree:

$$P_0^{\mathrm{am}} = P_0^{\mathrm{eu}}.$$

In other words, in the American call option case the exercise date is just at maturity.

[7]This fact holds whenever no dividends are payed (as considered in the present notes) and the instantaneous interest rate r is deterministic.

Let us prove that $P_0^{\rm am} \le P_0^{\rm eu}$. Since $e^{-\int_0^T r_s ds}(X_T - K)_+ \ge e^{-\int_0^T r_s ds}(X_T - K)$, for any $\tau \in \mathscr{T}_{[0,T]}$ one has

$$\begin{aligned} \mathbb{E}^*(e^{-\int_0^T r_s ds}(X_T - K)_+ \,|\, \mathscr{F}_\tau) &\ge \mathbb{E}^*(e^{-\int_0^T r_s ds}(X_T - K) \,|\, \mathscr{F}_\tau) \\ &= \mathbb{E}^*(e^{-\int_0^T r_s ds} X_T \,|\, \mathscr{F}_\tau) - e^{-\int_0^T r_s ds} K \\ = e^{-\int_0^\tau r_s ds} X_\tau - e^{-\int_0^T r_s ds} K &\ge e^{-\int_0^\tau r_s ds} X_\tau - e^{-\int_0^\tau r_s ds} K \end{aligned}$$

in which we have used the Optional Stopping Theorem to the $(\mathbb{P}^*, \mathscr{F}_t)$-martingale $e^{-\int_0^t r_s ds} X_t$ (recall that τ is a bounded stopping time w.r.t. $\mathscr{F}_t$) and the fact that $\tau \le T$. Since one obviously has $\mathbb{E}^*(e^{-\int_0^T r_s ds}(X_T - K)_+ \,|\, \mathscr{F}_\tau) \ge 0$, one can write

$$\begin{aligned} \mathbb{E}^*(e^{-\int_0^T r_s ds}(X_T - K)_+ \,|\, \mathscr{F}_\tau) &\ge \max\Big(0, e^{-\int_0^\tau r_s ds}(X_\tau - K)\Big) \\ &= e^{-\int_0^\tau r_s ds}(X_\tau - K)_+ \end{aligned}$$

and by passing to the expectation, for any $\tau \in \mathscr{T}_{[0,T]}$ one obtains

$$\mathbb{E}^*(e^{-\int_0^T r_s ds}(X_T - K)_+) \ge \mathbb{E}^*(e^{-\int_0^\tau r_s ds}(X_\tau - K)_+).$$

By passing to the sup as $\tau \in \mathscr{T}_{[0,T]}$ in the above r.h.s., one immediately obtains $P_0^{\rm eu} \ge P_0^{\rm am}$.

3.2 The behavior of the American put in the Black and Scholes model

Differently from the call option case, the price of an American put option differs from the price of the associated European one, and no closed form formulas are available even in the Black and Scholes framework. Let us put in this context and study the behavior of the American put. Suppose $d = 1$ and the risky underlying asset price X follows the Black and Scholes model. Under the risk neutral measure, one has

$$dX_t = rX_t dt + \sigma X_t dB_t^*, \quad X_0 = x$$

in which, as usual, r denotes the spot rate, σ stands for the volatility and B^* is the $\mathbb{P}^*$-Brownian motion. The price of the American put with maturity T and strike price equal to K, as seen at time $t \in [0,T]$, is then given by

$$V_t = \operatorname{ess\,sup}_{\tau \in \mathscr{T}_{[t,T]}} \mathbb{E}^*\Big(e^{-r(\tau - t)}\,(K - X_\tau)_+ \,|\, \mathscr{F}_t\Big).$$

Now, by using the Markov property, one immediately obtains $V_t = u(t, X_t)$, where

$$u(t,x) = \operatorname{ess\,sup}_{\tau \in \mathscr{T}_{[t,T]}} \mathbb{E}^*\Big(\Big(Ke^{-r(\tau-t)} - xe^{-\sigma^2(\tau-t)/2 + \sigma(B_\tau^* - B_t^*)}\Big)_+\Big). \quad (25)$$

For the sake of clearness, let us suppose that $t = 0$:

$$u(0,x) = \operatorname{ess\,sup}_{\tau \in \mathscr{T}_{[0,T]}} \mathbb{E}^*\Big(\Big(Ke^{-r\tau} - xe^{-\sigma^2\tau/2+\sigma B^*_\tau}\Big)_+\Big). \tag{26}$$

If $(W_t)_{t\geq 0}$ denotes a standard Brownian motion defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, we can write

$$\begin{aligned} u(0,x) &= \operatorname{ess\,sup}_{\tau \in \mathscr{T}_{[0,T]}} \mathbb{E}\Big(\Big(Ke^{-r\tau} - xe^{-\sigma^2\tau/2+\sigma W_\tau}\Big)_+\Big) \\ &\leq \operatorname{ess\,sup}_{\tau \in \mathscr{T}_{[0,+\infty]}} \mathbb{E}\Big(\Big(Ke^{-r\tau} - xe^{-\sigma^2\tau/2+\sigma W_\tau}\Big)_+ 1_{\{\tau<+\infty\}}\Big). \end{aligned} \tag{27}$$

in which $\mathscr{T}_{[0,+\infty]}$ denotes all the stopping times. The r.h.s of above Equation (27) can be interpreted as the value of the "perpetual" put, since in some sense it can be exercised at any time. Then, there exists a closed form formula for such a function:

Proposition 3.7. *The function*

$$u^\infty(x) = \operatorname{ess\,sup}_{\tau \in \mathscr{T}_{[0,+\infty]}} \mathbb{E}\Big(\Big(Ke^{-r\tau} - xe^{-\sigma^2\tau/2+\sigma W_\tau}\Big)_+ 1_{\{\tau<+\infty\}}\Big) \tag{28}$$

is given by:

$$u^\infty(x) = \begin{cases} K - x & \text{if } x \leq x^* \\ (K - x^*)\Big(\dfrac{x^*}{x}\Big)^\gamma & \text{if } x > x^* \end{cases}$$

in which $x^* = K\gamma/(1+\gamma)$ and $\gamma = 2r/\sigma^2$.

Before to continue with the proof, let us give some remarks. Let us come back to the American put, with finite maturity T. By developing similar arguments to the ones will be used for proving Proposition 3.7, one can see that for any $t \leq T$ there exists $x^*(t)$ such that

$$\begin{aligned} &\text{for any } x \leq x^*(t) \text{ then } u(t,x) = K - x \\ &\text{for any } x > x^*(t) \text{ then } u(t,x) > (K-x)_+ \end{aligned} \tag{29}$$

Moreover, by (27), we easily obtain that $x^*(t) \geq x^*$ for any $t \leq T$. Notice that if the underlying asset price is seen less than $x^*(t)$ at time t then the value of the perfect-hedging portfolio equals the payoff function, and the buyer of the option should exercise it immediately. In the opposite case, he should keep it and to wait for. Such a property justifies why the value $x^*(t)$ is usually called the "critical price" at time t.

Proof of Proposition 3.7. Formula (28) tells us the u^∞ satisfies the following properties:

- u^∞ is convex and decreasing;

- $u^\infty(x) \geq (K-x)_+$ and moreover, for any $T > 0$,

$$u^\infty(x) \geq \mathbb{E}\Big(\Big(Ke^{-rT} - xe^{-\sigma^2\tau/2+\sigma W_T}\Big)_+\Big),$$

which in turn implies $u^\infty(x) > 0$ for any x.

If we define

$$x^* = \inf\{x > 0 \,:\, u^\infty(x) = (K-x)_+\},$$

from the above properties it follows that $x^* < K$ and

$$\begin{array}{c} \text{for any } x \leq x^*,\ u^\infty(x) = K - x \\ \text{for any } x > x^*,\ u^\infty(x) > (K-x)_+ \end{array} \tag{30}$$

We have now to show that x^* is actually given by the formula $x^* = K\gamma/(1+\gamma)$ with $\gamma = 2r/\sigma^2$, and moreover the exact expression for u^∞ holds whenever $x > x^*$.

By the Optimal Stopping Theory and the Snell envelopes behavior, one can prove that there exists a stopping time giving the ess sup, that is for any x there exists τ_x such that

$$u^\infty(x) = \mathbb{E}\Big(\Big(Ke^{-r\tau_x} - xe^{-\sigma^2\tau/2+\sigma W_{\tau_x}}\Big)_+ 1_{\{\tau_x<\infty\}}\Big).$$

Moreover, setting $X_t^x = x\exp((r-\sigma^2/2)t + \sigma W_t)$, τ_x is given by

$$\tau_x = \inf\{t \geq 0 \,:\, u^\infty(X_t^x) = (K - X_t^x)_+\},$$

(recall the discussion at page 28). Now, from (30), it immediately follows that

$$\tau_x = \inf\{t \geq 0 \,:\, X_t^x \leq x^*\} = \inf\{t \geq 0 \,:\, (r-\sigma^2/2)t + \sigma W_t \leq \log(x^*/x)\}.$$

For $z > 0$, let us introduce the following stopping time τ_x^z and function $\phi(z)$:

$$\begin{array}{c} \tau_x^z = \inf\{t \geq 0 \,:\, (r-\sigma^2/2)t + \sigma W_t \leq \log(z/x)\} \\ \phi(z) = \mathbb{E}\Big(e^{-r\tau_x^z}(K - X^x_{\tau_x^z})_+ 1_{\{\tau_x^z<\infty\}}\Big). \end{array}$$

With these new notations, we have

$$\tau_x = \tau_x^{x^*} \quad \text{and} \quad u^\infty(x) = \phi(x^*).$$

Moreover, since $\tau_x^{x^*}$ is optimal, we know that the function ϕ attains its maximum at $z = x^*$. We are now going to compute ϕ explicitly, then we will maximize it to determine x^* and $\phi(x^*) = u^\infty(x)$.

If $z > x$, then $\tau_x^z = 0$, so that $\phi(z) = (K-x)_+$. If $z \leq x$, then by using the continuity of the paths of W we have

$$\tau_x^z = \inf\{t \geq 0 \,:\, (r-\sigma^2/2)t + \sigma W_t = \log(z/x)\} = \inf\{t \geq 0 \,:\, X_t^x = z\}$$

and $X^x_{\tau^z_x} = z$. Therefore,

$$\phi(z) = (K - z)_+ \mathbb{E}(e^{-r\tau^z_x} 1_{\{\tau^z_x < \infty\}}).$$

It then follows that we need to study the Laplace transform of τ^z_x. The following Lemma 3.8 states that, for $\alpha > 0$,

$$\mathbb{E}(e^{-\alpha T^\mu_b} 1_{\{T^\mu_b < \infty\}}) = \exp(\mu b - |b|\sqrt{\mu^2 + 2\alpha})$$

in which

$$T^\mu_b = \inf\{t \geq 0 \,:\, \mu t + W_t = b\}.$$

By substituting $\alpha = r$, $\mu = (r - \sigma^2/2)/\sigma$ and $b = \log(z/x)/\sigma$, straightforward computations give

$$\phi(z) = (K - z)_+ \left(\frac{z}{x}\right)^\gamma,$$

with $\gamma = 2r/\sigma^2$. Notice that, since

$$\phi'(z) = \frac{z^{\gamma-1}}{x^\gamma}(K\gamma - (\gamma + 1)z)$$

one has that the maximum is achieved for $z = K\gamma/(\gamma + 1)$.
By resuming, we have obtained

$$\phi(z) = \begin{cases} (K - x)_+ & \text{if } z > x \\ (K - z)\left(\frac{z}{x}\right)^\gamma & \text{if } z \leq x \text{ and } z \leq K \\ 0 & \text{if } z \leq x \text{ and } z \geq K \end{cases}$$

It then follows that if $x \leq K\gamma/(\gamma + 1)$ then $\max_z \phi(z) = \phi(x) = K - x$. If instead $x > K\gamma/(\gamma + 1)$ then $\max_z \phi(z) = \phi(K\gamma/(\gamma + 1))$. This gives the required formula, and the statement holds.

□

It remains to prove the following formula for the Laplace transform of the hitting time:

Lemma 3.8. *Let W denote a one-dimensional Brownian motion. For any $\mu, b \in \mathbb{R}$, set*

$$T^\mu_b = \inf\{t \geq 0 \,:\, \mu t + W_t = b\},$$

with the usual convention $\inf \emptyset = +\infty$. *Then, for any $\alpha > 0$,*

$$\mathbb{E}(e^{-\alpha T^\mu_b} 1_{\{T^\mu_b < \infty\}}) = \exp(\mu b - |b|\sqrt{2\alpha + \mu^2}).$$

Moreover, $T^\mu_b < \infty$ a.s. if and only if $\mu \cdot b \geq 0$. If $\mu \cdot b < 0$, then one has $\mathbb{P}(T^\mu_b < \infty) = \exp(2\mu b)$.

Proof. Without loss of generality, we can assume that $b \geq 0$ (otherwise, replace the Brownian motion W with the Brownian motion $-W$ and replace the drift μ with the drift $-\mu$).
Let us first show the above equality when $\mu = 0$. More precisely, we first show that $T_b^0 = \inf\{t \geq 0 : W_t = b\}$ is an a.s. finite stopping time and

$$\mathbb{E}(e^{-\alpha T_b^0}) = \exp(-|b|\sqrt{2\alpha}).$$

We can write

$$\{T_b^0 \leq t\} = \cap_{\varepsilon \in \mathbb{Q}_+} \{\sup_{s \leq t} W_s \geq b - \varepsilon\} = \cap_{\varepsilon \in \mathbb{Q}_+} \cup_{s \in \mathbb{Q}_+, s \leq t} \{W_s \geq b - \varepsilon\} \in \mathscr{F}_t,$$

(here, $\mathbb{Q}_+$ denotes the non negative rational numbers) so that T_b^0 is actually a stopping time. Set now $M_t = \exp(aW_t - \frac{a^2}{2}t)$, where a denotes a real number, and recall that M_t is a martingale. Now, we want to apply the Optional Sampling Theorem to T_b^0, but we do not know if it is bounded (actually, at the moment we do not know even if it is finite), so we consider the stopping time $T_b^0 \wedge t$, where t is a positive number. $T_b^0 \wedge t$ is now bounded, so that we can write

$$\mathbb{E}(M_{T_b^0 \wedge t}) = 1.$$

Now, $M_{T_b^0 \wedge t} = \exp(aW_{T_b^0 \wedge t} - \frac{a^2}{2}T_b^0 \wedge t) \leq \exp(ab)$. Moreover,

$$\lim_{t \to \infty} M_{T_b^0 \wedge t} = M_{T_b^0} 1_{\{T_b^0 < \infty\}} = \exp(ab - \frac{a^2}{2}T_b^0) 1_{\{T_b^0 < \infty\}}$$

(notice that the r.h.s. would be equal to $\exp(-ab - \frac{a^2}{2}T_b^0)1_{\{T_b^0 < \infty\}}$ if one had assumed $b < 0$).
By using the Lebesgue Dominated Convergence Theorem, we can state that

$$1 = \lim_{t \to \infty} \mathbb{E}(M_{T_b^0 \wedge t}) = \mathbb{E}(\exp(ab - \frac{a^2}{2}T_b^0) 1_{\{T_b^0 < \infty\}}),$$

i.e.

$$\mathbb{E}(\exp(-\frac{a^2}{2}T_b^0) 1_{\{T_b^0 < \infty\}}) = \exp(-ab).$$

If we take $a^2/2 = \alpha$, we obtain

$$\mathbb{E}(\exp(-\alpha T_b^0) 1_{\{T_b^0 < \infty\}}) = \exp(-b\sqrt{2\alpha}).$$

Now, if $\alpha \downarrow 0$, we obtain

$$\mathbb{P}(T_b^0 < \infty) = 1,$$

so that T_b^0 is a.s. finite and then, for any positive α,

$$\mathbb{E}(\exp(-\alpha T_b^0)) = \exp(-b\sqrt{2\alpha}).$$

Suppose now $\mu \neq 0$. By using arguments similar to the ones developed above, one easily obtains that T_b^μ is a stopping time w.r.t. $\mathscr{F}_t$. Now, let us write $T_b^\mu = T_b^\mu(W)$ to stress the dependence from W and let us write $B_t = W_t + \mu t$. By the Girsanov's Theorem, B is a Brownian motion under $\mathbb{Q}$, where $\mathbb{Q}$ is defined on $(\Omega, \mathscr{F}_t)$ as

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{F}_t} = e^{-\mu W_t - \frac{\mu^2}{2}t}.$$

Since $T_b^\mu(W) \wedge t$ is $\mathscr{F}_t$-measurable, we can write

$$\mathbb{E}(e^{-\alpha T_b^\mu(W)\wedge t}) = \mathbb{E}^{\mathbb{Q}}\Big(e^{-\alpha T_b^\mu(W)\wedge t}\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{F}_t}\Big) =$$
$$= \mathbb{E}^{\mathbb{Q}}\Big(e^{-\alpha T_b^\mu(W)\wedge t}e^{\mu W_t + \frac{\mu^2}{2}t}\Big) = \mathbb{E}^{\mathbb{Q}}\Big(e^{-\alpha T_b^0(B)\wedge t}e^{\mu B_t - \frac{\mu^2}{2}t}\Big)$$

in which we have substituted $W_t = B_t - \mu t$. Therefore, we can simply write

$$\mathbb{E}(e^{-\alpha T_b^\mu(W)\wedge t}) = \mathbb{E}\Big(e^{-\alpha T_b^0\wedge t}e^{\mu W_t - \frac{\mu^2}{2}t}\Big).$$

Now, by using the martingale property for $e^{\mu W_t - \frac{\mu^2}{2}t}$ and the Optional Sampling Theorem (recall that $T_b^0 \wedge t$ is a bounded stopping time), we obtain

$$\mathbb{E}\Big(e^{-\alpha T_b^0\wedge t}e^{\mu W_t - \frac{\mu^2}{2}t}\Big) = \mathbb{E}\Big(\mathbb{E}\Big(e^{-\alpha T_b^0\wedge t}e^{\mu W_t - \frac{\mu^2}{2}t}\,\Big|\,\mathscr{F}_{T_b^0\wedge t}\Big)\Big)$$
$$= \mathbb{E}\Big(e^{-\alpha T_b^0\wedge t}e^{\mu W_{T_b^0\wedge t} - \frac{\mu^2}{2}T_b^0\wedge t}\Big).$$

Therefore, the following equality holds:

$$\mathbb{E}(e^{-\alpha T_b^\mu(W)\wedge t}) = \mathbb{E}\Big(e^{-\alpha T_b^0\wedge t}e^{\mu W_{T_b^0\wedge t} - \frac{\mu^2}{2}T_b^0\wedge t}\Big)$$

Consider now the r.v. involved in the expectation in the r.h.s. above, that is $\beta_t = e^{-\alpha T_b^0\wedge t + \mu W_{T_b^0\wedge t} - \frac{\mu^2}{2}T_b^0\wedge t}$. One has $0 \leq \beta_t \leq e^\mu b$ and

$$\lim_{t\to\infty}\beta_t = e^{-\alpha T_b^0 + \mu b - \frac{\mu^2}{2}T_b^0} \quad \text{a.s.}$$

because we know $T_b^0 < \infty$ a.s. Therefore, we obtain

$$\lim_{t\to\infty}\mathbb{E}\Big(e^{-\alpha T_b^0\wedge t}e^{\mu W_{T_b^0\wedge t} - \frac{\mu^2}{2}T_b^0\wedge t}\Big) = e^{\mu b}\mathbb{E}(e^{-(\alpha+\mu^2/2)T_0^b}) = e^{\mu b - b\sqrt{2\alpha+\mu^2}}.$$

On the other hand, we have $e^{-\alpha T_b^\mu\wedge t} \downarrow e^{-\alpha T_b^\mu}1_{\{T_b^\mu<\infty\}}$ a.s. as $t \to \infty$, so that

$$\mathbb{E}(e^{-\alpha T_b^\mu}1_{\{T_b^\mu<\infty\}}) = \lim_{t\to\infty}\mathbb{E}(e^{-\alpha T_b^\mu\wedge t}) = e^{\mu b - b\sqrt{2\alpha+\mu^2}},$$

for any $\alpha > 0$, and the formula for the Laplace transform actually holds[8]. Now, if $\alpha \to 0$, then it follows that

$$\mathbb{P}(T_b^\mu < \infty) = e^{\mu b - |b\mu|}.$$

Then, in general, it is not true that $T_b^\mu < \infty$ a.s. In fact, we obtain

$$\mathbb{P}(T_b^\mu < \infty) = \begin{cases} 1 & \text{if } \mu \cdot b \geq 0 \\ e^{2\mu b} & \text{if } \mu \cdot b < 0 \end{cases}$$

The proof is now completed.

□

3.3 Price by no-arbitrage arguments

The aim of this section is to show that the price V_t given by Theorem 3.5 can be seen also as the price turning out by a "fair" game between seller and buyer similar to the one previously studied, that is, no arbitrage. Obviously, this will be a consequence of a suitable definition of an "arbitrage opportunity" in this new contest of American options.
Let us start with the following kind of strategies.

Definition 3.9. *A* buy-and-hold *strategy associated to an American option* $Z^a = (T, g)$ *is a pair* (c, τ)*, with* $\tau \in \mathscr{T}_{[0,T]}$ *standing for the exercise date and* $c > 0$ (long position) *or* $c < 0$ (short position) *standing for the number of units of the American security held at time* 0 *and then held in the portfolio up to the exercise date* τ*.*

Notice that in a buy-and-hold strategy, the American security is not traded after the initial date. Moreover, such a definition implicitly assumes the existence of an initial price P_0 for the American contingent claim, which of course represents what we aim to find and show to be equal to the quantity V_0 given in (21).
In order to proceed with our problem, we need to introduce more general self financing strategies, which take into account both trading-consumption and buy-and-hold strategies:

Definition 3.10. *A* self financing strategy for the American contingent claim $Z^a = (T, g)$ *is a collection* $\psi = (H, C, c, \tau)$ *such that:*

(i) (H, C) *is a self financing trading and consumption strategy;*

(ii) (c, τ) *is a buy-and-hold strategy;*

[8] Notice that, since $\alpha > 0$, one has $e^{-\alpha T_b^\mu} 1_{\{T_b^\mu < \infty\}} = e^{-\alpha T_b^\mu}$, so as a point of fact we have shown that $\mathbb{E}(e^{-\alpha T_b^\mu}) = e^{\mu b - b\sqrt{2\alpha + \mu^2}}$.

(iii) (H, C) *and* (c, τ) *are such that over* $(\tau, T]$ *one has*

$$H_t^i = 0, \; i = 1, \dots, N, \quad \text{and} \quad H_t^0 = \sum_{i=1}^N H_\tau^i \tilde{X}_\tau^i + H_\tau^0 + c\,\tilde{g}(\tau, X_\tau),$$

being $\tilde{g}(t, x) = e^{-\int_0^t r_s\,ds} g(t, x)$ *the associated discounted cash flows function.*

Notice that the portfolio on a self financing strategy $\psi = (H, C, c, \tau)$ depends also on the buy-and-hold strategy, so we will write $V_t(\psi)$. Moreover, it fulfills the following initial and final conditions:

$$\begin{array}{l} V_0(\psi) = \sum_{i=1}^N H_0^i X_0^i + H_0^0 + c\,P_0 \quad \text{and} \\ V_T(\psi) = e^{\int_\tau^T r_s\,ds}\Big(\sum_{i=1}^N H_\tau^i X_\tau^i + c\,g(\tau, X_\tau)\Big) + e^{\int_0^T r_s\,ds} H_\tau^0 \end{array} \tag{31}$$

Such kind of "American" strategies will be used in the notion of arbitrage in the American context (see below), where only the cases $c = +1$ (selling the option) and $c = -1$ (buying the option) are considered.
A closer look to the definition above shows after the exercise date τ, the portfolio holder closes the existing positions in the stocks and invests only in the benchmark asset (bond), in a quantity depending both on the value of the portfolio and the payoff of the American security at time τ. In some sense, trading following an American strategy means to add in the market the American contingent claim itself.
Let us introduce arbitrage strategies in this context:

Definition 3.11. *We say that* there is an arbitrage *in the market model for the American contingent claim* $Z^a = (T, g)$ *with initial price* U_0 *if one of the following conditions is fulfilled:*

(a) (long arbitrage) *there exists* $\tau \in \mathscr{T}_{[0,T]}$ *and a trading-consumption strategy* (H, C) *such that the American strategy* $\psi_+ = (H, C, +1, \tau)$ *satisfies:*

$$V_0(\psi_+) < 0 \quad \text{and} \quad V_T(\psi_+) \geq 0;$$

(b) (short arbitrage) *there exists a trading-consumption strategy* (H, C) *such that for any* $\tau \in \mathscr{T}_{[0,T]}$ *the American strategy* $\psi_- = (H, C, -1, \tau)$ *satisfies:*

$$V_0(\psi_-) < 0 \quad \text{and} \quad V_T(\psi_-) \geq 0.$$

Let us spend some words on conditions (a) and (b) above. A long arbitrage means that selling the American contingent claim ($c = +1$) implies the existence of a strategy and an exercise date giving a gain; in a short arbitrage, the gain is guaranteed when the American derivatives is bought ($c = -1$): in such a case, there exists a strategy in order to have a gain for any exercise date.

The *no-arbitrage price* of the American security is then defined as the value of P_0 which gives no arbitrage. This is well said: we are going to see that such a value does exist and is unique under $\mathbb{P}^*$ if the meaning of "an arbitrage opportunity" follows Definition 3.11.

Theorem 3.12. *There is no arbitrage (following Definition 3.11) if and only if the initial price P_0 of the American contingent claim $Z^a = (T, g)$ at time 0 is given by*

$$P_0 = \operatorname{ess\,sup}_{\tau \in \mathscr{T}_{[0,T]}} \mathbb{E}^*(e^{-\int_0^\tau r_s\,ds} g(\tau, X_\tau)) = \mathbb{E}^*(e^{-\int_0^{\tau_0} r_s\,ds} g(\tau_0, X_{\tau_0})),$$

where $\tau_0 = \inf\{t \in [0,T] \,:\, V_t = g(t, X_t)\}$ is the exercise date.
More generally, the no-arbitrage price P_t of the American contingent claim $Z^a = (T, g)$ as seen at time $t \in [0,T]$ is given by

$$\begin{aligned} P_t = &\ \operatorname{ess\,sup}_{\tau \in \mathscr{T}_{[t,T]}} \mathbb{E}^*(e^{-\int_t^\tau r_s\,ds} g(\tau, X_\tau) \,|\, \mathscr{F}_t) \\ = &\ \mathbb{E}^*(e^{-\int_t^{\tau_t} r_s\,ds} g(\tau_t, X_{\tau_t}) \,|\, \mathscr{F}_t), \end{aligned}$$

where $\tau_t = \inf\{s \in [t,T] \,:\, V_s = g(s, X_s)\}$ is the exercise date.

Proof. Let V_t as in (21) and let (H, C) denote the trading-consumption strategy given by Theorem 3.5, which is such that $V_t = V_t(H, C)$. We prove that the price of the American contingent claim has to be $P_0 = V_0$, otherwise there are arbitrage opportunities.
First, let us suppose $P_0 > V_0$: we will exhibit a short arbitrage. Indeed, take $\tau \in \mathscr{T}_{[0,T]}$ and $(\bar{H}, \bar{C})$ as follows: $\bar{C}_t = C_{t\wedge\tau}$ and

$$\begin{aligned} &\bar{H}_t^i = H_t^i\, 1_{[0,\tau]}(t), \quad i = 1, \dots, N, \quad \text{and} \\ &\bar{H}_t^0 = H_t^0 1_{[0,\tau]}(t) + \hat{H}_\tau^0 1_{(\tau,T]}(t) \\ &\text{with } \hat{H}_\tau^0 = H_\tau^0 + \sum_{i=1}^N e^{-\int_0^\tau r_s\,ds} H_\tau^i X_\tau^i - e^{-\int_0^\tau r_s\,ds} g(\tau, X_\tau). \end{aligned}$$

Let us give an idea on why the pair $(\bar{H}, \bar{C})$ is a self-financing trading and consumption strategy. On the event $\{t \le \tau\}$ one has $(\bar{H}, \bar{C}) = (H, C)$, so that $dV_t(\bar{H}, \bar{C}) = \sum_{i=0}^N \bar{H}_t^i\, dX_t^i - d\bar{C}_t$. On the event $\{t > \tau\}$ one has $V_t(\bar{H}, \bar{C}) = \hat{H}_\tau^0 X_t^0$, and then

$$dV_t(\bar{H}, \bar{C}) = H_\tau^0\, dX_t^0 = \sum_{i=0}^N \bar{H}_t^i\, dX_t^i - d\bar{C}_t$$

because $d\bar{C}_t = 0$ if $t > \tau$ (recall that $\bar{C}_t \equiv C_\tau$ for any $t > \tau$).
Consider now the American strategy $\bar{\psi}_- = (\bar{H}, \bar{C}, -1, \tau)$ and let $V_t(\psi_-)$ denote the associated portfolio. Notice that, since we have seen that $V_t \ge g(t, X_t)$ for any t, then $V_\tau \ge g(\tau, X_\tau)$. But since $V_T(\bar{\psi}_-) = V_\tau - g(\tau, X_\tau)$, one

has $V_T(\bar{\psi}_-) \geq 0$. Moreover, by (31) it follows that $V_0(\bar{\psi}_-) = V_0(H,C) - P_0 = V_0 - P_0 < 0$, that is $\bar{\psi}_-$ gives a short arbitrage.

Let us now suppose that $P_0 < V_0$: such a condition gives a long arbitrage. Let τ_0 be defined as in (23). Set $\bar{C}_t = C_{t\wedge\tau_0} \equiv 0$ and $\bar{H}$ defined as above but with τ replaced by τ_0. By developing arguments similar to the one previously used, one has that $(\bar{H}, \bar{C})$ is self-financing. Now, take $\bar{\psi}_+ = (-\bar{H}, -\bar{C}, +1, \tau_0) \equiv (-\bar{H}, 0, +1, \tau_0)$. Then $V_0(\bar{\psi}_+) = -V_0(H,C) + P_0 < 0$ and $V_T(\bar{\psi}_+) = -V_{\tau_0} + g(\tau_0, X_{\tau_0}) = 0$ because of (23) and (24). Thus, $\bar{\psi}_+$ gives a long arbitrage.

□

So, by summarizing, in the American case one needs to introduce more general kind of strategies (we had to introduce consumption processes and buy-and-hold strategies). But notice that this is just in order to achieve the price and the hedging portfolio: in some sense, this is more mathematics than practice. Indeed, as we have seen, the American contingent claim is well hedged and then priced by a trading-consumption strategy (buy-and-hold strategies are somehow technical things, although they have an interesting financial meaning, in order to precise the meaning of arbitrage in this new context). Now, since the consumption process nullifies up to the exercise date, it turns out that the trading-hedging strategy reduces to a standard one: once the exercise date is reached, the game between buyer and seller ends!

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