Control of Partial Differential Equations

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Article Outline

Glossary

$\mathbb{R}$ denotes the **real line**, $\mathbb{R}^n$ the n-dimensional Euclidean space, $x \cdot y$ stands for the Euclidean scalar product of $x, y \in \mathbb{R}^n$, and $|x|$ for the norm of x.

State variables quantities describing the state of a system; in this note they will be denoted by u; in the present setting, u will be either a function defined on a subset of $\mathbb{R} \times \mathbb{R}^n$, or a function of time taking its values in an Hilbert space H.

Space domain the subset of $\mathbb{R}^n$ on which state variables are defined.

Partial differential equation a differential equation containing the unknown function as well as its partial derivatives.

State equation a differential equation describing the evolution of the system of interest.

Control function an external action on the state equation aimed at achieving a specific purpose; in this note, control functions they will be denoted by f; f will be used to denote either a function defined on a subset of $\mathbb{R} \times \mathbb{R}^n$, or a function of time taking its values in an Hilbert space F. If the state equation is a partial differential equation of evolution, then a control function can be:

1. *distributed* if it acts on the whole space domain;
2. *locally distributed* if it acts on a subset of the space domain;
3. *boundary* if it acts on the boundary of the space domain;
4. *optimal* if it minimizes (together with the corresponding trajectory) a given cost;
5. *feedback* if it depends, in turn, on the state of the system.

Trajectory the solution of the state equation u_f that corresponds to a given control function f.

Distributed parameter system a system modeled by an evolution equation on an infinite dimensional space, such as a partial differential equation or a partial integro-differential equation, or a delay equation; unlike systems described by finitely many state variables, such as the ones modeled by ordinary differential equations, the information concerning these systems is "distributed" among infinitely many parameters.

$\mathbb{1}_A$ denotes the **characteristic function** of a set $A \subset \mathbb{R}^n$, that is,

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in \mathbb{R}^n \setminus A \end{cases}$$

∂_t , ∂_{x_i} denote **partial derivatives** with respect to t and x_i, respectively.

$L^2(\Omega)$ denotes the **Lebesgue space** of all real-valued square integrable functions, where functions that differ on sets of zero Lebesgue measure are identified.

$H_0^1(\Omega)$ denotes the **Sobolev space** of all real-valued functions which are square integrable together with their *first order* partial derivatives in the sense of distributions in Ω, and vanish on the boundary of Ω; similarly $H^2(\Omega)$ denotes the space of all functions which are square integrable together with their *second order* partial derivatives.

$H^{-1}(\Omega)$ denotes the dual of $H_0^1(\Omega)$.

$\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional **Hausdorff measure**.

H denotes a **normed spaces** over $\mathbb{R}$ with norm $\|\cdot\|$, as well as an **Hilbert space** with the scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

$L^2(0, T; H)$ is the space of all square integrable functions $f\colon [0, T] \to H$; $C([0, T]; H)$ (continuous functions) and $H^1(0, T; H)$ (Sobolev functions) are similarly defined .

Given Hilbert spaces F and H, $\mathcal{L}(F, H)$ denotes the (Banach) space of all bounded linear operators $\Lambda\colon F \to H$ with norm $\|\Lambda\| = \sup_{\|x\|=1} \|\Lambda x\|$ (when $F = H$, we use the abbreviated notation $\mathcal{L}(H)$); $\Lambda^*\colon H \to F$ denotes the adjoint of Λ given by $\langle \Lambda^* u, \phi\rangle = \langle u, \Lambda\phi\rangle$ for all $u \in H, \phi \in F$.

Definition of the Subject

Control theory (abbreviated, CT) is concerned with several ways of influencing the evolution of a given system by an external action. As such, it originated in the nineteenth century, when people started to use mathematics to analyze the perfomance of mechanical systems, even though its roots can be traced back to the calculus of variation, a discipline that is certainly much older. Since the second half of the twentieth century its study was pursued intensively to address problems in aerospace engineering, and then economics and life sciences. At the beginning, CT was applied to systems modeled by ordinary differential equations (abbreviated, ODE). It was a couple of decades after the birth of CT—in the late sixties, early seventies—that the first attempts to control models described by a partial differential equation (abbreviated, PDE) were made. The need for such a passage was unquestionable: too many interesting applications, from diffusion phenomena to elasticity models, from fluid dynamics to traffic flows on networks and systems biology, can be modeled by a PDE.

Because of its peculiar nature, control of PDE's is a rather deep and technical subject: it requires a good knowledge of PDE theory, a field of enormous interest in its own right, as well as familiarity with the basic aspects of CT for ODE's. On the other hand, the effort put into this research direction has been really intensive. Mathematicians and engineers have worked together in the construction of this theory: the results—from the stabilization of flexible structures to the control of turbulent flows—have been absolutely spectacular.

Among those who developed this subject are A. V. Balakrishnan, H. Fattorini, J. L. Lions, and D. L. Russell, but many more have given fundamental contributions.

Introduction

The basic examples of controlled partial differential equations are essentially two: the heat equation and the and the wave equation. In a bounded open domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary Γ the *heat equation*

$$\partial_t u = \Delta u + f \quad \text{in } Q_T \doteq (0,T)\times\Omega \tag{1}$$

describes the evolution in time of the temperature $u(t,x)$ at any point x of the body Ω. The term $\Delta u = \partial^2_{x_1} u + \cdots + \partial^2_{x_n} u$, called the Laplacian of u, accounts for heat diffusion in Ω, whereas the additive term f represents a heat source. In order to solve the above equation uniquely one needs to add further data, such as the initial distribution u_0 and the temperature of the boundary surface Γ of Ω. The fact that, for any given data $u_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$ Eq. (1) admits a unique weak solution u_f satisfying the boundary condition

$$u = 0 \quad \text{on } \Sigma_T \doteq (0,T)\times\Gamma \tag{2}$$

and the initial condition

$$u(0,x) = u_0(x) \quad \forall x = (x_1,\dots,x_n) \in \Omega \tag{3}$$

is well-known. So is the maximal regularity result ensuring that

$$u_f \in H^1\left(0,T;L^2(\Omega)\right) \cap C\left([0,T];H^1_0(\Omega)\right) \cap L^2\left(0,T;H^2(\Omega)\right) \tag{4}$$

whenever $u_0 \in H^1_0(\Omega)$. If problem (1)–(3) possesses a unique solution which depends continuously on data, then we say that the problem is *well-posed*.

Similarly, the *wave equation*

$$\partial^2_t u = \Delta u + f \quad \text{in} \quad Q_T \tag{5}$$

describes the vibration of an elastic membrane (when $n = 2$) subject to a force f. Here, $u(t,x)$ denotes the displacement of the membrane at time t in x. The initial condition now concerns both initial displacement and velocity:

$$\forall x \in \Omega \qquad \begin{cases} u(0,x) = u_0(x) \\ \partial_t u(0,x) = u_1(x)\,. \end{cases} \tag{6}$$

It is useful to treat the above problems as a first order *evolution equation* in a Hilbert space H

$$u'(t) = Au(t) + Bf(t) \quad t \in (0,T)\,, \tag{7}$$

where $f(t)$ takes its valued in another Hilbert space F, and $B \in \mathcal{L}(F,H)$. In this abstract set-up, the fact that (7) is related to a PDE translates into that the closed linear operator A is not defined on the whole space but only on a (dense) subspace $D(A) \subset H$, called the *domain* of A; such a property is often referred to as the *unboundedness* of A.

For instance, in the case of the heat equation (1), $H = L^2(\Omega) = F$, and A is defined as

$$\begin{cases} D(A) = H^2(\Omega) \cap H^1_0(\Omega) \\ Au = \Delta u\,, \quad \forall u \in D(A)\,, \end{cases} \tag{8}$$

whereas $B = I$.

As for the wave equation, since it is a second order differential equation with respect to t, the Hilbert space H should be given by the product $H^1_0(\Omega) \times L^2(\Omega)$. Then, problem (5) is turned into the first order equation

$$U'(t) = \mathcal{A}U(t) + Bf(t) \quad t \in (0,T)\,,$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad F = L^2(\Omega).$$

Accordingly, $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is given by

$$\begin{cases} D(\mathcal{A}) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega) \\ \mathcal{A}U = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} U = \begin{pmatrix} v \\ Au \end{pmatrix} \quad \forall U \in D(\mathcal{A}), \end{cases}$$

where A is taken as in (8).

Another advantage of the abstract formulation (7) is the possibility of considering locally distributed or boundary source terms. For instance, one can reduce to the same set-up the equation

$$\partial_t u = \Delta u + \mathbb{1}_\omega f \quad \text{in } Q_T, \tag{9}$$

where $\mathbb{1}_\omega$ denotes the characteristic function of an open set $\omega \subset \Omega$, or the nonhomogeneus boundary condition of Dirichlet type

$$u = f \quad \text{on } \Sigma_T, \tag{10}$$

or Neumann type

$$\frac{\partial u}{\partial \nu} = f \quad \text{on } \Sigma_T, \tag{11}$$

where ν is the outward unit normal to Γ. For Eq. (9), B reduces to multiplication by $\mathbb{1}_\omega$—a bounded operator on $L^2(\Omega)$; conditions (10) and (11) can also be associated to suitable linear operators B—which, in this case, turn out to be unbounded. Similar considerations can be adapted to the wave equations (5) and to more general problems.

Having an efficient way to represent a source term is essential in control theory, where such a term is regarded as an external action, the *control function*, exercised on the *state variable* u for a purpose, of which there are two main kinds:

- *positional:* $u(t)$ is to approach a given target in X, or attain it exactly at a given time $t > 0$;
- *optimal:* the pair (u, f) is to minimize a given functional.

The first criterion leads to *approximate* or *exact controllability* problems in time t, as well as to *stabilization* problems as $t \to \infty$. Here, the main tools will be provided by certain estimates for partial differential operators that allow to study the states that can be attained by the solution of a given controlled equation. These issues will be addressed in Sects. "Controllability" and "Stabilization" for linear evolution equations. Applications to the heat and wave equations will be discussed in the same sections.

On the other hand, *optimal control problems* require analyzing the typical issues of optimizations: existence results, necessary conditions for optimality, sufficient conditions, robustness. Here, the typical problem that has been successfully studied is the Linear Quadratic Regulator that will be discussed in Sect. "Linear Quadratic Optimal Control".

Control problems for nonlinear partial differential equations are extremely interesting but harder to deal with, so the literature is less rich in results and techniques. Nevertheless, among the problems that received great attention are those of fluid dynamics, specifically the *Euler equations*

$$\partial_t u + (u \cdot \nabla) u + \nabla p = 0$$

and the *Navier–Stokes equations*

$$\partial_t u - \mu \Delta u + (u \cdot \nabla) u + \nabla p = 0$$

subject to a *boundary control* and to the incompressibility condition $\operatorname{div} u = 0$.

Controllability

We now proceed to introduce the main notions of controllability for the evolution equation (7). Later on in this section we will give interpretations for the heat and wave equations.

In a given Hilbert space H, with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, let

$$A : D(A) \subset H \to H$$

be the *infinitesimal generator* of a *strongly continuous semigroup* e^{tA}, $t \geq 0$, of bounded linear operators on X. Intuitively, this amounts to saying that $u(t) \doteq e^{tA} u_0$ is the unique solution of the Cauchy problem

$$\begin{cases} u'(t) = Au(t) & t \geq 0 \\ u(0) = u_0, \end{cases}$$

in the classical sense for $u_0 \in D(A)$, and in a suitable generalized sense for all $u_0 \in H$. Necessary and sufficient conditions in order for an unbounded operator A to be the infinitesimal generator of a strongly continuous semigroup are given by the celebrated Hille–Yosida Theorem, see, e. g. [99] and [55].

Abstract Evolution Equations

Let F be another Hilbert space (with scalar product and norm denoted by the same symbols as for H), the so-called *control space*, and let $B\colon F \to H$ be a linear operator, that we will assume to be bounded for the time being. Then, given $T > 0$ and $u_0 \in H$, for all $f \in L^2(0,T;F)$ the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + Bf(t) & t \geq 0 \\ u(0) = u_0 \end{cases} \tag{12}$$

has a unique *mild solution* $u_f \in C([0,T];H)$ given by

$$u_f(t) = \mathrm{e}^{tA}u_0 + \int_0^t \mathrm{e}^{(t-s)A}Bf(s) \quad \forall t \geq 0 \tag{13}$$

Note 1 Boundary control problems can be reduced to the same abstract form as above. In this case, however, B in (12) turns out to be an unbounded operator related to suitable fractional powers of $-A$, see, e. g., [22].

For any $t \geq 0$ let us denote by $\Lambda_t\colon L^2(0,t;F) \to H$ the bounded linear operator

$$\Lambda_t f = \int_0^t \mathrm{e}^{(t-s)A}Bf(s)\,\mathrm{d}s \quad \forall f \in L^2(0,t;F)\,. \tag{14}$$

The *attainable* (or *reachable*) set from u_0 at time t, $\mathcal{A}(u_0,t)$ is the set of all points in H of the form $u_f(t)$ for some control function f, that is

$$\mathcal{A}(u_0,t) \doteq \mathrm{e}^{tA}u_0 + \Lambda_t L^2(0,t;F)\,.$$

We introduce below the main notions of controllability for (7). Let $T > 0$.

Definition 1 System (7) is said to be:

- **exactly controllable** in time T if $\mathcal{A}(u_0,T) = H$ for all $u_0 \in H$, that is, if for all $u_0, u_1 \in H$ there is a control function $f \in L^2(0,T;F)$ such that $u_f(T) = u_1$;
- **null controllable** in time T if $0 \in \mathcal{A}(u_0,T)$ for all $u_0 \in H$, that is, if for all $u_0 \in H$ there is a control function $f \in L^2(0,T;F)$ such that $u_f(T) = 0$;
- **approximately controllable** in time T if $\mathcal{A}(u_0,T)$ is dense in H for all $u_0 \in H$, that is, if for all $u_0, u_1 \in H$ and for any $\varepsilon > 0$ there is a control function $f \in L^2(0,T;F)$ such that $\|u_f(T) - u_1\| < \varepsilon$.

Clearly, if a system is exactly controllable in time T, then it is also null and approximately controllable in time T. Although these last two notions of controllability are strictly weaker than strong controllability, for specific problems—like when A generates a strongly continuous group—some of them may coincide.

Since controllability properties concern, ultimately, the range of the linear operator Λ_T defined in (14), it is not surprising that they can be characterized in terms of the adjoint operator $\Lambda_T^*\colon H \to L^2(0,T;F)$, which is defined by

$$\int_0^T \langle \Lambda_T^* u(s), f(s)\rangle\,\mathrm{d}s = \langle u_0, \Lambda_T f\rangle \qquad \forall u \in H\,,\ \forall f \in L^2(0,T;F)\,.$$

Such a characterization is the object of the following theorem. Notice that the above identity and (14) yield

$$\Lambda_T^* u(s) = B^*\,\mathrm{e}^{(T-s)A^*}u \quad \forall s \in [0,T]\,.$$

Theorem 1 *System (7) is:*

- ***exactly controllable*** *in time T if and only if there is a constant $C > 0$ such that*
$$\int_0^T \left\|B^*\,\mathrm{e}^{tA^*}u\right\|^2 \mathrm{d}t \geq C\|u\|^2 \quad \forall u \in H\,; \tag{15}$$
- ***null controllable*** *in time T if and only if there is a constant $C > 0$ such that*
$$\int_0^T \left\|B^*\,\mathrm{e}^{tA^*}u\right\|^2 \mathrm{d}t \geq C\left\|\mathrm{e}^{TA^*}u\right\|^2 \quad \forall u \in H\,; \tag{16}$$
- ***approximately controllable*** *in time T if and only if, for every $u \in H$,*
$$B^*\,\mathrm{e}^{tA^*}u = 0 \quad t \in [0,T] \ \text{a.e.} \implies u = 0\,. \tag{17}$$

To benefit the reader who is more familiar with optimization theory than abstract functional analysis, let us explain, by a variational argument, why estimate (16) implies null controllability. Consider, for every $\varepsilon > 0$, the penalized problem

$$\min\left\{J_\varepsilon(f) \,:\, f \in L^2(0,T;H)\right\},$$

where

$$J_\varepsilon(f) = \frac{1}{2}\int_0^T \|f(t)\|^2\,\mathrm{d}t + \frac{1}{2\varepsilon}\|u_f(T)\|^2 \qquad \forall f \in L^2(0,T;H)\,.$$

Since J_ε is strictly convex, it admits a unique minimum point f_ε. Set $u_\varepsilon = u_{f_\varepsilon}$. Recalling (13) we have, By Fermat's rule,

$$0 = J'_\varepsilon(f_\varepsilon)g = \int_0^T \langle f_\varepsilon(t), g(t)\rangle\,\mathrm{d}t + \frac{1}{\varepsilon}\langle u_\varepsilon(T), \Lambda_T g\rangle \quad \forall g \in L^2(0,T;H)\,. \tag{18}$$

Therefore, passing to the adjoint of Λ_T,

$$\int_0^T \left\langle f_\varepsilon(t) + \frac{1}{\varepsilon}\big(\Lambda_T^* u_\varepsilon(T)\big)(t), g(t)\right\rangle \mathrm{d}t = 0 \qquad \forall g \in L^2(0,T;H)\,,$$

whence, owing to (14),

$$f_\varepsilon(t) = -\frac{1}{\varepsilon}\big(\Lambda_T^* u_\varepsilon(T)\big)(t) = -B^* v_\varepsilon(t) \qquad \forall t \in [0,T]\,, \tag{19}$$

where $v_\varepsilon(t) \doteq \frac{1}{\varepsilon}\,\mathrm{e}^{(T-t)A^*} u_\varepsilon(T)$ is the solution of the *dual problem*

$$\begin{cases} v' + A^* v = 0 & t \in [0,T] \\ v(T) = \frac{1}{\varepsilon} u_\varepsilon(T)\,. \end{cases}$$

It turns out that

$$\frac{1}{2}\int_0^T \|f_\varepsilon(t)\|^2\,\mathrm{d}t + \frac{1}{\varepsilon}\|u_\varepsilon(T)\|^2 \le C\|u_0\|^2 \qquad \forall \varepsilon > 0 \tag{20}$$

for some positive constant C. Indeed, observe that, in view of (19),

$$\begin{cases} \big\langle u_\varepsilon' - Au_\varepsilon + BB^* v_\varepsilon\,,\, v_\varepsilon\big\rangle = 0\,, & u_\varepsilon(0) = u_0 \\ \big\langle v_\varepsilon' + A^* v_\varepsilon\,,\, u_\varepsilon\big\rangle = 0\,, & v_\varepsilon(T) = \frac{1}{\varepsilon} u_\varepsilon(T)\,. \end{cases}$$

So,

$$\int_0^T \left[\frac{\mathrm{d}}{\mathrm{d}t}\langle u_\varepsilon\,,\, v_\varepsilon\rangle + \left\|B^* v_\varepsilon\right\|^2\right] dt = 0\,,$$

whence

$$\frac{1}{\varepsilon}\|u_\varepsilon(T)\|^2 + \int_0^T \left\|B^* v_\varepsilon\right\|^2\,\mathrm{d}t = \langle u_0\,,\, v_\varepsilon(0)\rangle\,. \tag{21}$$

Now, apply estimate (16) with $u = \frac{u_\varepsilon(T)}{\varepsilon}$ and note that $v_\varepsilon(T-t) = \mathrm{e}^{tA^*}\frac{u_\varepsilon(T)}{\varepsilon}$ to obtain

$$\int_0^T \left\|B^* v_\varepsilon(t)\right\|^2\,\mathrm{d}t \ge C\,\|v_\varepsilon(0)\|^2$$

for some positive constant C. Hence, (20) follows from (21) and (19).

Finally, from (20) one deduces the existence of a weakly convergent subsequence f_{ε_j} in $L^2(0,T;F)$. Then, called f_0 the weak limit of f_{ε_j}, $u_{\varepsilon_j}(t) \to u_{f_0}(t)$ for all $t \in [0,T]$. So, owing to (20), $u_{f_0}(T) = 0$.

Heat Equation

It is not hard to see that the heat equation (9) with Dirichlet boundary conditions (2) fails to be exactly controllable. On the other hand, one can show that it is null controllable in any time $T > 0$, hence approximately controllable. Let ω be an open subset of Ω such that $\overline{\omega} \subset \Omega$.

Taking

$$H = L^2(\Omega) = F\,, \quad Bf = \mathbb{1}_\omega f \quad \forall f \in L^2(\Omega)$$

and A as in (8), one obtains that, for any $u_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$, the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta u + \mathbb{1}_\omega f & \text{in } Q_T \\ u = 0 & \text{on } \Sigma_T \\ u(0,x) = u_0(x) & x \in \Omega \end{cases} \tag{22}$$

has a unique mild solution $u_f \in C([0,T];L^2(\Omega))$. Moreover, multiplying both sides of equation (9) by u and integrating by parts, it is easy to see that

$$\partial_{x_i} u \in L^2(Q_T) \quad \forall i = 1,\dots,n\,. \tag{23}$$

Notice that the above property already suffices to explain why the heat equation cannot be exactly controllable: it is impossible to attain a state $u_1 \in L^2(\Omega)$ which fails to satisfy (23).

On the other hand, null controllability holds true in any positive time.

Theorem 2 *Let $T > 0$ and let ω be an open subset of Ω such that $\overline{\omega} \subset \Omega$. Then the heat equation (9) with homogeneous Dirichlet boundary conditions is null controllable in time T, i. e., for every initial condition $u_0 \in L^2(\Omega)$ there is a control function $f \in L^2(Q_T)$ such that the solution u_f of (22) satisfies $u_f(T,\cdot) \equiv 0$. Moreover,*

$$\iint_{Q_T} |f|^2\,\mathrm{d}x\,\mathrm{d}t \le C_T \int_\Omega |u_0|^2\,\mathrm{d}x$$

for some positive constant C_T.

The above property is a consequence of the abstract result in Theorem 1 and of concrete estimates for solutions of parabolic equations. Indeed, in order to apply Theorem 1 one has to translate (16) into an estimate for the heat operator. Now, observing that both A and B are self-adjoint, one promptly realizes that (16) reduces to

$$\int_0^T \int_\omega |v(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \ge C \int_\Omega |v(T,x)|^2\,\mathrm{d}x \tag{24}$$

for every solution v of the problem

$$\begin{cases} \partial_t v = \Delta v & \text{in } Q_T \\ v = 0 & \text{on } \Sigma_T\,. \end{cases} \tag{25}$$

Estimate (24) is called an *observability inequality* for the heat operator for obvious reasons: problem (25) is not well-posed since the initial condition is missing. Nevertheless, if, "observing" a solution v of such a problem on the "small" cylinder $(0, T) \times \omega$, you find that it vanishes, then you can conclude that $v(T, \cdot) \equiv 0$ in the whole domain Ω. Thus, $v(0, \cdot) \equiv 0$ by backward uniqueness.

In conclusion, as elegant as the abstract approach to null controllability may be, one is confronted by the difficult task of proving observability estimates. In fact, for the heat operator there are several ways to prove inequality (24). One of the most powerful, basically due to Fursikov and Imanuvilov [65], relies on global *Carleman estimates*. Compared to other methods that can be used to derive observability, such a technique has the advantage of applying to second order parabolic operators with variable coefficients, as well as to more general operators.

Global Carleman estimates are a priori estimates in weighted norms for solutions of the problem

$$\begin{cases} \partial_t v = \Delta v + f & \text{in } Q_T \\ v = 0 & \text{on } \Sigma_T \,. \end{cases} \qquad (26)$$

regardless of initial conditions. The weight function is usually of the form

$$\psi_r(t,x) \doteq \theta(t)\left(e^{2r\|\phi\|_{\infty,\Omega}} - e^{r\phi(x)}\right) \quad (t,x) \in Q_T\,, \quad (27)$$

where r is a positive constant, ϕ is a given function in $C^2(\overline{\Omega})$ such that

$$\nabla\phi(x) \neq 0 \quad \forall x \in \overline{\Omega}\,, \qquad (28)$$

and

$$\theta(t) \doteq \frac{1}{t(T-t)} \qquad 0 < t < T\,.$$

Note that

$$\theta > 0\,, \quad \theta(t) \to \infty \quad t \to 0\,, T$$

$$\psi_r > 0\,, \quad \psi_r(t,x) \to \infty \quad t \downarrow 0\,, t \uparrow T\,.$$

Using the above notations, a typical global Carleman estimate for the heat operator is the following result obtained in [65]. Let us denote by $\nu(x)$ the outword unit normal to Γ at a point $x \in \Gamma$, and by

$$\frac{\partial\phi}{\partial\nu}(x) = \nabla\phi(x) \cdot \nu(x)$$

the normal derivative of ϕ at x.

Theorem 3 *Let Ω be a bounded domain of $\mathbb{R}^n$ with boundary of class C^2, let $f \in L^2(Q_T)$, and let ϕ be a function satisfying (28). Let v be a solution of (26). Then there are positive constants r, s_0 and C such that, for any $s > s_0$,*

$$\begin{aligned} s^3 &\iint_{Q_T} \theta^3(t)|v(t,x)|^2 e^{-2s\psi_r}\,dx\,dt \\ &\leq C \iint_{Q_T} |f(t,x)|^2 e^{-2s\psi_r}\,dx\,dt \\ &+ Cs \int_0^T \theta(t)\,dt \\ &\times \int_\Gamma \frac{\partial\phi}{\partial\nu}(x) \left|\frac{\partial v}{\partial\nu}(t,x)\right|^2 e^{-2s\psi_r}\,d\mathcal{H}^{n-1}(x) \end{aligned} \qquad (29)$$

It is worth underlying that, thanks to the singular behavior of θ near 0 and T, the above result is independent of the initial value of v. Therefore, it can be applied, indifferently, to any solution of (26) as well as to any solution of the *backward problem*

$$\begin{cases} \partial_t v + \Delta v = f & \text{in } Q_T \\ v = 0 & \text{on } \Sigma_T \,. \end{cases}$$

Moreover, inequality (29) can be completed adding first and second order terms to its right-hand side, each with its own adapted power of s and θ.

Instead of trying to sketch the proof of Theorem 3, which would go beyond the scopes of this note, it is interesting to explain how it can be used to recover the observability inequality (24), which is what is needed to show that the heat equation is null controllable. The reasoning—not completely straightforward—is based on the following topological lemma, proved in [65].

Lemma 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary Γ of class C^k, for some $k \geq 2$, and let $\omega \subset \Omega$ be an open set such that $\overline{\omega} \subset \Omega$.*

Then there is function $\phi \in C^k(\overline{\Omega})$ such that

$$\begin{cases} (i) & \phi(x) = 0 \quad \text{and} \quad \frac{\partial\phi}{\partial\nu}(x) < 0 \quad \forall x \in \Gamma \\ (ii) & \{x \in \Omega \,|\, \nabla\phi(x) = 0\} \subset \omega\,. \end{cases} \qquad (30)$$

Now, given a solution v of (25) and an open set ω such that $\overline{\omega} \subset \Omega$, let $\omega' \subset\subset \omega'' \subset\subset \omega$ be subdomains with smooth boundary. Then the above lemma ensures the existence of a function ϕ such that

$$\{x \in \Omega \,|\, \nabla\phi(x) = 0\} \subset \omega'\,.$$

"Localizing" problem (25) onto $\Omega' \doteq \Omega \setminus \omega'$ by a cutoff function $\eta \in C^{\infty}(\mathbb{R}^n)$ such that

$$0 \le \eta \le 1\,, \quad \eta \equiv 1 \quad \text{on} \quad \mathbb{R}^n \setminus \omega''\,, \quad \eta \equiv 0 \quad \text{on} \quad \omega'\,,$$

that is, taking $w = \eta v$, gives

$$\begin{cases} \partial_t w = \Delta w + h & \text{in } Q'_T \doteq (0,T) \times \Omega' \\ w(t,\cdot) = 0 & \text{on } \partial\Omega' = \partial\Omega \cup \partial\omega'\,, \end{cases} \tag{31}$$

with $h := -v\Delta\eta + 2\nabla\eta \cdot \nabla u$. Since $\nabla\phi \neq 0$ on Ω', Theorem 3 can be applied to w on Q'_T to obtain

$$\begin{aligned} & s^3 \iint_{Q'_T} \theta^3 |w|^2 \mathrm{e}^{-2s\psi_r}\,\mathrm{d}x\,\mathrm{d}t \\ & \le C \iint_{Q'_T} |h|^2 \mathrm{e}^{-2s\psi_r}\,\mathrm{d}x\,\mathrm{d}t \\ & + Cs \int_0^T \theta\,\mathrm{d}t \int_{\Gamma} \frac{\partial\phi}{\partial\nu} \left|\frac{\partial w}{\partial\nu}\right|^2 \mathrm{e}^{-2s\psi_r}\, d\mathcal{H}^{n-1} \\ & + Cs \int_0^T \theta\,\mathrm{d}t \int_{\partial\omega'} \frac{\partial\phi}{\partial\nu} \left|\frac{\partial w}{\partial\nu}\right|^2 \mathrm{e}^{-2s\psi_r}\, d\mathcal{H}^{n-1} \\ & \le C \iint_{Q'_T} |h|^2 \mathrm{e}^{-2s\psi_r}\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for s sufficiently large. On the other hand, for any $0 < T_0 < T_1 < T$,

$$\begin{aligned} & s^3 \iint_{Q'_T} \theta^3 |w|^2 \mathrm{e}^{-2s\psi_r}\,\mathrm{d}x\,\mathrm{d}t \\ & \ge s^3 \int_{T_0}^{T_1} \mathrm{d}t \int_{\Omega\setminus\omega} \theta^3 |w|^2 \mathrm{e}^{-2s\psi_r}\,\mathrm{d}x\,\mathrm{d}t \\ & \ge \int_{T_0}^{T_1} \mathrm{d}t \int_{\Omega\setminus\omega} |v|^2\,\mathrm{d}x \end{aligned}$$

Therefore, recalling the definition of h,

$$\begin{aligned} & \int_{T_0}^{T_1} \mathrm{d}t \int_{\Omega\setminus\omega} |v|^2\,\mathrm{d}x \le C \iint_{Q'_T} |h|^2 \mathrm{e}^{-2s\psi_r}\,\mathrm{d}x\,\mathrm{d}t \\ & \le C \int_0^T \mathrm{d}t \int_{\omega''\setminus\omega'} \left[|\nabla^2\eta|^2 v^2 + |\nabla\eta|^2 |\nabla v|^2\right] \mathrm{e}^{-2s\psi_r}\,\mathrm{d}x \\ & \le C \int_0^T \mathrm{d}t \int_{\omega} |v|^2\,\mathrm{d}x + C \int_0^T \mathrm{d}t \\ & \qquad \times \int_{\omega''\setminus\omega'} |\nabla v|^2 \mathrm{e}^{-2s\psi_r}\,\mathrm{d}x\,. \end{aligned}$$

Now, fix $T_0 = T/3\,,\ T_1 = 2T/3$ and use Caccioppoli's inequality (a well-known estimate for solution of elliptic and parabolic PDE's)

$$\begin{aligned} & \int_0^T \mathrm{d}t \int_{\omega''\setminus\omega'} |\nabla v|^2 \mathrm{e}^{-2s\psi_r}\,\mathrm{d}x \\ & \qquad \le C \int_0^T \mathrm{d}t \int_{\omega} |v|^2 \mathrm{e}^{-2s\psi_r}\,\mathrm{d}x\,, \end{aligned}$$

to conclude that

$$\int_{T/3}^{2T/3} \mathrm{d}t \int_{\Omega\setminus\omega} |v|^2\,\mathrm{d}x \le C \int_0^T \mathrm{d}t \int_{\omega} |v|^2\,\mathrm{d}x$$

or

$$\int_{T/3}^{2T/3} \mathrm{d}t \int_{\Omega} |v|^2\,\mathrm{d}x \le (1+C) \int_0^T \mathrm{d}t \int_{\omega} |v|^2\,\mathrm{d}x$$

for some constant C. Then, the dissipativity of the heat operator (that is, the fact that $\int_\Omega |v(t,x)|^2\,\mathrm{d}x$ is decreasing with respect to t) implies that

$$\begin{aligned} \int_{\Omega} v^2(T,x)\,\mathrm{d}x & \le \frac{3}{T} \int_{T/3}^{2T/3} \mathrm{d}t \int_{\Omega} v^2(t,x)\,\mathrm{d}x \\ & \le (1+C)\frac{3}{T} \int_0^T \mathrm{d}t \int_{\omega} v^2(t,x)\,\mathrm{d}x\,, \end{aligned}$$

which is exactly (24).

Wave Equation

Compared to the heat equation, the wave equation (5) exhibits a quite different behavior from the point of view of exact controllability. Indeed, on the one hand, there is no obstruction to exact controllability since no regularizing effect is connected with wave propagation. On the other hand, due to the finite speed of propagation, exact controllability cannot be expected to hold true in arbitrary time, as null controllability does for the heat equation.

In fact, a typical result that holds true for the wave equation is the following, where a boundary control of Dirichlet type acts on a part $\Gamma_1 \subset \Gamma$, while homogeneous boundary conditions are imposed on $\Gamma_0 = \Gamma \setminus \Gamma_1$:

$$\begin{cases} \partial_t^2 u = \Delta u & \text{in } Q_T \\ u = f \mathbb{1}_{\Gamma_1} & \text{on } \Sigma_T \\ u(0,x) = u_0(x)\,,\, \partial_t u(0,x) = u_1(x) & x \in \Omega \end{cases} \tag{32}$$

Observe that problem (32) is well-posed taking

$$\begin{aligned} u_0 & \in L^2(\Omega)\,, \quad u_1 \in H^{-1}(\Omega) \\ f & \in L^2(0,T; L^2(\Gamma)) \\ u & \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))\,. \end{aligned}$$

Theorem 4 *Let Ω be a bounded domain of $\mathbb{R}^n$ with boundary of class C^2 and suppose that, for some point $x_0 \in \mathbb{R}^n$,*

$$\begin{cases} (x-x_0)\cdot\nu(x) > 0 & \forall x \in \Gamma_1 \\ (x-x_0)\cdot\nu(x) \leq 0 & \forall x \in \Gamma_0 \,. \end{cases}$$

Let

$$R = \sup_{x\in\Omega} |x - x_0| \,.$$

If $T > 2R$, then, for all $(u_0, u_1), (v_0, v_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there is a control function $f \in L^2(0,T; L^2(\Gamma))$ such that the solution u_f of (32) satisfies

$$u_f(T,x) = v_0(x)\,, \quad \partial_t u_f(T,x) = v_1(x)\,.$$

As we saw for abstract evolution equations, the above exact controllability property is proved to be equivalent to an observability estimate for the dual homogeneous problem using, for instance, the Hilbert Uniqueness Method (HUM) by J.-L. Lions [86].

Bibliographical Comments

The literature on controllability of parabolic equations and related topics is so huge, that no attempt to provide a comprehensive account of it would fit within the scopes of this note. So, the following comments have to be taken as a first hint for the interested reader to pursue further bibliographical research.

The theory of exact controllability for parabolic equations was initiated by the seminal paper [58] by Fattorini and Russell. Since then, it has experienced an enormous development. Similarly, the multiplier method to obtain observability inequalities for the wave equation was developed in [17,73,74,77,86]. Some fundamental early contributions were surveyed by Russell [108]. The next essential progress was made in the work by Lebeau and Robbiano [83] and then by Fursikov and Imanuvilov in a series of papers. In [65] one can find an introduction to global Carleman estimates, as well as applications to the controllability of several ODE's. In particular, the presentation of this paper as for observability inequalities and Carleman estimates for the heat operator is inspired by the last monograph. General perspectives for the understanding of global Carleman estimates and their applications to unique continuation and control problems for PDE's can be found in the works by Tataru [113,114,115,116]. Usually, the above approach requires coefficients to be sufficiently smooth. Recently, however, interesting adaptations of Carleman estimates to parabolic operators with discontinuous coefficients have been obtained in [21,82].

More recently, interest has focussed on control problems for nonlinear parabolic equations. Different approaches to controllability problems have been proposed in [57] and [44]. Then, null and approximate controllability results have been improved by Fernandez–Cara and Zuazua [61,62]. Techniques to produce insensitizing controls have been developed in [117]. These techniques have been successfully applied to the study of Navier–Stokes equations by several authors, see e. g. [63].

Fortunately, several excellent monographs are now available to help introduce the reader to this subject. For instance, the monograph by Zabczyk [121] could serve as a clean introduction to control and stabilization for finite- and infinite-dimensional systems. Moreover, [22,50,51], as well as [80,81] develop all the basic concepts of control and system theory for distributed parameter systems with special emphasis on abstract formulation. Specific references for the controllability of the wave equation by HUM can be found in [86] and [74]. More recent results related to series expansion and Ingham type methods can be found in [75]. For the control of Navier–Stokes equations the reader is referred to [64], as well as to the book by Coron [43], which contains an extremely rich collection of classical results and modern developments.

Stabilization

Stabilization of flexible structures such as beams, plates, up to antennas of satellites, or of fluids as, for instance, in aeronautics, is an important part of CT. In this approach, one wants either to derive feedback laws that will allow the system to autoregulate once they are implemented, or study the asymptotic behavior of the stabilized system i. e. determine whether convergence toward equilibrium states as times goes to infinity holds, determine its speed of convergence if necessary or study how many feedback controls are required in case of coupled systems.

Different mathematical tools have been introduced to handle such questions in the context of ODE's and then of PDE's. Stabilization of ODE's goes back to the work of Lyapunov and Lasalle. The important property is that trajectories decay along Lyapunov functions. If trajectories are relatively compact in appropriate spaces and the system is autonomous, then one can prove that trajectories converge to equilibria asymptotically. However, the construction of Lyapunov functions is not easy, in general.

This section will be concerned with some aspects of the stabilization of second order hyperbolic equations, our model problem being represented by the wave equation with distributed damping

$$\begin{cases} \partial_{tt}u - \Delta u + a(x)u_t = 0 & \text{in } \Omega \times \mathbb{R}\,, \\ u = 0 & \text{on } \Sigma = (0,\infty) \times \Gamma \\ (u, \partial_t u)(0) = (u^0, u^1) & \text{on } \Omega\,, \end{cases} \quad (33)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary Γ. For $n = 2$, $u(t, x)$ represents the displacement of point x of the membrane at time t. Therefore, equation (33) describes an elastic system. The energy of such a system is given by

$$E(t) = \frac{1}{2}\int_\Omega \left[|u_t(t,x)|^2 + |\nabla u(t,x)|^2\right] \mathrm{d}x\,.$$

When $a \geq 0$, the *feedback* term $a(x)u_t$ models friction: it produces a loss of energy through a dissipation phenomenon. More precisely, multiplying the equation in (33) by u_t and integrating by parts on Ω, it follows that

$$E'(t) = -\int_\Omega a(x)|u_t|^2\,\mathrm{d}x \leq 0\,, \quad \forall t \geq 0\,. \quad (34)$$

On the other hand, if $a \equiv 0$, then the system is *conservative*, i.e., $E(t) = E(0)$ for all $t \geq 0$.

Another well-investigated stabilization problem for the wave equation is when the feedback is localized on a part Γ_0 of the boundary Γ, that is,

$$\begin{cases} \partial_{tt}u - \Delta u = 0 & \text{in } \Omega \times \mathbb{R} \\ \frac{\partial u}{\partial \nu} + u_t = 0 & \text{on } \Sigma_0 = (0,\infty) \times \Gamma_0 \\ u = 0 & \text{on } \Sigma_1 = (0,\infty) \times (\Gamma \setminus \Gamma_0) \\ (u, \partial_t u)(0) = (u^0, u^1) & \end{cases} \quad (35)$$

In this case, the dissipation relation (34) takes the form

$$E'(t) = -\int_{\Gamma_0} |u_t|^2\,\mathrm{d}\mathcal{H}^{n-1} \leq 0\,, \quad \forall t \geq 0\,.$$

In many a situation—such as to improve the quality of an acoustic hall—one seeks to reduce vibrations to a minimum: this is why stabilization is an important issue in CT. We note that the above system has a unique stationary solution—or, *equilibrium*—given by $u \equiv 0$. Stabilization theory studies all questions related to the convergence of solutions to such an equilibrium: existence of the limit, rate of convergence, different effects of nonlinearities in both displacement and velocity, effects of geometry, coupled systems, damping effects due to memory in viscoelastic materials, and so on.

System (33) is said to be:

- **strongly stable** if $E(t) \to 0$ as $t \to \infty$;
- **(uniformly) exponentially stable** if $E(t) \leq C\,\mathrm{e}^{-\alpha t}E(0)$ for all $t \geq 0$ and some constants $\alpha > 0$ and $C \geq 0$, independent of u^0, u^1.

This note will focus on some of the above issues, such as geometrical aspects, nonlinear damping, indirect damping for coupled systems and memory damping.

Geometrical Aspects

A well-known property of the wave equation is the so-called *finite speed of propagation*, which means that, if the initial conditions u^0, u^1 have compact support, then the support of $u(t, \cdot)$ evolves in time at a finite speed. This explains why, for the wave equation, the geometry of Ω plays an essential role in all the issues related to control and stabilization.

The size and localization of the region in which the feedback is active is of great importance. In this paper such a region, denoted by ω, is taken as a subset of Ω of positive Lebesgue measure. More precisely, a is assumed to be continuous on $\overline{\Omega}$ and such that

$$a \geq 0 \quad \text{on} \quad \Omega \quad \text{and} \quad a \geq a_0 \quad \text{on} \quad \omega\,, \quad (36)$$

for some constant $a_0 > 0$. In this case, the feedback is said to be *distributed*. Moreover, it is said to be *globally* distributed if $\omega = \Omega$ and *locally* distributed if $\Omega \setminus \omega$ has positive Lebesgue measure.

Two main methods have been used or developed to study stabilization, namely the *multiplier method* and *microlocal analysis*. The one that gives the sharpest results is based on microlocal analysis. It goes back to the work of Bardos, Lebeau and Rauch [17], giving geodesics sufficient conditions on the region of active control for exact controllability to hold. These conditions say that each ray of geometric optics should meet the control region. Burq and Gérard [25] showed that these results hold under weaker regularity assumptions on the domain and coefficients of the operators (see also [26,27]). These geodesics conditions are not explicit, in general, but they allow to get decay estimates of the energy under very general hypotheses.

The multiplier method is an explicit method, based on energy estimates, to derive decay rates (as well as observability and exact controllability results). For boundary control and stabilization problems it was developed in the works of several authors, such as Ho [38,73], J.-L. Lions [86], Lasiecka–Triggiani, Komornik–Zuazua [76], and many others. Zuazua [123] gave an explicit geometric condition on ω for a semilinear wave equation subject to a locally distributed damping. Such a condition was then relaxed K. Liu [87] (see also [93]) who introduced the so-called piecewise multiplier method. Lasiecka and Triggiani [80,81] introduced a sharp trace regularity method which allow to estimate boundary terms in energy estimates. There also exist intermediate results between the geodesics conditions of Bardos–Lebeau–Rauch and the multiplier method, obtained by Miller [95] using differentiable escape functions.

Zuazua's multiplier geometric condition can be described as follows. If a subset O of $\overline{\Omega}$ is given, one can define an ε-neighborhood of O in $\overline{\Omega}$ as the subset of points of Ω which are at distance at most ε of O. Zuazua proved that if the set ω is such that there exists a point $x_0 \in \mathbb{R}^n$—an *observation* point—for which ω contains an ε-neighborhood of $\Gamma(x^0) = \{x \in \partial\Omega\,,\ (x - x^0)\cdot\nu(x) \geq 0\}$, then the energy decays exponentially. In this note, we refer to this condition as (MGC).

If a vanishes for instance in a neighborhood of the two poles of a ball Ω in $\mathbb{R}^n$, one cannot find an observation point x_0 such that (MGC) holds. K. Liu [87] (see also [93]) introduced a piecewise multiplier method which allows to choose several observation points, and therefore to handle the above case.

Introduce disjoint lipschitzian domains Ω_j of Ω, $j = 1, \ldots, J$, and observation points $x^j \in \mathbb{R}^N$, $j = 1, \ldots, J$ and define

$$\gamma_j(x^j) = \{x \in \partial\Omega_j\,,\ (x - x^j)\cdot\nu_j(x) \geq 0\}$$

Here ν_j stands for the unit outward normal vector to the boundary of Ω_j. Then the piecewise multiplier geometrical condition for ω is:

$$\omega \supset \mathcal{N}_\varepsilon\left(\cup_{j=1}^{J}\gamma_j(x^j) \cup \left(\Omega \backslash \cup_{j=1}^{J}\Omega_j\right)\right) \quad \text{(PWMGC)}$$

It will be denoted by (PWMGC) condition in the sequel.

Assume now that a vanishes in a neighborhood of the two poles of a ball in $\mathbb{R}^n$. Then, one can choose two subsets Ω_1 and Ω_2 containing, respectively, the two regions where a vanishes and apply the piecewise multiplier method with $J = 2$ and with the appropriate choices of two observation points and ε. The multiplier method consists of integrating by parts expressions of the form

$$\int_t^T\int_\Omega \left(\partial_t^2 u - \Delta u + a(x)u_t\right) Mu \,\mathrm{d}x\,\mathrm{d}t = 0 \qquad \forall 0 \leq t \leq T\,,$$

where u stands for a (strong) solution of (33), with an appropriate choice of Mu. Multipliers have generally the form

$$Mu = (m(x)\cdot\nabla u + c\,u)\,\psi(x)\,,$$

where m depends on the observation points and ψ is a cut-off function. Other multipliers of the form $Mu = \Delta^{-1}(\beta u)$, where β is a cut-off function and Δ^{-1} is the inverse of the Laplacian operator with homogeneous Dirichlet boundary conditions, have also be used.

The geometric conditions (MGC) or (PWMGC) serve to bound above by zero terms which cannot be controlled otherwise. One can then prove that the energy satisfies an estimate of the form

$$\int_t^T E(s)\,\mathrm{d}s \leq cE(t) + \int_t^T\left(\int_\Omega a(x)|u_t|^2 + \int_\omega |u_t|^2\right)\mathrm{d}s \qquad \forall t \geq 0\,. \quad (37)$$

Once this estimate is proved, one can use the dissipation relation to prove that the energy satisfies integral inequalities of Gronwall type. This is the subject of the next section.

Decay Rates, Integral Inequalities and Lyapunov Techniques

The Linear Feedback Case Using the dissipation relation (34), one has

$$\int_t^T\int_\Omega a|u_t|^2\,\mathrm{d}x\,\mathrm{d}s \leq \int_t^T -E'(s)\,\mathrm{d}s \leq E(t) \qquad \forall\, 0 \leq t \leq T\,.$$

On the other hand, thanks to assumption (36) on a

$$\int_t^T\int_\omega u_t^2\,\mathrm{d}x\,\mathrm{d}s \leq \frac{1}{a_0}\int_t^T\int_\Omega a|u_t|^2\,\mathrm{d}x\,\mathrm{d}s \leq \frac{1}{a_0}E(t) \quad \forall 0 \leq t \leq T\,.$$

By the above inequalities and (37), E satisfies

$$\int_t^T E(s)\,\mathrm{d}s \leq cE(t)\,, \quad \forall 0 \leq t \leq T\,. \quad (38)$$

Since E is a nonincreasing function and thanks to this integral inequality, Haraux [71] (see also Komornik [74]) proved that E decays exponentially at infinity, that is

$$E(t) \leq E(0)\exp\left(1 - t/c\right), \quad \forall t \geq c\,. \quad (39)$$

This proof is as follows. Define

$$\phi(t) = \exp(t/c)\int_t^\infty E(s)\,\mathrm{d}s \quad \forall t \geq 0\,.$$

Thanks to (38) ϕ is nonincreasing on $[0,\infty)$, so that

$$\phi(t) \le \phi(0) = \int_0^{\infty} E(s)\,\mathrm{d}s\,.$$

Using once again (38) with $t = 0$ in this last inequality and the definition of ϕ, one has

$$\int_t^{\infty} E(s)\,\mathrm{d}s \le cE(0)\exp(-t/c) \quad \forall t \ge 0\,.$$

Since E is a nonnegative and nonincreasing function

$$cE(t) \le \int_{t-c}^{t} E(s)\,\mathrm{d}s \le \int_{t-c}^{\infty} E(s)\,\mathrm{d}s$$
$$\le cE(0)\exp(-(t-c)/c)\,,$$

so that (39) is proved. One can remark that for $t \le c$, $E(t) \le E(0) \le \exp\left(1 - t/c\right)$.

An alternative method is to introduce a modified (or perturbed) energy E_ε which is equivalent to the natural one for small values of the parameter ε as in Komornik and Zuazua [76]. Then one shows that this modified energy satisfies a differential Gronwall inequality so that it decays exponentially at infinity. The exponential decay of the natural energy follows then at once. In this case, the modified energy is indeed a Lyapunov function for the PDE. The natural energy cannot be in general such a Lyapunov function due to the finite speed of propagation (consider initial data which have compact support compactly embedded in $\Omega \backslash \omega$).

There are also very interesting approaches using the frequency domain approach, or spectral analysis such as developed by K. Liu [87] Z. Liu and S. Zheng [88]. In the sequel, we concentrate on the integral inequality method. This method has been generalized in several directions and we present in this note some results concerning extensions to

- nonlinear feedback
- indirect or single feedback for coupled system
- memory type feedbacks

Generalizations to Nonlinear Feedbacks Assume now that the feedback term $a(x)u_t$ in (33) is replaced by a nonlinear feedback $a(x)\rho(u_t)$ where ρ is a smooth, increasing function satisfying $v\rho(v) \ge 0$ for $v \in \mathbb{R}$, linear at ∞ and with polynomial growth close to zero, that is: $\rho(v) = |v|^p$ for $|v| \le 1$ where $p \in (1,\infty)$.

Assume moreover that ω satisfies Zuazua's multiplier geometric condition (MGC) or Liu's piecewise multiplier method (PWMGC). Then using multipliers of the space and time variables defined as $E(s)^{(p-1)/2}\,Mu(x)$ where $Mu(x)$ are multipliers of the form described in section 5.1 and integrating by parts expressions of the form

$$\int_t^T E(s)^{(p-1)/2}$$
$$\times \int_\Omega \left(\partial_t^2 u - \Delta u + a(x)\rho(u_t)\right) Mu(x)\,\mathrm{d}x\,\mathrm{d}s = 0\,,$$

one can prove that the energy E of solutions satisfies the following inequality for all $0 \le t \le T$

$$\int_t^T E^{(p+1)/2}(s)\,\mathrm{d}t$$
$$\le cE^{(p+1)/2}(t) + c\int_t^T E^{(p-1)/2}(s)$$
$$\times \left(\int_\Omega \rho(u_t)^2 + \int_\omega |u_t|^2\right)\,.$$

One can remark than an additional multiplicative weight in time depending on the energy has to be taken. This weight is $E^{(p-1)/2}$. Then as in the linear case, but in a more involved way, thanks to the dissipation relation

$$E'(t) = -\int_\Omega a(x)u_t\rho(u_t)\,, \qquad (40)$$

one can prove that E satisfies the following nonlinear integral inequality

$$\int_t^T E^{(p+1)/2}(s)\,\mathrm{d}s \le cE(t)\,, \quad \forall 0 \le t \le T\,.$$

Thanks to the fact that E is nonincreasing, a well-known result Komornik [74] shows that E is polynomially decaying, as $t^{-2/(p-1)}$ at infinity. The above type results have been obtained by many authors under weaker form (see also [40,41,71,98,122]).

Extensions to nonlinear feedbacks without growth conditions close to zero have been studied by Lasiecka and Tataru [78], Martinez [93], W. Liu and Zuazua [89], Eller Lagnese and Nicaise [56] and Alabau–Boussouira [5]. We present the results obtained in this last reference since they provide *optimal* decay rates.

The method is as follows. Define respectively the linear and nonlinear kinetic energies

$$\begin{cases} \int_\omega |u_t|^2\,\mathrm{d}x\,, \\ \int_\Omega |a(x)\rho(u_t)|^2\,\mathrm{d}x\,, \end{cases}$$

and use a weight function in time $f(E(s))$ which is to be determined later on in an optimal way. Integrating by parts expressions of the form

$$\int_t^T f(E(s)) \int_\Omega \left(\partial_t^2 u - \Delta u + a(x)\rho(u_t)\right) Mu(x)\, dx\, ds = 0\,,$$

one can prove that the energy E of solutions satisfies the following inequality for all $0 \le t \le T$

$$\int_t^T E(s) f(E(s))\, ds \le c f(E(t)) + c \int_t^T f(E(s)) \times \left(\int_\Omega |a(x)\rho(u_t)|^2 + \int_\omega |u_t|^2 \right). \quad (41)$$

The difficulty is to determine the optimal weight under general growth conditions on the feedback close to 0, in particular for cases for which the feedback decays to 0 faster than polynomials.

Assume now that the feedback satisfies

$$g(|v|) \le |\rho(v)| \le C g^{-1}(|v|)\,, \quad \forall |v| \le 1\,, \quad (42)$$

where g is continuously differentiable on $\mathbb{R}$ strictly increasing with $g(0) = 0$ and

$$\begin{cases} g \in C^2([0, r_0])\,, r_0 \text{ sufficiently small}\,, \\ H(.) = \sqrt{.}\, g(\sqrt{.}) \text{ is strictly convex on } \left[0, r_0^2\right], \\ g \text{ is odd}\,. \end{cases}$$

Moreover, ρ is assumed to have a linear growth with respect to the second variable at infinity. We define the *optimal* weight function f as follows.

We first extend H to a function $\hat{H}$ define on all $\mathbb{R}$

$$\hat{H}(x) = \begin{cases} H(x) \text{ if } x \in \left[0, r_0^2\right], \\ +\infty \text{ otherwise}\,, \end{cases}$$

then, define a function F as follows:

$$F(y) = \begin{cases} \dfrac{\hat{H}^*(y)}{y} & \text{if } y \in (0, +\infty)\,, \\ 0 & \text{if } y = 0\,, \end{cases}$$

where $\hat{H}^*$ stands for the convex conjugate of $\hat{H}$, that is

$$\hat{H}^*(y) = \sup_{x \in \mathbb{R}} \{x\, y - \hat{H}(x)\}\,.$$

Then the *optimal* weight function f is determined in the following way

$$f(s) = F^{-1}(s/2\beta) \quad s \in \left[0, 2\beta r_0^2\right),$$

where β is of the form $\max(\eta_1, \eta_2 E(0))$, η_1 and η_2 being explicit positive constants.

One can prove that the above formulas make sense, and in particular that F is invertible and smooth. More precisely, F is twice continuously differentiable strictly increasing, one-to-one function from $[0, +\infty)$ onto $[0, r_0^2)$. Note that since the feedback is supposed to be linear at infinity, if one wants to obtain results for general growth types of the feedback, one can assume convexity of H only in a neighborhood of 0.

One can prove from (41) that there exists an (explicit) $T_0 > 0$ such that for all initial data, E satisfies the following nonlinear integral inequality

$$\int_t^T E(s) f(E(s))\, ds \le T_0 E(t) \quad \forall 0 \le t \le T\,. \quad (43)$$

This inequality is proved thanks to convexity arguments as follows. Thanks to the convexity of $\hat{H}$, one can use Jensen's inequality and 42), so that

$$\int_{\Omega_t} |a(x)\rho(u_t)|^2\, dx \le \gamma_1(t) \hat{H}^{-1} \times \left(\frac{1}{\gamma_1(t)} \int_\Omega a(x) u_t \rho(u_t)\, dx \right)$$

In a similar way, one proves that

$$\int_{\omega_t} |u_t|^2\, dx \le \gamma_2(t) \hat{H}^{-1} \left(\frac{1}{\gamma_2(t)} \int_\Omega a(x) u_t \rho(u_t)\, dx \right)$$

where Ω_t and ω_t are time-dependent sets of respective Lebesgue measures $\gamma_1(t)$ and $\gamma_2(t)$ on which the velocity $u_t(t, x)$ is sufficiently small. Using the above two estimates, together with the linear growth of ρ at infinity, one proves

$$\int_t^T f(E(s)) \left(\int_\Omega |a(x)\rho(u_t)|^2 + \int_\omega |u_t|^2 \right) \le \int_t^T f(E(s)) \hat{H}^{-1} \left(\frac{1}{c} \int_\Omega a(x) u_t \rho(u_t)\, dx \right)$$

Using then Young's inequality, together with the dissipation relation (40) in the above inequality, one obtains

$$\int_t^T f(E(s)) \left(\int_\Omega |a(x)\rho(u_t)|^2 + \int_\omega |u_t|^2 \right) \le C_1 \int_t^T \hat{H}^\star \left(f(E(s)\right)\, ds + C_2 E(t)\,, \quad (44)$$

where $C_i > 0$ $i = 1, 2$ is a constant independent of the initial data. Using the dissipation relation (40) in the above inequality, this gives for all $0 \le t \le T$

Combining this last inequality with (41) give

$$\int_t^T E(s) f(E(s))\, ds \le \beta \int_t^T (\hat{H})^\star \left(f(E(s)\right)\, ds + C_2 E(t)$$

where β is chosen of the form $\max(\eta_1, \eta_2 E(0))$, η_1 and η_2 being explicit positive constants to guarantee that the argument E of f stays in the domain of definition of f. Thus (43) is proved, thanks to the fact that the weight function has been chosen so that

$$\beta \hat{H}^\star(f(E(s)) = \frac{1}{2}E(s)f(E(s)) \quad \forall\, 0 \le s\,.$$

Therefore E satisfies a nonlinear integral inequality with a weight function $f(E)$ which is defined in a semi-explicit way in general cases of feedback growths.

The last step is to prove that a nonincreasing and nonnegative absolutely continuous function E satisfying a nonlinear integral inequality of the form (43) is decaying at infinity, and to establish at which rate this holds. For this, one proceeds as in [5].

Let $\eta > 0$ and $T_0 > 0$ be fixed given real numbers and F be a strictly increasing function from $[0, +\infty)$ on $[0, \eta)$, with $F(0) = 0$ and $\lim y \to +\infty F(y) = \eta$.

For any $r \in (0, \eta)$, we define a function K_r from $(0, r]$ on $[0, +\infty)$ by

$$K_r(\tau) = \int_\tau^r \frac{\mathrm{d}y}{yF^{-1}(y)}\,, \tag{45}$$

and a function ψ_r which is a strictly increasing onto function defined from $[\frac{1}{F^{-1}(r)}, +\infty)$ on $[\frac{1}{F^{-1}(r)}, +\infty)$ by

$$\psi_r(z) = z + K_r(F(\frac{1}{z})) \ge z, \quad \forall z \ge \frac{1}{F^{-1}(r)}\,, \tag{46}$$

Then one can prove that if E is a nonincreasing, absolutely continuous function from $[0, +\infty)$ on $[0, +\infty)$, satisfying $0 < E(0) < \eta$ and the inequality

$$\int_t^T E(s)F^{-1}(E(s))\,\mathrm{d}s \le T_0 E(S)\,, \quad \forall\, 0 \le t \le T\,. \tag{47}$$

Then E satisfies the following estimate:

$$E(t) \le F\left(\frac{1}{\psi_r^{-1}(\frac{t}{T_0})}\right), \quad \forall t \ge \frac{T_0}{F^{-1}(r)}\,, \tag{48}$$

where r is any real such that

$$\frac{1}{T_0}\int_0^{+\infty} E(\tau)F^{-1}(E(\tau))\,\mathrm{d}\tau \le r \le \eta\,.$$

Thus, one can apply the above result to E with $\eta = r_0^2$ and show that $\lim t \to +\infty E(t) = 0$, the decay rate being given by the estimate (48).

If g is polynomial close to zero, one gets back that the energy $E(t)$ decays as $t^{\frac{-2}{p-1}}$ at infinity. If $g(v)$ behaves as $\exp(-1/|v|)$ close to zero, then $E(t)$ decays as $1/(\ln(t))^2$ at infinity.

The usefulness of convexity arguments has been first pointed out by Lasiecka and Tataru [78] using Jensen's inequality and then in different ways by Martinez [93] (the weight function does not depend on the energy) and W. Liu and Zuazua [89] and Eller Lagnese and Nicaise [56]. Optimal decay rates have been obtained by Alabau–Boussouira [5,6] using a weight function determined through the theory of convex conjugate functions and Young's (named also as Fenchel–Moreau's) inequality. This argument was also used by W. Liu and Zuazua [89] in a slightly different way and combined to a Lyapunov technique. Optimality of estimates in [5] is proved in one-dimensional situation and for boundary dampings applying optimality results of Vancostenoble [119] (see also Martinez and Vancostenoble [118]).

Indirect Damping for Coupled Systems

Many complex phenomena are modelized through coupled systems. In stabilizing (or controlling) energies of the vector state, one has very often access only to some components of this vector either due to physical constraints or to cost considerations. In this case, the situation is to stabilize a full system of coupled equation through a reduced number of feedbacks. This is called indirect damping. This notion has been introduced by Russell [109] in 1993.

As an example, we consider the following system:

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases}$$

$$\text{in } \Omega \times \mathbb{R}\,, \quad u = 0 = v \quad \text{on } \partial\Omega \times \mathbb{R}\,. \tag{49}$$

Here, the first equation is damped through a linear distributed feedback, while no feedback is applied to the second equation. The question is to determine if this coupled system inherits any kind of stability for nonzero values of the coupling parameter α from the stabilization of the first equation only.

In the finite dimensional case, stabilization (or control) of coupled ODE's can be analyzed thanks to a powerful rank type condition named Kalman's condition. The situation is much more involved in the case of coupled PDE's.

One can show first show that the above system fails to be exponentially stable (see also [66] for related results). More generally, one can study the stability of the system

$$\begin{cases} u'' + A_1 u + Bu' + \alpha v = 0 \\ v'' + A_2 v + \alpha u = 0 \end{cases} \tag{50}$$

in a separable Hilbert space H with norm $|\cdot|$, where A_1, A_2 and B are self-adjoint positive linear operators in H. Moreover, B is assumed to be a bounded operator. So, our analysis applies to systems with internal damping supported in the whole domain Ω such as (49); the reader is referred to [1,2] for related results concerning boundary stabilization problems (see also Beyrath [23,24] for localized indirect dampings).

In light of the above observations, system (50) fails to be exponentially stable, at least when H is infinite dimensional and A_1 has a compact resolvent as in (49). Indeed it is shown in Alabau, Cannarsa and Komornik [8] that the total energy of sufficiently smooth solutions of (50) decays polynomially at infinity whenever $|\alpha|$ is small enough but nonzero. From this result we can also deduce that any solution of (50) is strongly stable regardless of its smoothness: this fact follows by a standard density argument since the semigroup associated with (50) is a contraction semigroup.

A brief description of the key ideas of the approach developed in [2,8] is as follows. Essentially, one uses a finite iteration scheme and suitable multipliers to obtain an estimate of the form

$$\int_0^T E(u(t), v(t))\,dt \le c\sum_{k=0}^{j} E(u^{(k)}(0), v^{(k)}(0)) \qquad \forall\, T \ge 0\,, \quad (51)$$

where j is a positive integer and E denotes the total energy of the system

$$E(u,v) = \frac{1}{2}\left(|A_1^{1/2}u|^2 + |u'|^2\right) + \frac{1}{2}\left(|A_2^{1/2}v|^2 + |v'|^2\right) + \alpha\langle u, v\rangle\,.$$

Once (51) is proved, an abstract lemma due to Alabau [1,2] shows that $E(u(t), v(t))$ decays polynomially at ∞. This abstract lemma can be stated as follows.

Let A be the infinitesimal generator of a continuous semi-group $\exp(tA)$ on an Hilbert space $\mathcal{H}$, and $D(A)$ its domain. For U^0 in $\mathcal{H}$ we set in all the sequel $U(t) = \exp(tA)U^0$ and assume that there exists a functional E defined on $C([0,+\infty), \mathcal{H})$ such that for every U^0 in $\mathcal{H}$, $E(\exp(.A))$ is a non-increasing, locally absolutely continuous function from $[0,+\infty)$ on $[0,+\infty)$. Assume moreover that there exist an integer $k \in \mathbb{N}^\star$ and nonnegative constants c_p for $p = 0, \dots k$ such that

$$\int_S^T E(U(t))\,dt \le \sum_{p=0}^{k} c_p E(U^{(p)}(S)) \qquad \forall 0 \le S \le T\,, \forall U^0 \in D(A^k)\,. \quad (52)$$

Then the following inequalities hold for every U^0 in $D(A^{kn})$ and all $0 \le S \le T$ where n is any positive integer:

$$\int_S^T E(U(\tau))\frac{(\tau - S)^{n-1}}{(n-1)!}\,d\tau \le c\sum_{p=0}^{kn} E(U^{(p)}(S))\,, \quad (53)$$

and

$$E(U(t)) \le c\sum_{p=0}^{kn} E(U^{(p)}(0))t^{-n} \qquad \forall t > 0\,, \quad \forall U^0 \in D(A^{kn})\,,$$

where c is a constant which depends on n.

First (53) is proved by induction on n. For $n = 1$, it reduces to the hypothesis (52). Assume now that (53) holds for n and let U^0 be given in $D(A^{k(n+1)})$. Then we have

$$\int_S^T\int_t^T E(U(\tau))\frac{(\tau - t)^{n-1}}{(n-1)!}\,d\tau\,dt \le c\sum_{p=0}^{kn}\int_S^T E(U^{(p)}(t))\,dt \qquad \forall\, 0 \le S \le T\,, \forall\, U^0 \in D(A^{kn}).$$

Since U^0 is in $D(A^{k(n+1)})$ we deduce that $U^{(p)}(0) = A^p U^0$ is in $D(A^k)$ for $p \in \{0, \dots kn\}$. Hence we can apply the assumption (52) to the initial data $U^{(p)}(0)$. This together with Fubini's Theorem applied on the left hand side of the above inequality give (53) for $n + 1$. Using the property that $E(U(t))$ is non increasing in (53) we easily obtain the last desired inequality.

Applications on wave-wave, wave-Petrowsky equations and various concrete examples hold.

The above results have been studied later on by Batkai, Engel, Prüss and Schnaubelt [18] using very interesting resolvent and spectral criteria for polynomial stability of abstract semigroups. The above abstract lemma in [2] has also been generalized using interpolation theory. One should note that this integral inequality involving higher order energies of solutions is not of differential nature contrarily to the Haraux's and Komornik's integral inequalities. Another approach based on decoupling techniques and for slightly different abstract systems have been introduced by Ammar Khodja Bader and Ben Abdallah [12].

Spectral conditions have also been studied by Z. Liu [88] and later on by Z. Liu and Rao [90], Loreti and Rao [92] for peculiar abstract systems and in general for coupled equations only of the *same* nature (wave-wave for instance), so that a dispersion relation for the eigenvalues of the coupled system can be derived. Also these last results are given for internal stabilization only. From the above limitations, Z. Liu–Rao and Loreti–Rao's results are less powerful in generality than the ones given by Alabau, Cannarsa and Komornik [8] and Alabau [2]. Moreover results through energy type estimates and integral inequalities can be generalized to include nonlinear indirect dampings as shown in [7]. On the other side spectral methods are very precise for the obtention of optimal decay rates provided that one can determine at which speed the eigenvalues approach the imaginary axis for high frequencies.

Memory Dampings

We consider the following model problem

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + \int_0^t \beta(t-s)\,\Delta u(s,x)\,\mathrm{d}s = \\ \qquad\qquad |u(t,x)|^{\gamma} u(t,x) \\ u(t,\cdot)_{|\partial\Omega} = 0 \\ (u(0,\cdot), u_t(0,\cdot)) = (u_0, u_1) \end{cases} \tag{54}$$

where $0 < \gamma \leq \frac{2}{N-2}$ holds. The second member is a source term. The damping

$$\int_0^t \beta(t-s)\Delta u(s,x)\,\mathrm{d}s$$

is of memory type.

The energy is defined by

$$\begin{aligned} E_u(t) = {} & \frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)}\,\mathrm{d}x \\ & + \frac{1}{2}\left(1 - \int_0^t \beta(s)\,\mathrm{d}s\right)\|\nabla u(t)\|^2_{L^2(\Omega)} \\ & - \frac{1}{\gamma+2}\|u(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)} \\ & + \frac{1}{2}\int_0^t \beta(t-s)\|\nabla u(t) - \nabla u(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \end{aligned}$$

The damping term produces dissipation of the energy, that is (for strong solutions)

$$\begin{aligned} E_u'(t) = {} & -\frac{1}{2}\beta(t)\|\nabla u(t)\|^2 \\ & + \frac{1}{2}\int_0^t \beta'(t)\|\nabla u(s) - \nabla u(t)\|^2\,\mathrm{d}s \leq 0 \end{aligned}$$

One can consider more general abstract equations of the form

$$u''(t) + Au(t) - \int_0^t \beta(t-s)Au(s)\,\mathrm{d}s = \nabla F(u(t)) \qquad t \in (0,\infty) \tag{55}$$

in a Hilbert space X, where $A\colon D(A) \subset X \to X$ is an accretive self-adjoint linear operator with dense domain, and ∇F denotes the gradient of a Gâteaux differentiable functional $F\colon D(A^{1/2}) \to \mathbb{R}$. In particular, equation (54) fits into this framework as well as several other classical equations of mathematical physics such as the linear elasticity system.

We consider the following assumptions.

Assumptions (H1)

1. A is a self-adjoint linear operator on X with dense domain $D(A)$, satisfying

$$\langle Ax, x\rangle \geq M\|x\|^2 \quad \forall x \in D(A) \tag{56}$$

for some $M > 0$.

2. $\beta\colon [0,\infty) \to [0,\infty)$ is a locally absolutely continuous function such that

$$\int_0^\infty \beta(t)\,\mathrm{d}t < 1 \beta(0) > 0 \quad \beta'(t) \leq 0 \qquad \text{for a.e. } t \geq 0\,.$$

3. $F\colon D(A^{1/2}) \to \mathbb{R}$ is a functional such that
   1. F is Gâteaux differentiable at any point $x \in D(A^{1/2})$;
   2. for any $x \in D(A^{1/2})$ there exists a constant $c(x) > 0$ such that

$$|DF(x)(y)| \leq c(x)\|y\|, \quad \text{for any } y \in D(A^{1/2}),$$

where $DF(x)$ denotes the Gâteaux derivative of F in x; consequently, $DF(x)$ can be extended to the whole space X (and we will denote by $\nabla F(x)$ the unique vector representing $DF(x)$ in the Riesz isomorphism, that is, $\langle \nabla F(x), y\rangle = DF(x)(y)$, for any $y \in X$);

3. for any $R > 0$ there exists a constant $C_R > 0$ such that

$$\|\nabla F(x) - \nabla F(y)\| \le C_R \|A^{1/2}x - A^{1/2}y\|$$

for all $x, y \in D(A^{1/2})$ satisfying $\|A^{1/2}x\|$, $\|A^{1/2}y\| \le R$.

Assumptions (H2)

1. There exist $p \in (2, \infty]$ and $k > 0$ such that

$$\beta'(t) \le -k\beta^{1+\frac{1}{p}}(t) \quad \text{for a.e. } t \ge 0$$

(here we have set $\frac{1}{p} = 0$ for $p = \infty$).

2. $F(0) = 0$, $\nabla F(0) = 0$, and there is a strictly increasing continuous function $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(0) = 0$ and

$$|\langle \nabla F(x), x\rangle| \le \psi(\|A^{1/2}x\|)\|A^{1/2}x\|^2 \quad \forall x \in D(A^{1/2}).$$

Under these assumptions, global existence for sufficiently small (resp. all) initial data in the energy space can be proved for nonvanishing (resp. vanishing) source terms.

It turns out that the above energy methods based on multiplier techniques combined with linear and nonlinear integral inequalities can be extended to handle memory dampings and applied to various concrete examples such as wave, linear elastodynamic and Petrowsky equations for instance. This allows to show in [10] that exponential as well as polynomial decay of the energy hold if the kernel decays respectively exponentially or polynomially at infinity.

The method is as follows. One evaluates expressions of the form

$$\int_t^T \langle u''(s) + Au(s) - \int_0^t \beta * Au(s) - \nabla F(u(s), Mu\rangle \, \mathrm{d}s$$

where the multipliers Mu are of the form $\phi(s)(c_1(\beta * u)(s) + c_2(s)u)$ with ϕ which is a differentiable, nonincreasing and nonnegative function, and c_1 being a suitable constant, whereas c_2 may be chosen dependent on β.

Integrating by parts the resulting relations and performing some involved estimates, one can prove that for all $t_0 > 0$ and all $T \ge t \ge t_0$

$$\int_t^T \phi(s)E(s)\,\mathrm{d}s \le C\phi(0)E(t) + \int_t^T \phi(s) \times \int_0^s \beta(s-\tau)\left\|A^{1/2}u(s) - A^{1/2}u(\tau)\right\|^2 \,\mathrm{d}\tau\,\mathrm{d}s\,,$$

If $p = \infty$, that is if the kernel β decays exponentially, one can easily bound the last term of the above estimate by $cE(t)$ thanks to the dissipation relation.

If $p \in (2, \infty)$, one has to proceed differently since the term

$$\int_t^T \phi(s)\int_0^s \beta(s-\tau)\left\|A^{1/2}u(s) - A^{1/2}u(\tau)\right\|^2 d\tau\,\mathrm{d}s$$

cannot be directly estimated thanks to the dissipation relation. To bound this last term, one can generalize an argument of Cavalcanti and Oquendo [37] as follows. Define, for any $m \ge 1$,

$$\varphi_m(t) := \int_0^t \beta^{1-\frac{1}{m}}(t-s)\|A^{1/2}u(s) - A^{1/2}u(t)\|^2\,\mathrm{d}s\,, \quad t \ge 0\,. \tag{57}$$

Then, we have for any $T \ge S \ge 0$

$$\int_S^T E_u^{\frac{m}{p}}(t)\int_0^t \beta(t-s)\|A^{1/2}u(s) - A^{1/2}u(t)\|^2\,\mathrm{d}s\,\mathrm{d}t \le CE_u^{\frac{p}{p+m}}(S)\left(\int_S^T E_u^{1+\frac{m}{p}}(t)\varphi_m(t)\,\mathrm{d}t\right)^{\frac{m}{p+m}} \tag{58}$$

for some constant $C > 0$. Then one proves Suppose that, if for some $m \ge 1$, the function φ_m defined in (57) is bounded. Then, for any $S_0 > 0$ there is a positive constant C such that

$$\int_S^\infty E_u^{1+\frac{m}{p}}(t)\,\mathrm{d}t \le C\Big(E_u^{\frac{m}{p}}(0) + \|\varphi_m\|_\infty^{\frac{m}{p}}\Big)E_u(S) \quad \forall\, S \ge S_0\,. \tag{59}$$

One uses this last result first with $m = 2$ noticing that φ_2 is bounded and $\phi = E^{2/p}$. This gives a first energy decay rate as $(t+1)^{-p/2}$. This estimate shows that φ_1 is bounded. Then one applies once again the last result with $m = 1$ and $\phi = E^{1/p}$. One deduces then that E decays as $(t+1)^{-p}$ which is the *optimal* decay rate expected.

Bibliographical Comments

For an introduction to the multiplier method, we refer the interested reader to the books of J.-L. Lions [86], Komornik [74] and the references therein. The celebrated result of Bardos Lebeau and Rauch is presented in [86]. A general abstract presentation of control problems for hyperbolic and parabolic equations can be found in the book of Lasiecka and Triggiani [80,81]. Results on spectral methods and the frequency domain approach can be found in the book of Z. Liu [88]. There also exists an interesting approach developed for bounded feedback operators by Haraux and extended to the case of unbounded feedbacks by Ammari and Tucsnak [11]. In this approach, the polynomial (or exponential) stability of the damped system is proved thanks to the corresponding observability for the undamped (conservative) system. Such observability results for weakly coupled undamped systems have been obtained for instance in [3].

Many other very interesting issues have been studied connected to semilinear wave equations [34,123] and the references therein, to the case of wave damped equations with nonlinear source terms [39].

Well-posedness and asymptotic properties for PDE's with memory terms have first been studied by Dafermos [53,54] for convolution kernels with past history (convolution up to $t = -\infty$), by Prüss [103] and Prüss and Propst [102] in which the efficiency of different models of dampings are compared to experiments (see also Londen Petzeltova and Prüss [91]). Decay estimates for the energy of solutions using multiplier methods combined with Lyapunov type estimates for an equivalent energy are proved in Munoz Rivera [97], Munoz Rivera and Salvatierra [96], Cavalcanti and Oquendo [37] and Giorgi Naso and Pata [67] and many other papers.

Optimal Control

As for positional control, also for optimal control problems it is convenient to adopt the abstract formulation introduced in Sect. "Abstract Evolution Equations". Let the state space be represented by the Hilbert space H, and the state equation be given in the form (12), that is

$$\begin{cases} u'(t) = Au(t) + Bf(t) \quad t \in [0, T] \\ u(0) = u_0 \, . \end{cases} \tag{60}$$

Recall that A is the infinitesimal of a strongly continuous semigroup, e^{tA}, in H, B is a (bounded) linear operator from F (the control space) to H, and u_f stands for the unique (mild) solution of (60) for a given control function $f \in L^2(0, T; H)$.

A typical optimal control problem of interest for PDE's is the *Bolza problem* which consists in

$$\begin{cases} \text{minimizing the } \textit{cost functional} \\ \quad J(f) \doteq \int_0^T L(t, u_f(t), f(t))\,\mathrm{d}t + \ell\left(u_f(T)\right) \\ \text{over all controls } \; f \in L^2(0, T; F) \, . \end{cases} \tag{61}$$

Here, T is a positive number, called the *horizon*, whereas L and ℓ are given functions, called the *running cost* and *final cost*, respectively. Such functions are usually assumed to be *bounded below*.

A control function $f_* \in L^2(0, T; F)$ at which the above minimum is attained is called an *optimal control* for problem (61) and the corresponding solution u_{f_*} of (60) is said to be an *optimal trajectory*. Alltogether, $\{u_{f_*}, f_*\}$ is called an *optimal (trajectory/control) pair*.

For problem (61) the following issues will be addressed in the sections below:

- *the existence* of controls minimizing functional J;
- *necessary conditions* that a candidate solution must satisfy;
- *sufficient conditions for optimality* provided by the dynamic programming method.

Other problems of particular interest to CT for PDE's are problems with an *infinite horizon* ($T = \infty$), problems with a *free horizon* T and a final *target*, and problems with constraints on both control variables and state variables. Moreover, the study of nonlinear variants of (60), including semilinear problems of the form

$$\begin{cases} u'(t) = Au(t) + h(t, u(t), f(t)) \quad t \in [0, T] \\ u(0) = u_0 \, , \end{cases} \tag{62}$$

is strongly motivated by applications. The discussion of all these variants, however, will not be here pursued in detail.

Traditionally, in optimal control theory, state variables are denoted by the letters $x, y, \ldots$, whereas $u, v, \ldots$ are reserved for control variables. For notational consistency, in this section $u(\cdot)$ will still denote the state of a given system and $f(\cdot)$ a control function, while ϕ will stand for a fixed element of control space F.

Existence of Optimal Controls

From the study of finite dimensional optimization it is a familiar fact that the two essential ingredients to guarantee the existence of minima are compactness and lower semicontinuity. Therefore, it is clear that, in order to obtain a solution of the optimal control problem (60)–(61), one has to make assumptions that allow to recover such properties. The typical hypotheses that are made for this purpose are the following:

- *coercivity*: there exist constants $c_0 > 0$ and $c_1 \in \mathbb{R}$ such that

$$\ell(\phi) \geq c_1 \quad \text{and} \quad L(t, u, \phi) \geq c_0 \|\phi\|^2 + c_1 \quad \forall (t, u, \phi) \in [0, T] \times H \times F \tag{63}$$

- *convexity*: for every $(t,u) \in [0,T] \times H$

$$\phi \mapsto L(t,u,\phi) \quad \text{is convex on} \quad F\,. \tag{64}$$

Under the above hypotheses, assuming lower semicontinuity of ℓ and of the map $L(t,\cdot,\phi)$, it is not hard to show that problem (60)–(61) has at least one solution. Indeed, assumption (63) allows to show that any minimizing sequence of controls $\{f_k\}$ is bounded in $L^2(0,T;H)$. So, it admits a subsequence, still denoted by $\{f_k\}$ which converges weakly in $L^2(0,T;H)$ to some function f. Then, by linearity, $u_{f_k}(t)$ converges to $u_f(t)$ for every $t \in [0,T]$. So, using assumption (64), it follows that f is a solution of (60)–(61).

The problem becomes more delicate when the Tonelli type coercivity condition (63) is relaxed, or the state equation is nonlinear as in (62). Indeed, the convergence of $u_{f_k}(t)$ is no longer ensured, in general. So, in order to recover compactness, one has to make further assumptions, such as the compactness of e^{tA}, or structural properties of L and h. For further reading, one may consult the monographs [22,85], and [79], for problems where the running and final costs are given by quadratic forms (the so-called Linear Quadratic problem), or [84] and [59] for more general optimal control problems.

Necessary Conditions

Once the existence of a solution to problem (60)–(61) has been established, the next important step is to provide conditions to detect a candidate solution, possibly showing that it is, in fact, optimal. By and large the optimality conditions of most common use are the ones known as Pontryagin's Maximum Principle, named after the Russian mathematician L.S. Pontryagin who greatly contributed to the development of control theory, see [100,101].

So, suppose $\{u_*, f_*\}$, where $u_* = u_{f_*}$ is a candidate optimal pair, and consider the so-called adjoint system

$$\begin{cases} -p'(t) = A^* p(t) + \partial_u L(t, u_*(t), f_*(t)) = 0 \\ \qquad\qquad\qquad\qquad\qquad t \in [0,T] \text{ a.e.} \\ p(T) = \partial \ell(u_*(T))\,, \end{cases}$$

where $\partial_u L(t,u,\phi)$ and $\partial \ell(u)$ denote the Fréchet gradients of the maps $L(t,\cdot,\phi)$ and ℓ at u, respectively. Observe that the above is a backward linear Cauchy problem with terminal condition, which can obviously be reduced to a forward one by the change of variable $t \to T - t$. So, it admits a unique mild solution, labeled p_*, which is called the *adjoint state* associated with $\{u_*, f_*\}$.

Pontryagin's Maximum Principle states that, if $\{u_*, f_*\}$ is optimal, then

$$\langle p_*(t), B f_*(t)\rangle + L(t, u_*(t), f_*(t)) = \min_{\phi \in F} \left[\langle p_*(t), B\phi\rangle + L(t, u_*(t), \phi)\right] \qquad t \in [0,T] \text{ a.e.} \tag{65}$$

The name Maximum Principle rather than Minimum Principle, as it would be more appropriate, is due to the fact that, traditionally, attention was focussed on the *maximization*—instead of minimization—of the functional in (61). Even today, in most models from economics, one is interested in maximizing payoffs, such as revenues, utility, capital and so on. In that case, (65) would still be true, with a "max" instead of a "min".

At first glance, it might be hard to understand the revelance of (65) to problem (61). To explain this, introduce the function, called the *Hamiltonian*,

$$\mathcal{H}(t,u,p) = \min_{\phi \in F} \left[\langle p, B\phi\rangle + L(t,u,\phi)\right] \qquad (t,u,p) \in [0,T] \times H \times H\,. \tag{66}$$

Then, Fermat's rule yields $B^* p + \partial_\phi L(t,u,\phi) = 0$ at every $\phi \in F$ at which the minimum in (66) is attained. Therefore, from (65) it follows that

$$B^* p_*(t) + \partial_\phi L(t, u_*(t), f_*(t)) = 0 \quad t \in [0,T] \text{ a.e.} \tag{67}$$

which provides a much-easier-to-use optimality condition.

There is a vast literature on necessary condition for optimality for distributed parameter systems. The set-up that was considered above can be generalized in several ways: one can consider nonlinear state equations as in (62), nonsmooth running and finals costs, constraints on both state and control, problems with infinite horizon or exit times. Further reading and useful references on most of these extensions can be found in the aforementioned monographs [22,79,84,85], and in [59] which is mainly concerned with time optimal control problems.

Dynamic Programming

Though useful as it may be, Pontryagin's Maximum Principle remains a necessary condition. So, without further information, it does not suffice to prove the optimality of a give trajectory/control pair. Moreover, even when the map $\phi \mapsto \partial_\phi L(t,u,\phi)$ turns out to be invertible, the best result identity (67) can provide, is a representation of $f_*(t)$ in terms of $u_*(t)$ and $p_*(t)$: not enough to determine $f_*(t)$, in general.

This is why other methods to construct optimal controls have been proposed over the years. One of the most interesting ones is the so-called *dynamic programming* method (abbreviated, DP), initiated by the work of R. Bellman [20]. Such a method will be briefly described below in the set-up of distributed parameter systems.

Fix $T > 0$, s such that $0 \le s \le T$, and consider the optimal control problem

to minimize

$$J^{s,v}(f) = \int_s^T L\left(t, u_f^{s,v}(t), f(t)\right) \mathrm{d}t + \ell\left(u_f^{s,v}(T)\right) \tag{68}$$

over all control functions $f \in L^2(s,T;F)$, where $u_f^{s,v}(t)$ is the solution of the controlled system

$$\begin{cases} u'(t) = Au(t) + Bf(t) & t \in [s,T] \\ u(s) = v\,. \end{cases} \tag{69}$$

The *value function* U associated to (68)-(69) is the real-valued function defined by

$$U(s,v) = \inf_{f \in L^2(s,T;F)} J^{s,v}(f) \quad \forall (s,v) \in [0,T] \times H\,. \tag{70}$$

A fundamental step of DP is the following result, known as Bellman's *optimality principle*.

Theorem 5 *For any* $(s,v) \in [0,T] \times H$ *and any* $f \in L^2(s,T;F)$

$$U(s,v) \le \int_s^r L\left(t, u_f^{s,v}(t), f(t)\right) \mathrm{d}t + U\left(r, u_f^{s,v}(r)\right) \qquad \forall r \in [s,T]\,.$$

Moreover, $f^*(\cdot)$ *is optimal if and only if*

$$U(s,v) = \int_s^r L\left(t, u_f^{s,v}(t), f(t)\right) \mathrm{d}t + U\left(r, u_f^{s,v}(r)\right) \qquad \forall r \in [s,T]\,.$$

The connection between DP and optimal control is based on the properties of the value function. Indeed, applying Bellman's optimality principle, one can show that, if U is Fréchet differentiable, then

$$\begin{cases} \partial_s U(s,v) + \langle Av, \partial_v U(s,v)\rangle + \mathcal{H}\left(s, v, \partial_v U(s,v)\right) = 0 \\ \qquad\qquad\qquad\qquad (s,v) \in (0,T) \times D(A) \\ U(T,v) = \ell(v) \quad v \in H \end{cases}$$

where $\mathcal{H}$ is the Hamiltonian defined in (66). The above equation is the celebrated *Hamilton–Jacobi equation* of DP. To illustrate its connections with the original optimal control problem, a useful formal argument—that can, however, be made rigorous—is the following. Consider a sufficiently smooth solution W of the above problem and let $(s,v) \in (0,T) \times D(A)$. Then, for any trajectory/control pair $\{u, f\}$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} W(t,u(t)) &= \partial_s W(t,u(t)) + \langle \partial_v W(t,u(t)), Au(t) \\ &\quad + Bf(t)\rangle \\ &= \langle \partial_v W(t,u(t)), Bf(t)\rangle \\ &\quad - \mathcal{H}\left(t, u(t), \partial_v W(t,u(t))\right) \\ &\ge -L(t,u(t),f(t)) \end{aligned} \tag{71}$$

by the definition of $\mathcal{H}$. Therefore, integrating from s to T,

$$\ell(u(T)) - W(s,v) \ge -\int_s^T L(t,u(t),f(t))\,\mathrm{d}t\,,$$

whence $J^{s,v}(f) \ge W(s,v)$. Thus, taking the infimum over all $f \in L^2(s,T;F)$,

$$W(s,v) \le U(s,v) \quad \forall (s,v) \in (0,T) \times D(A)\,. \tag{72}$$

Now, suppose there is a control function $f_* \in L^2(s,T;F)$ such that, for all $t \in [s,T]$,

$$\begin{aligned} \langle \partial_v W(t,u_*(t)), Bf_*(t)\rangle + L(t,u_*(t),f_*(t)) \\ = \mathcal{H}(t,u_*(t),\partial_v W(t,u_*(t)))\,, \end{aligned} \tag{73}$$

where $u_*(\cdot) = u_{f_*}^{s,v}(\cdot)$. Then, from (71) and (73) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t,u_*(t)) = -L(t,u(t),f(t))\,,$$

whence

$$W(s,v) = J^{s,v}(f_*) \ge U(s,v)\,.$$

From the above inequality and (72) it follows that $W(s,v) = U(s,v)$ for all $(s,v) \in (0,T) \times D(A)$, hence for all $(s,v) \in (0,T) \times H$ since $D(A)$ is dense in H. So, f_* is an **optimal control**.

Note 2 The above considerations lead to the following procedure to obtain optimal an optimal trajectory:

- find a smooth solution of the Hamilton–Jacobi equation;

- for every $(t,v) \in (0,T) \times D(A)$ provide a feedback $f(t,v)$ such that

$$\begin{aligned}\langle \partial_v W(t,v), Bf(t,v)\rangle + L(t,v,f(t,v)) \\ = \mathcal{H}(t,v,\partial_v W(t,v))\end{aligned}$$

- solve the so-called *closed loop equation*

$$\begin{cases} u'(t) = Au(t) + Bf(t,u(t)) & t \in [s,T] \\ u(s) = v \end{cases}$$

Notice that not only is trajectory u optimal, but the corresponding control f is given in feedback form as well.

Linear Quadratic Optimal Control

One of the most successful applications of DP is the so-called Linear Quadratic optimal control problem. Consider problem (68)–(69) with costs L and ℓ given by

$$\begin{aligned} L(t,u,\phi) = \langle M(t)u,u\rangle + \langle N(t)\phi,\phi\rangle \\ \forall (T,u,\phi) \in [0,T] \times H \times F \end{aligned}$$

and

$$\ell(u) = \langle Du, u\rangle \quad \forall u \in H\,,$$

where

- $M\colon [0,T] \to \mathcal{L}(H)$ is continuous, $M(t)$ is symmetric and $\langle M(t)u,u\rangle \geq 0$ for every $(t,u) \in [0,T] \times H$;
- $N\colon [0,T] \to F$ is continuous, $N(t)$ is symmetric and $\langle N(t)\phi,\phi\rangle \geq c_0|\phi|^2$ for every $(t,\phi) \in [0,T] \times F$ and some constant $c_0 > 0$;
- $D \in \mathcal{L}(H)$ is symmetric and $\langle Du,u\rangle \geq 0$ for every $u \in H$.

Then, assumptions (63) and (64) are satisfied. So, a solution to (68)–(69) does exist. Moreover, it is unique because of the strict convexity of functional $J^{s,v}$.

In order to apply DP, one computes the Hamiltonian

$$\begin{aligned}\mathcal{H}(t,u,p) &= \min_{\phi \in F}\Big[\langle p, B\phi\rangle + \langle M(t)u,u\rangle + \langle N(t)\phi,\phi\rangle\Big] \\ &= \langle M(t)u,u\rangle - \frac{1}{4}\langle BN^{-1}(t)B^*p,p\rangle\,,\end{aligned}$$

where the above minimum is attained at

$$\phi_*(t,p) = -\frac{1}{2}N^{-1}(t)B^*p\,. \tag{74}$$

Therefore, the Hamilton–Jacobi equation associated to the problem is

$$\begin{cases} \partial_s W(s,v) + \langle Av, \partial_v W(s,v)\rangle + \langle M(s)v,v\rangle \\ \quad -\frac{1}{4}\langle BN^{-1}(s)B^*\partial_v W(s,v), \partial_v W(s,v)\rangle = 0 \\ \qquad\qquad\qquad \forall (s,v) \in (0,T) \times D(A) \\ w(T,v) = \langle Dv,v\rangle \quad \forall v \in H \end{cases}$$

It is quite natural to search a solution of the above problem in the form

$$W(s,v) = \langle P(s)v,v\rangle \quad \forall (s,v) \in [0,T] \times H\,,$$

with $P\colon [0,T] \to \mathcal{L}(H)$ continuous, symmetric and such that $\langle P(t)u,u\rangle \geq 0$. Substituting into the Hamilton–Jacobi equation yields

$$\begin{cases} \langle P'(s)v,v\rangle + \langle [A^*P(s) + P(s)A]v,v\rangle + \langle M(s)v,v\rangle \\ \quad -\langle BN^{-1}(s)B^*P(s)v, P(s)v\rangle = 0 \\ \qquad\qquad\qquad \forall (s,v) \in (0,T) \times D(A) \\ \langle P(T)v,v\rangle = \langle Dv,v\rangle \quad \forall v \in H \end{cases}$$

Therefore, P must be a solution of the so-called *Riccati equation*

$$\begin{cases} P'(s) + A^*P(s) + P(s)A + M(s) \\ \quad -P(s)BN^{-1}(s)B^*P(s) = 0 \qquad \forall s \in (0,T) \\ P(T) = D \end{cases}$$

Once a solution $P(\cdot)$ the Riccati equation is known, the procedure described in Note 2 can be applied. Indeed, recalling (74) and the fact that $\partial_v W(t,v) = 2P(t)v$, one concludes that $f(t,v) = -N^{-1}(t)B^*P(t)v$ is a feedback law. So, solving the closed loop equation

$$\begin{cases} u'(t) = [A - BN^{-1}(t)B^*P(t)]u(t) & t \in (s,T) \\ u(s) = v \end{cases}$$

one obtains the unique optimal trajectory of problem (68)–(69).

In sum, by DP one reduces the original Linear Quadratic optimal control problem to the problem of finding the solution of the Riccati equation, which is easier to solve than the Hamilton–Jacobi equation.

Bibliographical Comments

Different variants of the Riccati equation have been successfully studied by several authors in connection with different state equations and cost functionals, including boundary control problems and problems for other functional equations, see [22,79] and the references therein. Sometimes, the solution of the Riccati equation related to a linearized model provides feedback stabilization for nonlinear problems as in [104].

Unfortunately, the DP method is hard to implement for general optimal control problems, because of several obstructions: nonsmoothness of solutions to Hamilton–Jacobi equations, selection problems that introduce discontinuities, unboundedness of the coefficients, numerical complexity. Besides the Linear Quadratic case, the so-called Linear Convex case is the other example that can be studied by DP under fairly general conditions, see [14]. For nonlinear optimal control problems some of the above difficulties have been overcome extending the notion of viscosity solutions to infinite dimensional spaces, see [45,46,47,48,49], see also [28,29,30,31,32,33] and [112]. Nevertheless, finding additional ideas to make a generalized use of DP for distributed parameter systems possible, remains a challenging problem for the next generations.

Future Directions

In addition to all considerations spread all over this article on promising developments of recent—as well as established—research lines, a few additional topics deserve to be mentioned.

The one subject that has received the highest attention, recently, is that of numerical approximation of control problems, from the point of view of both controllability and optimal control. Here the problem is that, due to high frequency spurious numerical solutions, stable algorithms for solving initial-boundary value problems do not necessarily yield convergent algorithms for computing controls. This difficulty is closely related to the existence of concentrated numerical solutions that escape the observation mechanisms. Nevertheless, some interesting results have been obtained so far, see, e. g., [124,125].

Several interesting results for nonlinear control problems have been obtained by the *return method*, developed initially by Coron [42] for a stabilization problem. This and other techniques have then been applied to fluid models ([68,69]), the Korteweg–de Vries equation ([105,106,107], and Schrödinger type equations ([19]), see also [43] and the references therein. It seems likely that these ideas, possibly combined with other techniques like Carleman estimates as in [70], will lead to new exiting results in the years to come.

A final comment on null controllability for *degenerate parabolic equations* is in order. Indeed, many problems that are relevant for applications are described by parabolic equation equations in divergence form

$$\partial_t u = \nabla\cdot(A(x)\nabla u) + b(x)\cdot\nabla u + c(t,x)u + f \quad \text{in} \quad Q_T ,$$

or in the general form

$$\partial_t u = \mathrm{Tr}\,[A(x)\nabla^2 u] + b(x)\cdot\nabla u + c(t,x)u + f \quad \text{in} \quad Q_T ,$$

where $A(x)$ is a symmetric matrix, positive definite in Ω but possibly singular on Γ. For instance, degenerate parabolic equations arise in fluid dynamics as suitable transformations of the Prandtl equations, see, e. g., [94]. They can also be obtained as Kolmogorov equations of diffusions processes on domains that are invariant for stochastic flows, see, e. g., [52]. The latter interpretation explains why they have been applied to biological problems, such as gene frequency models for population genetics (see, e. g., the Wright–Fischer model studied in [111]).

So far, null controllability properties of degenerate parabolic equations have been fully understood only in dimension one: for some kind of degeneracy, null controllability holds true (see [36] and [9]), but, in general, one can only expect regional null controllability (see [35]). Since very little is known on null controllability for degenerate parabolic equations in higher space dimensions, it is conceivable that such a topic will provide interesting problems for future developments.

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