Short Introduction to Rough Path theory (following Friz and Hairer)

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Contents

1 Introduction

The aim of these lectures is to give a short presentation of Lyons' theory of Rough Path. In the first part of the lectures we present the "classical" theory following the beautiful book of P. Friz and M. Hairer, "A Course on Rough Paths, with an introduction to regualrity structures". I will refer to this book as (FH).

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In the second part we present an alternative approach due to I. Bailleul which is based on the point of view of stochastic flows.

All over the text I will assume that one works on the time interval $(0,T)$, and smetimes we also assume that $T\leq 1$.

2 First presentation of the rough integral

2.1 Motivation

a Consider a one dimensional Brownian motion W and a bounded continuous function $F: R\rightarrow R$. Then one defines the Itô stochastic integral as limit of Riemann sums:

$$\int_0^t F(W_s)dW_s=\lim_{|\mathcal{P}|\rightarrow 0}\sum_{i-1}^{n}F(W_{s_{i-1}})(W_{s_i}-W_{s_{i-1}})$$

Here $\mathcal{P}=\{0=s_0<...<s_n=t\}$ is a partition of mesh $|\mathcal{P}|\rightarrow 0$ and, important, the limit is in $L^2(\Omega,P)$. In particular $\int_0^t F(W_s)dW_s$ is defined only $P(d\omega)$ almost surely, and the exception set depends on F. So, if one aims to define this integral for all possible F in the same time, with a **unique** exception set, this is not possible. However, if one replaces the Ito stochastic integral by the Lyon's "rough path integral" (that we present in the following) we are able to give a definition which holds for any coefficient F with an exception set which does not depend on F.

Moreover the definition of the rough integral is "pathwise": for each trajectory $s\rightarrow W_s(\omega)$ one is able to define $\int_0^t F(W_s(\omega))d\overline{W}_s(\omega)$ (I put $d\overline{W}_s(\omega)$ instead of $dW_s(\omega)$ in order to emphasise that I speak about the rough integral). Of course, this is not true for the Itô integral which is defined just as a class of equivalence of L^2.

b (flows) Consider the SDE

$$X_t(x)=x+\sum_{i=1}^{d}\int_0^t\sigma_i(X_s(x)dW_s^i+\int_0^t b(X_s(x))ds.$$

It is well knowen that if the coefficients σ_i and b are of class C_b^∞ then one may find a set $\Omega_{\sigma,b}\subset\Omega$ with $P(\Omega_{\sigma,b})=1$ such that for each $\omega\in\Omega_{\sigma,b}$ the application

$$(t,x)\rightarrow X_t(x,W(\omega))$$

is continuous with respect to t and of class C^∞ with respect to x. This is a crucial point in the the theory of stochastic flows developed by Kunita, Bismut, Ikeda Watanabe .... But the set $\Omega_{\sigma,b}$ depends on the coefficients σ and b. If one replaces the Itô integral dW_s by the rough path integral $d\overline{W}_s(\omega)$ one is able to find a universal set $\Omega_0\subset\Omega$ with $P(\Omega_0)=1$, such that for each $\omega\in\Omega_0$ the application $(t,x)\rightarrow X_t(x,\omega)$ is a smooth flow. So one may define a "universal map" which solves the equation.

c (continuity) See (FH) Proposition 1.1 and Exercise 5.21

The law of the Brownian motion W is supported by the space of continuous functions $C(0,T)$ which is a Banach space with the uniform norm$\|f\|_\infty=\sup_{t\leq T}|f_t|$. Take now $W_n(s)=W*\gamma_n(s)$ where is a smooth approximation of the Dirac mass: $\gamma_n\rightarrow\delta_0$ in distribution sense.

Since $s \to W_n(s)$ is a smooth function we may solve the ODS

$$X_t^n(x) = x + \sum_{i=1}^d \int_0^t \sigma_i(X_s^n(x) dW_n^i(s) + \int_0^t b(X_s^n(x))ds$$

where $dW_n^i(s)$ is now a Stieltjes integral. Then

$$\|W - W_n\|_\infty \to 0 \quad and \quad \|X - X^n\|_\infty \to 0.$$

To be precise, one has to take the Stratonovich integral instead of the Itô integral in the equation of X_t. But anyway since the coefficients are smooth, this is done by changing the drift b by a drift b' (I leave out this detail for the moment).

Now the question is

$$\|W - W_n\|_\infty \to 0 \quad \Rightarrow \quad \|X - X^n\|_\infty \to 0$$

for every sequence of smooth functions $W_n, n \in N$? This is the continuity property with respect to the uniform norm. But this is false: one may produce a sequence such that $\|W - W_n\|_\infty \to 0$ but $\lim_n \|X - X^n\|_\infty \neq 0$.

But we know that the law of the Brownian motion is concentrated on the space $C_\alpha(0,T)$ of Hölder continuous functions of order $\alpha < \frac{1}{2}$. And the Hölder norm $\|f\|_\alpha$ is stronger than the supremum norm. So one may hope that $\|W - W_n\|_\alpha \to 0 \Rightarrow \|X - X^n\|_\alpha \to 0$. But this is also false (one may produce a sequence which contradicts this). And finally one may prove that if one considers any Banach space $B \subset C(0,T)$ on which the law of the Brownian motion is supported then the assertion $\|W - W_n\|_B \to 0 \Rightarrow \|X - X^n\|_B \to 0$ is false. So there is no hope to get the continuity $W - W_n \to 0 \Rightarrow X - X^n \to 0$ does not matter which norm one employs. It turns out that in order to obtain $X_n \to X$ we need $W_n \to W$ but we also need some "more information" on the path W : this supplementary information is quantified in the 'rough path". (see Proposition 1.1 p 2 in (FH)).

2.2 The sewing lemma

This is the central instrument in our framework: it gives a way to prove that Euler scheme converges to the solution of some equation. We consider an abstract application $\Theta : R_+^2 \to R$. If Θ is additive, that is $\Theta_{s,t} = \Theta_{s,u} + \Theta_{u,t}$ for every $s < u < t$, then for every partition $\mathcal{P} = \{s = s_0 < ... < s_n = t\}$, we have

$$\Theta_{s,t}^{\mathcal{P}} := \sum_{i=1}^n \Theta_{s_{i-1},s_i} = \Theta_{s,t}.$$

But of course this is false if the additivity property fails. This is why we introduce the following "additivity error":

$$\delta_{s,u,t}(\Theta) = \Theta_{s,t} - \Theta_{s,u} - \Theta_{u,t}.$$

Lemma 2.1 *Suppose that, for some $\beta > 1$ and $C_{sew} \geq 1$*

$$|\delta_{s,u,t}(\Theta)| \leq C_{sew} |t - s|^\beta . \tag{2.1}$$

A *Then the following limit exists*

$$X_{s,t}(\Theta) = \lim_{|\mathcal{P}|\to 0} \Theta^{\mathcal{P}}_{s,t} = \lim_{|\mathcal{P}|\to 0} \sum_{i=1}^{n} \Theta_{s_{i-1},s_i}. \tag{2.2}$$

B *Moreover*

$$\left|X_{s,t}(\Theta) - \Theta^{\mathcal{P}}_{s,t}\right| \leq 2^{\beta}(t-s)\zeta(\beta)\left|\mathcal{P}\right|^{\beta-1} \tag{2.3}$$

with $\zeta(\beta) = \sum_{i=1}^{\infty} i^{-\beta}$. *And*

$$\left|X_{s,t}(\Theta) - \Theta_{s,t}\right| \leq C_{sew}\zeta(\beta)2^{\beta}(t-s)^{\beta}. \tag{2.4}$$

C X *is additive, and it is the unique additive process which verifies (2.4).*

Proof A Step 1. We fix a partiton $\mathcal{P} = \{s = s_0 < ... < s_n = t\}$. Note first that one may find i such that

$$s_{i+1} - s_{i-1} \leq \frac{2(t-s)}{n}.$$

We denote by $\mathcal{P}_i$ the partition $\mathcal{P}$ in which we have canceled i. Then

$$\Theta^{\mathcal{P}}_{s,t} - \Theta^{\mathcal{P}_i}_{s,t} = \Theta_{s_{i-1},s_i} + \Theta_{s_i,s_{i+1}} - \Theta_{s_{i-1},s_{i+1}} = \delta_{s_{i-1},s_i,s_{i+1}}(\Theta)$$

and consequentely

$$\left|\Theta^{\mathcal{P}}_{s,t} - \Theta^{\mathcal{P}_i}_{s,t}\right| \leq C_{sew}(s_{i+1} - s_{i-1})^{\beta} \leq C_{sew}\left(\frac{2(t-s)}{n}\right)^{\beta}.$$

We repeat this procedure in order to descend to partitions which are shorter and shorter, up to the trvial partition $\{s,t\}$. So we obtain

$$\left|\Theta^{\mathcal{P}}_{s,t} - \Theta_{s,t}\right| \leq C_{sew}\sum_{i=1}^{n}\left(\frac{2(t-s)}{i}\right)^{\beta} \leq C_{sew}\zeta(\beta)2^{\beta}(t-s)^{\beta}. \tag{2.5}$$

Step 2 We prove the Cauchy property

$$\lim_{|\mathcal{P}|\vee|\mathcal{P}'|\to 0}\left|\Theta^{\mathcal{P}}_{s,t} - \Theta^{\mathcal{P}'}_{s,t}\right| = 0.$$

We may suppose without loss of generality that $\mathcal{P}' \subset \mathcal{P}$ and we construct the partitions $\mathcal{P}_l$ in the following way: we consider the intervals (s'_j, s'_{j+1}) of the partition $\mathcal{P}'$ and we split the intervals with $j \leq l$ according to $\mathcal{P}$. But we leave the intervals with $j > l$ as they are in $\mathcal{P}'$. Then

$$\Theta^{\mathcal{P}}_{s,t} - \Theta^{\mathcal{P}'}_{s,t} = \sum_{l=1}^{n'}(\Theta^{\mathcal{P}_l}_{s,t} - \Theta^{\mathcal{P}_{l-1}}_{s,t})$$

so that

$$\begin{aligned}
\left|\Theta^{\mathcal{P}}_{s,t} - \Theta^{\mathcal{P}'}_{s,t}\right| &\leq \sum_{l=1}^{n'}\left|\Theta^{\mathcal{P}_l}_{s,t} - \Theta^{\mathcal{P}_{l-1}}_{s,t}\right| = \sum_{l=1}^{n'}\left|\Theta^{\mathcal{P}}_{s_l,s_{l+1}} - \Theta_{s_l,s_{l+1}}\right| \\
&\leq C_{sew}\zeta(\beta)2^{\beta}\sum_{l=1}^{n'}(s_{l+1} - s_l)^{\beta} \\
&\leq C_{sew}\zeta(\beta)2^{\beta}\left|\mathcal{P}'\right|^{\beta-1}(t-s) \to 0.
\end{aligned}$$

So we may define $X^{\mathcal{P}}_{s,t}(\Theta) = \lim_{|\mathcal{P}|\to 0} \Theta^{\mathcal{P}}_{s,t}$.

Step 3 B We pass to the limit with $|\mathcal{P}| \to 0$ and we get

$$\left| X_{s,t}(\Theta) - \Theta^{\mathcal{P}'}_{s,t} \right| \leq C_{sew}\zeta(\beta)2^{\beta} \left|\mathcal{P}'\right|^{\beta-1}(t-s).$$

Moreover passing to the limit in (2.5) we get

$$|X_{s,t}(\Theta) - \Theta_{s,t}| \leq C_{sew}\zeta(\beta)2^{\beta}(t-s)^{\beta}.$$

C The fact that $X_{s,t} = X_{s,u} + X_{u,t}$ is proved by passing to the limit and using that any partition of (s,t) may be split in a partition of (s,u) and of (u,t). □

2.2.1 Example: the Young integral.

Take $f, g : R_+ \to R$. If f has finite variation and g is continuous one may define the "Stiltjers integral" $\int_0^t g(s)df(s)$ as the limit of the Riemann sums. Young generalized this to functions which are just Hölder continuous. We denote $\mathcal{C}^{\alpha}$ the space of functions $f : R_+ \to R$ such that

$$\sup_{|t-s|>0} \frac{|f(t)-f(s)|}{|t-s|^{\alpha}} =: \|f\|_{\alpha} < \infty.$$

Notice that

$$\sup_{t\leq T} |f(t)| \leq |f(0)| + \|f\|_{\alpha} T^{\alpha}$$

In the following we will use the notation $f_{s,t} = f_t - f_s$ and then the above inequality concerns $f_{s,t}$: this reads $|f_{s,t}| \leq \|f\|_{\alpha}(t-s)^{\alpha}$. In a more general case, we will consider some functions $H : R_+^2 \to R$ and we say that $H \in \mathcal{C}^{\alpha}$ if

$$\sup_{|t-s|>0} \frac{|H_{s,t}|}{|t-s|^{\alpha}} =: \|H\|_{\alpha} < \infty.$$

Then we have the following

Lemma 2.2 *Suppose that $f \in \mathcal{C}^{\alpha}$ and $g \in \mathcal{C}^{\alpha'}$ with $\beta = \alpha + \alpha' > 1$. For a partition $\mathcal{P} = \{0 = s_0 < ... < s_n = t\}$ we denote*

$$S^{\mathcal{P}}_t = \sum_{i=1}^{n} g(s_{i-1})(f(s_i) - f(s_{i-1})).$$

Then the follwoing limit exists and is called the Young integral:

$$\int_0^t g(s)df(s) = \lim_{|\mathcal{P}|\to 0} S^{\mathcal{P}}_t.$$

And we have the following convergence rate:

$$\left| \int_0^t g(s)df(s) - S^{\mathcal{P}}_t \right| \leq C(t-s)\left|\mathcal{P}\right|^{\alpha+\alpha'-1}$$

Proof We denote

$$\Theta_{s,t} = g(s)(f(t) - f(s))$$

and we notice that

$$\delta_{s,u,t}(\Theta) = (g(u) - g(s))(f(t) - f(u)).$$

Under our hypothesis $|\delta_{s,u,t}(\Theta)| \leq C(t-s)^{\alpha+\alpha'}$ so that we may use the sewing lemma. Since $S_t^{\mathcal{P}} = \Theta_{s,t}^{\mathcal{P}}$ the proof is an immediate consequence of the sewing lemma. □

2.3 The rough integral.

We consider a $d-$dimensional Brownian motion $W = (W^1, ..., W^d)$ and we present the rough integral with respect to a path of W. To begin we consider a particular case (which is the one considered by Lyons in his first papers): we give a function $F \in C_b^2(R^d, R^d)$ and we would like to define

$$\int_0^t F(W_s)dW_s = \sum_{i=1}^d \int_0^t F^i(W_s)dW_s^i. \tag{2.6}$$

We may do it by using the Itô integral but, as we discussed in the first section, this is not "pathwise": we would like to do it for a given fixed trajectory $s \to W_s(\omega)$. Clearly we are not able to use the Yung integral because both $g(s) = F^i(W_s)$ and $f(s) = W_s^i$ are Hölder continuous of index $\alpha < \frac{1}{2}$ so that $\alpha + \alpha < 1$. In fact it turns out that we are not able to do it just by using the trajectory $s \to W_s(\omega)$, but we will need to add the information given by the Lévy areas. This is called the "enhanced" Brownian motion. Let us be more precise. We denote, for $i, j \in \{1, ..., d\}$,

$$W_{s,t}^i = W_t^i - W_s^i \quad and \quad \mathbb{W}_{s,t}^{i,j} = \int_s^t (W_r^i - W_s^i)dW_s^j \tag{2.7}$$

Finally, the couple $\mathbf{W} = (W, \mathbb{W})$ is called the "enhanced" Brownian motion. For $\alpha > 0$ we denote (as in the previous section)

$$\|W\|_\alpha = \sup_{|t-s|>0} \frac{|W_{s,t}|}{|t-s|^\alpha} \quad and \quad \|\mathbb{W}\|_\alpha = \sup_{|t-s|>0} \frac{|\mathbb{W}_{s,t}|}{|t-s|^\alpha}.$$

As a consequence of Kolmogorov's criterion (see (FH) Theorem 3.1) we have

Theorem 2.3 *For every $\alpha \in (\frac{1}{3}, \frac{1}{2})$*

$$\|W\|_\alpha < \infty \quad and \quad \|\mathbb{W}\|_{2\alpha} < \infty \tag{2.8}$$

almost surely. This means that we may find a set $\Omega_0 \subset \Omega$ such that the above assertion holds for every $\omega \in \Omega_0$ and $P(\Omega_0) = 1$. In the following we will suppose without any supplementary mention that we work with a trajectory corseponding to $\omega \in \Omega_0$ and so the above property holds.

Remark 2.4 *In (2.6) dW^i may represent the "Itô integral" or the "Stratonovich integral" $\circ dW^i$. In the second case we have to take $\mathbb{W}_{s,t}^{i,j} = \int_s^t (W_r^i - W_s^i) \circ dW_s^j$. In the following, in order to fix ideas, we take the Itô integral, and, along the presentation, we will precise what changes if we take the Stratonovich integral.*

We define now

$$\Theta_{s,t}=\sum_{i=1}^{d}F^i(W_s)W^i_{s,t}+\sum_{i,j=1}^{d}\partial_jF^i(W_s)\mathbb{W}^{i,j}_{s,t} \tag{2.9}$$

For a partititon $\mathcal{P}=\{0=s_0<...<s_n=t\}$ we denote

$$\Theta^{\mathcal{P}}_{s,t}=\sum_{i=1}^{n}\Theta_{s_{i-1},s_i}=\sum_{k=1}^{n}\left(\sum_{i=1}^{d}F^i(W_{s_{k-1}})W^i_{s_{k-1},s_k}+\sum_{i,j=1}^{d}\partial_jF^i(W_{s_{k-1}})\mathbb{W}^{i,j}_{s_{k-1},s_k}\right)$$

Then we have

Theorem 2.5 *Let us fix some $\alpha\in(\frac{1}{3},\frac{1}{2})$ and we fix some trajectory $\mathbf{W}(\omega)=(W(\omega),\mathbb{W}(\omega))$ with $\omega\in\Omega_0$. Then the following limit exists*

$$\int_s^t F(W_r)d\mathbf{W}_r:=\lim_{|\mathcal{P}|\to 0}\Theta^{\mathcal{P}}_{s,t} \tag{2.10}$$

and it is called the "rough integral" of $F(W_r)$ with respect to the rough path $\mathbf{W}$. Moreover $\left\|\int F(W_r)d\mathbf{W}_r\right\|_\alpha<\infty$.

Proof We want to use the sewing lemma so we have to estimate $\delta_{s,u,t}(\Theta)$. In the following we denote by $O(h^\alpha)$ a quantity wich is upper bounded by Ch^α. Let $s<u<t$. We have

$$F(W_u)=F(W_s+W_{s,u})=F(W_s)+\langle\nabla F(W_s),W_{s,u}\rangle+O((u-s)^{2\alpha})$$

and

$$\nabla F(W_u)=\nabla F(W_s)+O((u-s)^{\alpha})$$

so that

$$\begin{aligned}\Theta_{u,t} &= \sum_{i=1}^{d}F^i(W_u)W^i_{u,t}+\sum_{i,j=1}^{d}\partial_jF^i(W_u)\mathbb{W}^{i,j}_{u,t}\\ &= \sum_{i=1}^{d}F^i(W_s)W^i_{u,t}+\sum_{i,j=1}^{d}\partial_jF^i(W_s)W^i_{s,u}W^j_{u,t}+\sum_{i,j=1}^{d}\partial_jF^i(W_s)\mathbb{W}^{i,j}_{u,t}+O((t-s)^{3\alpha})\end{aligned}$$

This gives

$$\begin{aligned}&\Theta_{s,u}+\Theta_{u,t}\\ =&\sum_{i=1}^{d}F^i(W_s)(W^i_{s,u}+W^i_{u,t})+\sum_{i,j=1}^{d}\partial_jF^i(W_s)(\mathbb{W}^{i,j}_{s,u}+\mathbb{W}^{i,j}_{u,t}+W^i_{s,u}W^j_{u,t})\\ &+O((t-s)^{3\alpha}).\end{aligned}$$

Notice that we have the following identities (known as Chen relations)

$$\begin{aligned}W_{s,t} &= W_{s,u}+W_{u,t}\\ \mathbb{W}^{i,j}_{s,t} &= \mathbb{W}^{i,j}_{s,u}+\mathbb{W}^{i,j}_{u,t}+W^i_{s,u}W^j_{u,t}\end{aligned}$$

Inserting in the previous equality

$$\begin{aligned}\Theta_{s,u}+\Theta_{u,t} &= \sum_{i=1}^{d} F^i(W_s)W^i_{s,t}+\sum_{i,j=1}^{d}\partial_j F^i(W_s)\mathbb{W}^{i,j}_{s,t}+O((t-s)^{3\alpha})\\ &= \Theta_{s,t}+O((t-s)^{3\alpha})\end{aligned}$$

which means that $|\delta_{s,u,t}|\le C(t-s)^{3\alpha}$. Now we are able to use the sewing lemma and to obtain (2.10). Finally, using (2.4) we get

$$\left|\int_s^t F(W_r)d\mathbf{W}_r\right|\le|\Theta_{s,t}|+C(t-s)^{3\alpha}\le C(t-s)^{\alpha}.$$

□

Remark 2.6 ***(Link with the Itô integral)*** *The rough integral $\int_s^t F(W_r(\omega))d\mathbf{W}_r(\omega)$ coincides with the Itô inegral in the follwoing sense: the Itô integral $\int_s^t F(W_r)dW_r$ is defined as the limit in $L^2(\Omega)$ of the Riemann sums - so it is an element of $L^2(\Omega)$, that is a class of equivalence of elements which are almost surely equal each other. On the other hand $\int_s^t F(W_r(\omega))d\mathbf{W}_r(\omega)$ is defined for every $\omega\in\Omega_0$ as a limit of Euler schemes based on $\Theta_{s,t}$. Let us note that $\Theta^{\mathcal{P}}_{s,t}=S^{\mathcal{P}}_{s,t}+R^{\mathcal{P}}_{s,t}$ with*

$$S^{\mathcal{P}}_{s,t}=\sum_{i=1}^{n}\sum_{j=1}^{d}F^j(W_{s_{i-1}})W^j_{s_{i-1},s_i}\quad \textit{and}\quad R^{\mathcal{P}}_{s,t}=\sum_{i=1}^{n}\sum_{j,p=1}^{d}\partial_j F^p(W_{s_{i-1}})\mathbb{W}^{j,p}_{s_{i-1},s_i}$$

Notice that $\mathbb{W}_{s_{i-1},s_i}, i=1,...,n$ are independent and centred, and $E\left|\mathbb{W}^{j,p}_{s_{i-1},s_i}\right|^2\le C(s_i-s_{i-1})^{4\alpha}$. Then is easy to check that

$$E\left|R^{\mathcal{P}}_{s,t}\right|^2\le\|\nabla F\|_\infty\sum_{i=1}^{n}\sum_{j,p=1}^{d}E\left|\mathbb{W}^{j,p}_{s_{i-1},s_i}\right|^2\le C\|\nabla F\|_\infty|\mathcal{P}|^{4\alpha-1}\to 0.$$

So $\lim\Theta^{\mathcal{P}}_{s,t}=\lim S^{\mathcal{P}}_{s,t}=\int_s^t F(W_r(\omega))dW_r$ in $L^2(\Omega)$. This means that the rough integral produces a specific element in the class of equivalence of the Itô integral.

We discuss now the case of the **Stratonovich integral**. We recall that for two continuous martingales M and N we have

$$\int_0^t M\circ dN=\int_0^t MdN+\frac{1}{2}\langle M,N\rangle_t\,.$$

Suppose now that we consider the Lévy area with respect to the Stratonowich integral:

$$\overline{\mathbb{W}}^{i,j}_{s,t}=\int_s^t W^i_{s,r}\circ dW^t_r=\int_s^t W^i_{s,r}dW^t_r+\frac{1}{2}\delta_{i,j}(t-s).$$

We first notice that we still have, for $\alpha<\frac{1}{2}$ that $\|W\|_\alpha<\infty$ and $\left\|\overline{\mathbb{W}}\right\|_{2\alpha}<\infty$. If we define now

$$\overline{\Theta}_{s,t}=\sum_{i=1}^{d}F^i(W_s)W^i_{s,t}+\sum_{i,j=1}^{d}\partial_j F^i(W_s)\overline{\mathbb{W}}^{i,j}_{s,t}$$

then

$$\overline{\Theta}_{s,t} = \Theta_{s,t} + \frac{1}{2}\sum_{i=1}^{d} \partial_i F^i(W_s)(t-s).$$

This imediately implies that

$$\delta_{s,u,t}(\overline{\Theta}) = \delta_{s,u,t}(\Theta) + O(t-s)^{3\alpha}.$$

So the sewing lemma allows to construct

$$\int_s^t F(W_r)d\overline{\mathbf{W}}_r := \lim_{|\mathcal{P}|\to 0} \overline{\Theta}^{\mathcal{P}}_{s,t}.$$

Our aim now is to show that the above "Stratonovich rough integral" coresponds to the probabilistic Stratonovich integral. Notice that

$$\overline{\Theta}^{\mathcal{P}}_{s,t} = \Theta^{\mathcal{P}}_{s,t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{n} \partial_i F^i(W_{s_{k-1}})(s_k - s_{k-1})$$

By passing to the limit we get

$$\int_s^t F(W_r)d\overline{\mathbf{W}}_r = \int_s^t F(W_r)d\mathbf{W}_r + \frac{1}{2}\sum_{i=1}^{d}\int_s^t \partial_i F^i(W_r)dr.$$

Having in mind the previous remark, this means that $\int_s^t F(W_r)d\overline{\mathbf{W}}_r$ is a representer of

$$\sum_{i=1}^{d}\int_s^t F^i(W_r)dW^i_r + \frac{1}{2}\sum_{i=1}^{d}\int_s^t \partial_i F^i(W_r)dr = \sum_{i=1}^{d}\int_s^t F^i(W_r)\circ dW^i_r.$$

So, if we start with a Stratonovich Lévy area, we finish with a Stratonovich integral.

2.3.1 First generalisation: controlled path

We wolud like to be able to use a more general class of integrands. In order to do this we introduce the notion of "controlled path" which is due to Gublinelli. $Y : R_+ \to R^d$ which is α Hölder, is a "controlled path" if there exists $Y' : R_+ \to R^{d\times d}$ such that

$$Y_{s,t} = \left\langle Y'_s, W_{s,t}\right\rangle + O((t-s)^{2\alpha})$$

or, in other words, $R^Y_{s,t} := Y_{s,t} - \langle Y'_s, W_{s,t}\rangle$ is 2α Hölder. The obvious example is

$$Y_t = F(W_t) = F(W_s) + \langle \nabla F(W_s), W_{s,t}\rangle + o(t-s)^{2\alpha}$$

So, in this case $Y'_t = \nabla F(W_t)$. Having this in mind one says that Y' is the (Gubinelli) derivative of Y.

Uniqueness However, for a "general rough path" $(W, \mathbb{W})$ (see the following section) this derivative is not unique, so the notation is abusive, and we have to keep in mind the choice which has be done; so a "controlled paths" is the couple (Y, Y') and the constructions that we will do in the following depend on the choice of Y' (which has to be precised each time). We denote by $\mathcal{D}_W^\alpha$ the space of the α controlled (by W) path. So $(Y,Y')\in\mathcal{D}_W^\alpha$ if

$$R^Y_{s,t}:=Y_{s,t}-\left\langle Y'_s,W_{s,t}\right\rangle\in\mathcal{C}^{2\alpha}. \tag{2.11}$$

For $(Y,Y')\in\mathcal{D}_W^\alpha$ we denote (see (4.17) p56 in (FH))

$$\left\|Y,Y'\right\|_{W,2\alpha}=\left\|Y'\right\|_\alpha+\left\|R^Y\right\|_{2\alpha} \tag{2.12}$$

Notice that we also have the estimate (see (4.18) p56 in (FH))

$$\begin{aligned}\left\|Y\right\|_\alpha &\leq C(1+\left\|W\right\|_\alpha)(\left|Y'_0\right|+\left\|Y,Y'\right\|_{W,2\alpha}) &(2.13)\\ \left\|Y\right\|_\infty &\leq \left|Y_0\right|+\left\|Y\right\|_\alpha T^\alpha &(2.14)\end{aligned}$$

I recall that we work on the time interval $(0,T)$.

The lack of uniqueness does not stop the construction of rough integrals. However, it is much more pleasant to have a unique derivative; and this is true for standard examples of rough path, in particular for the one associated to the Brownian motion (and this is the case in this section). More precisely, suppose that we are in the one dimensional case $d=1$, and for every $0\leq s\leq T$

$$\overline{\lim_{t\downarrow s}}\frac{\left|W_{s,t}\right|}{(t-s)^{2\alpha}}=\infty. \tag{2.15}$$

Then Y' is uniquely determined. More precisely, it is given by the equality

$$Y'_s=\lim_{t\downarrow s}\frac{Y_{s,t}}{W_{s,t}}. \tag{2.16}$$

Indeed, by (2.15)

$$\frac{\left|R^Y_{s,t}\right|}{\left|W_{s,t}\right|}\leq\left\|R^Y\right\|_{2\alpha}\times\frac{\left|t-s\right|^{2\alpha}}{\left|W_{s,t}\right|}\to 0$$

so that, by using (2.11), the limit in (2.16) exists and is equal to Y'_s.

Friz and Hairer say that a rough path which verifies (2.15) is a "really rough" path. In fact, this condition is really necessary for getting uniqueness: suppose that the limit in (2.15) is finite uniformly. Then $W\in\mathcal{C}^{2\alpha}$ and consequently, for each real number η, we have the decomposion $Y_{s,t}=(Y'_s+\eta)W_{s,t}+(R^Y_{s,t}-\eta W_{s,t})$ and $R^Y_{s,t}-\eta W_{s,t}$ is a "good remainder". This means that $Y'_s+\eta$ may also be used as a derivative.

Let us check that the Brownian motion is "really rough". The iterated logarithm theorem says that, if W is the Brownian motion

$$\lim_{h\downarrow 0}\frac{\left|W_{s,s+h}\right|}{h^{1/2}\ln\ln(1/h)}=\sqrt{2} \tag{2.17}$$

almost surely. And one may construct an exeeption set which does not depend on s. So we may choose the trajectory W_t such that (2.17) holds for every s. Then

$$\frac{\left|W_{s,s+h}\right|}{h^{2\alpha}}=\frac{\left|W_{s,s+h}\right|}{h^{1/2}\ln\ln(1/h)}\times\frac{h^{1/2}\ln\ln(1/h)}{h^{2\alpha}}\to\infty$$

for every $\frac{1}{4}<\alpha$.

In the multi-dimensional case $d \geqslant 2$, one has the following definition of "truly (really) rough path" (see (FH- p 85 Definition 6.3):

$$\overline{\lim_{t \downarrow s}} \frac{\langle v, W_{s,t} \rangle}{|t-s|^{2\alpha}} = \infty \quad \forall v \in R^d, s \in (0,T).$$

And one proves that in this case the Gubinelli derivative Y' is unique (for any controlled rough path, of course). In this case we denote $D_W^j Y_s^i = (Y_s')^{i,j}$ so that

$$Y_t^i = Y_s^i + \sum_{j=1}^{d} D_W^j Y_s^i \times W_{s,t}^j + R_{s,t}^i$$

We come back to our problem (construction of the rough integral). With the concept of "Gubinelli derivative" at hand we obtain the following result. We define

$$\Theta_{s,t}(Y) = \sum_{i=1}^{d} Y_s^i W_{s,t}^i + \sum_{i,j=1}^{d} (Y_s')^{i,j} \mathbb{W}_{s,t}^{i,j}$$

In the case $Y_t = F(W_t)$ we have $Y_t' = \nabla F(W_t)$ and so the above definition coincides with (4.26).

For a partititon $\mathcal{P} = \{s = s_0 < ... < s_n = t\}$ we denote

$$\Theta_{s,t}^{\mathcal{P}}(Y) = \sum_{i=1}^{n} \Theta_{s_{i-1},s_i}(Y) = \sum_{k=1}^{n} \left(\sum_{i=1}^{d} Y_s^i W_{s,t}^i + \sum_{i,j=1}^{d} (Y_s')^{i,j} \mathbb{W}_{s,t}^{i,j} \right)$$

and we have

Theorem 2.7 *Let us fix some $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and we fix some trajectory $\mathbf{W}(\omega) = (W(\omega), \mathbb{W}(\omega))$ with $\omega \in \Omega_0$ (so that (2.8) holds) And let $(Y, Y') \in \mathcal{D}_W^\alpha$ be a $\alpha-$ controlled path (controlled by W). Then the following limit exists*

$$\int_s^t Y_r d\mathbf{W}_r := \lim_{|\mathcal{P}| \to 0} \Theta_{s,t}^{\mathcal{P}}(Y) \tag{2.18}$$

and it is called the "rough integral" of Y_r with respect to the rough path $\mathbf{W}$. Moreover $\left\| \int Y_r d\mathbf{W}_r \right\|_\alpha < \infty$. And $Z_{s,t} = \int_s^t Y_r d\mathbf{W}_r$ is the unique process such that

$$\left\| \int Y_r d\mathbf{W}_r - \Theta(Y) \right\|_{3\alpha} < \infty. \tag{2.19}$$

More precisely

$$\left| \int_s^t Y_r d\mathbf{W}_r - Y_s W_{s,t} - Y_s' \mathbb{W}_{s,t} \right| \leq C(\|W\|_\alpha \left\| R^Y \right\|_{2\alpha} + \|\mathbb{W}\|_{2\alpha} \left\| Y' \right\|_\alpha)(t-s)^{3\alpha} \tag{2.20}$$

Finally, Z is controlled by W and one has $Z' = Y$.

Proof The proof is quasi identical to the one concerning $Y_t = F(W_t)$, so we just sketch it. We want to use the sewing lemma so we have to estimate $\delta_{s,u,t}(\Theta)$. We first give an explicite expression wich will be used further on:

$$\Theta_{s,t} - \Theta_{s,u} - \Theta_{u,t} = \delta_{s,u,t}(\Theta) = \sum_{i=1}^{d} R^{Y,i}_{s,u} W^i_{u,t} + \sum_{i,j=1}^{d} (Y')^{i,j}_{s,u} \mathbb{W}^{i,j}_{u,t}. \tag{2.21}$$

We write

$$Y_u = Y_s + \left\langle Y'_s, W_{s,u} \right\rangle + R^Y_{s,t}$$

so that, by direct obvious computations

$$\begin{aligned} \Theta_{u,t} &= \sum_{i=1}^{d} Y^i_u W^i_{u,t} + \sum_{i,j=1}^{d} (Y')^{i,j}_u \mathbb{W}^{i,j}_{u,t} \\ &= \sum_{i=1}^{d} Y^i_s W^i_{u,t} + \sum_{i,j=1}^{d} (Y')^{i,j}_s W^i_{s,u} W^j_{u,t} + \sum_{i,j=1}^{d} (Y')^{i,j}_s \mathbb{W}^{i,j}_{u,t} + \delta_{s,u,t}(\Theta) \end{aligned}$$

with $\delta_{s,u,t}(\Theta)$ given in the right hand side of (2.21). Then, using Chen's relations

$$\begin{aligned} & \Theta_{s,u} + \Theta_{u,t} \\ = & \sum_{i=1}^{d} Y^i_s (W^i_{s,u} + W^i_{u,t}) + \sum_{i,j=1}^{d} (Y')^{i,j}_s (\mathbb{W}^{i,j}_{s,u} + \mathbb{W}^{i,j}_{u,t} + W^i_{s,u} W^j_{u,t}) + \delta_{s,u,t}(\Theta) \\ = & \Theta_{s,t} + \delta_{s,u,t}(\Theta). \end{aligned}$$

So we have proved (2.21), and since

$$|\delta_{s,u,t}(\Theta)| \le C(\left\|R^Y\right\|_{2\alpha} \|W\|_\alpha + \left\|Y'\right\|_\alpha \|\mathbb{W}\|_{2\alpha})(t-s)^{3\alpha}$$

so we are done. So we may use the sewing lemma in order to define the integral given by the limit in (2.18) and this integral verifies (2.19).

Let us check that the Gubinelli derivative of Z is given by Y. We have $\Theta_{s,t}(Y) = \langle Y_s, W_s \rangle + O((t-s)^{2\alpha})$ and $Z_{s,t} := \int_s^t Y_r d\mathbf{W}_r = \Theta_{s,t}(Y) + O((t-s)^{3\alpha})$. We conclude that $Z_{s,t} = \langle Y_s, W_s \rangle + O((t-s)^{2\alpha})$ and this means that $Z' = Y$. $\square$

Linearity Let $(Y, Y'), (\overline{Y}, \overline{Y}') \in \mathcal{D}^\alpha_W$. Then

$$(Y + \overline{Y})' = Y' + \overline{Y}', \quad and \quad R^{Y+\overline{Y}} = R^Y + R^{\overline{Y}}$$

and

$$\int Y dW + \int \overline{Y} dW = \int (Y + \overline{Y}) dW.$$

2.3.2 Second generalisation

Given $\alpha \in (\frac{1}{3}, \frac{1}{2})$ **we define** a α rough path to be a couple $\mathbf{W} = (W, \mathbb{W})$ with $W : R_+ \to R^d$ and $\mathbb{W} : R_+^2 \to R^{d\times d}$ such that

$$\|W\|_\alpha = \sup_{|t-s|>0} \frac{|W_{s,t}|}{|t-s|^\alpha} < \infty \quad and \quad \|\mathbb{W}\|_{2\alpha} = \sup_{|t-s|>0} \frac{|\mathbb{W}_{s,t}|}{|t-s|^{2\alpha}} < \infty$$

and for which "Chen relations" hold true:

$$\begin{aligned} W_{s,t} &= W_{s,u} + W_{u,t} \\ \mathbb{W}_{s,t}^{i,j} &= \mathbb{W}_{s,u}^{i,j} + \mathbb{W}_{u,t}^{i,j} + W_{s,u}^i W_{u,t}^j \end{aligned}$$

We speak about a **"geometric rough" path** if

$$\mathbb{W}_{s,t}^{i,j} + \mathbb{W}_{s,t}^{j,i} = W_{s,t}^i W_{s,t}^j.$$

For example, the "Stratonovich" rough path is a geometric rough path while the Itô rough path is not. The geometrical rough paths are important for two reasons: first, a first order calculus holds for them and second, rough intergals with respect to a geometrical rough path may be approximated by usual regularizations of the trajectory (see (FH) Proposition 2.5 p19).

Remark 2.8 *It is not possible to have the "second component" $\mathbb{W}_{s,t}^{i,j}$ which is symmetric, i.e. such that $\mathbb{W}_{s,t}^{i,j} = \mathbb{W}_{s,t}^{j,i}$. Indeed, the symmetry combined with the second Chen relation give $W_{s,u}^i W_{u,t}^j = W_{s,u}^j W_{u,t}^i$ which is clearly false. In particular one may not take $\mathbb{W}_{s,t}^{i,j} = W_{s,t}^i W_{s,t}^j$. However this is the quantity which multiplies $\partial_{i,j}^2 F(W_t)$ in the Taylor expension at order two.*

We denote by $\mathcal{C}^\alpha$ the space of α rough paths (some confusion with α Hölder path appears) and, for $\mathbf{W} = (W, \mathbb{W})$ we define (see (FH) p 15) the "homogenous rough path norm"

$$|||\mathbf{W}|||_\alpha = \|W\|_\alpha + \sqrt{\|\mathbb{W}\|_{2\alpha}} \tag{2.22}$$

and the "inhomogenuous rough path distance"

$$\rho_\alpha(\mathbf{W}, \widetilde{\mathbf{W}}) = \left\| W - \widetilde{W} \right\|_\alpha + \left\| \mathbb{W} - \widetilde{\mathbb{W}} \right\|_{2\alpha} \tag{2.23}$$

Remark 2.9 $|||\mathbf{W}|||_\alpha$ *is not a norm because it is null for non null costants, and similary for ρ_α. So, if we want that $\rho_\alpha(\mathbf{W}, \widetilde{\mathbf{W}}) = 0$ implies that $\mathbf{W} = \widetilde{\mathbf{W}}$ we have to ask the supplementary condtion $W_0 = \widetilde{W}_0$.*

The notion of "controlled path" is the same as before: $Y : R_+ \to R^d$ which is α Hölder, is a "controlled path" if there exists $Y' : R_+ \to R^{d\times d}$ such that

$$Y_t = \langle Y_s', W_{s,t} \rangle + O((t-s)^{2\alpha}).$$

The properties listed before (exception the notion of geometric rough path) are the ones which have been used in order to construct the rough integral so that we obtain the same result in this abstract framework: Given a α controlled path Y with Gubinelli derivative Y' we define

$$\Theta_{s,t}(Y) = \sum_{i=1}^d Y_s^i W_{s,t}^i + \sum_{i,j=1}^d (Y_s')^{i,j} \mathbb{W}_{s,t}^{i,j}$$

and for a partititon $\mathcal{P} = \{s = s_0 < ... < s_n = t\}$ we denote

$$\Theta^{\mathcal{P}}_{s,t}(Y) = \sum_{i=1}^{n} \Theta_{s_{i-1},s_i}(Y) = \sum_{k=1}^{n}\left(\sum_{i=1}^{d} Y^i_s W^i_{s,t} + \sum_{i,j=1}^{d} (Y'_s)^{i,j}\mathbb{W}^{i,j}_{s,t}\right)$$

and we have

Theorem 2.10 *Let us fix some $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and let $\mathbf{W} = (W, \mathbb{W})$ be a α rough path. And let (Y, Y') be a $\alpha-$controlled path. Then the following limit exists*

$$Z_{s,t} := \int_s^t Y_r d\mathbf{W}_r = \lim_{|\mathcal{P}|\to 0} \Theta^{\mathcal{P}}_{s,t}(Y)$$

and it is called the "rough integral" of Y_r with respect to the rough path $\mathbf{W}$. Moreover $\left\|\int Y_r d\mathbf{W}_r\right\|_\alpha < \infty$ and $Z' = Y$.

2.3.3 Stability for the rough integral

In this section we will use a specific distance that we define now. We consider two α rough paths $\mathbf{W}$ and $\widetilde{\mathbf{W}}$ and $(Y, Y') \in \mathcal{D}^{2\alpha}_W, (\widetilde{Y}, \widetilde{Y}') \in \mathcal{D}^{2\alpha}_{\widetilde{W}}$. Then we define the "distance"

$$d_{W,\widetilde{W},2\alpha}((Y, Y'), (\widetilde{Y}, \widetilde{Y}')) = \left\|Y' - \widetilde{Y}'\right\|_\alpha + \left\|R^Y - R^{\widetilde{Y}}\right\|_{2\alpha} \tag{2.24}$$

This is not really a distance because it is not null if $\widetilde{Y} - Y = c$.

Remark 2.11 *If Y and $\widetilde{Y}$ are controlled by the same rough path $\mathbf{W} = \widetilde{\mathbf{W}}$, then $Y' - \widetilde{Y}' = (Y - \widetilde{Y})'$ and $R^Y - R^{\widetilde{Y}} = R^{Y-\widetilde{Y}}$, so*

$$d_{W,\widetilde{W},2\alpha}((Y, Y'), (\widetilde{Y}, \widetilde{Y}')) = \left\|Y - \widetilde{Y}, (Y - \widetilde{Y})'\right\|_{W,2\alpha}$$

defined in (2.12). But here Y and $\widetilde{Y}$ are controlled by different rough paths and this is why we need a special, different, definition.

Now we fix $\frac{1}{3} < \alpha < \beta$. Such two indeces will appear in the fowlloning theorem and are used in order to perform an usefull computational trik.

We recall that in (2.23) we have defined

$$\rho_\alpha(\mathbf{W}, \widetilde{\mathbf{W}}) = \left\|W - \widetilde{W}\right\|_\alpha + \left\|\mathbb{W} - \widetilde{\mathbb{W}}\right\|_{2\alpha}$$

Finally we define the ball

$$\begin{aligned}\mathcal{A}_M &= \mathcal{A}_M(\alpha, \beta) \\ &= \{(Y, Y', \mathbf{W}) : \|W\|_\beta + \|\mathbb{W}\|_{2\beta} + \left\|Y'\right\|_\alpha + \left\|R^Y\right\|_{2\alpha} + |Y_0| + \left|Y'_0\right| \le M\}.\end{aligned}$$

We are now able to give our "local" stability result:

Theorem 2.12 *Let $\frac{1}{3} < \alpha \le \beta$ be given, and let $T \le 1$. Suppose that $(Y, Y') \in \mathcal{D}_W^{2\alpha}$, $(\widetilde{Y}, \widetilde{Y}') \in \mathcal{D}_{\widetilde{W}}^{2\alpha}$ and $(Y, Y', W) \in \mathcal{A}_M(\alpha, \beta)$ and $(\widetilde{Y}, \widetilde{Y}', \widetilde{\mathbf{W}}) \in \mathcal{A}_M(\alpha, \beta)$. Denote $Z_{s,t} = \int_s^t Y dW$ and $\widetilde{Z}_{s,t} = \int_s^t \widetilde{Y} d\widetilde{W}$ and recall that $Z' = Y$ and $\widetilde{Z}' = \widetilde{Y}$. Then there exists a constant $C = C(\alpha, \beta, M)$ such that the following holds:*

$$
\begin{aligned}
& d_{W,\widetilde{W},2\alpha}((Z, Z'), (\widetilde{Z}, \widetilde{Z}')) \le C(T^\alpha + T^{\beta-\alpha}) \\
& \times (d_{W,\widetilde{W},2\alpha}((Y, Y'), (\widetilde{Y}, \widetilde{Y}')) + \rho_\beta(\mathbf{W}, \widetilde{\mathbf{W}}) + \left| Y_0' - \widetilde{Y}_0' \right|)
\end{aligned} \tag{2.25}
$$

and

$$
\begin{aligned}
& \left\| Z - \widetilde{Z} \right\|_\alpha \le C(T^\alpha + T^{\beta-\alpha}) \\
& \times (d_{W,\widetilde{W},2\alpha}((Y, Y'), (\widetilde{Y}, \widetilde{Y}')) + \rho_\beta(\mathbf{W}, \widetilde{\mathbf{W}}) + \left| Y_0' - \widetilde{Y}_0' \right| + \left| Y_0 - \widetilde{Y}_0 \right|)
\end{aligned} \tag{2.26}
$$

Proof (2.26) follows (rather) easily from (2.25), so we concentrate on this one. We first compute

$$Z'_{s,t} - \widetilde{Z}'_{s,t} = Y_{s,t} - \widetilde{Y}_{s,t} = a + b + c$$

with

$$a = Y_s'(W_{s,t} - \widetilde{W}_{s,t}), \quad b = (Y_s' - \widetilde{Y}_s')\widetilde{W}_{s,t}, \quad c = R_{s,t}^Y - R_{s,t}^{\widetilde{Y}}.$$

Write $Y_s' = Y_0' + (Y_s' - Y_0')$. Then, for $Y \in \mathcal{A}_M(\alpha, \beta)$ one obtains $|Y_s'| \le M + MT^\alpha \le 2M$ so that

$$|a| \le 2M \left\| W - \widetilde{W} \right\|_\beta (t-s)^\beta \le 2M \left\| W - \widetilde{W} \right\|_\beta T^{\beta-\alpha}(t-s)^\alpha$$

Moreover

$$
\begin{aligned}
|b| &\le (\left| Y_0' - \widetilde{Y}_0' \right| + \left\| Y' - \widetilde{Y}' \right\|_\alpha (t-s)^\alpha) \left\| \widetilde{W} \right\|_\beta (t-s)^\alpha T^{\beta-\alpha} \\
&\le (\left| Y_0' - \widetilde{Y}_0' \right| + \left\| Y' - \widetilde{Y}' \right\|_\alpha) M (t-s)^\alpha T^{\beta-\alpha}
\end{aligned}
$$

Finally $|c| \le \left\| R^Y - R^{\widetilde{Y}} \right\|_{2\alpha} (t-s)^\alpha T^\alpha$. We conclude that

$$
\begin{aligned}
\left\| Z' - \widetilde{Z}' \right\|_\alpha &\le CM(T^\alpha + T^{\beta-\alpha}) \times \\
&\quad (\left| Y_0' - \widetilde{Y}_0' \right| + d_{W,\widetilde{W},2\alpha}((Y, Y'), (\widetilde{Y}, \widetilde{Y}')) + \rho_\beta(\mathbf{W}, \widetilde{\mathbf{W}}))
\end{aligned}
$$

so it fits in our estimate.

We deal now with the remainder. We write

$$
\begin{aligned}
R_{s,t}^Z &= Z_{s,t} - Z_s' W_{s,t} = Z_{s,t} - Y_s W_{s,t} - Y_s' \mathbb{W}_{s,t} + Y_s' \mathbb{W}_{s,t} \\
&= \int_s^t Y dW - \Theta_{s,t}(Y) + Y_s' \mathbb{W}_{s,t}
\end{aligned}
$$

with $\Theta_{s,t}(Y) = Y_s W_{s,t} + Y_s' \mathbb{W}_{s,t}$. And in the same way

$$R_{s,t}^{\widetilde{Z}} = \int_s^t \widetilde{Y} d\widetilde{W} - \Theta_{s,t}(\widetilde{Y}) + \widetilde{Y}_s' \widetilde{\mathbb{W}}_{s,t}.$$

We will use the sawing lemma for $\Gamma_{s,t} = \Theta_{s,t}(Y) - \Theta_{s,t}(\widetilde{Y})$. In order to do it we have to estimate $\delta_{s,u,t}(\Gamma)$. We look for $C_{sew}(\Gamma)$ which verifies (see (2.1))

$$\sup_{s<u<t} |\delta_{s,u,t}(\Gamma)| \leq C_{sew}(\Gamma)(t-s)^{3\alpha}.$$

Recall now that by (2.21)

$$\begin{aligned}
\delta_{s,u,t}(\Gamma) &= \sum_{i=1}^{d} R^{Y,i}_{s,u} W^i_{u,t} - \sum_{i=1}^{d} \widetilde{R}^{Y,i}_{s,u} \widetilde{W}^i_{u,t} \\
&\quad + \sum_{i,j=1}^{d} (Y')^{i,j}_{s,u} \mathbb{W}^{i,j}_{u,t} - \sum_{i,j=1}^{d} (\widetilde{Y}')^{i,j}_{s,u} \widetilde{\mathbb{W}}^{i,j}_{u,t}.
\end{aligned}$$

And standard computations (the same as above) give

$$\begin{aligned}
|\delta_{s,u,t}(\Gamma)| &\leq CM(T^{\alpha} + T^{\beta-\alpha}) \\
&\quad \times (d_{W,\widetilde{W},2\alpha}((Y,Y'),(\widetilde{Y},\widetilde{Y}')) + \rho_\beta(\mathbf{W},\widetilde{\mathbf{W}}) + \left|Y'_0 - \widetilde{Y}'_0\right|)(t-s)^{3\alpha}
\end{aligned}$$

which means that

$$C_{sew}(\Gamma) \leq CM(T^{\alpha} + T^{\beta-\alpha})(d_{W,\widetilde{W},2\alpha}((Y,Y'),(\widetilde{Y},\widetilde{Y}')) + \rho_\beta(\mathbf{W},\widetilde{\mathbf{W}}) + \left|Y'_0 - \widetilde{Y}'_0\right|)$$

Now we may use the sewing lemma. Notice that (with the notation in the first section) $X_{s,t}(\Gamma) = \int_s^t Y dW - \int_s^t \widetilde{Y} d\widetilde{W}$ so that, by the sewing lemma (see (2.4)) we have

$$\left|\int_s^t Y dW - \int_s^t \widetilde{Y} d\widetilde{W} - \Gamma_{s,t}\right| \leq 2^{3\alpha}\zeta(3\alpha) C_{sew}(\Gamma)(t-s)^{3\alpha}.$$

and this yields

$$\begin{aligned}
&\left\|\int Y dW - \int \widetilde{Y} d\widetilde{W} - \Gamma\right\|_{3\alpha} \leq CM(T^{\alpha} + T^{\beta-\alpha}) \\
&\times (d_{W,\widetilde{W},2\alpha}((Y,Y'),(\widetilde{Y},\widetilde{Y}')) + \rho_\beta(\mathbf{W},\widetilde{\mathbf{W}}) + \left|Y'_0 - \widetilde{Y}'_0\right|),
\end{aligned}$$

which is also choerent with our estimate.

We write now

$$\begin{aligned}
&\left|Y'_s \mathbb{W}_{s,t} - \widetilde{Y}'_s \widetilde{\mathbb{W}}_{s,t}\right| \leq \left|Y'_s - \widetilde{Y}'_s\right| \|\mathbb{W}_{s,t}\| + \left|Y'_s\right| \left|\mathbb{W}_{s,t} - \widetilde{\mathbb{W}}_{s,t}\right| \\
\leq\ & CM(\left|Y'_0 - \widetilde{Y}'_0\right| + d_{W,\widetilde{W},2\alpha}((Y,Y'),(\widetilde{Y},\widetilde{Y}')) + \rho_\beta(\mathbf{W},\widetilde{\mathbf{W}}))(t-s)^{2\alpha} T^{2(\beta-\alpha)}
\end{aligned}$$

We have used here $\|\mathbb{W}_{s,t}\|_{2\alpha} \leq \|\mathbb{W}\|_{2\beta} (t-s)^{2\alpha} T^{2(\beta-\alpha)}$. We have proved that

$$\begin{aligned}
&\left\|R^Z - R^{\widetilde{Z}}\right\|_{2\alpha} \leq CM(T^{2(\beta-\alpha)} + T^{\alpha}) \times \\
&\times (d_{W,\widetilde{W},2\alpha}((Y,Y'),(\widetilde{Y},\widetilde{Y}')) + \rho_\beta(\mathbf{W},\widetilde{\mathbf{W}}) + \left|Y'_0 - \widetilde{Y}'_0\right|).
\end{aligned}$$

This was the tricky step. $\square$

2.4 Itô's formula

2.4.1 The case of one forms

Iin this section we will give the Itô formula. We discuss first the simple case of the integrand $t \to F(W_t)$. This formula will include the rough integral of $Y_t = \nabla F(W_t)$ and also the Young integral of $Y_t' = \nabla^2 F(W_t)$ with respect to the "quadratic variation" process of W. But the problem is that, for the moment, we have no such a quadratic process, so we have to understand what it could be, and how it works. In order to do it we use second order Taylor expansion and we obtain

$$F(W_t) - F(W_s) = \sum_{i-1}^{d} \partial_i F(W_s) W_{s,t}^i + \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j F(W_s) W_{s,t}^i W_{s,t}^j + O(t-s)^{3\alpha}. \tag{2.27}$$

Since $\alpha > \frac{1}{3}$ we may ignor the last term. We look now to the second term. We denote

$$\mathbb{S}_{s,t}^{i,j} = \frac{1}{2}(\mathbb{W}_{s,t}^{i,j} + \mathbb{W}_{s,t}^{j,i})$$

the symmetric part of $\mathbb{W}$ and we define

$$\left\langle \mathbf{W}^{i,j} \right\rangle_{s,t} = W_{s,t}^i W_{s,t}^j - (\mathbb{W}_{s,t}^{i,j} + \mathbb{W}_{s,t}^{j,i}) = W_{s,t}^i W_{s,t}^j - 2\mathbb{S}_{s,t}^{i,j}. \tag{2.28}$$

Then

$$\frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j F(W_s) W_{s,t}^i W_{s,t}^j = \sum_{i,j=1}^{d} \partial_i \partial_j F(W_s) (\mathbb{S}_{s,t}^{i,j} + \frac{1}{2} \left\langle \mathbf{W}^{i,j} \right\rangle_{s,t}).$$

Coming back to (2.27) we get

$$\begin{aligned} F(W_t) - F(W_s) \quad = \quad & \sum_{i-1}^{d} \partial_i F(W_s) W_{s,t}^i + \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j F(W_s) (\mathbb{W}_{s,t}^{i,j} + \mathbb{W}_{s,t}^{j,i}) \\ & + \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j F(W_s) \left\langle \mathbf{W}^{i,j} \right\rangle_{s,t} + O(t-s)^{3\alpha}. \end{aligned}$$

Let us denote

$$Y_s^i = \partial_i F(W_s) \quad \textit{and} \quad (Y_s')^{i,j} = \partial_i \partial_j F(W_s).$$

Then

$$\begin{aligned} & \sum_{i-1}^{d} \partial_i F(W_s) W_{s,t}^i + \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j F(W_s) (\mathbb{W}_{s,t}^{i,j} + \mathbb{W}_{s,t}^{j,i}) \\ = \quad & \langle Y_s, W_{s,t} \rangle + \sum_{i,j=1}^{d} (Y_s')^{i,j} \times \mathbb{W}_{s,t}^{i,j} = \Theta_{s,t}(Y) \end{aligned} \tag{2.29}$$

reperesents the "one step Euler scheme" which allows to define the rough integral $\int Y_r d\mathbf{W}_r$.

What about the third term? We know that $s \to \partial_i \partial_j F(W_s)$ is α Hölder. It is also clear that $s \to \left\langle \mathbf{W}^{i,j} \right\rangle_s$ is 2α Hölder. Then $\partial_i \partial_j F(W_s) \left\langle \mathbf{W}^{i,j} \right\rangle_{s,t}$ will give a Young integral. This leads to the following

Theorem 2.13 *Consider a α rough path $\mathbf{W}$. Then for every $F \in C_b^3(R^d, R)$ one has*

$$F(W_T) = F(W_0) + \int_0^T \nabla F(W_r) d\mathbf{W}_r + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^T \partial_i \partial_j F(W_r) d \left\langle \mathbf{W}^{i,j} \right\rangle_{0,r} \tag{2.30}$$

where $\int_0^T \nabla F(W_r) d\mathbf{W}_r$ is the rough integral with respect to $\mathbf{W}$ and $\int_0^T \partial_i \partial_j F(W_r) d \left\langle \mathbf{W}^{i,j} \right\rangle_{0,r}$ is the Young integral with respect to $\left\langle \mathbf{W}^{i,j} \right\rangle$.

Remark 2.14 *The bracket $\left\langle \mathbf{W}^{i,j} \right\rangle$ is the term which is needed in order to pass from $W^i W^j$ (which appears in the terms of second order in the Taylor formula (2.27)) to $\frac{1}{2}(\mathbb{W}_{s,t}^{i,j} + \mathbb{W}_{s,t}^{j,i})$ which is the second order coefficient of $\partial_{i,j}^2 F = \partial_{j,i}^2 F$. In the case of "geometric rough path" (in particular for the Stratonovich integral) $\left\langle \mathbf{W}^{i,j} \right\rangle = 0$ so we find out the "standard" calculus rule.*

Remark 2.15 *One may be tempted to take $\mathbb{W}_{s,t}^{i,j} = W_{s,t}^i W_{s,t}^j$ in order to make the rough integral choerent whith the second order Taylor deelopment. But we have already noticed that this choice is not compatible with the second Chen relation.*

Proof We consider a partititon $\mathcal{P} = \{0 = s_0 < ... < s_n = T\}$ and we write

$$\begin{aligned} F(W_T) - F(W_0) &= \sum_{i=1}^{n} (F(W_{s_i}) - F(W_{s_{i-1}})) \\ &= \sum_{i=1}^{n} \Theta_{s_{i-1}, s_i}(\nabla F(W)) + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{i=1}^{n} \partial_i \partial_j F(W_{i-1}) \left\langle \mathbf{W}^{i,j} \right\rangle_{s_{i-1}, s_i}. \end{aligned}$$

And passing to the limit with $n \to \infty$ we obtain (2.30). $\square$

2.4.2 The case of Itô processes

We consider now the more general case when $Y_{s,t} \in R^d$ is (the analogous) of an Itô process. We consider $Y, Y', Y'' \in \mathcal{C}^\alpha$ such that $(Y, Y') \in \mathcal{D}_W^\alpha$ and $(Y', Y'') \in \mathcal{D}_W^\alpha$ are α controled (by W) paths. In order to be more explicit we denote $Y' = D_W Y = (D_W^1 Y, ..., D_W^d Y)$ and $Y'' = (D_W^{i,j} Y)_{i,j=1,...,d}$ and this means that

$$\begin{aligned} Y_{s,t} &= \sum_{i=1}^{d} (D_W^i Y_s) W_{s,t}^i + o(t-s)^{2\alpha} \\ D_W^j Y_{s,t} &= \sum_{i=1}^{d} (D_W^{i,j} Y_s) W_{s,t}^i + o(t-s)^{2\alpha}. \end{aligned}$$

Now we suppose that Y is an Itô type trajectory, that is

$$Y_{s,t} = \int_s^t Y'_{s,r} d\mathbf{W}_r + \Gamma_{s,t}$$

with $\Gamma \in \mathcal{C}^{2\alpha}$. Notice that the rough integral $\int Y' d\mathbf{W}$ is based on the rough path $\mathbf{W}$ and on the controled path $(Y', Y'') \in \mathcal{D}_W^\alpha$.

Theorem 2.16 *Consider a α rough path $\mathbf{W}$. Moreover let Y, Y', Y'' such that $(Y, Y') \in \mathcal{D}_W^\alpha$ and $(Y', Y'') \in \mathcal{D}_W^\alpha$. Then for every $F \in C_b^3(R^d, R)$ one has*

$$\begin{aligned} F(Y_t) &= F(Y_s) + \sum_{i=1}^d \int_s^t \partial_i F(Y_u) d\Gamma_u + \sum_{k,i=1}^d \int_s^t (\partial_i F)(Y_u) D_W^k Y_u^i d\mathbf{W}_u^k \\ &+ \frac{1}{2} \sum_{k,p=1}^d \sum_{i,j=1}^d \int_s^t \partial_j \partial_i F(Y_u)(D_W^k Y_u^i \times D_W^p Y_u^j) d \left\langle W^{k,p} \right\rangle_u \end{aligned}$$

where $\int_0^T \nabla F(W_r) D_W Y_r d\mathbf{W}_r$ is the rough integral with respect to $\mathbf{W}$ and $\int_0^T \nabla^2 F(W_r) D_W Y_r \otimes D_W Y_r d \left\langle \mathbf{W}^{i,j} \right\rangle_{0,r}$ is the Young integral with respect to $\left\langle \mathbf{W}^{i,j} \right\rangle$.

Remark 2.17 *In the case of "geometric rough path" $\left\langle \mathbf{W}^{i,j} \right\rangle = 0$ so the formula holds and coincides with the standard computation rule. In the case of the Itô Lévy area (coreponding to the Itô type integral) one has $\left\langle \mathbf{W}^{i,j} \right\rangle_t = \delta_{i,j} t$ and so one may use Itô's formula, which coincides with the classical one in stochastic calculus.*

Proof. Step 1 We use the notation $a \simeq_\beta b$ in order to say that $a_{s,t} = b_{s,t} + O(t-s)^\beta$. First we notice that, by the very definition of the rough integral we have

$$\begin{aligned} Y_{s,t} &\simeq {}_{3\alpha} Y_s' W_{s,t} + Y_s'' \mathbb{W}_{s,t} + \Gamma_{s,t} \\ &= \sum_{k-1}^d D_W^k Y_s W_{s,t}^k + \sum_{k,p-1}^d D_W^{k,p} Y_s \mathbb{W}_{s,t}^{k,p} + \Gamma_{s,t} \end{aligned}$$

Here $D_W^k Y_s = (Y_s')^k$ and $D_W^{k,p} Y_s = (Y_s'')^{k,p}$. The above equality gives

$$\begin{aligned} Y_{s,t}^i Y_{s,t}^j &\simeq {}_{3\alpha} \sum_{k,p} (D_W^k Y_s^i \times D_W^p Y_s^j) W_{s,t}^k W_{s,t}^p \\ &= \sum_{k,p} (D_W^k Y_s^i \times D_W^p Y_s^j)(\mathbb{W}_{s,t}^{k,p} + \mathbb{W}_{s,t}^{p,k} + \left\langle W^{k,p} \right\rangle_{s,t}) \end{aligned}$$

Step 2 Using Taylor expansion

$$F(Y_t) - F(Y_s) \simeq_{3\alpha} \sum_{i=1}^d \partial_i F(Y_s) Y_{s,t}^i + \frac{1}{2} \sum_{i,j=1}^d \partial_j \partial_i F(Y_s) Y_{s,t}^i Y_{s,t}^j = \sum_{r=1}^4 S_{s,t}^r$$

with

$$\begin{aligned} S_{s,t}^1 &= \sum_{i=1}^d \partial_i F(Y_s) \Gamma_{s,t}^i, \\ S_{s,t}^2 &= \sum_{k,p} \sum_{i,j=1}^d \partial_j \partial_i F(Y_s)(D_W^k Y_s^i \times D_W^p Y_s^j) \left\langle W^{k,p} \right\rangle_{s,t} \end{aligned}$$

and

$$
\begin{aligned}
S^3_{s,t} &= \sum_{i=1}^{d} \partial_i F(Y_s)(\sum_{k-1}^{d} D^k_W Y^i_s W^k_{s,t} + \sum_{k,p-1}^{d} D^{k,p}_W Y^i_s \mathbb{W}^{k,p}_{s,t}) \\
S^4_{s,t} &= \frac{1}{2}\sum_{i,j=1}^{d} \partial_j \partial_i F(Y_s) \sum_{k,p} (D^k_W Y^i_s \times D^p_W Y^j_s)(\mathbb{W}^{k,p}_{s,t} + \mathbb{W}^{p,k}_{s,t})
\end{aligned}
$$

Notice that, since $\partial_j \partial_i F(Y_s) = \partial_i \partial_j F(Y_s)$ we ahave

$$
S^4_{s,t} = \sum_{i,j=1}^{d} \partial_j \partial_i F(Y_s) \sum_{k,p} (D^k_W Y^i_s \times D^p_W Y^j_s) \mathbb{W}^{k,p}_{s,t}.
$$

And we have the convergence

$$
\begin{aligned}
\sum_{r=1}^{n} S^1_{s_{r-1},s_r} &\to \sum_{i=1}^{d} \int_s^t \partial_i F(Y_u) d\Gamma_u \\
\sum_{r=1}^{n} S^2_{s_{r-1},s_r} &\to \sum_{k,p=1}^{d} \sum_{i,j=1}^{d} \int_s^t \partial_j \partial_i F(Y_u)(D^k_W Y^i_u \times D^p_W Y^j_u) d\left\langle W^{k,p} \right\rangle_u
\end{aligned}
$$

both these integrals being Young integrals.

Step 3 Let

$$
G^{i,k}_t = (\partial_i F)(Y_t) D^k_W Y^i_t.
$$

Then, a direct computation shows that

$$
D^p_W G^{i,k}_s = (\partial_i F)(Y_s) D^p_W D^k_W Y^i_s + \sum_{j=1}^{d} (\partial_j \partial_i F)(Y_s) D^p_W Y^j_s D^k_W Y^i_s
$$

This formula may be obtained by "formal derivation", but this fits to the rigourous definition of Gubinelli derivative. Then, by the very definition of the rough integal

$$
\begin{aligned}
& \sum_{k,i=1}^{d} \int_s^t (\partial_i F)(Y_r) D^k_W Y^i_r d\mathbf{W}^k_r \\
\simeq & {}_{3\alpha} \sum_{k,i=1}^{d} (\partial_i F)(Y_t) D^k_W Y^i_t \times W^k_{s,t} \\
& + \sum_{k,i=1}^{d} \sum_{p=1}^{d} \left((\partial_i F)(Y_s) D^p_W D^k_W Y^i_s + \sum_{j=1}^{d} (\partial_j \partial_i F)(Y_s) D^p_W Y^j_s D^k_W Y^i_s \right) \mathbb{W}^{k,p}_{s,t} \\
= & S^3_{s,t} + S^4_{s,t}.
\end{aligned}
$$

We conclude that

$$
\sum_{r=1}^{n} (S^3_{s_{r-1},s_r} + S^4_{s_{r-1},s_r}) \to \sum_{k,i=1}^{d} \int_s^t (\partial_i F)(Y_u) D^k_W Y^i_u d\mathbf{W}^k_u
$$

And we have proved the formula

$$\begin{aligned} F(Y_t) &= F(Y_s) + \sum_{i=1}^{d} \int_s^t \partial_i F(Y_u) d\Gamma_u + \sum_{k,i=1}^{d} \int_s^t (\partial_i F)(Y_u) D_W^k Y_u^i d\mathbf{W}_u^k \\ &+ \sum_{k,p=1}^{d} \sum_{i,j=1}^{d} \int_s^t \partial_j \partial_i F(Y_u)(D_W^k Y_u^i \times D_W^p Y_u^j) \left\langle W^{k,p} \right\rangle_u \end{aligned}$$

□

3 Rough differential equations

3.1 Norms on controlled path and smooth functions

All along this section we assume, without special mention, that $T \leq 1$. We recall that (Y, Y') is a controlled path if

$$Y_{s,t} = Y'_s W_{s,t} + R^Y_{s,t}.$$

And one defines the norm

$$\left\| Y, Y' \right\|_{W,2\alpha} = \left\| Y' \right\|_\alpha + \left\| R^Y \right\|_{2\alpha}. \tag{3.1}$$

We will also use the notation

$$Q_\beta = 1 + \left\| W \right\|_\beta + \left\| \mathbb{W} \right\|_{2\beta}. \tag{3.2}$$

It is easy to check that

$$\left\| Y' \right\|_\infty \leq \left| Y'_0 \right| + \left\| Y, Y' \right\|_{W,2\alpha} T^\alpha \tag{3.3}$$

and, for $\alpha < \beta$,

$$\left\| Y \right\|_\alpha \leq C \left\| W \right\|_\beta (\left| Y'_0 \right| + \left\| Y, Y' \right\|_{W,2\alpha}) T^{\beta-\alpha} + \left\| Y, Y' \right\|_{W,2\alpha} T^\alpha, \tag{3.4}$$

$$\left\| Y \right\|_\infty \leq \left| Y_0 \right| + \left\| Y \right\|_\alpha \tag{3.5}$$

Remark 3.1 *In (3.4) $T^{\beta-\alpha}$ appears. This term will be useful for small T in order to destroy constants - for example when we will use a contraction argument for proving existence of solutions of rough equations. So, the fact that we introduce $\beta > \alpha$ is a useful trick.*

Lemma 3.2 *If $f \in C_b^2$ then (see Lemma 7.3 (FH))*

$$(f(Y))' = f'(Y)Y' \tag{3.6}$$

and

$$\left\| f(Y), f(Y)' \right\|_{W,2\alpha} \tag{3.7}$$

$$\leq C \left\| f \right\|_{2,\infty} Q_\alpha^2 (1 + \left| Y'_0 \right| + \left\| Y, Y' \right\|_{W,2\alpha})(\left| Y'_0 \right| + \left\| Y, Y' \right\|_{W,2\alpha}) \tag{3.8}$$

Proof One writes $Y'_s W_{s,t} = Y_{s,t} - R^Y_{s,t}$ so that

$$\begin{aligned} R_{s,t} &: = f(Y_t) - f(Y_s) - f'(Y_s) Y'_s W_{s,t} \\ &= f(Y_t) - f(Y_s) - f'(Y_s) Y_{s,t} + f'(Y_s) R^Y_{s,t}. \end{aligned}$$

Then

$$\|R\|_{2\alpha} \leq 2\,\|f\|_{2,\infty} (\|Y\|_\alpha^2 + \left\|R^Y\right\|_{2\alpha}) < \infty.$$

This already shows that $f'(Y_s)Y'_s = (f(Y))'$ and $R^{f(Y)} = R$. It is also easy to see that

$$\begin{aligned} \left\|(f(Y))'\right\|_\alpha &= \left\|f'(Y)Y'\right\|_\alpha \leq C\,\|f\|_{2,\infty} (\|Y\|_\alpha \left\|Y'\right\|_\infty + \left\|Y'\right\|_\alpha) \\ &\leq C\,\|f\|_{2,\infty} (\|Y\|_\alpha + \left\|Y'\right\|_\alpha)(1 + \left|Y'_0\right| + \left\|Y'\right\|_\alpha). \end{aligned}$$

Combining these two estimates with (3.4) (with $T \leq 1$ and $\alpha = \beta$) one obtains (3.7). □

We prove now that

$$\|f(Y)\|_\alpha \leq C\,\|f\|_{1,\infty} Q_\beta (1 + \left|Y'_0\right| + \left\|Y, Y'\right\|_{W,2\alpha})(T^{\beta-\alpha} + T^\alpha). \tag{3.9}$$

Indeed

$$\begin{aligned} |f(Y_t) - f(Y_s)| &\leq \|f\|_{1,\infty} |Y_{s,t}| \\ &\leq \|f\|_{1,\infty} (\left\|Y'\right\|_\infty \|W\|_\beta (t-s)^\beta + \left\|R^Y\right\|_{2\alpha} (t-s)^{2\alpha}) \\ &\leq C\,\|f\|_{1,\infty} (1 + \|W\|_\beta)(1 + \left|Y'_0\right| + \left\|Y, Y'\right\|_{W,2\alpha})(t-s)^\alpha \times (T^{\beta-\alpha} + T^\alpha). \end{aligned}$$

Finally we recall that, by the definition of the rough integral we have

$$\left|\int_s^t Y_r d\mathbf{W}_r - Y_s W_{s,t} - Y'_s \mathbb{W}_{s,t}\right| \leq C(\|W\|_\alpha \left\|R^Y\right\|_{2\alpha} + \|\mathbb{W}\|_{2\alpha} \left\|Y'\right\|_\alpha)(t-s)^{3\alpha} \tag{3.10}$$

And, if $Z = \int Y d\mathbf{W}$, then Z is controlled by W and with Gubinelli derivative $Z' = Y$.

We will still need the following:

Remark 3.3 *Let $G = (G^1, ..., G^d)$ and $H = (H^1, ..., H^d)$ with $G^i, H^i \in \mathcal{D}^{2\alpha}_W$. And let $Z := \langle G, H\rangle = \sum_{i=1}^d G^i H^i$. Then*

$$\left\|Z, Z'\right\|_{W,2\alpha} \leq C Q_\alpha^2 (|G_0| + \left|G'_0\right| + \left\|G, G'\right\|_{W,2\alpha})(|H_0| + \left|H'_0\right| + \left\|H, H'\right\|_{W,2\alpha}). \tag{3.11}$$

Proof Note first that

$$Z_{s,t} = \langle G_{s,t}, H_s\rangle + \langle G_t, H_{s,t}\rangle$$

and

$$Z'_t = \left\langle G'_t, H_t\right\rangle + \left\langle G_t, H'_t\right\rangle.$$

In particular

$$\left|Z'_{s,t}\right| \leq \left|G'_{s,t}\right| \|H\|_\infty + |H_{s,t}| \left\|G'\right\|_\infty + \left|H'_{s,t}\right| \|G\|_\infty + |G_{s,t}| \left\|H'\right\|_\infty$$

One also has (recall that $T \leq 1$ and use (3.3), (3.4) and (3.5))

$$\begin{aligned} \|H\|_\infty + \left\|H'\right\|_\infty &\leq C Q_\alpha (|H_0| + \left|H'_0\right| + \left\|H, H'\right\|_{W,2\alpha}) \\ |H_{s,t}| + \left|H'_{s,t}\right| &\leq C Q_\alpha (|H_0| + \left|H'_0\right| + \left\|H, H'\right\|_{W,2\alpha})(t-s)^\alpha \end{aligned}$$

and the same for G. We conclude that

$$\left\|Z'\right\|_\alpha \leq CQ_\alpha^2(|G_0| + \left|G_0'\right| + \left\|G, G'\right\|_{W,2\alpha})(|H_0| + \left|H_0'\right| + \left\|H, H'\right\|_{W,2\alpha}).$$

Let us now deal with the remainder

$$\begin{aligned} R_{s,t}^Z &= Z_{s,t} - Z_s' W_{s,t} \\ &= \langle G_{s,t}, H_s \rangle + \langle G_t, H_{s,t} \rangle - (\left\langle G_s', H_s \right\rangle + \left\langle G_s, H_s' \right\rangle) W_{s,t} \\ &= \left\langle G_{s,t} - G_s' W_{s,t}, H_s \right\rangle + \left\langle H_{s,t} - H_s' W_{s,t}, G_s \right\rangle + \langle G_{s,t}, H_{s,t} \rangle \\ &= \left\langle R_{s,t}^G, H_s \right\rangle + \left\langle R_{s,t}^H, G_s \right\rangle + \langle G_{s,t}, H_{s,t} \rangle . \end{aligned}$$

It follows that

$$\begin{aligned} \left\|R^Z\right\|_{2\alpha} &\leq \left\|R^G\right\|_{2\alpha} \|H\|_\infty + \left\|R^H\right\|_{2\alpha} \|G\|_\infty + \|G\|_\alpha \|H\|_\alpha \\ &\leq CQ_\alpha^2(|G_0| + \left|G_0'\right| + \left\|G, G'\right\|_{W,2\alpha})(|H_0| + \left|H_0'\right| + \left\|H, H'\right\|_{W,2\alpha}). \end{aligned}$$

□

Given $\xi \in R^d, \xi' \in R^{d\times d}$ and $M \geq 1$ we define the "ball"

$$\mathcal{B}_M(\xi, \xi') = \mathcal{B} = \{(Y, Y') \in \mathcal{D}_{W,2\alpha}(0, T), Y_0 = \xi, Y_0' = \xi', \left\|Y, Y'\right\|_{W,2\alpha} \leq M\}. \tag{3.12}$$

Lemma 3.4 ***(Locally Lipschitz)*** *Let $f \in C_b^3(R^d)$. We consider $(Y, Y'), (\overline{Y}, \overline{Y}') \in \mathcal{B}_M(\xi, \xi')$ and we denote $\Delta = Y - \overline{Y}$ and $\Delta(f) = f(Y) - f(\overline{Y})$. Then*

$$\left\|\Delta(f), \Delta'(f)\right\|_{W,2\alpha} \leq C_M Q_\alpha^2 \|f\|_{3,\infty} (\left\|f'\right\|_\infty + 1) \left\|\Delta, \Delta'\right\|_{W,2\alpha} \tag{3.13}$$

Proof In order to do it we write

$$\Delta_s(f) = \left\langle \int_0^1 \nabla f(\lambda Y_s + (1-\lambda)\overline{Y}_s) d\lambda, \Delta_s \right\rangle = \langle G_s, \Delta_s \rangle$$

with

$$G_s = g(Y_s, \overline{Y}_s), \quad g(y, \overline{y}) := \int_0^1 \nabla f(\lambda y + (1-\lambda)\overline{y}) d\lambda.$$

Using (3.11) and $\Delta_0 = Y_0 - \overline{Y}_0 = 0, \Delta_0' = Y_0' - \overline{Y}_0' = 0$ we get

$$\left\|\Delta(f), \Delta'(f)\right\|_{W,2\alpha} \leq C_M Q_\alpha^2 \left\|\Delta, \Delta')\right\|_{W,2\alpha} \times (|G_0| + \left|G_0'\right| + \left\|G, G'\right\|_{W,2\alpha}).$$

Now we use (3.7) with g instead of f, and we get

$$\left\|g(Y, \overline{Y}), g(Y, \overline{Y})'\right\|_{W,2\alpha} \leq C \|g\|_{2,\infty} (\|g\|_\infty + 1) \leq C \|f\|_{3,\infty} (\left\|f'\right\|_\infty + 1)$$

so (3.13) is proved. □

In the previous Lipschitzianity result, the two processes Y and $\overline{Y}$ are controlled by the same rough path. Now we discuss the case when each of them is controlled by a different rogh path - such a result is needed when we discuss the continuity of Lyons' map. So we come back to the framework from the section concerning the stability for the rough integral. We consider two α rough path $\mathbf{W}$ and $\widetilde{\mathbf{W}}$ and $(Y, Y') \in \mathcal{D}_W^{2\alpha}, (\widetilde{Y}, \widetilde{Y}') \in \mathcal{D}_{\widetilde{W}}^{2\alpha}$. Then we recall the "distance"

$$d_{W,\widetilde{W},2\alpha}((Y, Y'), (\widetilde{Y}, \widetilde{Y}')) = \left\|Y' - \widetilde{Y}'\right\|_\alpha + \left\|R^Y - R^{\widetilde{Y}}\right\|_{2\alpha}$$

We also recall that in (2.23) we have defined

$$\rho_\alpha(\mathbf{W},\widetilde{\mathbf{W}}) = \left\|W - \widetilde{W}\right\|_\alpha + \left\|\mathbb{W}-\widetilde{\mathbb{W}}\right\|_{2\alpha}.$$

Finally we will work on the ball

$$\mathcal{A}_M(\alpha,\alpha) = \{(Y, Y', \mathbf{W}) : \|W\|_\alpha + \|\mathbb{W}\|_{2\alpha} + \left\|Y'\right\|_\alpha + \left\|R^Y\right\|_{2\alpha} + |Y_0| + \left|Y_0'\right| \leq M\}.$$

Lemma 3.5 ***(Locally Lipschitz Bis -Th 7.5 (FH))*** *Let $(Y, Y') \in \mathcal{D}^{2\alpha}_W, (\widetilde{Y}, \widetilde{Y}') \in \mathcal{D}^{2\alpha}_{\widetilde{W}}$. We assume that $(Y, Y', \mathbf{W}) \in \mathcal{A}_M(\alpha,\alpha)$ and the same with tilde. Let $f \in C^3_b(R^d)$. We denote $Z = f(Y)$, so that $Z' = f'(Y)Y'$ and $\widetilde{Z} = f(\widetilde{Y})$, $\widetilde{Z}' = f'(\widetilde{Y})\widetilde{Y}'$. Then*

$$\begin{aligned} d_{W,\widetilde{W},2\alpha}((Z,Z'),(\widetilde{Z},\widetilde{Z}')) &\leq C_M(\rho_\alpha(\mathbf{W},\widetilde{\mathbf{W}}) + \left|Y_0 - \widetilde{Y}_0\right| + \left|Y_0' - \widetilde{Y}_0'\right| \\ &\quad + d_{W,\widetilde{W},2\alpha}((Y,Y'),(\widetilde{Y},\widetilde{Y}'))) \end{aligned} \tag{3.14}$$

with C_M a constant which depends on M, T, α and on $\|f\|_{3,\infty}$. Moreover

$$\begin{aligned} \left\|Z - \widetilde{Z}\right\|_\alpha &\leq C_M(\rho_\alpha(\mathbf{W},\widetilde{\mathbf{W}}) + \left|Y_0 - \widetilde{Y}_0\right| + \left|Y_0' - \widetilde{Y}_0'\right| \\ &\quad + d_{W,\widetilde{W},2\alpha}((Y,Y'),(\widetilde{Y},\widetilde{Y}'))) \end{aligned} \tag{3.15}$$

Proof The second inequality follows from the first one as soon as one notices that $\left|Z_0' - \widetilde{Z}_0'\right| \leq C_M(\left|Y_0 - \widetilde{Y}_0\right| + \left|Y_0' - \widetilde{Y}_0'\right|)$. So we focus on the first one.

In this proof we will use the following shorten notation: $\rho = \rho_\alpha(\mathbf{W},\widetilde{\mathbf{W}})$, $\varepsilon_0 = \left|Y_0 - \widetilde{Y}_0\right|$, $\varepsilon_0' = \left|Y_0' - \widetilde{Y}_0'\right|$ and $\varepsilon = d_{W,\widetilde{W},2\alpha}((Y,Y'),(\widetilde{Y},\widetilde{Y}'))$.

Step 1. We check first that

$$\left\|Y - \widetilde{Y}\right\|_\alpha \leq C_M(\rho + \varepsilon_0' + \varepsilon) =: \varepsilon_Y. \tag{3.16}$$

We write

$$Y_{s,t} = Y_s' W_{s,t} + R^Y_{s,t}, \quad \widetilde{Y}_{s,t} = \widetilde{Y}_s' \widetilde{W}_{s,t} + R^{\widetilde{Y}}_{s,t}$$

and taking the difference we obtain

$$\begin{aligned} \left\|Y - \widetilde{Y}\right\|_\alpha &\leq C(\left\|Y'\right\|_\infty \rho + M\left\|Y' - \widetilde{Y}'\right\|_\infty + \left\|R^Y - R^{\widetilde{Y}}\right\|_{2\alpha}) \\ &\leq C((\left|Y_0'\right| + M)\rho + M(\left|Y_0' - \widetilde{Y}_0'\right| + \left\|Y' - \widetilde{Y}'\right\|_\alpha T^\alpha) + \left\|R^Y - R^{\widetilde{Y}}\right\|_{2\alpha}) \\ &\leq C_M(\rho + \varepsilon_0' + \varepsilon) = \varepsilon_Y. \end{aligned}$$

As an immediate consequence we also have $\left\|Y - \widetilde{Y}\right\|_\infty \leq C(\varepsilon_0 + \varepsilon_Y)$ and

$$\left\|f'(Y) - f'(\widetilde{Y})\right\|_\alpha \leq C_M(\varepsilon_0 + \varepsilon_Y).$$

Indeed, one writes

$$f'(Y_t) - f'(Y_s) = \int_0^1 f''(\lambda Y_s + (1-\lambda)Y_t)d\lambda \times (Y_t - Y_s)$$

and the same with tilde. Then one takes differences and employs the estimates for $\left\|Y-\widetilde{Y}\right\|_{\alpha}$ and for $\left\|Y-\widetilde{Y}\right\|_{\infty}$. We notice that the constant C_M will depend on $\|f'''\|_{\infty}$. Finally we get

$$\left\|Z'-\widetilde{Z}'\right\|_{\alpha}=\left\|f'(Y)Y-f'(\widetilde{Y})\widetilde{Y}\right\|_{\alpha}\leq C_M(\varepsilon_0+\varepsilon_Y).$$

Step 2 We deal with R^Z. We write

$$R^Z_{s,t}=f(Y_t)-f(Y_s)-f'(Y_s)Y'_sW_{s,t}=\tau_1+\tau_2$$

with

$$\tau_1(s,t)=f(Y_t)-f(Y_s)-f'(Y_s)Y_{s,t},\quad \tau_2(s,t)=f'(Y_s)R^Y_{s,t}.$$

And we write the same decomposition for "tilda". We deal first with

$$\tau_1(s,t)=\int_0^1 f''(Y_s+\lambda Y_{s,t})(Y_{s,t},Y_{s,t})d\lambda.$$

We recall that $\left\|Y-\widetilde{Y}\right\|_{\infty}+\left\|Y-\widetilde{Y}\right\|_{\alpha}\leq C(\varepsilon_0+\varepsilon_Y)$ and $\|Y\|_{\infty}\leq CM$. Then elementary estimates leads to

$$|\tau_1(s,t)-\widetilde{\tau}_1(s,t)|\leq C_M\|f\|_{3,\infty}(\varepsilon_0+\varepsilon)(t-s)^{2\alpha}.$$

And it is not hard to check that

$$|\tau_2(s,t)-\widetilde{\tau}_2(s,t)|\leq C_M\|f\|_{2,\infty}(\varepsilon_0+\varepsilon)(t-s)^{2\alpha}.$$

□

3.2 Rough differential equations

The main theorem is the following:

Theorem 3.6 *Consider a β rough path $\mathbf{W}$, some $\frac{1}{3}<\alpha<\beta$ and a function $f:R^d\to R^{d\times d}$ in $C^3_b(R^d)$. Then the equation*

$$Y_t=\xi+\int_0^t f(Y_s)d\mathbf{W}_s$$

has a unique solution $(Y,Y')\in\mathcal{D}^{2\alpha}_W$. Here the integral with respect to $d\mathbf{W}$ is the rough integral defined in the previous sections for the integrand $(H,H')=(f(Y),(f(Y))')=(f(Y),f'(Y)Y')\in\mathcal{D}^{2\alpha}_W$.

Proof We fix $T\in(0,1)$ which will be chosen in the following - it will be sufficiently small in order to be able to use a fixed point argument on the interval $(0,T)$. Afterwords we will concatenate solution on these small intervals.

Given $(Y,Y')\in\mathcal{D}^{2\alpha}_W(0,T)$ We define the aplication

$$\mathcal{M}(Y,Y')=(\xi+\int_0^t f(Y_s)d\mathbf{W}_s,f(Y))=:(Z,Z')\in\mathcal{D}^{2\alpha}_W(0,T).$$

Notice that $f(Y)$ is the Gubinelli derivative of the rough integral. And the rough integral is constructed by using $f(Y)' = f'(Y)Y'$.

Step 1 We prove that $(Y, Y') \in \mathcal{B}_M(\xi, f(\xi))$ (see (3.12)) implies that $\mathcal{M}(Y, Y') \in \mathcal{B}_M(\xi, f(\xi))$. This will be true for sufficientelly small T, deppending on M and on $\|f\|_{3,\infty}$. Clearly the initial condition is preserved. Let us check that $\mathcal{M}(Y, Y') \in \mathcal{D}_W^{2\alpha}$. By (3.7), if $\|Y, Y'\|_{W,2\alpha} \leq M$ then

$$\begin{aligned} \left\|f(Y), f(Y)'\right\|_{W,2\alpha} &\leq CQ_\alpha^2 \left\|f\right\|_{2,\infty} (1 + \left|Y_0'\right| + \left\|Y, Y'\right\|_{W,2\alpha})^2 &(3.17)\\ &\leq C_M Q_\alpha^2 \left\|f\right\|_{2,\infty} (\left\|f\right\|_\infty + M)^2 = C_M(f) Q_\alpha^2. &(3.18)\end{aligned}$$

Here and in the following proof $C_M(f)$ stands for a constant which depends on M and on $\|f\|_{3,\infty}$. And also, with $Z = \int f(Y) d\mathbf{W}_r$, $Z' = f(Y)$

$$\left\|\int f(Y) d\mathbf{W}_r, f(Y)\right\|_{W,2\alpha} = \left\|Z, Z'\right\|_{W,2\alpha} = \left\|f(Y)\right\|_\alpha + \left\|R^Z\right\|_{2\alpha}.$$

One has

$$\begin{aligned} R_{s,t}^Z &= \int_s^t f(Y_r) d\mathbf{W}_r - f(Y_s) W_{s,t} \\ &= \int_s^t f(Y_r) d\mathbf{W}_r - f(Y_s) W_{s,t} - f'(Y_s) Y_s' \mathbb{W}_{s,t} + f'(Y_s) Y_s' \mathbb{W}_{s,t}. \end{aligned}$$

Since $\|Y, Y'\|_{W,2\alpha} \leq M$

$$\begin{aligned} &\left|\int_s^t f(Y_r) d\mathbf{W}_r - f(Y_s) W_{s,t} - f'(Y_s) Y_s' \mathbb{W}_{s,t}\right| \\ \leq\ & C(\|W\|_\alpha \left\|R^{f(Y)}\right\|_{2\alpha} + \|\mathbb{W}\|_{2\alpha} \left\|f'(Y) Y'\right\|_\alpha)(t-s)^{3\alpha} \\ \leq\ & C \left\|f(Y), (f(Y))'\right\|_{W,2\alpha} Q_\alpha (t-s)^{3\alpha} \\ \leq\ & C_M(f) Q_\alpha^3 (t-s)^{3\alpha} \leq C_M(f) Q_\alpha^3 (t-s)^{2\alpha} T^\alpha \end{aligned}$$

One also has

$$\begin{aligned} \left|f'(Y_s) Y_s' \mathbb{W}_{s,t}\right| &\leq \left|f'(Y_s) Y_s'\right| \|\mathbb{W}\|_{2\beta} (t-s)^{2\beta} \\ &\leq C \left\|f'\right\|_\infty (\left|f'(\xi)\right| + M) \|\mathbb{W}\|_{2\beta} (t-s)^{2\beta} \\ &= C_M(f) Q_\beta (t-s)^{2\alpha} T^{2(\beta-\alpha)}. \end{aligned}$$

We conclude that

$$\left\|R^Z\right\|_{2\alpha} \leq C_M(f) Q_\beta^3 (T^\alpha + T^{2(\beta-\alpha)}). \tag{3.19}$$

And by (3.9)

$$\|f(Y)\|_\alpha \leq C_M(f) Q_\beta \times (T^\alpha + T^{\beta-\alpha}).$$

So finally

$$\left\|\int f(Y) d\mathbf{W}_r, f(Y)\right\|_{W,2\alpha} \leq C_M(f) Q_\beta^3 (T^\alpha + T^{\beta-\alpha}) \leq M$$

the last inequality being true if we take sufficientely small T, depending on $C_M(f)$ and on Q_β.

Step 2 Contraction

Using (3.4) with $\Delta(f) = f(Y) - f(\overline{Y})$ instead of Y first and (3.13) next we get (recall that $\Delta'_0(f) = 0$)

$$\begin{aligned} \|\Delta(f)\|_\alpha &\leq CQ_\beta(T^\alpha + T^{\beta-\alpha}) \left\|\Delta(f), \Delta'(f)\right\|_{W,2\alpha} \\ &\leq C_M(f)(T^\alpha + T^{\beta-\alpha})Q_\beta^3 \left\|\Delta, \Delta'\right\|_{W,2\alpha} \end{aligned} \tag{3.20}$$

We write now

$$\mathcal{M}(Y, Y') - \mathcal{M}(\overline{Y}, \overline{Y}') = (\int_0^t \Delta_s(f) d\mathbf{W}_s, \Delta_s(f)) =: (Z, Z').$$

The same reasoning as in Step 1, see (3.19) (we also use $\Delta'_0(f) = 0$) gives (**exercise)**

$$\begin{aligned} \left\|R^Z\right\|_{2\alpha} &\leq C_M(f)Q_\beta^3 \left\|\Delta(f), \Delta'(f)\right\|_{W,2\alpha} (T^\alpha + T^{\beta-\alpha}) \\ &\leq C'_M(f)Q_\beta^3 \left\|\Delta, \Delta'\right\|_{W,2\alpha} (T^\alpha + T^{\beta-\alpha}) \end{aligned}$$

so that, using (3.4)

$$\begin{aligned} & \left\|\mathcal{M}(Y, Y') - \mathcal{M}(\overline{Y}, \overline{Y}')\right\|_{W,2\alpha} \\ = {} & \|\Delta(f)\|_\alpha + \left\|R^Z\right\|_{2\alpha} \\ \leq {} & C_M(f)Q_\beta^3(T^\alpha + T^{\beta-\alpha}) \left\|\Delta, \Delta'\right\|_{W,2\alpha} \\ \leq {} & \frac{1}{2} \left\|\Delta, \Delta'\right\|_{W,2\alpha} \end{aligned}$$

the last inequality being true if we take T sufficiently small in order to obtain $C_M(f)Q_\beta^3(T^\alpha + T^{\beta-\alpha}) \leq \frac{1}{2}$.

Conclusion $(Y, Y') \to \mathcal{M}(Y, Y')$ is a strict contraction on $\mathcal{B}_M(\xi, f(\xi))$ so it has a unique fixed point (one has to check that $\mathcal{B}_M(\xi, f(\xi))$ is complete with respect to $|Y_0| + |Y'_0| + \|Y, Y'\|_{W,2\alpha}$). This is the solution of our equation. And in order to go in long time, we concatenate.

Uniqueness: we leave the proof of uniqueness for later on because an a priory inequaltity is needed: see the Theorem concernng the continuity of the Lyon's map, in the following section. □

3.3 Continuity with respect to the driving signal

In order to prove a continuity result we will use the stability property of the rough integral and of the composition with regular functions. But these properties are "local" on $\mathcal{B}_M(\xi, \xi')$, and so we need that $\|Y, Y'\|_\alpha \leq M$. We do not want to ask such a restrictive condition on the solution Y itself, but we only accept the restriciton $\||\mathbf{W}|\|_\alpha \leq M$. So essentialy we need to prove that if Y is the solution of a rough equation then one controls $\|Y, Y'\|_\alpha$ by $\||\mathbf{W}|\|_\alpha := \|W\|_\alpha + \|\mathbb{W}\|_{2\alpha}^{1/2}$. This is the subject of the following "a priory" estimate (which represents the subtle point in the proof).

Theorem 3.7 *Consider a α rough path $\mathbf{W}$, for some $\frac{1}{3} < \alpha$ and a function $f \in C_b^3(R^d, R^{d\times d})$. And let $(Y, Y') \in \mathcal{D}_W^{2\alpha}$ be a solution of the equation $Y_t = \xi + \int_0^t f(Y_s)d\mathbf{W}_s$. There exists a universal constant C such that*

$$\|Y\|_\alpha + \|Y'\|_\alpha + \|R^Y\|_{2\alpha} \le C \|f\|_{3,\infty} |||\mathbf{W}|||_\alpha^2 \vee 1 \tag{3.21}$$

In particular if $|||\mathbf{W}|||_\alpha \le M \vee 1$ then $(Y, Y', W) \in \mathcal{A}_{M'}(\alpha, \alpha)$ with $M' = C\|f\|_3 M^2$.

Proof Step 1 We stress that all along the proof we use that, because of the equation, $Y' = f(Y)$. It follows that

$$\begin{aligned} \left|R^Y_{s,t}\right| &= \left|Y_{s,t} - Y'_s W_{s,t}\right| \\ &\le \left|\int_s^t f(Y)dW - f(Y_s)W_{s,t} - f'(Y_s)f(Y_s)\mathbb{W}_{s,t}\right| + \left|f'(Y_s)f(Y_s)\mathbb{W}_{s,t}\right| \end{aligned}$$

Notice that $f'(Y_s)f(Y_s) = f'(Y_s)Y'_s = (f(Y_s))'$ so that $\|f(Y)'\|_\alpha \le C_f \|Y\|_\alpha$. Then the sawing lemma gives

$$\begin{aligned} &\left|\int_s^t f(Y)dW - f(Y_s)W_{s,t} - f'(Y_s)f(Y_s)\mathbb{W}_{s,t}\right| \\ \le\ & (\|W\|_\alpha \left\|R^{f(Y)}\right\|_{2\alpha} + C_f \|\mathbb{W}\|_{2\alpha} \|Y\|_\alpha)(t-s)^{3\alpha} \end{aligned}$$

and moreover

$$\left|f'(Y_s)f(Y_s)\mathbb{W}_{s,t}\right| \le \|f\|_{1,\infty}^2 \|\mathbb{W}\|_{2\alpha} (t-s)^{2\alpha}.$$

In the following we will use the notation $\|W\|_{\alpha,h}$ when we take differences only for $t - s \le h$. With this notation convention we have proved that

$$\|R^Y\|_{2\alpha,h} \le C_f((\|W\|_{\alpha,h} \left\|R^{f(Y)}\right\|_{2\alpha,h} + \|\mathbb{W}\|_{2\alpha,h} \|Y\|_{\alpha,h})h^\alpha + \|\mathbb{W}\|_{2\alpha,h}) \tag{3.22}$$

We will now look to $\left\|R^{f(Y)}\right\|_{2\alpha,h}$. We write

$$\begin{aligned} R^{f(Y)}_{s,t} &= f(Y_t) - f(Y_s) - f'(Y_s)Y'_s W_{s,t} \\ &= f(Y_t) - f(Y_s) - f'(Y_s)Y_{s,t} + f'(Y_s)R^Y_{s,t} \end{aligned}$$

so that, using Taylor expansion

$$\left\|R^{f(Y)}\right\|_{2\alpha,h} \le \frac{1}{2}\|f\|_{2,\infty} \|Y\|_{\alpha,h}^2 + \|f\|_{1,\infty} \|R^Y\|_{2\alpha,h}.$$

We insert this in (3.22) and we get

$$\|R^Y\|_{2\alpha,h} \le C_f((\|W\|_{\alpha,h} (\|Y\|_{\alpha,h}^2 + \|R^Y\|_{2\alpha,h}) + \|\mathbb{W}\|_{2\alpha,h} \|Y\|_{\alpha,h})h^\alpha + \|\mathbb{W}\|_{2\alpha,h}).$$

We take now h small such that

$$C_f \|W\|_{\alpha,h} h^\alpha \le \frac{1}{2}, \quad C_f \|\mathbb{W}\|_{2\alpha,h}^{1/2} h^\alpha \le 2. \tag{3.23}$$

With such an h the above inequality reads

$$\begin{aligned}\left\|R^Y\right\|_{2\alpha,h} &\leq \frac{1}{2}(\|Y\|_{\alpha,h}^2+\left\|R^Y\right\|_{2\alpha,h})+2\,\|\mathbb{W}\|_{2\alpha,h}^{1/2}\,\|Y\|_{\alpha,h}+C_f\,\|\mathbb{W}\|_{2\alpha,h})\\ &\leq \frac{3}{2}\,\|Y\|_{\alpha,h}^2+\frac{1}{2}\left\|R^Y\right\|_{2\alpha,h}+(1+C_f)\,\|\mathbb{W}\|_{2\alpha,h}\end{aligned}$$

which finally gives

$$\left\|R^Y\right\|_{2\alpha,h}\leq 3\,\|Y\|_{\alpha,h}^2+2(1+C_f)\,\|\mathbb{W}\|_{2\alpha,h} \tag{3.24}$$

Now, since $Y_{s,t}=f(Y_s)W_{s,t}+R^Y_{s,t}$, the above inequality also gives (we also use (3.24) and with the restriction (3.23))

$$\begin{aligned}\|Y\|_{\alpha,h} &\leq \|f\|_\infty\,\|W\|_{\alpha,h}+\left\|R^Y\right\|_{2\alpha,h}h^\alpha\\ &\leq \|f\|_\infty\,\|W\|_{\alpha,h}+(3\,\|Y\|_{\alpha,h}^2+2(1+C_f)\,\|\mathbb{W}\|_{2\alpha,h})h^\alpha\\ &\leq C_f'(\|W\|_{\alpha,h}+\|\mathbb{W}\|_{2\alpha,h}^{1/2})+3\,\|Y\|_{\alpha,h}^2\,h^\alpha.\\ &= C_f'\,|||\mathbf{W}|||_{\alpha,h}+3\,\|Y\|_{\alpha,h}^2\,h^\alpha\end{aligned}$$

Then we denote

$$\lambda_h=3C_f'\,|||\mathbf{W}|||_{\alpha,h}\,h^\alpha,\quad \psi_h=3\,\|Y\|_{\alpha,h}\,h^\alpha$$

and the above inequality reads

$$\psi_h\leq\lambda_h+\psi_h^2. \tag{3.25}$$

In the following step we will use the above inequality in order to find some h_0 such that

$$\psi_{h_0}\leq C\lambda_{h_0}.$$

And this gives for every $h\leq h_0$

$$\|Y\|_{\alpha,h}\leq C_f\,|||\mathbf{W}|||_\alpha\,.$$

Using (3.24) we also get $\left\|R^Y\right\|_{\alpha,h}\leq C_f\,|||\mathbf{W}|||_\alpha^2$. Finally $\|Y'\|_{\alpha,h}=\|f(Y)\|_{\alpha,h}\leq\|f\|_{1,\infty}\,\|Y\|_{\alpha,h}\leq C_f\,|||\mathbf{W}|||_\alpha$.

And we notice that for $a\geq 1$, we have $\|Y\|_{\alpha,h}\leq a\,\|Y\|_{\alpha,h/a}$. This allows to get $\|Y\|_\alpha\leq C\,\|Y\|_{\alpha,h}$ and to elliminate h.

Step 2 We choose h_0 such that, for $h\leq h_0$ one has $\lambda_h\leq\lambda_{h_0}:=\frac{5}{36}<\frac{1}{4}$. Then the equation $\psi_h=\lambda_h+\psi_h^2$ has two distinct solutions

$$\begin{aligned}\psi_+ &= \frac{1}{2}(1+\sqrt{1-4\lambda_h})\geq\frac{1}{2}\\ \psi_- &= \frac{1}{2}(1-\sqrt{1-4\lambda_h})\leq\frac{1}{6}.\end{aligned}$$

Since we know that (3.25) holds true, it follows that for all $h\leq h_0$ one has $\psi_h\geq\frac{1}{2}$ or $\psi_h\leq\frac{1}{6}$. We also know (by the very definition of ψ_h) that $\psi_h\downarrow 0$ as $h\downarrow 0$ so that for sufficiently small h we have $\psi_h\leq\frac{1}{6}$. We want to prove that this is the case for every $h\leq h_0$. Notice first that

$$\|Y\|_{\alpha,h}\leq 3\,\|Y\|_{\alpha,h/3}\leq 3\lim_{g\uparrow h}\|Y\|_{\alpha,g}$$

which gives $\psi_h\leq 3\lim_{g\uparrow h}\psi_g$. And in a similar way $\psi_h\geq\frac{1}{3}\lim_{g\downarrow h}\psi_g$.

Take now $h_* = \sup\{h : \psi_h \leq \frac{1}{6}\}$. Since $h \to \psi_h$ is an increasing function we havr $\psi_h \leq \frac{1}{6}$ for $h < h_*$ and $\psi_h \geq \frac{1}{2}$ for $h > h_*$. We also have $\psi_{h_*} \leq 3 \lim_{g \uparrow h} \psi_g \leq 3 \times \frac{1}{6} = \frac{1}{2}$, so $\psi_{h_*} \leq \frac{1}{6}$. Suppose now that $h_* < h_0$. Then, for every $g \in (h_*, h_0)$ we have $\psi_g \geq \frac{1}{2}$ (because we do not have $\psi_g \leq \frac{1}{6}$) and consequentely $\psi_{h_*} \geq \frac{1}{3} \lim_{g \downarrow h_*} \psi_g > \frac{1}{6}$ which is in contradiction with $\psi_{h_*} \leq \frac{1}{6}$. We conclude that $\psi_h \leq \frac{1}{6}$ for every $h \leq h_0$. Coming now back to (3.25) we get $\psi_h \leq \lambda_h + \frac{1}{6}\psi_h$ and then

$$\psi_h \leq \frac{6}{5}\lambda_h.$$

□

We are now able to give the continuity result for Lyon's map. We recall that the distance $d_{W,\widetilde{W},2\alpha}$ is given in (2.24) and ρ_α is defined in (2.23). The important trick that will be used in the proof of the theorem below is the following: if we work on $(0,T)$ and we have $\alpha < \beta$ then

$$d_{W,\widetilde{W},2\alpha}(Y, Y', \widetilde{Y}, \widetilde{Y}') \leq d_{W,\widetilde{W},2\beta}(Y, Y', \widetilde{Y}, \widetilde{Y}') \times T^{\beta-\alpha}. \tag{3.26}$$

The reason is that $\|U\|_\alpha \leq \|U\|_\beta T^{\beta-\alpha}$.

Theorem 3.8 ***A*** *Let $f \in C_b^3$ and let $\mathbf{W}$ and $\widetilde{\mathbf{W}}$ be two β rought path. We denote by Y a solution of the rough differential equation $dY = f(Y)d\mathbf{W}$,$Y_0 = \xi$ and $\widetilde{Y}$ a solution of $d\widetilde{Y} = f(\widetilde{Y})d\widetilde{\mathbf{W}}$,$\widetilde{Y}_0 = \widetilde{\xi}$. Suppose that*

$$\left\|\left|\mathbf{W}\right|\right\|_\beta + \left\|\left|\widetilde{\mathbf{W}}\right|\right\|_\beta \leq M. \tag{3.27}$$

Then, for every $\frac{1}{3} < \alpha < \beta$ there exists C depending on $M, \|f\|_{3,\infty}, \alpha$ and β such that

$$d_{W,\widetilde{W},2\alpha}(Y, f(Y), \widetilde{Y}, f(\widetilde{Y})) \leq C(\rho_\alpha(\mathbf{W}, \widetilde{\mathbf{W}}) + \left|\xi - \widetilde{\xi}\right|) \tag{3.28}$$

and

$$\left\|Y - \widetilde{Y}\right\|_\alpha \leq C(\rho_\alpha(\mathbf{W}, \widetilde{\mathbf{W}}) + \left|\xi - \widetilde{\xi}\right|). \tag{3.29}$$

B *As a consequence, taking $\mathbf{W} = \widetilde{\mathbf{W}}$ and $\xi = \widetilde{\xi}$ we obtain the uniqueness of the solution of the rough differential equation.*

Proof We will use the inequalities (3.14) and (2.25) which are verified if $(Y, Y', \mathbf{W}) \in \mathcal{A}_M(\alpha, \beta)$ and $(\widetilde{Y}, \widetilde{Y}', \widetilde{\mathbf{W}}) \in \mathcal{A}_M(\alpha, \beta)$. Using the a priory inequality from the previous theorem, and the hypothesis (3.27) this is true (this is the reason of being of the a priory estimate (3.21)).

Step 1 Let $Z = \int f(Y)d\mathbf{W}$ and $\widetilde{Z} = \int f(\widetilde{Y})d\widetilde{\mathbf{W}}$. We recall that $Y' = Z' = f(Y)$ and $\widetilde{Y}' = \widetilde{Z}' = f(\widetilde{Y})$, because of the integral and of the equation. One also has the equation $Y = \xi + Z$ and similar with tilda.

Then the stablity property for the rough integral (2.25) gives, with $C_1(T) = C_1(T^\alpha + T^{b-\alpha})$,

$$\begin{aligned} d_{W,\widetilde{W},2\alpha}(Y, f(Y), \widetilde{Y}, f(\widetilde{Y})) &= d_{W,\widetilde{W},2\alpha}(Z, Z', \widetilde{Z}, \widetilde{Z}') && (3.30) \\ &\leq C_1(T)(d_{W,\widetilde{W},2\alpha}(f(Y), f(Y)', f(\widetilde{Y}), f(\widetilde{Y})') + \rho_\beta(\mathbf{W}, \widetilde{\mathbf{W}}) + \left|\xi - \widetilde{\xi}\right|) \\ &\leq C_2(T)(d_{W,\widetilde{W},2\alpha}(Y, Y', \widetilde{Y}, \widetilde{Y}') + \rho_\beta(\mathbf{W}, \widetilde{\mathbf{W}}) + \left|\xi - \widetilde{\xi}\right|) && (3.31) \end{aligned}$$

The last inequality is obtained by using the Lipschitz property of the composition with regular functions (3.14). Taking T sufficiently small we have $C_2(T) \leq \frac{1}{2}$ and this yields

$$d_{W,\widetilde{W},2\alpha}(Y, f(Y), \widetilde{Y}, f(\widetilde{Y})) \leq \rho_\beta(\mathbf{W}, \widetilde{\mathbf{W}}) + \left|\xi - \widetilde{\xi}\right|$$

The above estimate holds for small $T \leq T_*$. In order to extend it for a general T one concatanates T/T_* (so a constant will appear). $\square$

4 Construction of abstract flows

We begin with some definitions and notation. We work on R^d and we denote by $C_b^k(R^d)$ the space of the k time differentiable functions $\phi : R^d \to R^d$ which are bounded and have bounded derivatives. For $\phi \in C_b^k(R^d)$ we denote

$$\left|\phi\right|_{1,k} = \sum_{i=1}^{k} \sum_{|\alpha|=i} \sup_{x \in R^d} |\partial^\alpha \phi(x)| \quad and \quad \left|\phi\right|_k = \left|\phi\right|_{1,k} + \left\|\phi\right\|_\infty . \tag{4.1}$$

Here $|\partial^\alpha \phi(x)|^2 = \sum_{i=1}^d \left|\partial^\alpha \phi^i(x)\right|^2$. Notice also that if $k = 0$, this is just the uniform norm: $\left|\phi\right|_0 = \left\|\phi\right\|_\infty$.

We work with compositions of functions on R^d and we use the (abusive) multiplicative notation: for $f : R^d \to R^d$ and $g : R^d \to R^d$ we denote

$$fg = f \circ g.$$

First we have to establish some formulas for the computation of the derivatives of composed functions. Let $f : R^d \to R^d$ and $g : R^d \to R^d$ be smooth functions. One may prove by recurence that for a multi index α with $|\alpha| \geq 1$

$$\partial^\alpha [f \circ g] = \sum_{1 \leq |\beta| \leq |\alpha|} (\partial^\beta f)(g) P_{\alpha,\beta}(g) \tag{4.2}$$

with

$$P_{\alpha,\beta}(g) = \sum c_{\alpha,\beta}((\gamma_1, j_1)...(\gamma_k, j_k)) \prod_{i=1}^{k} \partial^{\gamma_i} g^{j_i} \tag{4.3}$$

with the sum over $k = 1, ..., |\alpha|, j_1, ..., j_k \in \{1, ..., d\}$ and γ_i multi-indexes with $1 \leq |\gamma_i| \leq |\alpha|$. And $c_{\alpha,\beta}((\gamma_1, j_1)...(\gamma_k, j_k))$ are some universal coefficients (the precise expression is not of interest in these notes). In particular we have, for some universal constant $C(k)$

$$\left|fg\right|_{1,k} \leq C(k) \left|f\right|_{1,k} \left|g\right|_{1,k}^k . \tag{4.4}$$

One also has $\left|\varphi f - \overline{\varphi} f\right|_0 \leq \left|\varphi - \overline{\varphi}\right|_0$ and, for a multi-index α with $|\alpha| = k \geq 1$,

$$|\partial^\alpha(\varphi f) - \partial^\alpha(\overline{\varphi} f)| \leq C \left|\varphi - \overline{\varphi}\right|_{1,k} \left|f\right|_{1,k}^k .$$

So

$$\left|\varphi f - \overline{\varphi} f\right|_k \leq C \left|\varphi - \overline{\varphi}\right|_k (1 + \left|f\right|_{1,k}^k).$$

Using Taylor expansion of order one

$$\varphi f - \varphi \overline{f} = \int_0^1 \left\langle \nabla\varphi(\lambda f + (1-\lambda)\overline{f}), f - \overline{f} \right\rangle d\lambda$$

which gives

$$\left|\varphi f - \varphi \overline{f}\right|_k \leq C \left|\varphi\right|_{1,k+1} (1 + \left|f\right|_{1,k}^k + \left|\overline{f}\right|_{1,k}^k) \left|f - \overline{f}\right|_k .$$

Combining these we get

$$\left|\varphi f - \overline{\varphi} \overline{f}\right|_k \leq C(1 + \left|f\right|_{1,k}^k + \left|\overline{f}\right|_{1,k}^k)(\left|\varphi - \overline{\varphi}\right|_k + \left|\varphi\right|_{1,k+1} \left|f - \overline{f}\right|_k .) \tag{4.5}$$

and by iterating this inequality

$$\begin{aligned} \left|\varphi f \psi - \overline{\varphi} \overline{f} \overline{\psi}\right|_k &\leq C'(k) \times C_k(f, \psi, \overline{f}, \overline{\psi}) & (4.6) \\ &\quad (\left|\varphi - \overline{\varphi}\right|_k + \left|\varphi\right|_{1,k+1} \left|f - \overline{f}\right|_k + \left|\varphi\right|_{1,k+1} \left|f\right|_{1,k+1} \left|\psi - \overline{\psi}\right|_k) & (4.7) \end{aligned}$$

with $C'(k)$ a universal constant depending on k and

$$C_k(f, \psi, \overline{f}, \overline{\psi}) = (1 + \left|f\right|_{1,k}^{2k} + \left|\overline{f}\right|_{1,k}^{2k} + \left|\psi\right|_{1,k}^{2k^2} + \left|\overline{\psi}\right|_{1,k}^{2k^2})^2. \tag{4.8}$$

We consider now a function $X : [0,T]^2 \to C_b^k(R^d)$ and for $\alpha > 0$ we define the Hölder type norms by

$$\left|X\right|_{k,\alpha} = \sup_{s<t} \frac{\left|X_{s,t}\right|_k}{\left|t-s\right|^\alpha}. \tag{4.9}$$

In the following we will also use the following localized variant: we fix $\varepsilon_* \in (0,\infty]$ and we define

$$\left|X\right|_{\varepsilon_*,k,\alpha} = \sup_{s<t<s+\varepsilon_*} \frac{\left|X_{s,t}\right|_k}{\left|t-s\right|^\alpha}. \tag{4.10}$$

This means that we take into acount only t, s such that $|t-s| \leq \varepsilon_*$. For $\varepsilon_* = \infty$ we have $\left|X\right|_{\varepsilon_*,k,\alpha} = \left|X\right|_{k,\alpha}$.

Definition 4.1 *We say that an application $X : [0,T]^2 \to C_b^k(R^d)$ is a $(k,\alpha)-$ flow if $\left|X\right|_{1,k,\alpha} < \infty$ and, for every $s < u < t$, we have*

$$X_{u,t} X_{s,u} = X_{s,t}. \tag{4.11}$$

Here and in the sequel $X_{u,t}X_{s,u}$ denotes the composition: $X_{u,t}X_{s,u} = X_{u,t} \circ X_{s,u}$ (we travel from s to u and then from u to t). The basic example is given by the flow associated to a rough differential equation (in short RDE), or the stochastic flow associated to a stochastic differential equation (in short SDE). But for the moment we keep in an abstract, deterministic framwork.

Our aim is to approximate $X_{s,t}$ by using an application $\Theta : [0,T]^2 \to C_b^k(R^d)$ which does not verify the flow condition (4.11) but only an "approximative" flow condition. Let us introduce some notation. We fix $s < t$ and we consider a grid $\mathcal{P} = \mathcal{P}(s,t) = \{s = s_0 < ... < s_m = t\}$. We may also think to $\mathcal{P}$ as being a partition of $[s,t]$. We denote $|\mathcal{P}| = \max_{i=1,...,m-1}(s_{i+1} - s_i)$ the mesh of $\mathcal{P}$.

Moreover we construct the "Euler scheme" associated to $\mathcal{P}$

$$X^{\mathcal{P}}_{s,s_j}(\Theta) \quad : \quad =\prod_{i=1}^{j}\Theta_{s_{i-1},s_i}=\Theta_{s_{j-1},s_j}\circ ....\circ\Theta_{s,s_1} \quad and \tag{4.12}$$

$$X^{\mathcal{P}}_{s,t}(\Theta) \quad = \quad X^{\mathcal{P}}_{s,s_m}(\Theta)=\prod_{i=1}^{m}\Theta_{s_{i-1},s_i}=\Theta_{s_{m-1},t}\circ ....\circ\Theta_{s,s_1}. \tag{4.13}$$

Let $\mathcal{P}=\{0=s_0<...<s_m=T\}$ be a partition of $(0,T)$. We consider $0\leq s<t\leq T$ and we suppose that $s_{i_0}\leq s<s_{i_0+1}$ and $s_{j_0}\leq t<s_{j_0+1}$. Then we denote by $\mathcal{P}(s,t)$ the trace of $\mathcal{P}$ on (s,t) more precisely $\mathcal{P}(s,t)=\{s<s_{i_0+1}<...<s_{j_0}<t\}$. So we take the points of $\mathcal{P}$ which are in (s,t) and we add as initial point s and as final point t. So $s=s_{i_0}$ and $t=s_{j_0+1}$.

We denote $n_{\mathcal{P}}(s,t)=j_0-i_0$, this is the number of the points of $\mathcal{P}$ which fall in the interval (s,t). And we also define

$$X^{\mathcal{P}}_{s,t}(\Theta)=X^{\mathcal{P}(s,t)}_{s,s_{j_0+1}}(\Theta)=\prod_{i=i_0+1}^{j_0+1}\Theta_{s_{i-1},s_i}=\Theta_{s_{j_0},t}\circ ....\circ\Theta_{s,s_{i_0+1}}. \tag{4.14}$$

We also define

$$L_{s,t}(x)=\Theta_{s,t}(x)-x \tag{4.15}$$

and we use L in order to give an alternative expression of $X^{\mathcal{P}}_{s,s_j}(\Theta)$:

$$X^{\mathcal{P}}_{s,s_j}(\Theta)(x)=x+\sum_{i=1}^{j}L_{s_{i-1},s_i}(X^{\mathcal{P}}_{s,s_{i-1}}(\Theta)(x)).$$

We give now our assumptions on Θ. First we assume that $\Theta_{t,t}(x)=x$ and for some $\alpha>0$,

$$\left|L\right|_{1,k+1,\alpha}=\sup_{s<t}\frac{\left|L_{s,t}\right|_{1,k+1}}{\left|t-s\right|^{\alpha}}<\infty. \tag{4.16}$$

Moreover, for $s<u<t$, we denote

$$\delta\Theta_{s,u,t}=\Theta_{s,t}-\Theta_{u,t}\Theta_{s,u} \tag{4.17}$$

which quantifies the "error" with respect to the flow property (4.11) for Θ. And for $\beta>1$, we assume that

$$\left\|\delta\Theta\right\|_{k,\beta}:=\sup_{s<u<t}\frac{\left|\delta\Theta_{s,u,t}\right|_{k}}{\left|t-s\right|^{\beta}}<\infty. \tag{4.18}$$

If we take $\sup_{s<u<t<s+\varepsilon_*}$ we denote $\left\|\delta\Theta\right\|_{\varepsilon_*,k,\beta}$. Finally we define

$$\left\|\Theta\right\|_{k,\alpha,\beta}=\left|L\right|_{1,k+1,\alpha}+\left\|\delta\Theta\right\|_{k,\beta} \tag{4.19}$$

Definition 4.2 *We denote by $\mathcal{C}^{\alpha,\beta}_{k}([0,T])$ the space of the applications $\Theta:[0,T]^2\rightarrow C^k_b(R^d)$, with $\Theta_{t,t}(x)=x$ which verify (4.16) and (4.18) so that $\left\|\Theta\right\|_{k,\alpha,\beta}<\infty$. And an element of $\mathcal{C}^{\alpha,\beta}_{k}([0,T])$ will be called a $(k,\alpha,\beta)-$ semi flow.*

Lemma 4.3 *Let $\Theta \in \mathcal{C}_k^{\alpha,\beta}([0,T])$ for some $k \in N$ and let*

$$C_*(k) := \max\{3^k C(k),\ 2^{\beta+1}\zeta(\beta)\left\|\delta\Theta\right\|_{k,\beta}, 10^{k^2+k}C'(k)\left|L\right|_{1,k+1,\alpha}\} \tag{4.20}$$

where $C(k)$ (respectively $C'(k)$) is the universal constant in (4.4) (respectively in (4.6)). And $\zeta(\beta) = \sum_{i=1}^{\infty} i^{-\beta}$. Take also some $1 < \beta' < \beta$ and take $\varepsilon_* > 0$ which satisfies

$$C_*(k) \times \varepsilon_*^{\alpha\wedge(\beta-\beta')\wedge\beta'} \leq 1. \tag{4.21}$$

We fix $s < t \leq T$ with $t - s < \varepsilon_*$. Then, for every partition $\mathcal{P}$ of (s,t) we obtain

$$\left|X_{s,t}^{\mathcal{P}}(\Theta) - \Theta_{s,t}\right|_k \leq C_k^*(\Theta)(t-s)^{\beta'} \tag{4.22}$$

with

$$C_k^*(\Theta) = 2^{\beta'+1}\zeta(\beta)\left\|\delta\Theta\right\|_{k,\beta}. \tag{4.23}$$

Finally, for every $s < t \leq T$ such that $t - s \leq \varepsilon_*$ and every partition $\mathcal{P}$ we have

$$\left|X_{s,t}^{\mathcal{P}}(\Theta)\right|_{1,k} \leq 3. \tag{4.24}$$

Proof. To begin we notice that (4.24) is an immediate consequence of (4.22). Indeed, by our choice of ε_*we have

$$\left|X_{s,t}^{\mathcal{P}}(\Theta) - \Theta_{s,t}\right|_k \leq C_k^*(\Theta)(t-s)^{\beta'} \leq 1$$

and moreover since $t - s \leq \varepsilon_*$

$$\left|\Theta_{s,t}\right|_{1,k} \leq 1 + \left|L_{s,t}\right|_{1,k} \leq 1 + \left|L\right|_{1,k+1,\alpha}(t-s)^{\alpha} \leq 2.$$

which gives (4.24) .

We prove (4.22) by recurence on $r = n_{\mathcal{P}}(s,t)$, the number of points of $\mathcal{P}$ which fall inside the interval (s,t). For $r = 1$ and $\mathcal{P}(s,t) = \{s < s_1 < t\}$ we have $X_{s,t}^{\mathcal{P}} = \Theta_{s_1,t}\Theta_{s,s_1}$ so that $\Theta_{s,t} - X_{s,t}^{\mathcal{P}} = \Theta_{s,t} - \Theta_{s_1,t}\Theta_{s,s_1}$ and then

$$\left|\Theta_{s,t} - X_{s,t}^{\mathcal{P}}\right|_k \leq \left\|\delta\Theta\right\|_{k,\beta}(t-s)^{\beta}. \tag{4.25}$$

Suppose now that (4.22) (and consequetely (4.24)) is true for partitions of length $n_{\mathcal{P}}(s,t)$ less or equal to $r-1$ and let us prove it for a partition of length $n_{\mathcal{P}}(s,t) = r$.

Step 1. We fix $\mathcal{P} = \{s = s_0 < s_1 < ... < s_r = t\}$ and we consider some other partition $\mathcal{P}' = \{s = s_0 < s_1' < ... < s_{r'}' = t\} \subset \mathcal{P}$ (the partition $\mathcal{P}'$ is a sub partition of $\mathcal{P}$ and such sub partitions appear when we use "the sewing argument" in the Step 2 of the proof). Then we define $Y_i = X_{s,s_i'}^{\mathcal{P}}(\Theta), i = 1, ..., r'$ (so, this is the Euler scheme associated to the partition $\mathcal{P}$ but which is considered just in the points s_i' of the partition $\mathcal{P}'$). Then we define

$$Y_{s,s_i'}^{\mathcal{P}'}(x) = x + \sum_{j=1}^{i} L_{s_{j-1}',s_j'}(Y_{j-1}(x)), \quad i = 1, ..., r'.$$

Notice that $Y_{s,s_i'}^{\mathcal{P}'} \neq X_{s,s_i'}^{\mathcal{P}'}(\Theta)$ beacause $Y_{j-1} \neq X_{s,s_{j-1}'}^{\mathcal{P}'}(\Theta)$. We fix now $i_0 \in \{1, ..., r'\}$ and we denote by $\mathcal{P}_{i_0}'$ the partition $\mathcal{P}'$ in which we have cancelled i_0. Then

$$Y_{s,t}^{\mathcal{P}'} - Y_{s,t}^{\mathcal{P}_{i_0}'} = L_{s_{i_0-1}',s_{i_0}'}(Y_{i_0-1}) + L_{s_{i_0}',s_{i_0+1}'}(Y_{i_0}) - L_{s_{i_0-1}',s_{i_0+1}'}(Y_{i_0-1}) = A + B$$

with

$$
\begin{aligned}
A &= L_{s'_{i_0-1},s'_{i_0}}(Y_{i_0-1}) + L_{s'_{i_0},s'_{i_0+1}}(\Theta_{s'_{i_0-1},s'_{i_0}}(Y_{i_0-1})) - L_{s'_{i_0-1},s'_{i_0+1}}(Y_{i_0-1}),\\
B &= L_{s'_{i_0},s'_{i_0+1}}(Y_{i_0}) - L_{s'_{i_0},s'_{i_0+1}}(\Theta_{s'_{i_0-1},s'_{i_0}}(Y_{i_0-1})).
\end{aligned}
$$

We estimate first A. We have

$$
\begin{aligned}
&Y_{i_0-1} + L_{s'_{i_0-1},s'_{i_0}}(Y_{i_0-1}) + L_{s'_{i_0},s'_{i_0+1}}(\Theta_{s'_{i_0-1},s'_{i_0}}(Y_{i_0-1}))\\
= \;&\Theta_{s'_{i_0-1},s'_{i_0}}(Y_{i_0-1}) + L_{s'_{i_0},s'_{i_0+1}}(\Theta_{s'_{i_0-1},s'_{i_0}}(Y_{i_0-1})) = \Theta_{s'_{i_0},s'_{i_0+1}}\Theta_{s'_{i_0-1},s'_{i_0}}(Y_{i_0-1})
\end{aligned}
$$

and $Y_{i_0-1} + L_{s'_{i_0-1},s'_{i_0+1}}(Y_{i_0-1}) = \Theta_{s'_{i_0-1},s'_{i_0+1}}(Y_{i_0-1})$ so that

$$
A = (\Theta_{s'_{i_0},s'_{i_0+1}}\Theta_{s'_{i_0-1},s'_{i_0}} - \Theta_{s'_{i_0-1},s'_{i_0+1}})(Y_{i_0-1})) = \delta_{s'_{i_0-1},s'_{i_0},s'_{i_0+1}}(\Theta)(Y_{i_0-1})).
$$

Notice that $n_{\mathcal{P}}(s,s'_{i_0-1}) \leq n_{\mathcal{P}}(s,t)-1 \leq r-1$, so we may use the reccurence hypothesis. Recall that $Y_{i_0-1} = X^{\mathcal{P}}_{s,s'_{i_0-1}}$ and notice that $s'_{i_0-1}-s \leq t-s \leq \varepsilon_*$. Then by our reccurence hypothesis and (4.24) we get $|Y_{i_0-1}|_{1,k} \leq 3$. Using (4.4) with $f = \delta_{s'_{i_0-1},s'_{i_0},s'_{i_0+1}}(\Theta)$ and $g = Y_{i_0-1}$ ($C(k)$ is the constant in that inequality)

$$
\begin{aligned}
|A|_{1,k} &\leq \|\delta\Theta\|_{k,\beta}\,(s'_{i_0+1}-s'_{i_0-1})^{\beta} C(k)\,|Y_{i_0-1}|^k_{1,k}\\
&\leq 3^k C(k)\,\|\delta\Theta\|_{k,\beta}\,(s'_{i_0+1}-s'_{i_0-1})^{\beta}\\
&\leq 3^k C(k)(s'_{i_0+1}-s'_{i_0-1})^{\beta-\beta'} \times \|\delta\Theta\|_{k,\beta}\,(s'_{i_0+1}-s'_{i_0-1})^{\beta'}\\
&\leq \|\delta\Theta\|_{k,\beta}\,(s'_{i_0+1}-s'_{i_0-1})^{\beta'}
\end{aligned}
$$

the last inequality being a consequence of (4.21) because $3^k C(k)\varepsilon_*^{\beta-\beta'} \leq 1$. Also

$$
|A|_0 \leq \left\|\delta_{s'_{i_0-1},s'_{i_0},s'_{i_0+1}}(\Theta)\right\|_\infty \leq \|\delta\Theta\|_{k,\beta}\,(s'_{i_0+1}-s'_{i_0-1})^{\beta}
$$

so finally

$$
|A|_k \leq 2\,\|\delta\Theta\|_{k,\beta}\,(s'_{i_0+1}-s'_{i_0-1})^{\beta'}
$$

We treat now B. Notice that

$$
Y_{i_0} = X^{\mathcal{P}(s'_{i_0-1},s'_{i_0})}_{s'_{i_0-1},s'_{i_0}}(Y_{i_0-1})
$$

where $\mathcal{P}(s'_{i_0-1},s'_{i_0})$ is the trace of the partition $\mathcal{P}$ on (s'_{i_0-1},s'_{i_0}) that is: if $s'_{i_0-1}=s_j$ and $s_{j'}=s'_{i_0}$, then $\mathcal{P}(s'_{i_0-1},s'_{i_0}) = \{s'_{i_0-1}=s_j<\ldots<s_{j'}=s'_{i_0}\}$. Notice that $n_{\mathcal{P}}(s'_{i_0-1},s'_{i_0}) \leq n_{\mathcal{P}}(s,t)-1 \leq r-1$ so we are able to use the recurence hypothesis.

Then we write

$$
B = L_{s'_{i_0},s'_{i_0+1}}(X^{\mathcal{P}(s'_{i_0-1},s'_{i_0})}_{s'_{i_0-1},s'_{i_0}}(Y_{i_0-1})) - L_{s'_{i_0},s'_{i_0+1}}(\Theta_{s'_{i_0-1},s'_{i_0}}(Y_{i_0-1})).
$$

We will use (4.6) with $\varphi = \overline{\varphi} = L_{s'_{i_0},s'_{i_0+1}}$, $f = X^{\mathcal{P}(s'_{i_0-1},s'_{i_0})}_{s'_{i_0-1},s'_{i_0}}$, $\overline{f} = \Theta_{s'_{i_0-1},s'_{i_0}}$ and $\psi = \overline{\psi} = Y_{i_0-1}$. This gives

$$
|B|_k = \left|\varphi f g - \varphi \overline{f} g\right|_k \leq C_k(f,\overline{f},\psi)\,|\varphi|_{1,k+1}\left|f-\overline{f}\right|_k
$$

with

$$C_k(f,\overline{f},\psi)=C(k)(1+|f|_{1,k}+\left|\overline{f}\right|_{1,k}+|\psi|_{1,k})^{k^2+k}\leq C(k)\times 10^{k^2+k}$$

with $C(k)$ the universal constant from (4.6). The last inequality is a consequence of the reccurence hypothesis (4.24) (which we may use).

Morover, by our hypothesis (4.16)

$$|\varphi|_{1,k+1}=\left|L_{s'_{i_0},s'_{i_0+1}}\right|_{1,k+1}\leq |L|_{1,k+1,\alpha}\,(s'_{i_0+1}-s'_{i_0})^{\alpha}.$$

And, by the reccurence hypothesis, $\left|f-\overline{f}\right|_k\leq C^*_k(\Theta)(s'_{i_0+1}-s'_{i_0})^{\beta'}$. So finally we obtain

$$\begin{aligned}|B_k|_k &\leq C'(k)\times 10^{k^2+k}\,|L|_{1,k+1,\alpha}\,C^*_k(\Theta)\times(s'_{i_0+1}-s'_{i_0})^{\alpha}\times(s'_{i_0+1}-s'_{i_0})^{\beta'}\\ &= C'(k)\times 10^{k^2+k}\,|L|_{1,k+1,\alpha}\,2^{\beta+1}\zeta(\beta)\varepsilon_*^{\alpha}\times\|\delta\Theta\|_{k,\beta}\,(s'_{i_0+1}-s'_{i_0})^{\beta'}\\ &\leq \|\delta\Theta\|_{k,\beta}\,(s'_{i_0+1}-s'_{i_0})^{\beta'}.\end{aligned}$$

We come back and we get

$$\left|Y^{\mathcal{P}'}_{s,t}-Y^{\mathcal{P}'_{i_0}}_{s,t}\right|_k\leq 3\,\|\delta\Theta\|_{k,\beta}\,(s'_{i_0+1}-s'_{i_0-1})^{\beta'}. \tag{4.26}$$

Step 2 (the sewing argument). We come back to our partition $\mathcal{P}=\{s=s_0<...<s_r=t\}$ and we take i_0 such that

$$s_{i_0+1}-s_{i_0-1}\leq\frac{2}{r-1}(t-s).$$

Such an i always exists (if we have the converse inequality for every $i_0\in\{1,...,r-1\}$ then $2(t-s)\geq\sum_{i=1}^{r-1}(s_{i+1}-s_{i-1})>2(t-s)$). We denote by $\mathcal{P}_{i_0}$ the partition where we have dropped out s_{i_0} and we use (4.26) in order to get

$$\left|Y^{\mathcal{P}}_{s,t}-Y^{\mathcal{P}_{i_0}}_{s,t}\right|_k\leq 3\,\|\delta\Theta\|_{k,\beta}\,(s_{i_0+1}-s_{i_0-1})^{\beta'}\leq\frac{2^{2+\beta'}}{(r-1)^{\beta'}}\,\|\delta\Theta\|_{k,\beta}\,(t-s)^{\beta'}.$$

We repeat this argument for $\mathcal{P}'=\mathcal{P}_{i_0}$ (notice that we still have (4.26)) and so on (in order to exhaust $\mathcal{P}$) we obtain

$$\left|Y^{\mathcal{P}}_{s,t}-\Theta_{s,t}\right|_k\leq 2^{\beta'+2}\zeta(\beta)\,\|\delta\Theta\|_{k,\beta}\,(t-s)^{\beta'}$$

with $\zeta(\beta)=\sum_{i=0}^{\infty}\frac{1}{r^{\beta}}$. $\square$

We need to generalize (4.24) to any $0\leq s<t\leq T$. We define

$$p_k(T)=1+T\times(C_*(k)\,|L|_{1,k+1,\alpha})^{\frac{1}{\alpha\wedge(\beta-\beta')\wedge\beta'}}\quad and\quad C_k(T)=(3C(k))^{k^{p_k(T)}}. \tag{4.27}$$

Corollary 4.4 *We are under the assumptions of the previous Lemma. Then for every partition* $\mathcal{P}$ *of* $[0,T]$ *and every* $0\leq s<t\leq T$

$$\left|X^{\mathcal{P}}_{s,t}(\Theta)\right|_{1,k}\leq C_k(T). \tag{4.28}$$

Proof Let $\varepsilon_* = (C_*(k)\,|L|_{1,k+1,\alpha})^{-\frac{1}{\alpha\wedge(\beta-\beta')\wedge\beta'}}$. Then (4.24) holds if $t-s\leq\varepsilon_{*}$. So we split the interval $[0,T]$ in T/ε_* intervals and use (4.24) on each of these sub intervals. Using recursively (4.4) we get (4.24). $\square$

We are now able to give our main result: this is the so called "sewing lemma" (introduced simultaneously by Gubinelli and by De la Pradelle and Fayel) adapted to our framework.

Lemma 4.5 ***A (Existence and unicity)*** *Let $k\in N$ and ε_* be given in (4.21). Let $\Theta\in\mathcal{C}_{k+1}^{\alpha,\beta}([0,T])$ with $\beta>1$. Then for every $1<\beta'<\beta$ there exists a unique $X:[0,T]^2\to C_b^k(R^d)$ which has the flow propery (4.11), and such that for every $0\leq s<t\leq T$ with $t-s\leq\varepsilon_*$*

$$\left|\Theta_{s,t}-X_{s,t}\right|_k\leq C_{k+1}(T)C_k(\Theta)\left|t-s\right|^{\beta'} \tag{4.29}$$

with $C_{k+1}(T)$ given in (4.27) Uniqueness holds in the following sense: if X and $\overline{X}$ are two flows such that $\left|\Theta-X\right|_{\varepsilon_*,k,\beta'}<\infty$ and $\left|\Theta-\overline{X}\right|_{\varepsilon_*,k,\beta'}<\infty$, then $X=\overline{X}$. We denote by $X_{s,t}(\Theta)$ the unique flow which verifies (4.29).

B (approximation) *We have the following error estimate. For every $1<\beta'<\beta$, every partition with $|\mathcal{P}|\leq\varepsilon_*$ and every $s<t$, with $s,t\in\mathcal{P}$*

$$\left|X_{s,t}^{\mathcal{P}}(\Theta)-X_{s,t}(\Theta)\right|_k\leq 2^{\beta+5}C_{k+1}(T)\zeta(\beta)\left\|\Theta\right\|_{k,\alpha,\beta}\left|\mathcal{P}\right|^{\beta'-1}(t-s). \tag{4.30}$$

Remark 4.6 *Notice that in order to obtain the estimate (4.29) in norm $|\circ|_k$ we need that $\|\Theta\|_{k+1,\alpha,\beta}<\infty$, instead of the hypothesis $\|\Theta\|_{k,\alpha,\beta}<\infty$ in the previous lemma. This is because we need to obtain*

$$\left|X^{\mathcal{P}}(\Theta)\right|_{1,k+1}\leq C_{k+1}(T) \tag{4.31}$$

in (4.24), and this is crucial in the proof of the lipschizianity of $X^{\mathcal{P}}(\Theta)$ (which represents a major difficulty in the proof). This shows that the calculus which is behind, essentially involves derivatives, and this is a strong reason of working with norms of type $|\circ|_k$ which do control derivatives.

Remark 4.7 *The basic existence and uniqueness result coresponds to $k=0$, so we need that $\Theta\in\mathcal{C}_1^{\alpha,\beta}([0,T])$. And this implies that $\Theta_{s,t}\in C_b^2(R^d)$.*

Remark 4.8 *The estimate (4.30) is written for $s,t\in\mathcal{P}$, because, if they do not belong to $\mathcal{P}$, the Euler scheme $X_{s,t}^{\mathcal{P}}(\Theta)$ is not defined. But we may define in a natural way an extension of $X_{s,t}^{\mathcal{P}}(\Theta)$ to every $s<t$. This is done as follows: we denote by $\mathcal{P}_{(s,t)}$ the partition $\mathcal{P}$ to which we have added the times s and t. Then we defin $\widetilde{X}_{s,t}^{\mathcal{P}}(\Theta):=X_{s,t}^{\mathcal{P}_{(s,t)}}(\Theta)$. With this definition (4.30) holds for $\widetilde{X}_{s,t}^{\mathcal{P}}(\Theta)$. In particular, since $t-s\leq|\mathcal{P}|$, for every $1<\beta'<\beta$ one has $|\mathcal{P}|^{\beta-1}(t-s)\leq|\mathcal{P}|^{\beta-\beta'}(t-s)^{\beta'}$ so that (4.30) gives an estimate of the β' Hölder norm:*

$$\left|\widetilde{X}^{\mathcal{P}}(\Theta)-X(\Theta)\right|_{k,\beta'}\leq C\left\|\delta\Theta\right\|_{\varepsilon_*,k,\beta}\left\|\Theta\right\|_{k+1,\alpha,\beta}^{2(1+k+k^2)/\rho}\left|\mathcal{P}\right|^{\beta-\beta'}. \tag{4.32}$$

Remark 4.9 *In the following section we will consider RDE (rough differential equations). In this framework we prove that $X_{s,t}(\Theta)$ is a solution of a rough differential equation associated to Θ iff (4.29) holds. Such a characterization coincides with Devie's deffinition for the solution of $RDE's$. So our approach appears as an abstract varient of Devie's approach at the level of flows.*

Proof. We want to define

$$X_{s,t} = \lim_{|\mathcal{P}|\to 0} X^{\mathcal{P}}_{s,t}(\Theta).$$

In the above limit $\mathcal{P}$ is a partition of (s,t). In order to do it we have to check that the above limit exists, so we prove that, for any two partitions $\mathcal{P}$ and $\overline{\mathcal{P}}$ then

$$\lim_{|\mathcal{P}|\vee\left|\overline{\mathcal{P}}\right|\to 0} \left| X^{\mathcal{P}}_{s,t}(\Theta) - X^{\overline{\mathcal{P}}}_{s,t}(\Theta)\right|_k = 0.$$

We may assume (without loss of generality) that $|\mathcal{P}| \leq \varepsilon_*/2$ and $\left|\overline{\mathcal{P}}\right| \leq \varepsilon_*/2$. We also assume that $\overline{\mathcal{P}}$ is a refinement of $\mathcal{P}$ so we write $\mathcal{P} = \{s = s_0 < ... < s_r = t\}$ and $\overline{\mathcal{P}} = \cup_{i=1}^r \mathcal{P}^i(s_{i-1}, s_i)$ for some sub-partitions $\mathcal{P}^i = \mathcal{P}^i(s_{i-1}, s_i) = \{s_{i-1} = s_i^0 < ... < s_i^{r_i} = s_i\}$. Moreover we denote

$$\overline{\mathcal{P}}_l = \cup_{i=1}^l \mathcal{P}^i(s_{i-1}, s_i) \cup \{s_l < ... < s_r = t\}.$$

So $\overline{\mathcal{P}}_l$ is the partiton in which we refine the intervals $(s_{i-1}, s_i), i = 1, ..., l$ according to $\overline{\mathcal{P}}$ but we keep the intervals $(s_{i-1}, s_i), i = l+1, ..., r$ as they are in $\mathcal{P}$. Finally we write

$$\left| X^{\mathcal{P}}_{s,t}(\Theta) - X^{\overline{\mathcal{P}}}_{s,t}(\Theta)\right|_k \leq \sum_{l=0}^{r} \left| X^{\overline{\mathcal{P}}_{l+1}}_{s,t}(\Theta) - X^{\overline{\mathcal{P}}_l}_{s,t}(\Theta)\right|_k .$$

We compute

$$X^{\overline{\mathcal{P}}_{l+1}}_{s,t}(\Theta) - X^{\overline{\mathcal{P}}_l}_{s,t}(\Theta) = \varphi_l X^{\mathcal{P}^l}_{s_l,s_{l+1}}(\Theta)\psi_l - \varphi_l \Theta_{s_l,s_{l+1}}\psi_l$$

with $\varphi_l = X^{\overline{\mathcal{P}}}_{s_{l+1},t}, \quad \psi_l = X^{\mathcal{P}}_{s,s_l}$. We will use (4.6) so we will need

$$|\varphi_l|_{1,k+1} + |\psi_l|_{1,k} + \left| X^{\mathcal{P}^l}_{s_l,s_{l+1}}(\Theta)\right|_{1,k+1} + |\Theta|_{1,k} \leq 4C_{k+1}(T)$$

whith $C_{k+1}(T)$ defined in (4.27) (Here comes on the fact that we need $k+1$ derivatives). Then by (4.6) first and (4.22) next (notice that $s_{i+1} - s_{i-1} \leq 2|\mathcal{P}| \leq \varepsilon_*$)

$$\begin{aligned}\left| X^{\overline{\mathcal{P}}_{l+1}}_{s,t}(\Theta) - X^{\overline{\mathcal{P}}_l}_{s,t}(\Theta)\right|_k &\leq 4C_{k+1}(T)\left| X^{\mathcal{P}^l}_{s_l,s_{l+1}}(\Theta) - \Theta_{s_l,s_{l+1}}\right|_k \\ &\leq 2^{\beta+5}C_{k+1}(T)\zeta(\beta)\|\Theta\|_{k+1,\alpha,\beta}(s_{l+1} - s_l)^{\beta'}\end{aligned}$$

We conclude that

$$\begin{aligned}\left| X^{\mathcal{P}}_{s,t}(\Theta) - X^{\overline{\mathcal{P}}}_{s,t}(\Theta)\right|_k &\leq 2^{\beta+5}C_{k+1}(T)\zeta(\beta)\|\Theta\|_{k+1,\alpha,\beta} \times \sum_{l=0}^{r} |s_{l+1} - s_l|^{\beta'} \\ &\leq 2^{\beta+5}C_{k+1}(T)\zeta(\beta)\|\Theta\|_{k+1,\alpha,\beta} \times |\mathcal{P}|^{\beta'-1}(t-s) \to 0.\end{aligned}$$

So $\lim_{|\mathcal{P}|\to 0} X^{\mathcal{P}}_{s,t}(\Theta) =: X_{s,t}$ exists in $C_b^k(R^d)$ and $X_{s,t} \in C_b^k(R^d)$ is well defined. And passing to the limit in the above estimate (with $\left|\overline{\mathcal{P}}\right| \to 0$) we obtain (4.30).

Step 3. The fact that $X_{s,t} = X_{s,u}X_{u,t}$ is true because the concatenation of a partition of (s,u) with a partition of (u,t) gives a partition of (s,t). Finally, by passing to the limit in (4.22) we obtain (4.29).

Uniquness is obvious. □

5 Rough path

In this section we will deal with semi flows associated to a rough path. Before we come back and give some definitions and notation. For $F:[0,T]\times R^d\to R^d$ which is k time differentiable with respect to x we define the norm (the same as in (4.9))

$$\left|F\right|_{k,\alpha}=\sup_{s<t}\frac{\left|F(t,\cdot)-F(s,\cdot)\right|_k}{(t-s)^\alpha}$$

and, if we consider just $s<t<s+\varepsilon$ then we denote $\left|F\right|_{\varepsilon,k,\alpha}$. We also denote $\mathcal{C}^\alpha_{(k)}$ the space of the applications $F:[0,T]\times R^d\to R^d$ such that $\left|F\right|_{k,\alpha}<\infty$.

In the case where F does not depend on the space variable x, more precisely $F:[0,T]\to R$, we denote just $\mathcal{C}^\alpha$ and

$$\left\|F\right\|_\alpha=\sup_{s<t}\frac{\left|F(t)-F(s)\right|}{(t-s)^\alpha}$$

the usual Hölder norm.

We recall now the definition of a **"rough path"**. Given $\alpha\in(\frac{1}{3},\frac{1}{2})$ **we define** a α rough path to be a couple $\mathbf{W}=(W,\mathbb{W})$ with $W:R_+\to R^d$ and $\mathbb{W}:R_+^2\to R^{d\times d}$ such that

$$\left\|W\right\|_\alpha=\sup_{|t-s|>0}\frac{\left|W_{s,t}\right|}{\left|t-s\right|^\alpha}<\infty\quad and\quad\left\|\mathbb{W}\right\|_{2\alpha}=\sup_{|t-s|>0}\frac{\left|\mathbb{W}_{s,t}\right|}{\left|t-s\right|^{2\alpha}}<\infty$$

and for which "Chen relations" hold true:

$$\begin{aligned}W_{s,t}&=W_{s,u}+W_{u,t}\\ \mathbb{W}^{i,j}_{s,t}&=\mathbb{W}^{i,j}_{s,u}+\mathbb{W}^{i,j}_{u,t}+W^i_{s,u}W^j_{u,t}\end{aligned}$$

We will consider the "norm"

$$\left\|\mathbf{W}\right\|_\alpha=\left\|W\right\|_\alpha+\left\|\mathbb{W}\right\|_{2\alpha}.$$

We define now the **"controlled path".** This is analogous with the previous definition but now we have function valued processes. We denote $\Gamma=\{1,...,d\}$. We say that $F:[0,T]\to C^k_b(R^d,R^d)$ is controlled by W if there exist $\gamma_j\in\mathcal{C}^\alpha_{(k)},j\in\Gamma$ and $R^F\in\mathcal{C}^{2\alpha}_{(k)}$ such that

$$F(t,x)-F(s,x)=\sum_{j\in\Gamma}\gamma_j(s,x)W^j_{s,t}+R^F_{s,t}(x).$$

We denote $D^j_WF=\gamma_j$ the "Gubinelli derivative". This is not unique, so the notation is abusive - we will precise in each case which is our choice. Following [FH] we denote $\mathcal{D}^\alpha_{k,W}$ the space of the couples Y,Y' such that $Y\in\mathcal{C}^\alpha_{(k)}$ is controlled by Y' (so $Y'=DY$). In the case of "real rough path" we have uniquenees of $Y'=:D_WY$. And the above relation reads

$$Y(t,x)-Y(s,x)=\sum_{j\in\Gamma}D^j_WY(s,x)W^j_{s,t}+R^Y_{s,t}(x).$$

Moreover, we will use the following computational rule. Consider $F,G\in\mathcal{C}^\alpha_{(k)}$ which are controlled by W and suppose that one may find some version of Gubinelli derivative D_WF which is Lipschitz continuous with respect to $x \in R^n$, uniformly in $t \in [0,T]$. Then $F(t, G(t,x))$ is controlled by W and , for every version $D_W G$, one has

$$D_W^j F(t, G(t,x)) = (D_W^j F)(t, G(t,x)) + \sum_{k=1}^{n} (\partial_k F)(t, G(t,x)) D_W^j G^k(t,x).$$

The proof is straightforward and we skip it.

We define now $\Theta_{s,t}(\sigma, W)(x) = x + L_{s,t}(\sigma, W)(x)$ with

$$L_{s,t}(\sigma, W)(x) = \sum_{j\in\Gamma} \sigma_j(s,x) W_{s,t}^j + \sum_{j,j'\in\Gamma} \sigma_{j,j'}(s,x) \mathbb{W}_{s,t}^{j,j'}.$$

Here $\sigma_j \in \mathcal{D}^{\alpha}_{(k+3),W}$ and

$$\sigma_{j,j'} = D_W^{j'} \sigma_j + D_\sigma^{j'} \sigma_j \quad with \quad D_\sigma^{j'} \sigma_j = \sum_{k=1}^{d} \sigma_{j'}^k \partial_{x_k} \sigma_j.$$

We recall that in the previous section we have defined $\delta(\Theta)$ (see (4.17)), $\|\delta(\Theta)\|_{k,\beta}$ (see (4.18)) and $\|\Theta\|_{k,\alpha,\beta}$ (see (4.19)). Concerning these quantities we will need some bound and stability estimates. In order to give these estimates we have to precise the following notation:

$$\begin{aligned} \sigma_j(t,x) - \sigma_j(s,x) - \sum_{j'\in\Gamma} D_W^{j'} \sigma_j(s,x) W_{s,t}^{j'} &= R_{s,t}^j(x) \\ \sigma_{j,j'}(t,x) - \sigma_{j,j'}(s,x) &= r_{s,t}^{j,j'}(x) \end{aligned}$$

and we konw that $R^j \in \mathcal{C}^{2\alpha}_{(k+1)}$ and $r^{j,j'} \in \mathcal{C}^{\alpha}_{(k+1)}$.

Now we are able to give our estimates:

Lemma 5.1 *let* $\mathbf{W}$ *be* α *rough path. We consider some coefficients* σ *which are controlled by* W *and such that* $|\sigma|_{k+2} < \infty$. *Then*

$$\|\Theta(\sigma, \mathbf{W})\|_{k,\alpha,3\alpha} = |L(\sigma, \mathbf{W})|_{1,k+1,\alpha} + \|\delta(\Theta(\sigma, \mathbf{W}))\|_{k,3\alpha} < \infty. \tag{5.1}$$

Proof We denote $\sigma_{j,s}(x) = \sigma_j(s,x)$. We compute first

$$\begin{aligned} & \sum_{j\in\Gamma} (\sigma_{j,s} W_{s,t}^j - \sigma_{j,s} W_{s,u}^j - \sigma_{j,u} W_{u,t}^j) \\ = \quad & -\sum_{j\in\Gamma} (\sigma_{j,u} - \sigma_{j,s}) W_{u,t}^j = -\sum_{j,j'\in\Gamma} D_W^{j'} \sigma_{j,s} W_{s,u}^{j'} W_{u,t}^j + o_1(s,u,t) \end{aligned}$$

with

$$o_1(s,u,t) = \sum_{j\in\Gamma} R_{s,u}^j W_{u,t}^j = O(t-s)^{3\alpha}.$$

Moreover, let

$$o_2(s,u,t) = \sum_{j,j'\in\Gamma} (\sigma_{j,j',u} - \sigma_{j,j',s}) \mathbb{W}_{u,t}^{j,j'} = \sum_{j,j'\in\Gamma} r_{s,u}^{j,j'} \mathbb{W}_{u,t}^{j,j'} = O(t-s)^{3\alpha}.$$

We have (Chen relations)

$$\begin{aligned}
& \sum_{j,j'\in\Gamma}(\sigma_{j,j',s}\mathbb{W}^{j,j'}_{s,t}-\sigma_{j,j',s}\mathbb{W}^{j,j'}_{s,u}-\sigma_{j,j',u}\mathbb{W}^{j,j'}_{u,t} \\
= \;& o_2(s,u,t)+\sum_{j,j'\in\Gamma}\sigma_{j,j',s}(\mathbb{W}^{j,j'}_{s,t}-\mathbb{W}^{j,j'}_{s,u}-\mathbb{W}^{j,j'}_{u,t}) \\
= \;& o_2(s,u,t)+\sum_{j,j'\in\Gamma}\sigma_{j,j',s}W^{j'}_{s,u}W^{j}_{u,t} \\
= \;& o_2(s,u,t)+\sum_{j,j'\in\Gamma}D^{j'}_{W}\sigma_{j,s}W^{j'}_{s,u}W^{j}_{u,t}+\sum_{j,j'\in\Gamma}D^{j'}_{\sigma}\sigma_{j,s}W^{j'}_{s,u}W^{j}_{u,t}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
& L_{s,t}-L_{s,u}-L_{u,t} \\
= \;& \sum_{j\in\Gamma}\sigma_{j,s}W^{j}_{s,t}+\sum_{j,j'\in\Gamma}\sigma_{j,j',s}\mathbb{W}^{j,j'}_{s,t} \\
& -(\sum_{j\in\Gamma}\sigma_{j,s}W^{j}_{s,u}+\sum_{j,j'\in\Gamma}\sigma_{j,j',s}\mathbb{W}^{j,j'}_{s,u}) \\
& -(\sum_{j\in\Gamma}\sigma_{j,u}W^{j}_{u,t}+\sum_{j,j'\in\Gamma}\sigma_{j,j',u}\mathbb{W}^{j,j'}_{u,t}) \\
= \;& \sum_{j\in\Gamma}R^{j}_{s,u}W^{j}_{u,t}+\sum_{j,j'\in\Gamma}r^{j,j'}_{s,u}\mathbb{W}^{j,j'}_{u,t}+\sum_{j,j'\in\Gamma}(D^{j'}_{\sigma}\sigma_j)_sW^{j'}_{s,u}W^{j}_{u,t} \\
= \;& \sum_{j,j'\in\Gamma}D^{j'}_{\sigma}\sigma_{j,s}W^{j'}_{s,u}W^{j}_{u,t}+O(t-s)^{3\alpha}
\end{aligned}$$

Write now $\langle\nabla L_{u,t}(x),L_{s,u}(x)\rangle$ in an explicite way. We emphsise the "main terms" and we get

$$\langle\nabla L_{u,t}(x),L_{s,u}(x)\rangle=\alpha_3(s,u,t)+\sum_{j,j'\in\Gamma}(D^{j'}_{\sigma}\sigma_j)_sW^{j'}_{s,u}W^{j}_{u,t}$$

with

$$\begin{aligned}
\alpha_3(s,u,t) \;=\; & \sum_{j\in\Gamma}\sum_{p,p'\in\Gamma}\sigma_{j,u}\sigma_{p,p',s}(W^{j}_{u,t}\mathbb{W}^{p,p'}_{s,u}+W^{j}_{s,u}\mathbb{W}^{p,p'}_{u,t}) \\
& +\sum_{j,j'\in\Gamma}\sum_{p,p'\in\Gamma}\sigma_{j,j',u}\sigma_{p,p',s}\mathbb{W}^{j,j'}_{s,u}\mathbb{W}^{p,p'}_{u,t}) \\
& +\sum_{j,j'\in\Gamma}\left\langle\sigma_{j',s},\sigma_{j,u}-\sigma_{j,s}\right\rangle W^{j}_{u,t}W^{j'}_{s,u}=O(t-s)^{3\alpha}
\end{aligned}$$

We conclude that

$$L_{s,t}-L_{s,u}-L_{u,t}=\langle\nabla L_{u,t},L_{s,u}\rangle+O(t-s)^{3\alpha}.$$

This finally gives

$$\begin{aligned}
& \Theta_{s,t}(x)-\Theta_{u,t}\Theta_{s,u}(x) \\
= \;& L_{s,t}(x)-L_{s,u}(x)-L_{u,t}(x+L_{s,u}(x)) \\
= \;& L_{s,t}(x)-L_{s,u}(x)-L_{u,t}(x)-(L_{u,t}(x+L_{s,u}(x))-L_{u,t}(x)) \\
= \;& \langle\nabla L_{u,t}(x),L_{s,u}(x)\rangle-(L_{u,t}(x+L_{s,u}(x))-L_{u,t}(x))+O(t-s)^{3\alpha}.
\end{aligned}$$

Then, using Taylor expansion we upper bound the above term by

$$\|\nabla L_{u,t}\|_\infty \|\nabla L_{s,u}\|_\infty^2 = O(t-s)^{3\alpha}.$$

□

The above estimates show that $\Theta(\sigma, \mathbf{W})$ is a (α, β) semi-flow, with $\beta = 3\alpha$. Then we may use the sewing lemma 4.5 and obtain:

Lemma 5.2 ***(Existence and uniqueness)*** *Suppose that $\sigma_j \in \mathcal{C}^\alpha_{(k+3)}$. For every $1 < \beta' < \beta = 3\alpha$ there exists a unique $X : [0,T]^2 \to C_b^k(R^d)$ which has the flow propery and such that for every $0 \le s < t \le T$ with $t - s \le \varepsilon_*$*

$$|\Theta_{s,t}(\sigma, \mathbf{W}) - X_{s,t}|_k \le C_{k+1}(T) C_k(\Theta) |t-s|^{\beta'} \tag{5.2}$$

with $C_{k+1}(T)$ a suitable constant. Uniqueness holds in the following sense: if X and $\overline{X}$ are two flows which verifie (5.2) then $X = \overline{X}$. We denote by $X_{s,t}(\Theta)$ the unique flow which verifies (5.2).

Remark 5.3 *Let $X_t(x) = X_{0,t}(x)$. We have $D^j_W X_t(x) = \sigma_j(X_t(x))$.*

Proof. Using(5.2)

$$\begin{aligned} X_t(x) - X_s(x) &= X_{s,t}(X_s(x)) - X_s(x) = \Theta_{s,t}(X_s(x)) - X_s(x) + o(t-s)^{\beta'} \\ &= \sum_{j\in\Gamma} \sigma_j(X_s(x)) W^j_{s,t} + o(t-s)^{2\alpha}. \end{aligned}$$

□

5.1 Rough integrals and rough differential equations

Our aim now is to define the rough integral with respect to $\mathbf{W}$ and then to construct solutions of rough differential equations.

To begin we precise the definition of the rough integral. We consider $Y_j : [0,T] \to R^d, j \in \Gamma$ (it plays the role of σ_j from the previous section, but it does not depend on x). We may consider $Y_j(s)$ as a function on R^d which is constant with respect to $x \in R^d$ and to try to use the results from the previous section. First we assume that Y is α Hölder in the usual sense. This will give that Y is also α Hölder in $|\circ|_k$ for every k. We also suppose that Y is derivable in Gubinelli sense, with derivative DY. We fix $(Y_j, DY_j) \in \mathcal{D}^\alpha_{\mathbf{W}}, j \in \Gamma$ and we define

$$L^Y_{s,t} = \sum_{j\in\Gamma} Y_{j,s} W^j_{s,t} + \sum_{j,j'\in\Gamma} D^{j'}_W Y_{j,s} \mathbb{W}^{j,j'}_{s,t} \quad \textit{and} \quad \Theta^Y_{s,t}(x) = x + L^Y_{s,t}.$$

Since Y does not depend on x, we have $D_Y Y = 0$ so we have $D^{j'}_W Y_{j,s} = \sigma_{j,j',s}$ from the previous section. Consequentely Θ^Y is a $(\alpha, 3\alpha) - semi - flow$ and we may use Lemma 5.2 from the previous section in order to associate a flow $X_{s,t}(\Theta^Y)$.

Definition 5.4 *We define the rough integral*

$$I^Y_{s,t} := \int_s^t \langle Y_r, d\mathbf{W}_r \rangle := X_{s,t}(\Theta^Y)(x) - x. \tag{5.3}$$

Remark 5.5 *By the very construction of $X_{s,t}(\Theta^Y)(x)$ we get*

$$\int_s^t \langle Y_r, d\mathbf{W}_r\rangle = \lim_{|\mathcal{P}|\to 0} \sum_{(u,v)\in\mathcal{P}} (\sum_{j\in\Gamma} Y_{j,u} W^j_{u,v} + \sum_{j,j'\in\Gamma} D^{j'}_W Y_{j,u} \mathbb{W}^{j,j'}_{u,v}). \tag{5.4}$$

So we find out the definition of the rough integral in the usual rough calculus.

Remark 5.6 *We stress that the above integral depends on the Goubinelli derivative $D_W Y$.*

Remark 5.7 *The flow property for $X_{s,t}(\Theta^Y)$ reads $I^Y_{s,t} = I^Y_{s,u} + I^Y_{u,t}$. So we come back to the "classical" framework.*

Remark 5.8 *I^Y is characterized by*

$$\begin{aligned} & I^Y_{s,t} - \langle Y_s, W_{s,t}\rangle - \langle D_W Y_s, \mathbb{W}_{s,t}\rangle \\ = \quad & X_{s,t}(\Theta^Y)(x) - \Theta^Y_{s,t}(x) = o(t-s)^{3\alpha}. \end{aligned} \tag{5.5}$$

In particular this means that

$$D^j I^Y_t = Y_j(t). \tag{5.6}$$

We discuss now the rough differential equations. We give now some coefficients $\sigma_j : [0,T] \to C^3_b(R^d), j \in \Gamma$ such that $\sigma_j \in \mathcal{D}^\alpha_{k+3,W}$, with $\alpha > \frac{1}{3}$ and we consider the rough differential equation

$$dX = \langle \sigma(X), d\mathbf{W}\rangle .$$

This equation may be undersood in two different senses. First we may consider the semi flow $\Theta_{s,t} = \Theta_{s,t}(\sigma, \mathbf{W})$ and use the result from the previous section (Lemma 5.2) in order to construct the flow $X_{s,t} = X_{s,t}(\Theta(\sigma, \mathbf{W}))$. This coincides with Devie's definition of the solution of the rough equation above. So we call it a D flow solution. Secondly we may consider the "classical" rough equation based on the definition of the rough integral. We call the solution of this equation a R flow solution. We will check that the two definitions coincide. More precisely

Definition 5.9 *Let $X : [0,T]^2 \times R^d \to R^d$ belong to $\mathcal{C}^\alpha$. We say that X is a R flow solution of the rough differential equation $dX = \langle \sigma(X), d\mathbf{W}\rangle$ if $X_{s,t} = X_{u,t}X_{s,u}$ for every $s < u < t$ (it is a flow), $X_{s,t} : R^d \to R^d$ is a bijection and $X_{s,t}(x)$ satisfis*

$$X_{s,t}(x) = x + \int_s^t \langle \sigma(X_{s,r}(x)), d\mathbf{W}_r\rangle . \tag{5.7}$$

with the rough integral associated, for each fixed $s \geq 0$ and $x \in R^d$, to

$$Y^{s,x}_{j,r} = \sigma_j(X_{s,r}(x)) \quad \text{and} \quad D^{j'}_W Y^{s,x}_{j,r} = D^{j'}_\sigma \sigma_j(X_{s,r}(x)). \tag{5.8}$$

Notice that if X solves tha above equation then $D^{j'}_W X^{j'}_{s,t}(x) = \sigma^{j'}_j(X_{s,t}(x))$. This is why we get $D^{j'}_W Y^{s,x}_{j,r} = D^{j'}_\sigma \sigma_j(X_{s,r}(x))$.

In order to construct a solution of the above equation we consider the *semi − flow*

$$\Theta_{s,t}(x) = \Theta_{s,t}(\sigma, \mathbf{W})(x) = x + \sum_{j\in\Gamma} \sigma_j(x) W^j_{s,t} + \sum_{j,j'\in\Gamma} D^{j'}_\sigma \sigma_j(x) \mathbb{W}^{j,j'}_{s,t}.$$

Theorem 5.10 *Suppose that $\sigma_j \in \mathcal{C}_2^\alpha, j \in \Gamma$ and that $\mathbf{W} = (W, \mathbb{W})$ is a rough path. Then $X_{s,t}(\Theta(\sigma, \mathbf{W}))$ (constructed in the previous section, Lemma 5.2) is the unique R flow solution of the equation (5.7).*

Proof. We briefy denote $\Theta = \Theta(\sigma, \mathbf{W})$ and we consider the D flow solution $X_{s,t}(\Theta)$ produced by Lemma 5.2. We first prove that $X_{s,t}(\Theta)$ solves (5.7) We fix $x_0 \in R^d$ and $Y_j(r) = Y_j^{0,x_0}(r) = \sigma_j(X_{0,r}(\Theta)(x_0))$ We have already checked that $D_W^{j'} X_{s,t}^j(\Theta) = \sigma_{j'}^j(X_{s,t}(\Theta))$. Then

$$D_W^{j'} Y_j(r) = (D_\sigma^{j'} \sigma_j)(r, X_{0,r}(\Theta)(x_0)$$

so that

$$\begin{aligned} L_{s,t}^Y &= \sum_{j\in\Gamma} Y_j(s) W_{s,t}^j + \sum_{j,j'\in\Gamma} D_W^{j'} Y_j(s) \mathbb{W}_{s,t}^{j,j'} \\ &= \sum_{j\in\Gamma} \sigma_j(X_{0,s}(\Theta)(x_0)) W_{s,t}^j + \sum_{j,j'\in\Gamma} (D_\sigma^{j'} \sigma_j)(X_{0,s}(\Theta)(x_0)) \mathbb{W}_{s,t}^{j,j'} \\ &= \Theta_{s,t}(X_{0,s}(\Theta)(x_0)) - X_{0,s}(\Theta)(x_0) = L_{s,t}(X_{0,s}(\Theta)(x_0)) \end{aligned}$$

Recall first that I^Y is charachterized by

$$\left| I_t^Y - I_s^Y - L_{s,t}^Y \right| \leq C(t-s)^{3\alpha} \tag{5.9}$$

and $X_{s,t}(\Theta)$ verifies

$$\sup_x |X_{s,t}(\Theta)(x) - x - L_{s,t}(\Theta)(x)| \leq C(t-s)^{3\alpha}$$

Notice that

$$L_{s,t}^Y = L_{s,t}(\Theta)(X_{0,s}(x_0)).$$

In particular, taking $x = X_{0,s}(x_0)$ (here comes on the fact that $x \to X_{0,s}(x_0)$ is a bijection) the above inequaltity gives

$$\begin{aligned} & |X_{s,t}(\Theta)(X_{0,s}(x_0)) - X_{0,s}(x_0) - L_{s,t}(\Theta)(X_{0,s}(x_0))| \\ = & \left| X_{s,t}(\Theta)(X_{0,s}(x_0)) - X_{0,s}(x_0) - L_{s,t}^Y \right| \leq C(t-s)^{3\alpha} \end{aligned}$$

and this guarantess that

$$I_t^Y - I_s^Y = X_{s,t}(\Theta)(X_{0,s}(x_0)) - X_{0,s}(x_0)$$

which, if $s = 0$ reads

$$I_t^Y = X_{0,t}(\Theta)(x_0) - x_0$$

and this is equation (5.7).□