

# Hölder estimates in space-time for viscosity solutions of Hamilton-Jacobi equations

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## Abstract

It is well-known that solutions to the basic problem in the calculus of variations may fail to be Lipschitz continuous when the Lagrangian depends on  $t$ . Similarly, for viscosity solutions to time-dependent Hamilton-Jacobi equations one cannot expect Lipschitz bounds to hold uniformly with respect to the regularity of coefficients. This phenomenon raises the question whether such solutions satisfy uniform estimates in some weaker norm. We will show that this is the case for a suitable Hölder norm, obtaining uniform estimates in  $(x, t)$  for solutions to first and second order Hamilton-Jacobi equations. Our results apply to degenerate parabolic equations and require superlinear growth at infinity, in the gradient variables, of the Hamiltonian. Proofs are based on comparison arguments and representation formulas for viscosity solutions, as well as weak reverse Hölder inequalities. © 2000 Wiley Periodicals, Inc.

## 1 Introduction

The object of this paper is the regularity of solutions to the Hamilton-Jacobi equation

$$(1.1) \quad u_t(x, t) - \text{Tr}(a(x, t)D^2u(x, t)) + H(x, t, Du(x, t)) = 0 \quad \text{in } \mathbb{R}^N \times (0, T)$$

where  $H$  and  $a$  will be assumed to satisfy the following hypotheses:

- there are real numbers  $q > 2$ ,  $\delta > 1$  and  $\eta_{\pm} \geq 0$  such that

$$(1.2) \quad \frac{1}{\delta}|z|^q - \eta_- \leq H(x, t, z) \leq \delta|z|^q + \eta_+ \quad \forall (x, t, z) \in \mathbb{R}^N \times (0, T) \times \mathbb{R}^N;$$

- $a = \sigma\sigma^*$  for some locally Lipschitz continuous map  $\sigma : (x, t) \mapsto \sigma(x, t)$ , with values in the  $N \times D$  real matrices ( $D \geq 1$ ), such that  $\|\sigma(x, t)\| \leq \delta$  for all  $(x, t) \in \mathbb{R}^N \times (0, T)$ .

Note that no initial condition is needed for our analysis, nor convexity of  $H$  in  $Du$ .

For a given viscosity solution  $u$  of (1.1), the kind of regularity properties we are interested in are uniform continuity estimates in  $(x, t)$  that do not depend on

the smoothness of coefficients but just on the constants that appear in (1.2), and on the sup-norm of  $u$ . Another important feature of our approach is that the above equation will not be assumed to be uniformly parabolic. Indeed, it will be allowed to degenerate to the point of reducing to the first order equation

$$(1.3) \quad u_t + H(x, t, Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, T),$$

in which case we will just require  $q > 1$  in (1.2).

The typical form of our results ensures that any bounded continuous viscosity solution  $u$  of (1.1) satisfies, for positive time, the uniform Hölder estimate

$$(1.4) \quad |u(x, t) - u(y, s)| \leq C \left[ |x - y|^{\frac{\theta - p}{\theta - 1}} + |t - s|^{\frac{\theta - p}{\theta}} \right]$$

where  $p$  is the conjugate exponent of  $q$ , and  $\theta > p$  depends only on the aforementioned constants. The above result may take a specific form according to the problem we will consider. For instance, for second order equations we suppose that the Hamiltonian  $H$  is super-quadratic ( $q > 2$ ), whereas for problem (1.3) we just need super-linear growth ( $q > 1$ ). Moreover, for both first and second order problems we can also give a local version of our result, that is, an estimate that applies to solutions in an open set  $\mathcal{O} \subset \mathbb{R}_x^N \times \mathbb{R}_t$ .

In order to better understand the problem under investigation it is convenient to start the analysis with first order equations. In this case, when  $H(x, t, z)$  is convex in  $z$ , the viscosity solutions of (1.3) can be represented as the value functions of problems in the calculus of variations. Consequently, the regularity of  $u$  is connected with that of minimizers. As is well-known, minimizers are Lipschitz continuous in the autonomous case (see [8], [2], [10], [13]), so that solutions turn out to be locally Lipschitz when  $H = H(x, z)$ . On the other hand, for nonautonomous problems, the Lipschitz regularity of minimizers is no longer true as is shown in [1], and (1.4) is the optimal Hölder estimate that can be expected, see the example in section 5.1 and Remark 5.6 of this paper. A class of nonautonomous first order problems for which such an estimate can be obtained is studied in [7]. Unlike the above references, however, our present results do not require  $H$  to be convex in  $z$ .

As for second order problems, Hölder regularity results for solutions of *uniformly* parabolic equations have been the object of a huge literature for both linear and nonlinear problems. However, very few results can be found in connection with the present context, where we drop uniform parabolicity and allow for unbounded Hamiltonians. In the *stationary* case, Lipschitz bounds for solutions of uniformly elliptic equations with a super-quadratic Hamiltonian were obtained in [19]. More recently, Hölder estimates have been proved in [6] for viscosity subsolutions of fully nonlinear degenerate elliptic equations with super-quadratic growth in the gradient.

An interesting point of the above elliptic results is that they suggest super-quadratic growth ( $q > 2$ ) should be a sharp condition for regularity. For instance, as explained in [19],  $q = 2$  plays the role of a regularity threshold for the equation  $-\Delta u + |Du|^q + \lambda u = f$  (where  $\lambda > 0$  and  $f$  is bounded) in a bounded domain

$\Omega \subset \mathbb{R}^N$ . If  $q \in (1, 2]$ , then the above equation admits solutions which tend to  $+\infty$  as  $x \rightarrow \partial\Omega$ . On the contrary, if  $q > 2$ , then every solution which is bounded below is also bounded above and can be continuously extended up to the boundary. One way to look at this phenomenon is to say that, as far as regularity is concerned, diffusion is the dominant term in the equation if  $q \in (1, 2]$ , whereas, for  $q > 2$ , the leading term is  $|Du|^q$ . Since this paper focusses on degenerate diffusions, regularity has to come from the gradient term, whence the condition  $q > 2$ .

Our work is mainly motivated by homogenization theory, where such uniform estimates are necessary to study the limiting behavior of solutions and/or to prove the existence of correctors (see, e.g., [21] and [23]). For instance, estimate (1.4) could be applied to equations of the form

$$u_t^\varepsilon(x, t) - \text{Tr} \left( a \left( x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) D^2 u^\varepsilon(x, t) \right) + H \left( x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, Du^\varepsilon(x, t) \right) = 0$$

where  $a(x, t, \cdot, \cdot)$  and  $H(x, t, \cdot, \cdot, z)$  are periodic in  $\mathbb{R}^N \times \mathbb{R}$ .

A brief comment of the structure of the proof is now in order. Our reasoning involves three main steps:

- (1) construction of suitable arcs along which super-solutions exhibit a sort of monotone behavior;
- (2) one-sided Hölder bound for sub-solutions;
- (3) application of a weak reverse Hölder inequality result.

Let us be more specific on the above points in the simpler case of first order equations. Our first step consists in showing that, if  $u$  is a super-solution of (1.1), then for any point  $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, T]$  there is an arc  $\xi \in W^{1,p}([0, \bar{t}]; \mathbb{R}^N)$ , satisfying  $\xi(\bar{t}) = \bar{x}$ , such that

$$(1.5) \quad u(\bar{x}, \bar{t}) \geq u(\xi(t), t) + C \int_t^{\bar{t}} |\xi'(s)|^p ds - \eta_+( \bar{t} - t) \quad \forall t \in [0, \bar{t}]$$

for some constant  $C > 0$ . Second, using Hopf's formula, we obtain the following one-sided bound for any sub-solution  $u$  of (1.1):

$$(1.6) \quad u(\bar{x}, s) \leq u(y, t) + C(s-t)^{1-p} |y - \bar{x}|^p + \eta_-(s-t) \quad \forall y \in \mathbb{R}^N, \forall s > t.$$

So, choosing  $y = \xi(t)$  in (1.6) and combining such an estimate with (1.5), we derive

$$\frac{1}{\bar{t}-t} \int_t^{\bar{t}} |\xi'(s)|^p ds \leq C_1 \left( \frac{|\xi(t) - \bar{x}|}{\bar{t}-t} \right)^p + C_0 \quad \forall t \in [0, \bar{t}]$$

which yields, in turn, the weak reverse Hölder inequality

$$(1.7) \quad \frac{1}{\bar{t}-t} \int_t^{\bar{t}} |\xi'(s)|^p ds \leq C_1 \left( \frac{1}{\bar{t}-t} \int_t^{\bar{t}} |\xi'(s)| ds \right)^p + C_0 \quad \forall t \in [0, \bar{t}].$$

Observe that (1.7) is weaker than the classical reverse Hölder inequality used to improve the integrability of functions (see, e.g., [15]). Nevertheless, we prove that,

$$\int_t^{\bar{t}} |\xi'(s)| ds \leq C (\bar{t}-t)^{1-\frac{1}{\theta}} \quad \forall t \in [0, \bar{t}]$$

for some exponent  $\theta > p$  depending only on structural constants. Finally, we show that the above inequalities imply estimate (1.4) with exactly the same exponent  $\theta$ .

One of the interesting aspects of our approach is that, using essentially the same ideas we have just described, we manage to study the second order problem (1.1). As it should be clear from the above discussion, such a transposition requires a certain familiarity with some techniques that are typical of stochastic analysis. For instance, the role of  $\xi$  will be now played by the controlled diffusion process which satisfies  $dX_t = \zeta_t dt + \sigma(X_t, t) dW_t$ , where  $W$  is a standard  $N$  dimensional Brownian motion and  $\zeta$  is a  $p$ -summable adapted control. Moreover, the one-sided Hölder bound of step 2 will be recovered by the use of a suitable Brownian bridge. Furthermore, the stochastic version of our reverse Hölder inequality result will require

$$\mathbb{E} \left[ \frac{1}{t-\bar{t}} \int_{\bar{t}}^t |\zeta_s|^p ds \right] \leq C_1 \mathbb{E} \left[ \left( \frac{1}{t-\bar{t}} \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] + \frac{C_0}{(t-\bar{t})^{\frac{p}{2}}} \quad \forall t \in (\bar{t}, T]$$

to yield the conclusion that

$$\mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] \leq C(t-\bar{t})^{p-\frac{p}{\theta}} (\|\zeta\|_p^p + B) \quad \forall t \in (\bar{t}, T]$$

for some  $\theta \in (p, 2)$ .

The outline of this paper is the following. In section 2 we fix notation and recall preliminaries from stochastic analysis, including the basic properties of Brownian bridges. Section 3 is devoted to weak reverse Hölder inequalities. Then, we present our main results: we study the Hölder continuity of solutions to first order equations in section 4, while second order problems are investigated in section 6 (for both problems we give a global and a local version of our results). In between (section 5), we discuss counterexamples to higher regularity.

## 2 Notation and preliminaries

We denote by  $x \cdot y$  the Euclidean scalar product of two vectors  $x, y \in \mathbb{R}^N$  and by  $|x|$  the Euclidean norm of  $x$ . For any  $x_0 \in \mathbb{R}^N$  and  $r > 0$ , we denote by  $B(x_0, r)$  the open ball of radius  $r$ , centered at  $x_0 \in \mathbb{R}^n$ , and we set  $B_r = B(0, r)$ .

Let  $D \geq 1$  be an integer. We denote by  $\mathbb{R}^{N \times D}$  the space of all  $N \times D$  real matrices equipped with the norm  $\|\sigma\| = \sqrt{\text{Tr}(\sigma\sigma^*)}$ , where  $\sigma^*$  denotes the transpose of  $\sigma$  and  $\text{Tr}(A)$  the trace of  $A \in \mathbb{R}^{N \times N}$ .

We denote by  $C(\mathbb{R}^N \times [0, T])$  the space of all continuous functions defined on  $\mathbb{R}^N \times [0, T]$ .

For any nonempty set  $S \subset \mathbb{R}^N$  let  $S^c = \mathbb{R}^N \setminus S$ . We denote  $d_S$  the Euclidean distance function from  $S$ , that is,  $d_S(x) = \inf_{y \in S} |x - y|$  for all  $x \in \mathbb{R}^N$ . It is well-known that  $d_S$  is a Lipschitz function of constant 1.

For  $1 \leq p < \infty$  we denote by  $L^p(a, b; \mathbb{R}^N)$  the space of all  $p$ -summable (with respect to the Lebesgue measure) Borel vector-valued functions  $\xi : [a, b] \rightarrow \mathbb{R}^N$ ,

and we use the shorter notation  $L^p(a, b)$  if  $N = 1$ . Similarly,  $W^{1,p}([a, b]; \mathbb{R}^N)$  stands for the Sobolev space of all absolutely continuous arcs  $\xi : [a, b] \rightarrow \mathbb{R}^N$  such that  $\dot{\xi} \in L^p(a, b; \mathbb{R}^N)$ .

Let now  $(\Omega, \mathcal{F}, \mathbb{P})$  be a stochastic basis, i.e., a measure space where  $\mathbb{P}$  is a probability measure. We denote by  $L^p(\Omega \times [a, b]; \mathbb{R}^N)$  the space of all measurable functions (with respect to the product measure)  $\xi : \Omega \times [a, b] \rightarrow \mathbb{R}^N$ , again suppressing the arrival set when  $N = 1$ . In all the above cases, we denote by  $\|\xi\|_p$  the standard  $L^p$ -norm of  $\xi$ .

Let  $(\mathcal{F}_t)$  be a filtration on  $\Omega$ . We denote by  $L^p_{\text{ad}}(\Omega \times [a, b]; \mathbb{R}^N)$  the space of  $p$ -summable stochastic processes, adapted to  $(\mathcal{F}_t)$ .

We will repeatedly use, in the sequel, the following classical estimate for solutions of the stochastic differential equation

$$(2.1) \quad dY_t = \zeta_t dt + \sigma(Y_t, t) dW_t,$$

where  $(W_t)$  is a  $D$ -dimensional Brownian motion adapted to  $(\mathcal{F}_t)$ .

**Lemma 2.1.** *Let  $\sigma : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}^{N \times D}$  be a Lipschitz continuous map such that  $\|\sigma\| \leq \delta$ , let  $\zeta \in L^p_{\text{ad}}(\Omega \times [0, T]; \mathbb{R}^N)$  ( $p > 1$ ), and let  $Y$  be a solution of (2.1). Then, for every  $r \in (0, p]$  there is a positive constant  $C(r)$  such that*

$$(2.2) \quad \mathbb{E}[|Y_t - Y_s|^r] \leq C(r) \left\{ \mathbb{E} \left[ \left| \int_s^t \zeta_\tau d\tau \right|^r \right] + \delta^r |t - s|^{\frac{r}{2}} \right\} \quad \forall s, t \in [0, T].$$

*Proof.* For every  $r \in (0, p]$  and any  $s, t \in [0, T]$  we have

$$\begin{aligned} \mathbb{E}[|Y_t - Y_s|^r] &= \mathbb{E} \left[ \left| \int_s^t \zeta_\tau d\tau + \int_s^t \sigma(Y_\tau, \tau) dW_\tau \right|^r \right] \\ &\leq C(r) \left\{ \mathbb{E} \left[ \left| \int_s^t \zeta_\tau d\tau \right|^r \right] + \mathbb{E} \left[ \left| \int_s^t \sigma(Y_\tau, \tau) dW_\tau \right|^r \right] \right\} \end{aligned}$$

where  $C(r) = 2^{[r-1]_+}$  (notice that  $r$  may be  $< 1$ ). Moreover, by the Burkholder-Davis-Gundy inequality and the bound on  $\sigma$ ,

$$\begin{aligned} \mathbb{E} \left[ \left| \int_s^t \sigma(Y_\tau, \tau) dW_\tau \right|^r \right] \\ \leq C(r) \mathbb{E} \left[ \left( \int_s^t \text{Tr}(\sigma(Y_\tau, \tau) \sigma^*(Y_\tau, \tau)) d\tau \right)^{\frac{r}{2}} \right] \leq C(r) \delta^r |t - s|^{\frac{r}{2}} \end{aligned}$$

for some constant  $C(r) \geq 0$ . The conclusion follows combining the above estimates.  $\square$

Let us finally recall some properties of Brownian bridges, which are one of the main ingredients of our method.

**Lemma 2.2.** *Let  $p \in (1, 2)$  and let  $\sigma : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}^{N \times D}$  be a Lipschitz continuous map such that  $\|\sigma\| \leq \delta$ . Then, for any  $x, y \in \mathbb{R}^N$  there is a process  $\zeta \in L^p_{\text{ad}}(\Omega \times [0, T]; \mathbb{R}^N)$  such that the solution to*

$$\begin{cases} dY_t = \zeta_t dt + \sigma(Y_t, t) dW_t \\ Y_0 = y \end{cases}$$

satisfies  $Y_T = x$  ( $\mathbb{P}$  a.s.) and

$$(2.3) \quad \mathbb{E} \left[ \int_0^T |\zeta_t|^p dt \right] \leq C(p, \delta) \left( T^{1-p} |y - x|^p + T^{1-p/2} \right).$$

Following [14],  $(Y_t)$  is called a Brownian bridge between  $(y, 0)$  and  $(x, T)$ . Estimate (2.3) can be found, e.g., in [20]. We give a proof of Lemma 2.2 for completeness.

*Proof.* Without loss of generality, we can assume that  $x = 0$ . Having fixed  $\alpha \in (1 - 1/p, 2)$  (for instance  $\alpha = 3/4 + 1/(2p)$ ), let  $Y_t$  be the solution to

$$\begin{cases} dY_t = -\alpha \frac{Y_t}{T-t} dt + \sigma(Y_t, t) dW_t \\ Y_0 = y \end{cases}$$

We claim that

$$(2.4) \quad Y_t = T^{-\alpha} (T-t)^\alpha y + (T-t)^\alpha \int_0^t \sigma(Y_s, s) (T-s)^{-\alpha} dW_s$$

and that (2.3) holds for  $\zeta_t \doteq -\alpha Y_t / (T-t)$ . Indeed, let

$$Z_t = T^{-\alpha} (T-t)^\alpha y + (T-t)^\alpha \int_0^t \sigma(Y_s, s) (T-s)^{-\alpha} dW_s.$$

Then  $Z_0 = y = Y_0$  and

$$\begin{aligned} dZ_t &= \left( -\alpha T^{-\alpha} (T-t)^{\alpha-1} y - \alpha (T-t)^{\alpha-1} \int_0^t \sigma(Y_s, s) (T-s)^{-\alpha} dW_s \right) dt \\ &\quad + (T-t)^\alpha \sigma(Y_t, t) (T-t)^{-\alpha} dW_t \\ &= -\alpha (T-t)^{-1} Z_t dt + \sigma(Y_t, t) dW_t \end{aligned}$$

Hence,  $Z_t = Y_t$  by uniqueness. Equality (2.4) also implies that  $Y_T = 0$  ( $\mathbb{P}$  a.s.). Let us now show that (2.3) holds. We have

$$\zeta_t = -\alpha \frac{Y_t}{T-t} = -\alpha T^{-\alpha} (T-t)^{\alpha-1} y - \alpha (T-t)^{\alpha-1} \int_0^t \sigma(Y_s, s) (T-s)^{-\alpha} dW_s$$

Therefore,

$$\begin{aligned}
& \mathbb{E} \left[ \int_0^T |\zeta_t|^p dt \right] \\
& \leq 2^{p-1} \alpha^p T^{-\alpha p} |y|^p \int_0^T (T-t)^{p(\alpha-1)} dt \\
& \quad + 2^{p-1} \alpha^p \int_0^T (T-t)^{p(\alpha-1)} \mathbb{E} \left[ \left( \int_0^t \sigma(Y_s, s) (T-s)^{-\alpha} dW_s \right)^p \right] dt \\
& \leq C(p) T^{1-p} |y|^p \\
& \quad + 2^{p-1} C(p) \alpha^p \int_0^T (T-t)^{p(\alpha-1)} \\
& \quad \quad \mathbb{E} \left[ \left( \int_0^t \text{Tr}(\sigma(Y_s, s) \sigma^*(Y_s, s)) (T-s)^{-2\alpha} ds \right)^{p/2} \right] dt \\
& \leq C(p) T^{1-p} |y|^p + C(p, \delta) T^{(1-2\alpha)p/2} \int_0^T (T-t)^{p(\alpha-1)} dt \\
& \leq C(p) T^{1-p} |y|^p + C(p, \delta) T^{1-p/2},
\end{aligned}$$

the second estimate above being justified by the Burkholder-Davis-Gundy inequality.  $\square$

### 3 Weak reverse Hölder inequalities

Though sharing the same flavor of most results of common use, the following reverse Hölder inequality lemma, obtained in [7], exhibits important differences in both assumptions and conclusion. Since this is absolutely essential to our approach, we will give a new proof of it which exploits a technique due to [12].

**Lemma 3.1.** *Let  $p > 1$  and let  $\phi \in L^p(a, b)$  be a nonnegative function such that*

$$(3.1) \quad \frac{1}{t-a} \int_a^t \phi^p(s) ds \leq A \left( \frac{1}{t-a} \int_a^t \phi(s) ds \right)^p \quad \forall t \in (a, b]$$

for some constant  $A > 1$ . Then, there are constants  $\theta = \theta(p, A) > p$  and  $C = C(p, A) \geq 0$  such that

$$(3.2) \quad \int_a^t \phi(s) ds \leq C(t-a)^{1-\frac{1}{\theta}} (b-a)^{\frac{1}{\theta}-\frac{1}{p}} \|\phi\|_p \quad \forall t \in [a, b].$$

*Remark 3.2.* Observe that, by Hölder's inequality,

$$\int_a^t \phi(s) ds \leq (t-a)^{1-\frac{1}{p}} \|\phi\|_p \quad \forall t \in [a, b].$$

So, the interest of the above lemma lies in the fact that (3.2) provides the exponent  $1 - 1/\theta$  for  $(t-a)$ , which is higher than  $1 - 1/p$ .

*Proof.* Without loss of generality we can assume  $a = 0$  and  $b = 1$ , the general form of the result being easy to recover by a rescaling argument. Let us further assume

$$(3.3) \quad \phi(t) \leq \phi_0 \quad t \in [0, t_0] \quad \text{a.e.}$$

for some constant  $\phi_0 \geq 0$  and some  $t_0 \in (0, 1)$ . Define

$$f(s) = \frac{1}{s} \int_0^s \phi(t) dt \quad \forall s \in (0, 1]$$

and observe that, just like  $\phi$ ,  $f$  is bounded in a neighborhood of 0. Now, let  $\theta > p$  and recall Hardy's inequality (see, e.g., [16])

$$(3.4) \quad \left( \frac{\theta}{\theta - 1} \right)^p \int_0^1 s^{\frac{p}{\theta} - 1} \phi^p(s) ds \geq \int_0^1 s^{\frac{p}{\theta} - 1} f^p(s) ds.$$

Moreover, observe that, in view of (3.1),

$$(3.5) \quad \begin{aligned} \int_0^1 s^{\frac{p}{\theta} - 1} f^p(s) ds &\geq \frac{1}{A} \int_0^1 s^{\frac{p}{\theta} - 2} \left( \int_0^s \phi^p(t) dt \right) ds \\ &= \frac{1}{A} \int_0^1 \phi^p(t) \left( \int_t^1 s^{\frac{p}{\theta} - 2} ds \right) dt = \frac{\theta}{(p - \theta)A} \left( \int_0^1 \phi^p(t) dt - \int_0^1 t^{\frac{p}{\theta} - 1} \phi^p(t) dt \right). \end{aligned}$$

Then, combine (3.4) and (3.5) to obtain

$$\left[ \frac{\theta}{(p - \theta)A} + \left( \frac{\theta}{\theta - 1} \right)^p \right] \int_0^1 s^{\frac{p}{\theta} - 1} \phi^p(s) ds \geq \frac{\theta}{(p - \theta)A} \int_0^1 \phi^p(s) ds$$

or

$$\left[ \frac{\theta}{(\theta - p)A} - \left( \frac{\theta}{\theta - 1} \right)^p \right] \int_0^1 s^{\frac{p}{\theta} - 1} \phi^p(s) ds \leq \frac{\theta}{(\theta - p)A} \int_0^1 \phi^p(s) ds.$$

Finally, choose  $\theta = \theta(p, A) > p$  such that

$$\frac{\theta}{(\theta - p)A} > \left( \frac{\theta}{\theta - 1} \right)^p$$

to deduce that, for some constant  $C = C(p, A)$ ,

$$(3.6) \quad \int_0^1 s^{\frac{p}{\theta} - 1} \phi^p(s) ds \leq C \int_0^1 \phi^p(s) ds.$$

At this point, the conclusion follows from Hölder's inequality and (3.6): denoting by  $q$  (resp.  $\theta'$ ) the conjugate exponent of  $p$  (resp.  $\theta$ ), we have

$$\begin{aligned} \int_0^t \phi(s) ds &\leq \left( \int_0^t s^{p(\frac{1}{\theta} - \frac{1}{p})} \phi^p(s) ds \right)^{\frac{1}{p}} \left( \int_0^t s^{q(\frac{1}{p} - \frac{1}{\theta})} ds \right)^{\frac{1}{q}} \\ &= \left( \int_0^1 s^{\frac{p}{\theta} - 1} \phi^p(s) ds \right)^{\frac{1}{p}} \left( \frac{q}{\theta'} \right)^{\frac{1}{q}} t^{1 - \frac{1}{\theta}} \leq C^{\frac{1}{p}} \left( \frac{q}{\theta'} \right)^{\frac{1}{q}} t^{1 - \frac{1}{\theta}} \|\phi\|_p \end{aligned}$$

To complete the proof it remains to dispose of assumption (3.3). For any  $\tau \in (0, 1]$ , set

$$\phi_\tau(s) = \begin{cases} \frac{1}{\tau} \int_0^\tau \phi(t) dt & \text{if } s \in [0, \tau] \\ \phi(s) & \text{otherwise.} \end{cases}$$

Then,  $\phi_\tau$  is bounded near 0 and  $\phi_\tau \in L^p(0, 1)$ . We claim that (3.1) is still true for  $\phi_\tau$ . Indeed, this is obvious if  $t \in [0, \tau]$ . On the other hand, for any  $t \in (\tau, 1]$ ,

$$\begin{aligned} \frac{1}{t} \int_0^t \phi_\tau^p(s) ds &= \frac{\tau}{t} \left( \frac{1}{\tau} \int_0^\tau \phi(s) ds \right)^p + \frac{1}{t} \int_\tau^t \phi^p(s) ds \leq \frac{1}{t} \int_0^t \phi^p(s) ds \\ &\leq A \left( \frac{1}{t} \int_0^t \phi(s) ds \right)^p = A \left( \frac{1}{t} \int_0^t \phi_\tau(s) ds \right)^p. \end{aligned}$$

Therefore, owing to the first part of the proof,

$$\int_0^t \phi_\tau(s) ds \leq C t^{1-\frac{1}{\theta}} \|\phi_\tau\|_p \quad \forall t \in [0, 1].$$

Letting  $\tau \rightarrow 0^+$  gives (3.2).  $\square$

We now give an adaptation of Lemma 3.1 that will be used in what follows.

**Lemma 3.3.** *Let  $p > 1$  and let  $\phi \in L^p(a, b)$  be a nonnegative function such that*

$$(3.7) \quad \frac{1}{b-t} \int_t^b \phi^p(s) ds \leq A \left( \frac{1}{b-t} \int_t^b \phi(s) ds \right)^p + B \quad \forall t \in [a, b]$$

for some constants  $A > 1$  and  $B \geq 0$ .

Then, there are constants<sup>1</sup>  $\theta = \theta(p, A) > p$  and  $C = C(p, A) \geq 0$  such that

$$(3.8) \quad \int_t^b \phi(s) ds \leq C(b-t)^{1-\frac{1}{\theta}} \left\{ (b-a)^{\frac{1}{\theta}-\frac{1}{p}} \|\phi\|_p + B^{1/p} (b-a)^{\frac{1}{\theta}} \right\}$$

for every  $t \in [a, b]$ .

*Proof.* Let  $\psi(s) \doteq \phi(s) + k$  where  $k \doteq B^{1/p}/(A^{1/p} - 1)$ . In view of (3.7), we have

$$\begin{aligned} \left( \frac{1}{b-t} \int_t^b \psi^p(s) ds \right)^{1/p} &\leq \left( \frac{1}{b-t} \int_t^b \phi^p(s) ds \right)^{1/p} + k \\ &\leq \frac{A^{1/p}}{b-t} \int_t^b \phi(s) ds + B^{1/p} + k = \frac{A^{1/p}}{b-t} \int_t^b \psi(s) ds \end{aligned}$$

for every  $t \in [a, b]$ . Therefore, Lemma 3.1 applied to  $\psi$  yields—after a change of variable—the existence of constants  $\theta > p$  and  $C \geq 0$ , depending on  $A$  and  $p$  only, such that

$$\int_t^b \phi(s) ds = \int_t^b \psi(s) ds - k(b-t) \leq C(b-t)^{1-\frac{1}{\theta}} (b-a)^{\frac{1}{\theta}-\frac{1}{p}} \|\psi\|_p.$$

<sup>1</sup> These are the same constants given by Lemma 3.1.

The proof can now be completed noting that  $\|\psi\|_p \leq \|\phi\|_p + k(b-a)^{1/p}$ .  $\square$

We conclude this section with a generalization of Lemma 3.1 to stochastic processes, which will be needed to study second order problems.

**Lemma 3.4.** *Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space. Let  $p \in (1, 2)$ ,  $r \in (0, 1)$  and let us assume that  $\xi \in L^p(\Omega \times (a, b))$  is a nonnegative function such that*

$$(3.9) \quad \mathbb{E} \left[ \frac{1}{t-a} \int_a^t \xi_s^p ds \right] \leq A \mathbb{E} \left[ \left( \frac{1}{t-a} \int_a^t \xi_s ds \right)^p \right] + \frac{B}{(t-a)^r} \quad \forall t \in (a, b]$$

for some positive constants  $A$  and  $B$ . Then there are constants  $\theta \in (p, 2 \wedge (p/r))$  and  $C > 0$ , depending only on  $p, A$  and  $r$ , such that

$$\mathbb{E} \left[ \left( \int_a^t \xi_s ds \right)^p \right] \leq C(t-a)^{p-\frac{p}{\theta}} \left\{ (b-a)^{\frac{p}{\theta}-1} \|\xi\|_p^p + B(b-a)^{\frac{p}{\theta}-r} \right\} \quad \forall t \in (a, b].$$

*Proof.* Under the extra assumption that  $\xi_t$  is bounded ( $\mathbb{P}$  a.s.) for a.e.  $t$  near  $a$ , say for a.e.  $t \in (a, t_0)$ , let us define

$$z_t = \frac{1}{t-a} \int_a^t \xi_s ds \quad (\mathbb{P} \text{ a.s.}) \quad \forall t \in (a, b].$$

Then, for any  $\theta \in (p, 2 \wedge \frac{p}{r})$ , Hardy's inequality (3.4) yields

$$(3.10) \quad \left( \frac{\theta}{\theta-1} \right)^p \int_a^b (t-a)^{\frac{p}{\theta}-1} \xi_t^p dt \geq \int_a^b (t-a)^{\frac{p}{\theta}-1} z_t^p dt \quad (\mathbb{P} \text{ a.s.})$$

Owing to assumption (3.9), we have

$$(3.11) \quad \begin{aligned} \mathbb{E} \left[ \int_a^b (t-a)^{\frac{p}{\theta}-1} z_t^p dt \right] &= \int_a^b (t-a)^{\frac{p}{\theta}-1} \mathbb{E} \left[ \left( \frac{1}{t-a} \int_a^t \xi_s ds \right)^p \right] dt \\ &\geq \frac{1}{A} \int_a^b (t-a)^{\frac{p}{\theta}-1} \left\{ \mathbb{E} \left[ \frac{1}{t-a} \int_a^t \xi_s^p ds \right] - \frac{B}{(t-a)^r} \right\} dt \\ &= \frac{1}{A} \mathbb{E} \left[ \int_a^b (t-a)^{\frac{p}{\theta}-2} \int_a^t \xi_s^p ds dt \right] - \frac{B}{A} \int_a^b (t-a)^{\frac{p}{\theta}-r-1} dt \\ &= \frac{\theta}{(p-\theta)A} \mathbb{E} \left[ \int_a^b \left\{ (b-a)^{\frac{p}{\theta}-1} - (s-a)^{\frac{p}{\theta}-1} \right\} \xi_s^p ds \right] \\ &\quad - \frac{\theta B}{(p-r\theta)A} (b-a)^{\frac{p}{\theta}-r} \end{aligned}$$

Now, taking the expectation of both sides of (3.10) and chaining the resulting estimate with (3.11) we obtain

$$\begin{aligned} &\left\{ \frac{\theta}{(p-\theta)A} + \left( \frac{\theta}{\theta-1} \right)^p \right\} \mathbb{E} \left[ \int_a^b (t-a)^{\frac{p}{\theta}-1} \xi_t^p dt \right] \\ &\geq \frac{\theta}{(p-\theta)A} (b-a)^{\frac{p}{\theta}-1} \mathbb{E} \left[ \int_a^b \xi_t^p dt \right] - \frac{\theta B}{(p-r\theta)A} (b-a)^{\frac{p}{\theta}-r} \end{aligned}$$

or

$$\begin{aligned} & \left\{ \frac{\theta}{(\theta-p)A} - \left( \frac{\theta}{\theta-1} \right)^p \right\} \mathbb{E} \left[ \int_a^b (t-a)^{\frac{p}{\theta}-1} \xi_t^p dt \right] \\ & \leq \frac{\theta}{(\theta-p)A} (b-a)^{\frac{p}{\theta}-1} \mathbb{E} \left[ \int_a^b \xi_t^p dt \right] + \frac{\theta B}{(p-r\theta)A} (b-a)^{\frac{p}{\theta}-r} \end{aligned}$$

Thus, choosing  $\theta = \theta(p, A, r) \in (p, 2 \wedge (p/r))$  such that

$$\frac{\theta}{(\theta-p)A} > \left( \frac{\theta}{\theta-1} \right)^p$$

we deduce that, for some positive constant  $C = C(p, A, r)$ ,

$$(3.12) \quad \mathbb{E} \left[ \int_a^b (t-a)^{\frac{p}{\theta}-1} \xi_t^p dt \right] \leq C \left\{ (b-a)^{\frac{p}{\theta}-1} \mathbb{E} \left[ \int_a^b \xi_t^p dt \right] + B(b-a)^{\frac{p}{\theta}-r} \right\}.$$

By Hölder's inequality and (3.12), we have, for all  $t \in (a, b]$ ,

$$\begin{aligned} \mathbb{E} \left[ \left( \int_a^t \xi_s ds \right)^p \right] &= \mathbb{E} \left[ \left( \int_a^t (s-a)^{\frac{1}{p}-\frac{1}{\theta}} (s-a)^{\frac{1}{\theta}-\frac{1}{p}} \xi_s ds \right)^p \right] \\ &\leq \mathbb{E} \left[ \left( \int_a^t (s-a)^{\frac{q}{p}-\frac{q}{\theta}} \right)^{\frac{p}{q}} \int_a^t (s-a)^{\frac{p}{\theta}-1} \xi_s^p ds \right] \\ &\leq C(t-a)^{p-\frac{p}{\theta}} \left\{ (b-a)^{\frac{p}{\theta}-1} \mathbb{E} \left[ \int_a^b \xi_t^p dt \right] + B(b-a)^{\frac{p}{\theta}-r} \right\} \end{aligned}$$

which in turn implies the conclusion. The extra assumption that  $\xi_t$  is bounded near  $a$  can be removed arguing as in last part of the proof of Lemma 3.1.  $\square$

## 4 First order equations

In this section we consider the first order Hamilton-Jacobi equation

$$(4.1) \quad u_t + H(x, t, Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, T)$$

where  $u_t = \frac{\partial u}{\partial t}$  and  $Du = \left( \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \right)$ . The data  $H$  and  $u$  will be assumed to satisfy the following hypotheses:

- $H : \mathbb{R}^N \times (0, T) \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a continuous function such that

$$(4.2) \quad \frac{1}{\delta} |z|^q - \eta_- \leq H(x, t, z) \leq \delta |z|^q + \eta_+ \quad \forall (x, t, z) \in \mathbb{R}^N \times (0, T) \times \mathbb{R}^N$$

for some constants  $q > 1, \delta > 1$  and  $\eta_{\pm} \geq 0$ ;

- $u : \mathbb{R}^N \times (0, T) \rightarrow \mathbb{R}$  is a bounded continuous viscosity solution of (4.1).

Moreover,  $p$  denotes the conjugate exponent of  $q$ , i.e.,  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $M$  is a positive constant such that

$$(4.3) \quad |u(x, t)| \leq M \quad \forall (x, t) \in \mathbb{R}^N \times (0, T)$$

(e.g.  $M = \sup_{(x,t) \in \mathbb{R}^N \times (0,T)} |u(x,t)|$ ).

In what follows, a (*universal*) *constant* is a positive number depending on the given data  $q, \delta, N, \eta_{\pm}$  and  $M$  only. Universal constants will be typically labeled with  $C$ , but also with different letters (e.g.,  $\theta, A, \dots$ ). Dependence on extra quantities will be accounted for by using parentheses (e.g.,  $C(r)$  denotes a constant depending also on  $r$ ).

**Theorem 4.1.** *Let  $u \in C(\mathbb{R}^N \times [0, T])$  be a viscosity solution of (4.1) satisfying (4.3). Then there is a universal constant  $\theta > p$  such that, for any  $\tau \in (0, T]$ ,*

$$(4.4) \quad |u(x, t) - u(y, s)| \leq C(\tau) \left[ |x - y|^{\frac{\theta-p}{\theta-1}} + |t - s|^{\frac{\theta-p}{\theta}} \right]$$

for all  $(x, t), (y, s) \in \mathbb{R}^N \times [\tau, T]$  and some constant  $C(\tau) > 0$ .

*Remark 4.2.* The main point of the above theorem is that estimate (4.4) holds uniformly with respect to  $H$  and  $u$ , as long as conditions (4.2) and (4.3) hold true. In particular,  $C(\tau)$  is independent of the continuity modulus of  $H$ .

#### 4.1 Proof of Theorem 4.1

Let us set

$$H_+(z) = \delta|z|^q + \eta_+ \quad \text{and} \quad H_-(z) = \frac{1}{\delta}|z|^q - \eta_- \quad z \in \mathbb{R}^N.$$

We begin the analysis with a kind of optimality principle for super-solutions.

**Lemma 4.3.** *Let  $u \in C(\mathbb{R}^N \times [0, T])$  be a viscosity super-solution of*

$$(4.5) \quad u_t + H_+(Du) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, T)$$

*satisfying (4.3). Then, for any  $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, T]$  there is an arc  $\xi \in W^{1,p}([0, \bar{t}]; \mathbb{R}^N)$ , satisfying the end-point condition  $\xi(\bar{t}) = \bar{x}$ , such that*

$$(4.6) \quad u(\bar{x}, \bar{t}) \geq u(\xi(t), t) + C_+ \int_t^{\bar{t}} |\xi'(s)|^p ds - \eta_+(\bar{t} - t) \quad \forall t \in [0, \bar{t}]$$

where

$$(4.7) \quad C_+ = \frac{\delta^{-p/q}}{pq^{p/q}}.$$

*Proof.* The requested arc will be provided by an approximation procedure. Let  $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, T]$ . For any positive integer  $n$  let us set

$$\tau_n = \frac{\bar{t}}{n}, \quad t_k = k\tau_n \quad (k \in \{0, \dots, n\}).$$

We shall first construct a finite set of points  $(x_k)_{k=0}^n$  such that  $x_n = \bar{x}$  and

$$(4.8) \quad u(x_k, t_k) \geq u(x_{k-1}, t_{k-1}) + C_+ \tau_n \left| \frac{x_{k-1} - x_k}{\tau_n} \right|^p - \eta_+ \tau_n$$

with  $C_+$  given by (4.7). Having set  $x_n = \bar{x}$ , we proceed as follows to construct  $x_{k-1}$  from  $x_k$  that we assume given for some  $k \in \{1, \dots, n\}$ . Let  $v_k$  be the viscosity solution of

$$(4.9) \quad \begin{cases} v_t + H_+(Dv) = 0 & \text{in } \mathbb{R}^N \times (t_{k-1}, T) \\ v(x, t_{k-1}) = u(x, t_{k-1}) & x \in \mathbb{R}^N. \end{cases}$$

As is well-known,  $v_k$  is given by Hopf's formula

$$v_k(x, t) = \min_{y \in \mathbb{R}^N} \left\{ (t - t_{k-1}) H_+^* \left( \frac{y - x}{t - t_{k-1}} \right) + u(y, t_{k-1}) \right\} \quad \forall (x, t) \in \mathbb{R}^N \times (t_{k-1}, T],$$

where  $H_+^*$  is the convex conjugate of  $H_+$ , i.e.,

$$H_+^*(w) = \max_{z \in \mathbb{R}^N} \{ z \cdot w - H_+(z) \} = C_+ |w|^p - \eta_+$$

with  $C_+$  given by (4.7). Since  $u$  is a super-solution of (4.9), the comparison principle yields  $u(\cdot, t) \geq v_k(\cdot, t)$  for any  $t \in [t_{k-1}, T]$ . In particular, for  $t = t_k$ , we obtain that, for some point  $x_{k-1} \in \mathbb{R}^N$ ,

$$u(x_k, t_k) \geq v_k(x_k, t_k) = u(x_{k-1}, t_{k-1}) + C_+ \tau_n \left| \frac{x_{k-1} - x_k}{\tau_n} \right|^p - \eta_+ \tau_n.$$

The construction of  $(x_k)_{k=0}^n$  can thus be completed by finite backward induction.

Next, for any positive integer  $n$ , let  $\xi_n : [0, \bar{t}] \rightarrow \mathbb{R}^N$  be the piecewise linear interpolation of the above set  $(x_k)_{k=0}^n$  such that  $\xi_n(t_k) = x_k$  for any  $k = 0, \dots, n$ . We note that (4.8) can be rewritten as

$$u(\xi_n(t_k), t_k) \geq u(\xi_n(t_{k-1}), t_{k-1}) + C_+ \int_{t_{k-1}}^{t_k} |\xi_n'(s)|^p ds - \eta_+(t_k - t_{k-1}) \quad (k = 1, \dots, n).$$

Summing up the above inequalities gives

$$(4.10) \quad u(\bar{x}, \bar{t}) \geq u(\xi_n(t_k), t_k) + C_+ \int_{t_k}^{\bar{t}} |\xi_n'(s)|^p ds - \eta_+(\bar{t} - t_k) \quad (k = 0, \dots, n).$$

Since  $u$  is bounded,  $(\xi_n)_{n \in \mathbb{N}}$  is bounded in  $W^{1,p}([0, \bar{t}]; \mathbb{R}^N)$ . Then, there is a subsequence of  $(\xi_n)_{n \in \mathbb{N}}$  which weakly converges in  $W^{1,p}([0, \bar{t}]; \mathbb{R}^N)$  (hence converges uniformly) to some limit arc  $\xi$  which satisfies  $\xi(\bar{t}) = \bar{x}$ . Passing to the limit in (4.10) for such a subsequence gives (4.6).  $\square$

*Remark 4.4.* Observe that, owing to (4.3), any arc  $\xi$  which satisfies (4.6) can be bounded above as follows

$$(4.11) \quad \int_0^{\bar{t}} |\xi'(s)|^p ds \leq \frac{2M + \eta_+ \bar{t}}{C_+}.$$

We now turn to the analysis of sub-solutions of

$$(4.12) \quad u_t + H_-(Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, T)$$

**Lemma 4.5.** *Let  $u \in C(\mathbb{R}^N \times [0, T])$  be a viscosity sub-solution of (4.12). Then, for any  $s, t \in [0, T]$ , with  $s > t$ ,*

$$u(x, s) \leq u(y, t) + C_- (s-t)^{1-p} |y-x|^p + \eta_-(s-t) \quad \forall x, y \in \mathbb{R}^N,$$

where

$$(4.13) \quad C_- = \frac{\delta^{p/q}}{pq^{p/q}}.$$

*Proof.* Let  $\bar{v}$  be the viscosity solution of

$$(4.14) \quad \begin{cases} v_t + H_-(Dv) = 0 & \text{in } \mathbb{R}^N \times (t, T) \\ v(x, t) = u(x, t) & x \in \mathbb{R}^N. \end{cases}$$

By Hopf's formula,

$$(4.15) \quad \bar{v}(x, s) = \min_{y \in \mathbb{R}^N} \left\{ (s-t) H_-^* \left( \frac{y-x}{s-t} \right) + u(y, t) \right\} \quad \forall (x, s) \in \mathbb{R}^N \times (t, T),$$

where  $H_-^*$ , the convex conjugate of  $H_-$ , is given by

$$H_-^*(w) = \max_{z \in \mathbb{R}^N} \{ z \cdot w - H_-(z) \} = C_- |w|^p + \eta_-.$$

Since  $u$  is a sub-solution of (4.14), by comparison  $u(\cdot, s) \leq \bar{v}(\cdot, s)$  for all  $s \in [t, T]$ . So, (4.15) yields

$$u(x, s) \leq \bar{v}(x, s) \leq C_- (s-t)^{1-p} |y-x|^p + \eta_-(s-t) + u(y, t)$$

for all  $(y, s) \in \mathbb{R}^N \times (t, T)$ , as desired.  $\square$

Next, we derive a weak reverse Hölder inequality for the arcs that satisfy (4.6).

**Lemma 4.6.** *Let:*

- $u \in C(\mathbb{R}^N \times [0, T])$  be a viscosity sub-solution of (4.12) satisfying (4.3);
- $\xi \in W^{1,p}([0, \bar{t}]; \mathbb{R}^N)$  be an arc satisfying (4.6) with  $\bar{x} = \xi(\bar{t})$ .

Then

$$(4.16) \quad \frac{1}{\bar{t}-t} \int_t^{\bar{t}} |\xi'(s)|^p ds \leq C_1 \left( \frac{1}{\bar{t}-t} \int_t^{\bar{t}} |\xi'(s)| ds \right)^p + C_0 \quad \forall t \in [0, \bar{t}]$$

where

$$(4.17) \quad C_0 = \frac{\eta_+ + \eta_-}{C_+} \quad \text{and} \quad C_1 = \delta^{2p/q} > 1.$$

*Proof.* Let  $t \in [0, \bar{t}]$ . By Lemma 4.5,

$$\begin{aligned} u(\bar{x}, \bar{t}) &\leq u(\xi(t), t) + (\bar{t}-t) C_- \left| \frac{\xi(t) - \bar{x}}{\bar{t}-t} \right|^p + \eta_-(\bar{t}-t) \\ &\leq u(\xi(t), t) + (\bar{t}-t) C_- \left( \frac{1}{\bar{t}-t} \int_t^{\bar{t}} |\xi'(s)| ds \right)^p + \eta_-(\bar{t}-t). \end{aligned}$$

Combining (4.6) with the above inequality we obtain

$$C_+ \int_t^{\bar{t}} |\xi'(s)|^p ds - \eta_+(\bar{t}-t) \leq (\bar{t}-t) C_- \left( \frac{1}{\bar{t}-t} \int_t^{\bar{t}} |\xi'(s)| ds \right)^p + \eta_-(\bar{t}-t),$$

which in turns implies (4.16).  $\square$

In view of the above results, Lemma 3.3 yields the following.

**Corollary 4.7.** *Let  $u \in C(\mathbb{R}^N \times [0, T])$  be a viscosity solution of*

$$(4.18) \quad u_t + H(x, t, Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, T)$$

*satisfying (4.3), and let  $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, T]$ . Then there is an arc  $\xi \in W^{1,p}([0, \bar{t}]; \mathbb{R}^N)$ , satisfying the end-point condition  $\xi(\bar{t}) = \bar{x}$  and inequality (4.6), and a constant  $\theta > p$ , depending only on  $q$  and  $\delta$ , such that*

$$(4.19) \quad \int_t^{\bar{t}} |\xi'(s)| ds \leq C(\tau) (\bar{t}-t)^{1-\frac{1}{\theta}} \quad \forall t \in [0, \bar{t}]$$

*for some constant  $C(\tau) \geq 0$ .*

*Proof.* First observe that, owing to (4.2),  $u$  is a super-solution of (4.5) and a sub-solution of (4.12). Then, Lemma 4.3 can be applied to construct an arc  $\xi$  satisfying (4.6) together with  $\xi(\bar{t}) = \bar{x}$ , while Lemma 4.6 ensures that (4.16) holds true for  $C_0, C_1$  given by (4.17). So, Lemma 3.3 implies the existence of constant  $\theta > p$  and  $C_3 > 0$ , depending only on  $C_1 = \delta^{2p/q}$ , such that

$$\int_t^{\bar{t}} |\xi'(s)| ds \leq C_3 \left\{ \|\xi\|_p + k\bar{t}^{\frac{1}{p}} \right\} \frac{(\bar{t}-t)^{1-\frac{1}{\theta}}}{\bar{t}^{\frac{1}{p}-\frac{1}{\theta}}} \quad \forall t \in [0, \bar{t}],$$

where  $k = C_0^{1/p} / (C_1^{1/p} - 1)$ . Using the definition of  $C_0, C_1$  and upper bound (4.11) for  $\|\xi\|_p$  yields (4.19).  $\square$

*Proof of Theorem 4.1.* We will obtain two Hölder estimates in space and time, respectively, each of which will be uniform in the other variable.

*Space regularity.* Fix  $\tau \in (0, T]$ . Let  $\bar{t} \in [\tau, T]$  and let  $x, \bar{x} \in \mathbb{R}^N, x \neq \bar{x}$ . From Lemma 4.5,  $u(x, \bar{t})$  is bounded from above by

$$u(x, \bar{t}) \leq u(y, t) + C_-(\bar{t}-t)^{1-p} |y-x|^p + \eta_-(\bar{t}-t) \quad \forall y \in \mathbb{R}^N,$$

for all  $t \in [0, \bar{t})$ . Taking, in such an expression,  $y = \xi(t)$ , where  $\xi$  is the arc provided by the conclusion of Corollary 4.7, yields, owing to (4.6),

$$\begin{aligned} u(x, \bar{t}) &\leq (\bar{t}-t) C_- \left| \frac{\xi(t) - x}{\bar{t}-t} \right|^p + \eta_-(\bar{t}-t) + u(\xi(t), t) \\ &\leq (\bar{t}-t)^{1-p} C_- [|\xi(t) - \bar{x}| + |\bar{x} - x|]^p + u(\bar{x}, \bar{t}) + \eta(\bar{t}-t) \end{aligned}$$

for every  $t \in [0, \bar{t})$ , where  $\eta = \eta_- + \eta_+$ . Therefore, since

$$|\xi(t) - \bar{x}| \leq \int_t^{\bar{t}} |\xi'(s)| ds \leq C(\tau) (\bar{t}-t)^{1-\frac{1}{\theta}} \quad \forall t \in [0, \bar{t}],$$

on account of (4.19), we obtain

$$(4.20) \quad u(x, \bar{t}) - u(\bar{x}, \bar{t}) \leq (\bar{t} - t)^{1-p} C_- \left\{ C(\tau) (\bar{t} - t)^{1-\frac{1}{\theta}} + |\bar{x} - x| \right\}^p + \eta(\bar{t} - t)$$

for every  $t \in [0, \bar{t}]$ . Now, suppose  $|\bar{x} - x| < \min\{1, \tau^{1-1/\theta}\}$ . Then there is a number  $t \in [0, \bar{t}]$  such that

$$\bar{t} - t = |\bar{x} - x|^{\frac{\theta}{\theta-1}}.$$

Hence, (4.20) yields

$$u(x, \bar{t}) - u(\bar{x}, \bar{t}) \leq C(\tau) |\bar{x} - x|^{\frac{\theta-p}{\theta-1}}$$

for some new constant  $C(\tau) > 0$ . On the other hand, the above inequality is trivial for  $|\bar{x} - x| \geq \min\{1, \tau^{1-1/\theta}\}$  since  $u$  is bounded. Moreover, the reasoning is symmetric with respect to  $x$  and  $\bar{x}$ . So, we have shown that

$$(4.21) \quad |u(x, \bar{t}) - u(\bar{x}, \bar{t})| \leq C(\tau) |\bar{x} - x|^{\frac{\theta-p}{\theta-1}} \quad \forall x, \bar{x} \in \mathbb{R}^N.$$

*Time regularity.* Let  $\bar{x} \in \mathbb{R}^N$  and let  $\tau \leq t < \bar{t} \leq T$ . Applying Lemma 4.5 at  $x = \bar{x} = y$ , we obtain

$$(4.22) \quad u(\bar{x}, \bar{t}) - u(\bar{x}, t) \leq \eta_- (\bar{t} - t).$$

To estimate the above left-hand side from below, let  $\xi$  be as in the first part of the proof. Then, owing to (4.6),

$$(4.23) \quad u(\xi(t), t) \leq u(\bar{x}, \bar{t}) + \eta_+ (\bar{t} - t).$$

On the other hand, in view of (4.21) and (4.19),

$$(4.24) \quad \begin{aligned} u(\xi(t), t) &\geq u(\bar{x}, t) - C(\tau) |\xi(t) - \bar{x}|^{\frac{\theta-p}{\theta-1}} \\ &\geq u(\bar{x}, t) - C(\tau) \left( \int_t^{\bar{t}} |\xi'(s)| ds \right)^{\frac{\theta-p}{\theta-1}} \geq u(\bar{x}, t) - C(\tau) (\bar{t} - t)^{\frac{\theta-p}{\theta}}. \end{aligned}$$

Combining (4.23) and (4.24) we conclude that

$$u(\bar{x}, \bar{t}) - u(\bar{x}, t) \geq -\eta_+ (\bar{t} - t) - C(\tau) (\bar{t} - t)^{\frac{\theta-p}{\theta}}.$$

Since  $(\theta - p)/\theta < 1$ , recalling (4.22) we finally get

$$(4.25) \quad |u(\bar{x}, \bar{t}) - u(\bar{x}, t)| \leq C(\tau) |\bar{t} - t|^{(\theta-p)/\theta} \quad \forall \bar{x} \in \mathbb{R}^N, \forall t, \bar{t} \in [\tau, T].$$

The conclusion follows from (4.21) and (4.25).  $\square$

## 4.2 Local regularity for first order equations

The Hölder regularity result of the previous section can be given a “local version”, that is, a form that applies to solutions of the Hamilton-Jacobi equation

$$(4.26) \quad u_t + H(x, t, Du) = 0 \quad \text{in } \mathcal{O},$$

where  $\mathcal{O}$  is an open domain of  $\mathbb{R}_x^N \times \mathbb{R}_t$ . The Hamiltonian  $H : \mathcal{O} \times \mathbb{R}^N \rightarrow \mathbb{R}$  is still assumed to satisfy the growth condition

$$(4.27) \quad \frac{1}{\delta} |z|^q - \eta_- \leq H(x, t, z) \leq \delta |z|^q + \eta_+ \quad \forall (x, t, z) \in \mathcal{O} \times \mathbb{R}^N$$

for some constants  $q > 1, \delta \geq 1$  and  $\eta_{\pm} \geq 0$ . Recall that  $p$  is conjugate to  $q$ , and set

$$\mathcal{O}_\rho = \{(x, t) \in \mathcal{O} : d_{\mathcal{O}^c}(x, t) > \rho\} \quad \forall \rho > 0.$$

**Theorem 4.8.** *Let  $u$  be a continuous viscosity solution of (4.26) satisfying  $|u| \leq M$  in  $\mathcal{O}$  for some  $M \geq 0$ . Then there is a universal constant  $\theta > p$  and, for any  $\rho > 0$ , a constant  $C(\rho) \geq 0$  such that*

$$(4.28) \quad |u(x, t) - u(y, s)| \leq C(\rho) \left[ |x - y|^{\frac{\theta-p}{\theta-1}} + |t - s|^{\frac{\theta-p}{\theta}} \right] \quad \forall (x, t), (y, s) \in \mathcal{O}_\rho.$$

*Proof.* We will obtain the above Hölder bound on any sufficiently small parabolic cylinder contained in  $\mathcal{O}$ . Specifically, we will suppose that such a cylinder has the form  $B_\rho \times (\rho, T)$ , where  $T = 4\rho$  and  $\rho \in (0, 1/4]$  is chosen so as to satisfy (4.33) below. We will also assume that  $\bar{B}_{4\rho} \times [0, T] \subset \mathcal{O}$  since this implies no loss of generality. Again without loss of generality, we can—and will—assume that the Hamiltonian has been extended to  $\mathbb{R}_x^N \times \mathbb{R}_t \times \mathbb{R}^N$ , and that such an extension (still labeled by  $H$ ) coincides with the original Hamiltonian on  $B_{4\rho} \times (0, T)$  and satisfies (4.2) on the whole space with the same constants that appear in (4.27).

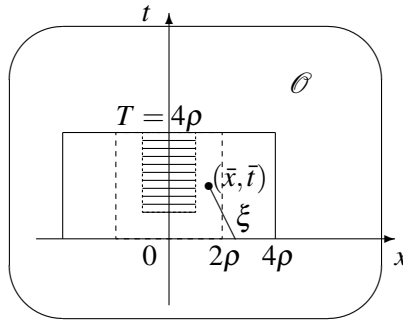


FIGURE 4.1. local Hölder continuity

**Step 1:** Let us show that there is a universal constant  $\alpha > 0$  such that, for any  $(\bar{x}, \bar{t}) \in B_{2\rho} \times (0, T)$ , there is an arc  $\xi \in W^{1,p}([0, \bar{t}]; \mathbb{R}^N)$ , with  $\xi(\bar{t}) = \bar{x}$ , satisfying

$$(4.29) \quad |\xi(t) - \bar{x}| \leq \rho$$

and

$$(4.30) \quad u(\bar{x}, \bar{t}) \geq u(\xi(t), t) + C_+ \int_t^{\bar{t}} |\xi'(s)|^p ds - \eta_+(\bar{t} - t)$$

for all  $t \in [(\bar{t} - \alpha \rho^{p/(p-1)})_+, \bar{t}]$ , where  $C_+$  is defined by (4.7).

*Proof of Step 1.* Let  $\phi(x) = \phi(|x|)$  be a smooth function defined on  $\mathbb{R}^N$  such that  $|\phi| \leq M$  in  $\mathbb{R}^N$ ,  $\phi \equiv M$  in  $B_{3\rho}$ , and  $\phi \equiv -M$  in  $B_{4\rho}^c$ . Since  $\phi$  is a trivial super-solution of (4.5), the function  $\bar{u} : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}$  defined by

$$\bar{u} = \begin{cases} u \wedge \phi & \text{in } B_{4\rho} \times [0, T] \\ -M & \text{in } B_{4\rho}^c \times [0, T] \end{cases}$$

is also a super-solution of (4.5) satisfying, thanks to (4.3),  $\bar{u} \equiv u$  in  $B_{3\rho} \times (0, T)$ . So, applying Lemma 4.3 to  $\bar{u}$  we deduce that for any  $(\bar{x}, \bar{t}) \in B_{2\rho} \times (0, T)$  there is an arc  $\xi \in W^{1,p}([0, \bar{t}]; \mathbb{R}^N)$ , with  $\xi(\bar{t}) = \bar{x}$ , such that

$$(4.31) \quad \bar{u}(\bar{x}, \bar{t}) \geq \bar{u}(\xi(t), t) + C_+ \int_t^{\bar{t}} |\xi'(s)|^p ds - \eta_+(\bar{t} - t) \quad \forall t \in [0, \bar{t}].$$

Moreover, recalling Remark 4.4,

$$\int_t^{\bar{t}} |\xi'(s)|^p ds \leq \frac{2M + \eta_+}{C_+} \quad \forall t \in [0, \bar{t}]$$

since  $\rho \leq 1/4$ . So, by Hölder's inequality,

$$(4.32) \quad |\xi(t) - \bar{x}| \leq \left( \frac{2M + \eta_+}{C_+} \right)^{\frac{1}{p}} (\bar{t} - t)^{1 - \frac{1}{p}} \quad \forall t \in [0, \bar{t}].$$

Therefore, taking

$$\alpha = \left( \frac{C_+}{2M + \eta_+} \right)^{\frac{1}{p-1}},$$

inequalities (4.31) and (4.32), combined with the fact that  $\bar{u} = u$  in  $B_{3\rho} \times (0, T)$ , give (4.29) and (4.30).  $\square$

Hereafter we will assume, without loss of generality, that  $\rho > 0$  is such that

$$(4.33) \quad \alpha \rho^{p/(p-1)} < \rho.$$

**Step 2:** Let  $(\bar{x}, \bar{t}) \in B_{2\rho} \times (\rho, T)$ . We will show that there exists universal constants  $\theta > p$  and  $C > 0$  such that, if  $\xi$  is an arc in  $W^{1,p}([0, \bar{t}]; \mathbb{R}^N)$  satisfying (4.29), (4.30) and  $\xi(\bar{t}) = \bar{x}$  (as in Step 1), then

$$(4.34) \quad \int_t^{\bar{t}} |\xi'(s)| ds \leq C \rho^{-\frac{\theta-p}{\theta(p-1)}} (\bar{t} - t)^{1 - \frac{1}{\theta}} \quad \forall t \in [\bar{t} - \alpha \rho^{p/(p-1)}, \bar{t}].$$

Moreover, for any  $(x, t) \in B_{3\rho} \times [0, \bar{t}]$  and  $y \in B_{3\rho}$ ,

$$(4.35) \quad u(x, \bar{t}) \leq u(y, t) + C_- (\bar{t} - t)^{1-p} |y - x|^p + C \rho^{-\frac{p}{p-1}} (\bar{t} - t).$$

where  $C_-$  is defined by (4.13).

*Proof of Step 2.* Let  $\phi$  be a function as in Step 1 such that  $\|D\phi\|_\infty \leq C/\rho$  for some universal constant  $C$ . Then,  $-\phi$  is a stationary sub-solution of

$$(4.36) \quad w_t + \tilde{H}_-(Dw) = 0 \quad \text{in } \mathbb{R}^N \times (0, T).$$

where

$$\begin{cases} \tilde{H}_-(z) = \frac{1}{\delta}|z|^q - \tilde{\eta}_- & z \in \mathbb{R}^N \\ \tilde{\eta}_- = \max\{\eta_-, C^q/(\delta\rho^q)\}. \end{cases}$$

Let us set

$$\tilde{u} = \begin{cases} u \wedge (-\phi) & \text{in } B_{4\rho} \times (0, T) \\ M & \text{in } B_{4\rho}^c \times (0, T). \end{cases}$$

Note that  $\tilde{u}$  is a sub-solution of (4.36) such that  $\tilde{u} = u$  in  $B_{3\rho} \times (0, T)$ , because  $u$  is a sub-solution of (4.12) and  $|u| \leq M$  in  $B_{4\rho} \times (0, T)$ . Let us now apply Lemma 4.6 to  $\tilde{u}$ ,  $(\bar{x}, \bar{t})$  and  $\xi$ : since  $\tilde{u} = u$  in  $B_{3\rho} \times (0, T)$ ,  $\xi$  satisfies

$$\tilde{u}(\bar{x}, \bar{t}) \geq \tilde{u}(\xi(t), t) + C_+ \int_t^{\bar{t}} |\xi'(s)|^p ds - \eta_+(\bar{t} - t) \quad \forall t \in [\bar{t} - \alpha\rho^{p/(p-1)}, \bar{t}],$$

we have

$$\frac{1}{\bar{t} - t} \int_t^{\bar{t}} |\xi'(s)|^p ds \leq C_1 \left( \frac{1}{\bar{t} - t} \int_t^{\bar{t}} |\xi'(s)| ds \right)^p + C_0 \quad \forall t \in [\bar{t} - \alpha\rho^{p/(p-1)}, \bar{t}]$$

for some constants  $C_0 = (\eta_+ + \tilde{\eta}_-)/C_+ = C_0'/\rho^q$  and  $C_1 = \delta^{2p/q} > 1$ . Then, Lemma 3.3 yields the existence of universal constants  $\theta > p$  and  $C'' > 0$  such that

$$\int_t^{\bar{t}} |\xi'(s)| ds \leq C'' \left( \|\xi\|_p + k\alpha^{\frac{1}{p}}\rho^{\frac{1}{p-1}} \right) \frac{(\bar{t} - t)^{1 - \frac{1}{\theta}}}{\alpha^{\frac{\theta-p}{\theta p}} \rho^{\frac{\theta-p}{\theta(p-1)}}} \quad \forall t \in [\bar{t} - \alpha\rho^{p/(p-1)}, \bar{t}],$$

where

$$k = \frac{C_0^{\frac{1}{p}}}{C_1^{\frac{1}{p}} - 1} = \frac{C}{\rho^{\frac{q}{p}}} = \frac{C}{\rho^{\frac{1}{p-1}}} \quad \text{and} \quad \|\xi\|_p \leq [(2M + \eta_+)/C_+]^{\frac{1}{p}} \leq C$$

for some universal constant  $C$ . Estimate (4.34) follows from the above inequality. Moreover,  $\tilde{u}$  being a sub-solution of (4.36), Lemma 4.5 ensures that

$$\tilde{u}(x, \bar{t}) \leq \tilde{u}(y, t) + (\bar{t} - t)^{1-p} C_- |y - x|^p + \tilde{\eta}_-(\bar{t} - t)$$

for any  $(x, t) \in \mathbb{R}^N \times [0, \bar{t}]$ ,  $y \in \mathbb{R}^N$ . Since  $\tilde{u} = u$  in  $B_{3\rho} \times (0, T)$ , (4.35) follows for some constant  $C$ .  $\square$

**Step 3:** We can now complete the proof of Theorem 4.8.

*Space regularity:* Let  $\bar{t} \in [\rho, T]$ , let  $x, \bar{x} \in B_{2\rho}$  be such that  $x \neq \bar{x}$ , and let  $\xi$  be the arc of Step 1. Taking  $t \in [\bar{t} - \alpha\rho^{p/(p-1)}, \bar{t}]$  and  $y = \xi(t)$  in (4.35) yields

$$\begin{aligned} u(x, \bar{t}) &\leq u(\xi(t), t) + (\bar{t} - t)^{1-p} C_- |\xi(t) - x|^p + C\rho^{-\frac{p}{p-1}} (\bar{t} - t) \\ &\leq u(\bar{x}, \bar{t}) + (\bar{t} - t)^{1-p} C_- [|\xi(t) - \bar{x}| + |\bar{x} - x|]^p + C'\rho^{-\frac{p}{p-1}} (\bar{t} - t) \end{aligned}$$

for some universal constant  $C'$ . Hence, in view of (4.34),

$$(4.37) \quad \begin{aligned} u(x, \bar{t}) - u(\bar{x}, \bar{t}) &\leq (\bar{t} - t)^{1-p} C_- \left\{ C\rho^{-\frac{\theta-p}{\theta(p-1)}} (\bar{t} - t)^{1-\frac{1}{\theta}} + |\bar{x} - x| \right\}^p + C'\rho^{-\frac{p}{p-1}} (\bar{t} - t) \end{aligned}$$

Now, suppose  $|\bar{x} - x| < \alpha^{(\theta-1)/\theta} \rho$ . Then there is a number  $t \in [\bar{t} - \alpha\rho^{p/(p-1)}, \bar{t}]$  such that

$$\bar{t} - t = \rho^{\frac{\theta-p}{(\theta-1)(p-1)}} |\bar{x} - x|^{\frac{\theta}{\theta-1}}.$$

So, owing to (4.37),

$$u(x, \bar{t}) - u(\bar{x}, \bar{t}) \leq C'' \left( \frac{|\bar{x} - x|}{\rho} \right)^{\frac{\theta-p}{\theta-1}} + C' \left( \frac{|\bar{x} - x|}{\rho} \right)^{\frac{\theta}{\theta-1}} \leq C''' \left( \frac{|\bar{x} - x|}{\rho} \right)^{\frac{\theta-p}{\theta-1}}$$

for some new universal constants  $C'', C''' > 0$ . Therefore,

$$|u(x, \bar{t}) - u(\bar{x}, \bar{t})| \leq C \left( \frac{|\bar{x} - x|}{\rho} \right)^{\frac{\theta-p}{\theta-1}}$$

for all  $x, \bar{x} \in B_{2\rho}$  and  $\bar{t} \in [\rho, T]$  with  $|\bar{x} - x| \leq \alpha^{\frac{\theta-1}{\theta}} \rho$ .

*Time regularity:* Let  $\bar{x} \in B_\rho$  and  $\rho \leq t < \bar{t} \leq T$ . Applying inequality (4.35) at  $x = \bar{x} = y$ , we obtain

$$u(\bar{x}, \bar{t}) - u(\bar{x}, t) \leq C\rho^{-\frac{p}{p-1}} (\bar{t} - t).$$

To estimate the above left-hand side from below, let  $\xi$  be given by Step 1. Then

$$u(\xi(t), t) \leq u(\bar{x}, \bar{t}) + \eta_+(\bar{t} - t).$$

Arguing as in Step 1, we can find a universal constant  $\beta \in (0, \alpha)$  such that

$$|\xi(t) - \bar{x}| \leq \alpha^{\frac{\theta-1}{\theta}} \rho \quad \forall t \in [\bar{t} - \beta\rho^{p/(p-1)}, \bar{t}].$$

Then, using the space regularity estimate we have just shown and (4.34), we obtain

$$\begin{aligned} u(\xi(t), t) &\geq u(\bar{x}, t) - C \left( \frac{|\xi(t) - \bar{x}|}{\rho} \right)^{\frac{\theta-p}{\theta-1}} \\ &\geq u(\bar{x}, t) - C \left( \frac{1}{\rho} \int_t^{\bar{t}} |\xi'(s)| ds \right)^{\frac{\theta-p}{\theta-1}} \geq u(\bar{x}, t) - C\rho^{-\frac{p(\theta-p)}{\theta(p-1)}} (\bar{t} - t)^{\frac{\theta-p}{\theta}}. \end{aligned}$$

Thus,

$$|u(\bar{x}, \bar{t}) - u(\bar{x}, t)| \leq C\rho^{-\frac{p(\theta-p)}{\theta(p-1)}} (\bar{t} - t)^{\frac{\theta-p}{\theta}}$$

for all  $\bar{x} \in B_\rho$  and all  $\bar{t}, t \in [\rho, T]$  satisfying  $|\bar{t} - t| \leq \beta \rho^{p/(p-1)}$ .  $\square$

*Remark 4.9.* A simple analysis of the above proof allows to compute the dependence on  $\rho$  of the constant in (4.28) as follows

$$|u(x, s) - u(y, t)| \leq C \left[ \rho^{-\frac{\theta-p}{\theta-1}} |y-x|^{\frac{\theta-p}{\theta-1}} + \rho^{-\frac{p(\theta-p)}{\theta(\theta-1)}} (t-s)^{\frac{\theta-p}{\theta}} \right]$$

for all  $(x, s), (y, t) \in \mathcal{O}_\rho$  such that  $|\bar{x} - x| \leq k\rho$  and  $|t - s| \leq k\rho^{\frac{p}{p-1}}$ , where  $C, k > 0$  are universal constants.

## 5 Examples

In this section we investigate two questions naturally arising from Theorems 4.1 and 4.8. First, one may wonder whether the solutions of (4.26) satisfy stronger a priori estimates than (4.28), independent of the regularity of  $H$ . We address such a question with an example showing that uniform Lipschitz estimates cannot be expected even for a simple Hamilton-Jacobi equation in one space dimension. Second, one may ask if the local Hölder estimates for solutions in an open domain can be extended up to the boundary. Surprisingly—and in stark contrast to the stationary setting (see [6])—this is not the case: we will exhibit a solution of a first order Hamilton-Jacobi equation with constant coefficients which turns out to be discontinuous at the boundary of the domain.

### 5.1 Counterexample to Lipschitz continuity

The following example is inspired by [1]. In particular, Lemma 5.1 and Proposition 5.2 could also be deduced from the results of the above paper.

Let us fix  $\gamma \in (2 - \sqrt{2}, 1)$  and define

$$\xi_0(t) = t^\gamma \quad \forall t \in [0, 1].$$

**Lemma 5.1.** *For every  $t \in [0, 1]$*

$$(5.1) \quad \int_t^{t+h} |\xi_0'(s)|^2 ds < \frac{2}{h} |\xi_0(t+h) - \xi_0(t)|^2 \quad \forall h \in (0, 1-t].$$

*Proof.* Let  $t \in [0, 1]$  and define

$$X_t(h) = \int_t^{t+h} |\xi_0'(s)|^2 ds - \frac{2}{h} |\xi_0(t+h) - \xi_0(t)|^2 \quad \forall h \in (0, 1-t].$$

Observe that  $X_t(h) < 0$  for  $h > 0$  small enough, since  $\lim_{h \downarrow 0} X_t(h)/h < 0$ . In order to obtain that  $X_t(h) < 0$  for every  $h \in (0, 1-t]$ , let us show that  $X_t(\cdot)$  is decreasing. Indeed, for any  $h \in (0, 1-t]$ ,

$$(5.2) \quad X_t'(h) = |\xi_0'(t+h)|^2 - \frac{4}{h} (\xi_0(t+h) - \xi_0(t)) \xi_0'(t+h) \\ + \frac{2}{h^2} |\xi_0(t+h) - \xi_0(t)|^2 = \left( \frac{\xi_0(t+h) - \xi_0(t)}{h} \right)^2 [Y_t(h)^2 - 4Y_t(h) + 2]$$

where

$$Y_t(h) \doteq \frac{h \xi_0'(t+h)}{\xi_0(t+h) - \xi_0(t)}.$$

Now, since  $\xi_0$  is an increasing concave function,  $Y_t(h) \leq 1$ . Moreover,

$$Y_t(h) \geq \frac{h \xi_0'(t+h)}{\xi_0(t+h)} = \frac{\gamma(t+h)^{\gamma-1}}{(t+h)^{\gamma-1}} = \gamma > 2 - \sqrt{2}.$$

Since  $y^2 - 4y + 2 < 0$  for every  $y \in (2 - \sqrt{2}, 1]$ ,  $X_t'(h) < 0$  owing to (5.2).  $\square$

Now, define

$$a(x, t) = \begin{cases} 1 & \text{if } x = \xi_0(t) \\ 2 & \text{if } x \neq \xi_0(t), \end{cases} \quad \forall (x, t) \in \mathbb{R} \times [0, 1]$$

and

$$g(x) = \begin{cases} 0 & \text{if } x = 1 \\ G & \text{if } x \neq 1 \end{cases} \quad \forall x \in \mathbb{R}$$

where  $G$  is a real number such that

$$(5.3) \quad G > \frac{\gamma^2}{2\gamma - 1}.$$

Let us consider the functional

$$J[\xi] = \int_0^1 a(\xi(t), t) |\xi'(t)|^2 dt + g(\xi(1)) \quad \forall \xi \in W^{1,2}([0, 1]).$$

**Proposition 5.2.**  $\xi_0$  is the unique solution of the variational problem

$$(5.4) \quad \min\{J[\xi] : \xi \in W^{1,2}([0, 1]), \xi(0) = 0\}.$$

*Proof.* To begin with, let us note that the minimum in (5.4) does exist owing to well-known existence results for functionals with lower semicontinuous data (see, e.g., [4, section 3.2]). So, let  $\xi_*$  be a solution of (5.4) and observe that  $\xi_*(1) = 1$  since otherwise

$$J[\xi_*] \geq g(\xi_*(1)) = G > \frac{\gamma^2}{2\gamma - 1} = J[\xi_0].$$

Now, suppose that the open set  $\{t \in (0, 1) : \xi_*(t) \neq \xi_0(t)\}$  is nonempty and let  $(t_1, t_2)$  be a connected component of such a set. Then,  $\xi_*(t_i) = \xi_0(t_i)$  for  $i = 1, 2$ . Define

$$\xi_1(t) = \begin{cases} \xi_*(t) & \text{if } t \in [0, t_1] \cup [t_2, 1] \\ \xi_0(t) & \text{if } t \in (t_1, t_2). \end{cases}$$

Then  $\xi_1 \in W^{1,2}([0, 1])$  satisfies  $\xi_1(0) = 0$  and  $\xi_1(1) = 1$ . Moreover, in view of Lemma 5.1,

$$\begin{aligned} J[\xi_1] &= \int_0^{t_1} a(\xi_*(t), t) |\xi_*'(t)|^2 dt + \int_{t_1}^{t_2} |\xi_0'(t)|^2 dt + \int_{t_2}^1 a(\xi_*(t), t) |\xi_*'(t)|^2 dt + g(1) \\ &< \int_0^{t_1} a(\xi_*(t), t) |\xi_*'(t)|^2 dt + 2 \frac{|\xi_0(t_2) - \xi_0(t_1)|^2}{t_2 - t_1} + \int_{t_2}^1 a(\xi_*(t), t) |\xi_*'(t)|^2 dt + g(1) \\ &= \int_0^{t_1} a(\xi_*(t), t) |\xi_*'(t)|^2 dt + 2 \frac{|\xi_1(t_2) - \xi_1(t_1)|^2}{t_2 - t_1} + \int_{t_2}^1 a(\xi_*(t), t) |\xi_*'(t)|^2 dt + g(1) \\ &\leq J[\xi_*] \end{aligned}$$

in contrast with the optimality of  $\xi_*$ . Therefore,  $\xi_* \equiv \xi_0$ .  $\square$

Let us now fix two sequences

$$a_n : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \quad \text{and} \quad g_n : \mathbb{R} \rightarrow \mathbb{R} \quad (n \geq 1)$$

of continuous functions such that

$$\begin{cases} \frac{1}{2} \leq a_n(x, t) \leq 2 & \forall n \geq 1 \\ a_n(x, t) \uparrow a(x, t) & n \rightarrow \infty \end{cases} \quad \forall (x, t) \in \mathbb{R} \times [0, 1]$$

and

$$\begin{cases} 1 \leq g_n(x) \leq g(x) & \forall n \geq 1 \\ g_n(x) \uparrow g(x) & n \rightarrow \infty \end{cases} \quad \forall x \in \mathbb{R}.$$

For instance, one can take

$$(5.5) \quad a_n(x, t) = \min \left\{ 2, n|x - \xi_0(t)| + \sum_{k=1}^n \frac{1}{2^k} \right\} \quad \forall (x, t) \in \mathbb{R} \times [0, 1]$$

and

$$(5.6) \quad g_n(x) = \min \{ G, n|x - 1| \} \quad \forall x \in \mathbb{R}.$$

Define, for all  $n \in \mathbb{N}$ ,

$$J_n[\xi] = \int_0^1 a_n(\xi(t), t) |\xi'(t)|^2 dt + g_n(\xi(1)) \quad \forall \xi \in W^{1,2}([0, 1]).$$

**Proposition 5.3.** *For every  $n \in \mathbb{N}$  let  $\xi_n$  be a solution of the variational problem*

$$\min \{ J_n[\xi] : \xi \in W^{1,2}([0, 1]), \xi(0) = 0 \}.$$

*Then  $\xi_n \rightharpoonup \xi_0$  in  $W^{1,2}([0, 1])$  as  $n \rightarrow \infty$ .*

*Proof.* Since  $(\xi_n)_n$  is bounded in  $W^{1,2}([0, 1])$ , we can assume, without loss of generality, that  $(\xi_n)_n$  weakly converges to some limit  $\xi_*$  in  $W^{1,2}([0, 1])$ . Consequently,  $\xi_n \rightarrow \xi_*$  uniformly as  $n \rightarrow \infty$ .

Now, observe that,

$$(5.7) \quad \limsup_{n \rightarrow \infty} J_n[\xi_n] \leq J[\xi_0]$$

since  $J_n[\xi_n] \leq J_n[\xi_0]$  and, by monotone convergence,  $J_n[\xi_0] \rightarrow J[\xi_0]$  as  $n \rightarrow \infty$ . Moreover, for any fixed  $n \geq 1$  and all  $m \geq n$ ,  $J_m[\xi_m] \geq J_n[\xi_m]$  in view of the monotonicity of  $a_m$  and  $g_m$ . Therefore, recalling (5.7),

$$J[\xi_0] \geq \liminf_{m \rightarrow \infty} J_m[\xi_m] \geq \liminf_{m \rightarrow \infty} J_n[\xi_m] \geq J_n[\xi_*]$$

by the lower semicontinuity of  $J_n$ . Since, by monotone convergence,  $J_n[\xi_*] \rightarrow J[\xi_*]$  as  $n \rightarrow \infty$ , we conclude that  $J[\xi_*] \leq J[\xi_0]$ . But we know that  $\xi_0$  is the unique solution of (5.4). So,  $\xi_* = \xi_0$  as requested.  $\square$

Since  $\xi_0$  is just Hölder continuous with exponent  $\gamma$ , and  $\xi_n \rightarrow \xi_0$  uniformly in  $[0, 1]$ , the above result implies that  $(\xi_n)_n$  cannot be equi-Lipschitz.

**Proposition 5.4.** *Let  $0 < \tau < 1$ . Then the sequence of (value) functions*

$$(5.8) \quad u_n(x, t) = \inf \left\{ \int_t^1 a_n(\xi(s), s) |\xi'(s)|^2 ds + g_n(\xi(1)) : \xi \in W^{1,2}([t, 1]), \xi(t) = x \right\}$$

*is not equi-Lipschitz in  $\mathbb{R} \times [0, \tau]$ .*

*Proof.* Let  $0 < \tau < 1$  and suppose  $u_n$  is Lipschitz continuous in  $\mathbb{R} \times [0, \tau]$ , with the same constant  $K \geq 0$  for every  $n \geq 1$ . Let  $\xi_n$  be as in Proposition 5.3. Then the optimality principle ensures that

$$u_n(0, 0) = \int_0^t a_n(\xi_n(s), s) |\xi_n'(s)|^2 ds + u(t, \xi_n(t)) \quad \forall t \in [0, 1].$$

Therefore,

$$\int_0^t |\xi_n'(s)|^2 ds \leq u_n(0, 0) - u(t, \xi_n(t)) \leq K(t + |\xi_n(t)|) \quad \forall t \in [0, \tau],$$

whence

$$\left| \frac{\xi_n(t)}{t} \right|^2 \leq \frac{1}{t} \int_0^t |\xi_n'(s)|^2 ds \leq K \left( 1 + \left| \frac{\xi_n(t)}{t} \right| \right) \quad \forall t \in (0, \tau].$$

The above inequality in turn implies that  $|\xi_n(t)| \leq c(K)t$  for every  $t \in (0, \tau]$ , uniformly for  $n \geq 1$ , in contrast with the fact that  $\xi_n \rightarrow \xi_0$  uniformly in  $[0, 1]$ .  $\square$

Since  $u_n$  above is the (unique) viscosity solution of the corresponding Hamilton-Jacobi equation, from Proposition 5.4 we directly obtain the following corollary, which answers (negatively) the first question at the beginning of section 5.

**Corollary 5.5.** *For any integer  $n \geq 1$  let  $a_n$  and  $g_n$  be given by (5.5) and (5.6), respectively, and let  $u_n$  be the viscosity solution of*

$$\begin{cases} -u_t + \frac{|u_x|^2}{4a_n(x, t)} = 0 & \text{in } \mathbb{R} \times (0, 1) \\ u(x, 1) = g_n(x) & x \in \mathbb{R}. \end{cases}$$

Then  $(u_n)_n$  is not equi-Lipschitz in  $\mathbb{R} \times [0, \tau]$ , for any  $0 < \tau < 1$ .

Observe that the above equation is of the form (4.26), after the change of variable  $t \mapsto 1 - t$ , and satisfies condition (4.2) uniformly in  $n$ .

*Remark 5.6.* A careful examination of the proof of Proposition 5.4 actually shows that no uniform Hölder bound can be true for  $(u_n)_n$  on  $\mathbb{R} \times [0, \tau]$  with a Hölder exponent in the  $x$  variable (resp.  $t$  variable) greater than  $1 - 1/\sqrt{2}$  (resp.  $3 - 2\sqrt{2}$ ). Note that such an optimal exponent has the form  $(\theta - 2)/(\theta - 1)$  (resp.  $(\theta - 2)/\theta$ ) for  $\theta = 1 + \sqrt{2}$  in agreement with (4.4).

## 5.2 Counterexample to boundary continuity

Our next example gives a negative reply to the second question raised at the beginning of section 5. Let  $\mathbb{R}_+ = (0, \infty)$  and consider the Hamilton-Jacobi equation

$$(5.9) \quad u_t + \frac{1}{4} \left| \frac{\partial u}{\partial x} \right|^2 = 0 \quad \text{in } \mathcal{O} = \mathbb{R}_+^2.$$

Assumption (4.2) is obviously satisfied with  $q = 2$ . Now, define  $u : \mathcal{O} \rightarrow \mathbb{R}$  by

$$u(x, t) \doteq \min \left\{ 1, \frac{x^2}{(t-1)_+} \right\} = \begin{cases} 1 & \text{if } x^2 \geq t-1 \\ \frac{x^2}{t-1} & \text{if } x^2 < t-1 \end{cases}$$

Then,  $u$  is a continuous function in  $\mathcal{O}$  satisfying  $0 \leq u \leq 1$ . Moreover, it is easily checked that  $u$  is a solution of the above equation in  $\mathcal{O} \setminus \Gamma$ , where  $\Gamma$  is the arc of parabola

$$\Gamma = \{(x, t) \in \mathcal{O} : x^2 = t - 1\}.$$

So, since  $u$  is locally semiconcave in  $\mathcal{O}$  (see [5] for details),  $u$  is a viscosity solution of (5.9). On the other hand,  $u$  is discontinuous at  $(0, 1) \in \partial \mathcal{O}$  because, for instance,

$$\lim_{x \rightarrow 0^+} u(x, 1) = 1 \quad \text{while} \quad \lim_{x \rightarrow 0^+} u(x, 1 + 2x^2) = \frac{1}{2}.$$

## 6 Second order equations

This section is concerned with equations of the form

$$(6.1) \quad u_t(x, t) - \text{Tr}(a(x, t)D^2u(x, t)) + H(x, t, Du(x, t)) = 0 \quad \text{in } \mathbb{R}^N \times (0, T)$$

where

$$D^2u(x, t) = \left( \frac{\partial^2 u}{\partial x_i \partial x_j} (x, t) \right)_{1 \leq i, j \leq N}$$

is the Hessian matrix of  $u$ . The following hypotheses will be assumed:

- $H : \mathbb{R}^N \times (0, T) \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a continuous function such that

$$(6.2) \quad \frac{1}{\delta} |z|^q - \eta_- \leq H(x, t, z) \leq \delta |z|^q + \eta_+ \quad \forall (x, t, z) \in \mathbb{R}^N \times (0, T) \times \mathbb{R}^N,$$

for some constants  $q > 2$ ,  $\delta > 1$  and  $\eta_{\pm} \geq 0$  (*super-quadratic growth*);

- there is a locally Lipschitz map  $\sigma : \mathbb{R}^N \times (0, T) \rightarrow \mathbb{R}^{N \times D}$  such that
- $$(6.3) \quad a(x, t) = \sigma(x, t)\sigma^*(x, t) \quad \text{and} \quad \|\sigma(x, t)\| \leq \delta \quad \forall (x, t) \in \mathbb{R}^N \times (0, T);$$
- $u : \mathbb{R}^N \times (0, T) \rightarrow \mathbb{R}$  is a continuous viscosity solution of (6.1) such that  $|u| \leq M$  in  $\mathbb{R}^N \times [0, T]$ .

As before, a *universal constant* will be a positive number depending on the given data  $q, \delta, M, \eta_-, \eta_+$  and  $N$  only. Recall that  $p$  is the conjugate exponent of  $q$ .

The main result of this section is the following Hölder estimate.

**Theorem 6.1.** *Let  $u \in C(\mathbb{R}^N \times [0, T])$  be a viscosity solution of (6.1) such that  $|u| \leq M$  in  $\mathbb{R}^N \times [0, T]$ . Then there is a universal constant  $\theta > p$  such that, for every  $\tau > 0$ ,*

$$(6.4) \quad |u(x_1, t_1) - u(x_2, t_2)| \leq C(\tau) \left[ |x_1 - x_2|^{(\theta-p)/(\theta-1)} + |t_1 - t_2|^{(\theta-p)/\theta} \right]$$

for all  $(x_1, t_1), (x_2, t_2) \in \mathbb{R}^N \times [\tau, T]$  and some constant  $C(\tau) > 0$ .

As for Theorem 4.8, the main point of the above result is the fact that (6.4) holds true uniformly with respect to  $H$  and  $a$ , as long as conditions (6.2) and the bound  $|u| \leq M$  are satisfied. In particular,  $\theta$  and  $C(\tau)$  are independent of the continuity moduli of  $H$  and  $a$ .

## 6.1 Some preliminary results

For notational simplicity, we prefer to replace (6.1) with the backward equation

$$(6.5) \quad -u_t(x, t) - \text{Tr}(a(x, t)D^2u(x, t)) + H(x, t, Du(x, t)) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, T)$$

(which should be coupled with a *terminal condition*). Note that the change of variable  $t \mapsto T - t$  turns any solution of (6.1) into a solution of (6.5), provided  $a(x, t)$  and  $H(x, t, z)$  are replaced by  $a(x, T - t)$  and  $H(x, T - t, z)$ .

Throughout this section we shall need to keep track of the constants  $\eta_+$  and  $\eta_-$ : indeed such a dependence is essential for the proof of Theorem 6.7. For this purpose, we will denote simply by  $C$  (or  $C_0, C_1$ ) constants which depend only on  $\delta, M, p, T$  and  $N$ . Dependence with respect to  $\tau$  and  $\eta_{\pm}$  will be made explicit by the use of parentheses.

Let us begin with some estimates for super/sub-solutions of (6.5).

**Lemma 6.2.** *Let  $u \in C(\mathbb{R}^N \times [0, T])$  be a super-solution of*

$$(6.6) \quad -u_t - \text{Tr}(aD^2u) + \delta|Du|^q + \eta_+ = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, T)$$

satisfying  $|u| \leq M$  in  $\mathbb{R}^N \times (0, T)$ . Then, for any  $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, T)$  there is a stochastic basis  $(\Omega, \mathcal{F}, \mathbb{P})$ , a filtration  $(\mathcal{F}_t)_{t \geq \bar{t}}$ , a  $D$ -dimensional Brownian motion  $(W_t)_{t \geq \bar{t}}$  adapted to the filtration  $(\mathcal{F}_t)$ , and a process  $\zeta \in L_{\text{ad}}^p(\Omega \times [\bar{t}, T]; \mathbb{R}^N)$  such that the solution to

$$(6.7) \quad \begin{cases} dX_t = \zeta_t dt + \sqrt{2}\sigma(X_t, t)dW_t \\ X_{\bar{t}} = \bar{x} \end{cases}$$

satisfies

$$(6.8) \quad u(\bar{x}, \bar{t}) \geq \mathbb{E} \left[ u(X_t, t) + C_+ \int_{\bar{t}}^t |\zeta_s|^p ds \right] - \eta_+(t - \bar{t}) \quad \forall t \in [\bar{t}, T]$$

where  $C_+ > 0$  is the universal constant given by (4.7).

*Proof.* Let us fix a  $D$ -dimensional Brownian motion  $W$  on some probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  and, for any  $\bar{t} \leq t \leq t' \leq T$ , let us denote by  $\mathcal{F}_{t,t'}$  the  $\sigma$ -field

$$\mathcal{F}_{t,t'} = \sigma(W_s, t < s \leq t') \cup \mathcal{N},$$

where  $\mathcal{N}$  is the set of subsets of  $\Omega$  with nul probability. Throughout the proof we denote by  $L_{\text{ad}}^p(\Omega \times [t, t']; \mathbb{R}^N)$  the set of measurable controls  $\zeta : \Omega \times [t, t'] \rightarrow \mathbb{R}^N$ , adapted to the filtration  $(\mathcal{F}_{t,s})_{s \in [t, t']}$ , such that

$$\mathbb{E} \left[ \int_t^{t'} |\zeta_s|^p ds \right] < +\infty.$$

Finally, for any  $x \in \mathbb{R}^N$  and  $\zeta \in L_{\text{ad}}^p(\Omega \times [t, T]; \mathbb{R}^N)$ ,  $Y^{x,t,\zeta}$  stands for the solution of

$$\begin{cases} dY_s = \zeta_s ds + \sqrt{2} \sigma(Y_s, s) dW_s \\ Y_t = x \end{cases}$$

Let  $n$  be a large integer and set

$$\tau = (T - \bar{t})/n \quad \text{and} \quad t_k = \bar{t} + k\tau \quad \text{for } k \in \{0, \dots, n\}.$$

Let us fix an initial condition  $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times [0, T]$ . We are going to build a control  $\zeta^n \in L_{\text{ad}}^p(\Omega \times [\bar{t}, T]; \mathbb{R}^N)$  and a process  $Y^n = Y^{\bar{x}, \bar{t}, \zeta^n}$  such that

$$(6.9) \quad u(\bar{x}, \bar{t}) \geq \mathbb{E} \left[ u(Y_{t_k}^n, t_k) + C_+ \int_{\bar{t}}^{t_k} |\zeta_s^n|^p ds - (\eta_+ + \tau)(t_k - \bar{t}) \right]$$

for every  $k \in \{0, \dots, n-1\}$ . Let  $k \in \{1, \dots, n\}$  and let  $v^k$  be the solution of (6.6), defined on the time interval  $[0, t_k]$ , with terminal condition  $u(\cdot, t_k)$ . From a classical representation formula (see, for instance, [9]) we have

$$v^k(x, t) = \inf_{\zeta \in L_{\text{ad}}^p(\Omega \times [t, t_k]; \mathbb{R}^N)} \mathbb{E} \left[ u(Y_{t_k}^{x,t,\zeta}, t_k) + C_+ \int_t^{t_k} |\zeta_s|^p ds - \eta_+(t_k - t) \right]$$

for any  $(x, t) \in \mathbb{R}^N \times [0, t_k]$ . Since  $u(\cdot, t_k)$  is continuous, one can build, thanks to the measurable selection theorem (see [3]), a Borel measurable map  $x \rightarrow Z^{x,k}$  from  $\mathbb{R}^N$  into  $L_{\text{ad}}^p(\Omega \times [t_{k-1}, t_k]; \mathbb{R}^N)$  such that

$$v^k(x, t_{k-1}) \geq \mathbb{E} \left[ u(Y_{t_k}^{x, t_{k-1}, Z^{x,k}}, t_k) + C_+ \int_{t_{k-1}}^{t_k} |Z_s^{x,k}|^p ds - (\eta_+ + \tau)\tau \right] \quad \forall x \in \mathbb{R}^N.$$

Note that

$$\begin{aligned} \mathbb{E} \left[ u(Y_{t_k}^{x,t_{k-1},Z^{x,k}}, t_k) + C_+ \int_{t_{k-1}}^{t_k} |Z_s^{x,k}|^p ds \right] \\ = \mathbb{E} \left[ u(Y_{t_k}^{x,t_{k-1},Z^{x,k}}, t_k) + C_+ \int_{t_{k-1}}^{t_k} |Z_s^{x,k}|^p ds \mid \mathcal{F}_{\bar{t}, t_{k-1}} \right] \end{aligned}$$

because  $Y_{t_k}^{x,t_{k-1},Z^{x,k}}$  and  $Z_s^{x,k}$  are  $\mathcal{F}_{t_{k-1}, t_k}$ -measurable. So, for any  $x \in \mathbb{R}^N$ , we have

$$(6.10) \quad v^k(x, t_{k-1}) \geq \mathbb{E} \left[ u(Y_{t_k}^{x,t_{k-1},Z^{x,k}}, t_k) + C_+ \int_{t_{k-1}}^{t_k} |Z_s^{x,k}|^p ds - (\eta_+ + \tau)\tau \mid \mathcal{F}_{\bar{t}, t_{k-1}} \right].$$

We now construct  $Y^n$  and  $\zeta^n$  by induction on the time intervals  $[t_k, t_{k+1})$ . On  $[\bar{t}, t_1)$  we set  $\zeta_t^n = Z_t^{\bar{x}, \bar{t}}$  and  $Y^n = Y^{\bar{x}, \bar{t}, \zeta^n}$ . Assume that  $\zeta^n$  and  $Y^n$  have been built on  $[\bar{t}, t_{k-1})$ . Then we set

$$\zeta^n = Z_{t_{k-1}}^{Y_{t_{k-1}}^n, k} \quad \text{and} \quad Y^n = Y^{\bar{x}, \bar{t}, \zeta^n} \quad \text{on} \quad [t_{k-1}, t_k).$$

Fix  $k \in \{1, \dots, n\}$ . Since  $Y_{t_{k-1}}^n$  is  $\mathcal{F}_{\bar{t}, t_{k-1}}$ -measurable, putting  $x = Y_{t_{k-1}}^n$  in (6.10) and using the fact that  $Y_{t_k}^{Y_{t_{k-1}}^n, t_{k-1}, \zeta^n} = Y_{t_k}^n$  while  $\zeta^n = Z_{t_{k-1}}^{Y_{t_{k-1}}^n, k}$  on  $[t_{k-1}, t_k)$  gives

$$u(Y_{t_k}^n, t_k) \geq \mathbb{E} \left[ u(Y_{t_{k+1}}^n, t_{k+1}) + C_+ \int_{t_k}^{t_{k+1}} |\zeta_s^n|^p ds - (\eta_+ + \tau)\tau \mid \mathcal{F}_{t_k} \right] \quad (\mathbb{P} \text{ a.s.})$$

Taking the expectation of both sides of the above inequality and summing up gives (6.9).

In order to pass to the limit in (6.9) as  $n \rightarrow +\infty$ , we first note that, since  $u$  is bounded, inequality (6.9) implies that

$$(6.11) \quad \mathbb{E} \left[ \int_{\bar{t}}^T |\zeta_s^n|^p ds \right] \leq C \quad \forall n \geq 0,$$

for some universal constant  $C$ . Let us set

$$\Lambda_t^n \doteq \int_{\bar{t}}^t \zeta_s^n ds \quad \text{and} \quad M_t^n \doteq Y_t^n - \Lambda_t^n - \bar{x} = \sqrt{2} \int_0^t \sigma(Y_s^n, s) ds \quad \forall t \in [\bar{t}, T].$$

We claim that the sequence of probability measures  $(\mathbb{P}^{(Y^n, \Lambda^n)})$ , defined on the space  $\mathcal{C}([\bar{t}, T], \mathbb{R}^{2N})$  of continuous maps from  $[\bar{t}, T]$  into  $\mathbb{R}^{2N}$  with the topology of uniform convergence, is tight. Since  $(Y_{\bar{t}}^n, \Lambda_{\bar{t}}^n) = (\bar{x}, 0)$  for any  $n$ , [18, Theorem 4.10] ensures that the sequence  $(\mathbb{P}^{(Y^n, \Lambda^n)})$  is tight if and only if

$$(6.12) \quad \lim_{r \rightarrow 0^+} \sup_n \mathbb{P} \left[ \sup_{|t-s| \leq r} |(Y_t^n, \Lambda_t^n) - (Y_s^n, \Lambda_s^n)| \geq \varepsilon \right] = 0 \quad \forall \varepsilon > 0.$$

By the triangle inequality, we have

$$(6.13) \quad \begin{aligned} & \mathbb{P} \left[ \sup_{|t-s| \leq r} |(Y_t^n, \Lambda_t^n) - (Y_s^n, \Lambda_s^n)| \geq \varepsilon \right] \\ & \leq \mathbb{P} \left[ \sup_{|t-s| \leq r} |M_t^n - M_s^n| \geq \varepsilon/4 \right] + \mathbb{P} \left[ \sup_{|t-s| \leq r} |\Lambda_t^n - \Lambda_s^n| \geq \varepsilon/4 \right]. \end{aligned}$$

Since, for any  $s, t \in [\bar{t}, T]$ , with  $|t-s| \leq r$ ,

$$|\Lambda_t^n - \Lambda_s^n| \leq |t-s|^{1-1/p} \left( \int_0^T |\zeta_\tau^n|^p d\tau \right)^{1/p} \leq r^{1-1/p} \left( \int_0^T |\zeta_\tau^n|^p d\tau \right)^{1/p},$$

we have

$$\begin{aligned} & \mathbb{P} \left[ \sup_{|t-s| \leq r} |\Lambda_t^n - \Lambda_s^n| \geq \varepsilon/4 \right] \leq \mathbb{P} \left[ \left( \int_0^T |\zeta_\tau^n|^p d\tau \right)^{1/p} \geq \varepsilon/(4r^{1-1/p}) \right] \\ & \leq \mathbb{P} \left[ \int_0^T |\zeta_\tau^n|^p d\tau \geq (\varepsilon/(4r^{1-1/p}))^p \right] \leq \frac{4^p r^{p-1}}{\varepsilon^p} \mathbb{E} \left[ \int_0^T |\zeta_\tau^n|^p d\tau \right] \leq C \frac{4^p r^{p-1}}{\varepsilon^p} \end{aligned}$$

thanks to Bienaymé-Chebyshev inequality. So,

$$(6.14) \quad \lim_{r \rightarrow 0^+} \sup_n \mathbb{P} \left[ \sup_{|t-s| \leq r} |\Lambda_t^n - \Lambda_s^n| \geq \varepsilon/4 \right] = 0 \quad \forall \varepsilon > 0.$$

On another hand, from the Burkholder-Davis-Gundy inequality and the uniform bound on  $\sigma$ , we have

$$\mathbb{E} [|M_t^n - M_s^n|^4] \leq C|t-s|^2 \quad \forall s, t \in [\bar{t}, T],$$

where  $C$  is a universal constant. The Kolmogorov criterion (see, e.g., Problem 4.12 of [18]) then ensures that the sequence  $(\mathbb{P}_{M^n})$  is tight in  $\mathcal{C}([\bar{t}, T], \mathbb{R}^N)$ , i.e.,

$$\lim_{r \rightarrow 0^+} \sup_n \mathbb{P} \left[ \sup_{|t-s| \leq r} |M_t^n - M_s^n| \geq \varepsilon \right] = 0 \quad \forall \varepsilon > 0.$$

Combining the above expression with (6.13) and (6.14) implies that (6.12) holds.

Since the sequence  $(\mathbb{P}_{(Y^n, \Lambda^n)})$  is tight, Prohorov's theorem (Theorem 4.7 of [18]) states that we can find a subsequence of  $(Y^n, \Lambda^n)$  (again labeled  $(Y^n, \Lambda^n)$ ) and a measure  $\mu$  on  $\mathcal{C}([\bar{t}, T], \mathbb{R}^{2N})$  such that  $(\mathbb{P}_{(Y^n, \Lambda^n)})$  weakly converges to  $\mu$ . Then, Skorokhod's embedding theorem (Theorem 2.4 of [11]) implies that we can find random variables  $(\bar{Y}^n, \bar{\Lambda}^n)$  and  $(\bar{Y}, \bar{\Lambda})$  defined on a new probability space  $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}})$ , such that  $(\bar{Y}^n, \bar{\Lambda}^n)$  has the same law as  $(Y^n, \Lambda^n)$  for any  $n$ , the law of  $(\bar{Y}, \bar{\Lambda})$  is  $\mu$  and,  $\bar{\mathbb{P}}$ -almost surely, the sequence  $(\bar{Y}^n, \bar{\Lambda}^n)$  converges to  $(\bar{Y}, \bar{\Lambda})$  in  $\mathcal{C}([\bar{t}, T], \mathbb{R}^{2N})$ . Since  $t \rightarrow \Lambda_t^n$  is absolutely continuous  $\mathbb{P}$  a.s. and  $\bar{\Lambda}^n$  has the same law as  $\Lambda^n$ ,  $t \rightarrow \bar{\Lambda}_t^n$  is absolutely continuous ( $\bar{\mathbb{P}}$  a.s.). Let us set  $\bar{\zeta}_s^n = \frac{d}{ds} \bar{\Lambda}_s^n$ . Then, by (6.11),  $\mathbb{E}[\int_{\bar{t}}^T |\bar{\zeta}_s^n|^p ds] \leq C$  for all  $n \geq 0$ . Therefore, (up to a subsequence again labeled in the same way)  $(\bar{\zeta}^n)$  converges weakly in  $L^p(\Omega \times [\bar{t}, T])$  to some limit,  $\bar{\zeta}$ , which ( $\bar{\mathbb{P}}$

a.s.) satisfies  $\bar{\Lambda}_t = \int_{\bar{t}}^t \bar{\zeta}_s ds$  for all  $t \in [\bar{t}, T]$ . Note that  $\bar{M}_t^n \doteq \bar{Y}_t^n - \bar{x} - \bar{\Lambda}_t^n$  has the same law as  $M^n$ , so that  $\bar{M}^n$  is a continuous martingale satisfying

$$\bar{A}_{ij,t}^n \doteq \langle \bar{M}_i^n, \bar{M}_j^n \rangle_t = 2 \int_{\bar{t}}^t (\sigma \sigma^*)_{ij}(\bar{Y}_s^n, s) ds \quad \forall s \in [\bar{t}, T], \forall 1 \leq i, j \leq N.$$

Let us set  $\bar{M}_t \doteq \bar{Y}_t - \bar{x} - \bar{\Lambda}_t$  and  $\bar{A}_{ij,t} = 2 \int_{\bar{t}}^t (\sigma \sigma^*)_{ij}(\bar{Y}_s, s) ds$ . Since  $(\bar{M}^n, \bar{A}^n)$  converges to  $(\bar{M}, \bar{A})$   $\mathbb{P}$  a.s. with  $\bar{A}_0 = 0$  and  $\bar{A}$  continuous, [22, Theorem 12] implies that  $\bar{M}$  is a continuous martingale with  $\langle \bar{M}, \bar{M} \rangle = \bar{A}$  ( $\mathbb{P}$  a.s.). Owing to the martingale representation theorem (see, e.g., [18, Theorem 3.4.2]), there is a  $D$ -dimensional Brownian motion  $\tilde{W} = \{\tilde{W}_t, \tilde{\mathcal{F}}_t, t \in [\bar{t}, T]\}$ , defined on an extension  $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbf{P}})$  of  $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbf{P}})$ , such that

$$\bar{Y}_t - \bar{x} - \bar{\Lambda}_t = \bar{M}_t = \sqrt{2} \int_{\bar{t}}^t \sigma(\bar{Y}_s, s) d\tilde{W}_s \quad \forall t \in [\bar{t}, T].$$

Consequently,  $\bar{Y}_t = \bar{x} + \int_{\bar{t}}^t \bar{\zeta}_s ds + \sqrt{2} \int_{\bar{t}}^t \sigma(\bar{Y}_s, s) d\tilde{W}_s$  for all  $t \in [\bar{t}, T]$ . So, recalling (6.9), a classical lower semicontinuity argument yields

$$u(\bar{x}, \bar{t}) \geq \mathbb{E} \left[ u(\bar{Y}_t, t) + C_+ \int_{\bar{t}}^t |\bar{\zeta}_s|^p ds - \eta_+(t - \bar{t}) \right] \quad \forall t \in [\bar{t}, T],$$

which concludes the proof.  $\square$

**Lemma 6.3.** *Let  $u \in C(\mathbb{R}^N \times [0, T])$  be a sub-solution of*

$$(6.15) \quad -u_t - \text{Tr}(aD^2u) + \frac{1}{\delta} |Du|^q - \eta_- = 0 \quad \text{in } \mathbb{R}^N \times (0, T)$$

*satisfying  $|u| \leq M$ . Then, for all  $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, T)$  and all  $(y, t) \in \mathbb{R}^N \times (\bar{t}, T)$ ,*

$$(6.16) \quad u(\bar{x}, \bar{t}) \leq u(y, t) + C \left\{ |y - \bar{x}|^p (t - \bar{t})^{1-p} + (t - \bar{t})^{1-p/2} \right\} + \eta_-(t - \bar{t})$$

*for some universal constant  $C > 0$ .*

*Remark 6.4.* In particular, if  $u$  is a sub-solution of the stationary equation

$$-\text{Tr}(aD^2u) + \frac{1}{\delta} |Du|^q - \delta = 0 \quad \text{in } \mathbb{R}^N,$$

then inequality (6.16) implies that, for any  $x, y \in \mathbb{R}^N$  and any  $\tau > 0$ ,

$$u(x) \leq u(y) + C \left\{ |y - x|^p \tau^{1-p} + \tau^{1-p/2} \right\} + \eta_- \tau,$$

for some constant  $C$ . Taking  $\tau = |x - y|^2$  yields  $u(x) \leq u(y) + C |y - x|^{2-p}$ , which shows that  $u$  is Hölder continuous, partially recovering one of the results in [6].

*Proof.* Let us fix  $t \in (\bar{t}, T)$ . Let  $v$  be the solution of equation (6.15) with terminal condition  $u(\cdot, t)$ , and let  $(W_t)_{t \geq \bar{t}}$  be a  $D$ -dimensional Brownian motion on some

stochastic basis  $(\Omega, \mathcal{F}, \mathbb{P})$ , with associated filtration  $(\mathcal{F}_t)_{t \geq \bar{t}}$ . Then, by a classical representation formula (see, e.g., [9]),

$$v(\bar{x}, \bar{t}) = \inf_{\zeta \in L_{\text{ad}}^p(\Omega \times [\bar{t}, t]; \mathbb{R}^N)} \mathbb{E} \left[ u(X_t^{\bar{x}, \bar{t}, \zeta}, t) + C_- \int_{\bar{t}}^t |\zeta_s|^p ds + \eta_-(t - \bar{t}) \right]$$

where  $C_-$  is the constant given by (4.13) and  $X^{\bar{x}, \bar{t}, \zeta}$  is the solution of (6.7). Owing to Lemma 2.2, we can choose  $\zeta \in L_{\text{ad}}^p(\Omega \times [\bar{t}, t]; \mathbb{R}^N)$  so that  $X^{\bar{x}, \bar{t}, \zeta}$  is a Brownian bridge between  $(\bar{x}, \bar{t})$  and  $(y, t)$  which satisfies

$$\mathbb{E} \left[ \int_{\bar{t}}^t |\zeta_s|^p ds \right] \leq C \left\{ |y - \bar{x}|^p (t - \bar{t})^{1-p} + (t - \bar{t})^{1-p/2} \right\}$$

for some constant  $C$  depending only on  $p$  and  $\delta$ . Since  $u$  is a sub-solution of (6.15), the comparison principle yields

$$\begin{aligned} u(\bar{x}, \bar{t}) \leq v(\bar{x}, \bar{t}) &\leq \mathbb{E} \left[ u(y, t) + C_- \int_{\bar{t}}^t |\zeta_s(\omega)|^p ds + \eta_-(t - \bar{t}) \right] \\ &\leq u(y, t) + C \left\{ |y - \bar{x}|^p (t - \bar{t})^{1-p} + (t - \bar{t})^{1-p/2} \right\} + \eta_-(t - \bar{t}) \end{aligned}$$

for some new constant  $C$  (depending only on  $p$  and  $\delta$ ).  $\square$

**Lemma 6.5.** *Let  $u \in C(\mathbb{R}^N \times [0, T])$  be a sub-solution of (6.15) satisfying  $|u| \leq M$ . Fix  $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, T)$  and  $\zeta \in L_{\text{ad}}^p(\bar{t}, T; \mathbb{R}^N)$ , and let  $X$  be the solution of (6.7). Then, for any  $x \in \mathbb{R}^N$  and  $t \in (\bar{t}, T)$ ,*

$$(6.17) \quad \begin{aligned} &u(x, \bar{t}) - \mathbb{E}[u(X_t, t)] \\ &\leq C \left\{ (t - \bar{t})^{1-p} \left( \mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] + |\bar{x} - x|^p \right) + (t - \bar{t})^{1-p/2} \right\} + \eta_-(t - \bar{t}) \end{aligned}$$

for some constant  $C > 0$ .

*Proof.* Fix  $t \in (\bar{t}, T)$  and apply Lemma 6.3 to  $y = X_t(\omega)$ . Then, for a.e.  $\omega \in \Omega$ ,

$$u(x, \bar{t}) \leq u(X_t(\omega), t) + C \left\{ |X_t(\omega) - x|^p (t - \bar{t})^{1-p} + (t - \bar{t})^{1-p/2} \right\} + \eta_-(t - \bar{t}).$$

Hence,

$$\begin{aligned} u(x, \bar{t}) &\leq \mathbb{E}[u(X_t, t)] \\ &\quad + C \left\{ (\mathbb{E}[|X_t - \bar{x}|^p] + |\bar{x} - x|^p) (t - \bar{t})^{1-p} + (t - \bar{t})^{1-p/2} \right\} + \eta_-(t - \bar{t}). \end{aligned}$$

Since, on account of Lemma 2.1,

$$\mathbb{E}[|X_t - \bar{x}|^p] \leq C \left\{ \mathbb{E} \left[ \left( \int_{\bar{t}}^t |\xi_s| ds \right)^p \right] + \delta^p (t - \bar{t})^{\frac{p}{2}} \right\},$$

the conclusion follows.  $\square$

**Lemma 6.6.** *Let  $u \in C(\mathbb{R}^N \times [0, T])$  be a sub-solution of (6.15) such that  $|u| \leq M$  and let  $\tau \in (0, T)$ . Then there is a universal constant  $\theta \in (p, 2)$  and a constant  $C(\tau, \eta_{\pm}) > 0$  such that, for every  $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, T - \tau)$ , every  $\zeta \in L_{ad}^p(\bar{t}, T; \mathbb{R}^N)$ , and every solution  $X$  of (6.7) satisfying (6.8), we have*

$$\mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] \leq C(\tau, \eta_{\pm})(t - \bar{t})^{p - \frac{p}{\theta}} \quad \forall t \in (\bar{t}, T].$$

*Proof.* First, observe that, by Lemma 6.5 applied to  $x = \bar{x}$ ,

$$u(\bar{x}, \bar{t}) \leq \mathbb{E}[u(X_t, t)] + C \left\{ (t - \bar{t})^{1-p} \mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] + (t - \bar{t})^{1-p/2} \right\} + \eta_-(t - \bar{t})$$

for all  $t \in [\bar{t}, T]$ . Moreover, in view of (6.8),

$$\mathbb{E}[u(X_t, t)] \leq u(\bar{x}, \bar{t}) - C_+ \mathbb{E} \left[ \int_{\bar{t}}^t |\zeta_s|^p ds \right] + \eta_+(t - \bar{t}) \quad \forall t \in [\bar{t}, T].$$

Hence,

$$\mathbb{E} \left[ \int_{\bar{t}}^t |\zeta_s|^p ds \right] \leq C_0 (t - \bar{t})^{1-p} \mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] + C_1(\eta_{\pm})(t - \bar{t})^{1-p/2} \quad \forall t \in [\bar{t}, T]$$

for some constants  $C_0$  and  $C_1(\eta_{\pm})$ . Then, owing to Lemma 3.4, there are constants  $\theta \in (p, 2)$  and  $C(\eta_{\pm}) > 0$  such that

$$\mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] \leq C(\eta_{\pm}) (\|\zeta\|_p^p + 1) \frac{(t - \bar{t})^{p - \frac{p}{\theta}}}{(T - \bar{t})^{1 - \frac{p}{\theta}}} \quad \forall t \in (\bar{t}, T].$$

Since  $u$  is bounded by  $M$ , assumption (6.8) implies that  $\|\zeta\|_p \leq C(\eta_+)$ . Finally,

$$\mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] \leq C(\tau, \eta_{\pm})(t - \bar{t})^{p - \frac{p}{\theta}} \quad \forall t \in (\bar{t}, T],$$

because  $\bar{t} \leq T - \tau$ .  $\square$

## 6.2 Proof of Theorem 6.1

We are now ready to prove Theorem 6.1. As above, we will work with the backward equation (6.5) instead of the forward one. Since  $\eta_+$  and  $\eta_-$  are here fixed, we shall omit the dependence on such variables of all constants in the proof.

Let  $u$  be a bounded continuous solution of (6.5). Thanks to the growth assumption for  $H$ ,  $u$  is a super-solution of (6.6) and a sub-solution of (6.15).

*Space regularity:* Fix  $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, T - \tau)$  and let  $x \in \mathbb{R}^N$ . By Lemma 6.2 there is a process  $\zeta \in L_{ad}^p(\Omega \times [\bar{t}, T]; \mathbb{R}^N)$  and a solution  $X$  to (6.7) such that (6.8) holds. So,

$$(6.18) \quad u(\bar{x}, \bar{t}) \geq \mathbb{E}[u(X_t, t)] - \eta_+(t - \bar{t}) \quad \forall t \in [\bar{t}, T].$$

Also, Lemma 6.6 ensures that

$$(6.19) \quad \mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] \leq C(\tau)(t - \bar{t})^{p - \frac{p}{\theta}} \quad \forall t \in (\bar{t}, T]$$

for some universal constant  $\theta \in (p, 2)$  and some constant  $C(\tau) > 0$ . Furthermore, applying Lemma 6.5, for any  $t \in (\bar{t}, T]$  we have that

$$\begin{aligned} u(x, \bar{t}) - \mathbb{E}[u(X_t, t)] \\ \leq C \left\{ (t - \bar{t})^{1-p} \mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] + |\bar{x} - x|^p (t - \bar{t})^{1-p} + (t - \bar{t})^{1-p/2} \right\} \\ + \eta_-(t - \bar{t}). \end{aligned}$$

Plugging (6.18) and (6.19) into the above inequality leads to

$$\begin{aligned} u(x, \bar{t}) \leq u(\bar{x}, \bar{t}) + \eta(t - \bar{t}) \\ + C(\tau)(t - \bar{t})^{(\theta-p)/\theta} + C|\bar{x} - x|^p (t - \bar{t})^{1-p} + C(t - \bar{t})^{1-p/2} \end{aligned}$$

for any  $t \in (\bar{t}, T)$ , where  $\eta = \eta_+ + \eta_-$ . Since  $1 > 1 - \frac{p}{2} > \frac{\theta-p}{\theta}$  (recall that  $\theta < 2$ ),

$$u(x, \bar{t}) \leq u(\bar{x}, \bar{t}) + C(\tau)(t - \bar{t})^{(\theta-p)/\theta} + C|\bar{x} - x|^p (t - \bar{t})^{1-p}.$$

Then, for  $|x - \bar{x}|$  sufficiently small, choose  $t = \bar{t} + |x - \bar{x}|^{\theta/(\theta-1)}$  to obtain

$$u(x, \bar{t}) \leq u(\bar{x}, \bar{t}) + C(\tau)|x - \bar{x}|^{(\theta-p)/(\theta-1)}.$$

*Time regularity :* Let now  $t \in (0, T - \tau)$ . Then, in light of (6.18),

$$u(\bar{x}, \bar{t}) \geq \mathbb{E}[u(X_t, t)] - \eta_+(t - \bar{t}).$$

Now, applying the space regularity result we have just proved, we obtain

$$\mathbb{E}[u(X_t, t)] \geq u(\bar{x}, t) - C(\tau) \mathbb{E} \left[ |X_t - \bar{x}|^{\frac{\theta-p}{\theta-1}} \right].$$

Moreover, by Lemma 2.1,

$$\mathbb{E} \left[ |X_t - \bar{x}|^{\frac{\theta-p}{\theta-1}} \right] \leq \mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^{\frac{\theta-p}{\theta-1}} \right] + C(t - \bar{t})^{\frac{\theta-p}{2(\theta-1)}}.$$

Also, by Hölder's inequality and (6.19),

$$\mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^{\frac{\theta-p}{\theta-1}} \right] \leq C \left\{ \mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] \right\}^{\frac{\theta-p}{p(\theta-1)}} \leq C(\tau)(t - \bar{t})^{\frac{\theta-p}{\theta}}.$$

Note that  $\frac{\theta-p}{2(\theta-1)} > \frac{\theta-p}{\theta}$  since  $\theta < 2$ . So,

$$u(\bar{x}, \bar{t}) \geq u(\bar{x}, t) - C(\tau)(t - \bar{t})^{\frac{\theta-p}{\theta}}.$$

To derive the reverse inequality, one just needs to apply Lemma 6.3 with  $y = \bar{x}$  to get

$$u(\bar{x}, \bar{t}) \leq u(\bar{x}, t) + C(t - \bar{t})^{1-\frac{p}{2}} + \eta_-(t - \bar{t}).$$

This leads to the desired result since  $1 - \frac{p}{2} > \frac{\theta-p}{\theta}$ .

### 6.3 Local regularity for second order equations

We will now obtain a local version of Theorem 6.1. Let  $\mathcal{O}$  be a non-empty open subset of  $\mathbb{R}_x^N \times \mathbb{R}_t$  and consider the second order Hamilton-Jacobi equation

$$(6.20) \quad u_t(x, t) - \text{Tr}(a(x, t)D^2u(x, t)) + H(x, t, Du(x, t)) = 0 \quad \text{in } \mathcal{O}.$$

As above,  $H : \mathcal{O} \times \mathbb{R}^N \rightarrow \mathbb{R}$  satisfies the super-quadratic growth condition

$$\frac{1}{\delta}|z|^q - \eta_- \leq H(x, t, z) \leq \delta|z|^q + \eta_+ \quad \forall (x, t, z) \in \mathcal{O} \times \mathbb{R}^N$$

for some constants  $q > 2$ ,  $\delta > 1$  and  $\eta_{\pm} \geq 0$ , and  $a = \sigma\sigma^*$  where  $\sigma : \mathcal{O} \rightarrow \mathbb{R}^{N \times D}$  ( $D \geq 1$ ) is assumed to be bounded by  $\delta$  and locally Lipschitz continuous.

For any  $\rho > 0$ , let us set  $\mathcal{O}_\rho = \{(x, t) \in \mathcal{O} : d_{\mathcal{O}^c}(x, t) > \rho\}$ .

**Theorem 6.7.** *Let  $u$  be a continuous viscosity solution of (6.20) satisfying  $|u| \leq M$  in  $\mathcal{O}$  for some  $M \geq 0$ . Then there exists a universal constant  $\theta > p$  and, for any  $\rho > 0$ , a constant  $C(\rho) > 0$  such that*

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C(\rho) \left\{ |x_1 - x_2|^{\frac{\theta-p}{\theta-1}} + |t_1 - t_2|^{\frac{\theta-p}{\theta}} \right\}$$

for all  $(x_1, t_1), (x_2, t_2) \in \mathcal{O}_\rho$ .

*Proof.* As in the proof of Theorem 4.8, for  $\rho > 0$  small enough we will develop our reasoning on a parabolic cylinder contained in  $\mathcal{O}$ , that can be taken of the form  $\overline{B_{4\rho}} \times [0, T]$  where  $T = 4\rho$ . Changing  $t$  to  $T - t$ , we can assume that  $u$  solves the backward equation

$$-u_t(x, t) - \text{Tr}(a(x, t)D^2u(x, t)) + H(x, t, Du(x, t)) = 0 \quad \text{in } B_{4\rho} \times (0, T),$$

which allows certain simplifications in the integrals below. Moreover, we can also extend  $H$  and  $\sigma$  outside of  $\overline{B_{4\rho}} \times [0, T]$  so that (6.2) holds in  $\mathbb{R}^N \times [0, T]$ , and  $\sigma$  is bounded by  $\delta$  and locally Lipschitz continuous in  $\mathbb{R}^N \times [0, T]$ . We shall estimate the Hölder semi-norm of  $u$  on the cylinder  $B_\rho \times (0, T - \rho)$ .

Let  $\phi = \phi(|x|)$  be a smooth function such that  $|\phi| \leq M$  in  $\mathbb{R}^N$ ,  $\phi = M$  in  $B_{3\rho}$ , and  $\phi = -M$  in  $B_{4\rho}^c$ . Let  $\tilde{\eta}_+(\rho) > \eta_+$  and  $\tilde{\eta}_-(\rho) > \eta_-$  be such that  $\phi$  is a (stationary) super-solution of

$$(6.21) \quad -w_t(x, t) - \text{Tr}(a(x, t)D^2w(x, t)) + \delta|Dw(x, t)|^q + \tilde{\eta}_+(\rho) = 0$$

in  $\mathbb{R}^N \times (0, T)$ , and  $-\phi$  is a sub-solution of

$$(6.22) \quad -w_t(x, t) - \text{Tr}(a(x, t)D^2w(x, t)) + \frac{1}{\delta}|Dw(x, t)|^q - \tilde{\eta}_-(\rho) = 0$$

in the same strip. Then, the map  $\bar{u}$  defined by

$$\bar{u} = \begin{cases} u \wedge \phi & \text{in } B_{4\rho} \times (0, T) \\ -M & \text{in } B_{4\rho}^c \times (0, T) \end{cases}$$

is a super-solution of (6.21) which satisfies  $\bar{u} = u$  in  $B_{3\rho} \times (0, T)$ , whereas

$$\tilde{u} \doteq \begin{cases} u \vee (-\phi) & \text{in } B_{4\rho} \times (0, T) \\ M & \text{in } B_{4\rho}^c \times (0, T) \end{cases}$$

is a sub-solution of (6.22) such that  $\tilde{u} = u$  in  $B_{3\rho} \times (0, T)$ .

Recall that, on account of Lemma 6.2, for every  $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, T)$  there is a stochastic basis  $(\Omega, \mathcal{F}, \mathbb{P})$ , a filtration  $(\mathcal{F}_t)_{t \geq \bar{t}}$ , a  $D$ -dimensional Brownian motion  $(W_t)_{t \geq \bar{t}}$  adapted to  $(\mathcal{F}_t)$ , and a process  $\zeta \in L_{\text{ad}}^p(\Omega \times [\bar{t}, T]; \mathbb{R}^N)$  such that the solution  $X$  of (6.7) satisfies

$$(6.23) \quad \bar{u}(\bar{x}, \bar{t}) \geq \mathbb{E} \left[ \bar{u}(X_t, t) + C_+ \int_{\bar{t}}^t |\zeta_s|^p ds \right] - \bar{\eta}_+(\rho)(t - \bar{t}) \quad \forall t \in [\bar{t}, T]$$

with  $C_+$  given by (4.7).

CLAIM 1: Let  $(\bar{x}, \bar{t}) \in B_{2\rho} \times (0, T)$  and let  $X$  be as above. Then

$$(6.24) \quad \mathbb{P}[|X_t - \bar{x}| \geq \rho] \leq C(\rho)(t - \bar{t})^{p-1} \quad \forall t \in [\bar{t}, T]$$

(hereafter,  $C(\rho)$  denotes a constant depending only on  $N, q, \delta, M, \eta_{\pm}$  and  $\rho$ ). Consequently,

$$(6.25) \quad |\mathbb{E}[\bar{u}(X_t, t)] - \mathbb{E}[\tilde{u}(X_t, t)]| \leq C(\rho)(t - \bar{t})^{p-1} \quad \forall t \in [\bar{t}, T].$$

*Proof.* Since  $\bar{u}$  is bounded by  $M$ , (6.23) implies that

$$(6.26) \quad \mathbb{E} \left[ \int_{\bar{t}}^t |\zeta_s|^p ds \right] \leq C(\rho) \quad \forall t \in [\bar{t}, T].$$

Now, by the Bienaymé-Chebyshev inequality and Lemma 2.1 we obtain

$$\mathbb{P}[|X_t - \bar{x}| \geq \rho] \leq \frac{\mathbb{E}[|X_t - \bar{x}|^p]}{\rho^p} \leq \frac{C}{\rho^p} \left\{ \mathbb{E} \left[ \left| \int_{\bar{t}}^t |\zeta_s| ds \right|^p \right] + (t - \bar{t})^{\frac{p}{2}} \right\}.$$

Also, by Hölder's inequality and (6.26),

$$\mathbb{E} \left[ \left| \int_{\bar{t}}^t |\zeta_s| ds \right|^p \right] \leq (t - \bar{t})^{p-1} \mathbb{E} \left[ \int_{\bar{t}}^t |\zeta_s|^p ds \right] \leq C(\rho)(t - \bar{t})^{p-1}.$$

Since  $\frac{p}{2} > p - 1$ , (6.24) follows. To complete the proof of the claim, recall that  $\bar{u} = \tilde{u} = u$  in  $B_{3\rho} \times (0, T)$ , with  $\bar{u}$  and  $\tilde{u}$  bounded by  $M$ . Hence, owing to (6.24),

$$\begin{aligned} |\mathbb{E}[\bar{u}(t, X_t)] - \mathbb{E}[\tilde{u}(t, X_t)]| &\leq 2M\mathbb{P}[X_t \notin B_{3\rho}] \\ &\leq 2M\mathbb{P}[|X_t - \bar{x}| \geq \rho] \leq C(\rho)(t - \bar{t})^{p-1} \end{aligned}$$

which shows (6.25).  $\square$

CLAIM 2: Let  $r = (p-1) \wedge (1 - \frac{p}{2})$ . There is a universal constant  $\theta \in (p, 2 \wedge \frac{p}{r})$  such that, for any  $(\bar{x}, \bar{t}) \in B_{2\rho} \times (0, T - \rho)$  and any  $\zeta \in L^p_{\text{ad}}(\Omega \times [\bar{t}, T]; \mathbb{R}^N)$  such that the solution  $X$  to (6.7) satisfies (6.23), we have

$$(6.27) \quad \mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] \leq C(\rho)(t - \bar{t})^{p - \frac{p}{\theta}} \quad \forall t \in [\bar{t}, T].$$

Moreover, for any  $x \in B_{2\rho}$  and any  $t \in (\bar{t}, T)$ ,

$$(6.28) \quad \begin{aligned} & \tilde{u}(x, \bar{t}) - \mathbb{E}[\tilde{u}(X_t, t)] \\ & \leq C(\rho) \left\{ (t - \bar{t})^{1-p} \left( \mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] + |\bar{x} - x|^p \right) + (t - \bar{t})^{1 - \frac{p}{2}} \right\} + \tilde{\eta}_-(t - \bar{t}). \end{aligned}$$

*Proof.* Inequality (6.28) is a straightforward application of Lemma 6.5. Combining (6.23) and (6.25) gives, for all  $t \in [\bar{t}, T]$ ,

$$\tilde{u}(\bar{x}, \bar{t}) \geq \mathbb{E} \left[ \tilde{u}(X_t, t) + C_+ \int_{\bar{t}}^t |\zeta_s|^p ds \right] - \tilde{\eta}_+(\rho)(t - \bar{t}) - C(\rho)(t - \bar{t})^{p-1}.$$

Putting together the above inequality with inequality (6.28), where  $x = \bar{x}$ , we obtain

$$\mathbb{E} \left[ \int_{\bar{t}}^t |\zeta_s|^p ds \right] \leq C(t - \bar{t})^{1-p} \mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] + C(\rho)(t - \bar{t})^r.$$

We can now apply Lemma 3.4 to deduce the existence of constants  $\theta \in (p, 2 \wedge \frac{p}{r})$  and  $C(\rho) > 0$  such that (6.27) holds.  $\square$

Without loss of generality, in what follows we will assume that

$$(6.29) \quad 1 - \frac{p}{\theta} \leq p - 1.$$

*Space regularity:* Let  $(\bar{x}, \bar{t}) \in B_{2\rho} \times (0, T - \rho)$  and let  $x \in B_{2\rho}$ . Thanks to Claim 1 we can find  $\zeta \in L^p_{\text{ad}}(\Omega \times [\bar{t}, T]; \mathbb{R}^N)$  and a solution  $X$  to (6.7) satisfying (6.23). Combining (6.23), (6.27) and (6.28) with (6.25) yields, for any  $t \in (\bar{t}, T)$ ,

$$\begin{aligned} u(x, \bar{t}) & \leq u(\bar{x}, \bar{t}) \\ & + C(\rho) \left\{ (t - \bar{t})^{1-p} \left( \mathbb{E} \left[ \left( \int_{\bar{t}}^t |\zeta_s| ds \right)^p \right] + |\bar{x} - x|^p \right) + (t - \bar{t})^{1 - \frac{p}{2}} + (t - \bar{t})^{p-1} \right\} \\ & \leq u(\bar{x}, \bar{t}) + C(\rho) \left\{ (t - \bar{t})^{1 - \frac{p}{\theta}} + |\bar{x} - x|^p (t - \bar{t})^{1-p} \right\}, \end{aligned}$$

where we have used (6.29) and the fact that  $1 - \frac{p}{\theta} < 1 - \frac{p}{2}$ . Then, choosing (for  $|x - \bar{x}|$  sufficiently small)  $t = \bar{t} + |x - \bar{x}|^{\theta/(\theta-1)}$  gives

$$u(x, \bar{t}) \leq u(\bar{x}, \bar{t}) + C(\rho)|x - \bar{x}|^{\frac{\theta-p}{\theta-1}}.$$

*Time regularity:* Let now  $\bar{x} \in B_\rho$ . In view of (6.23) we have that

$$u(\bar{x}, \bar{t}) \geq \mathbb{E}[\tilde{u}(X_t, t)] - \tilde{\eta}_+(t - \bar{t})$$

for all  $\bar{t} \in [0, T - \rho)$  and  $t \in [\bar{t}, T]$ . Since  $\bar{u}$  and  $u$  are bounded and coincide on  $B_{3\rho}$ ,

$$\begin{aligned} \mathbb{E}[\bar{u}(X_t, t)] &\geq \mathbb{E}[u(X_t, t)\mathbf{1}_{X_t \in B_{2\rho}}] - M\mathbb{P}[X_t \notin B_{2\rho}] \\ &\geq \mathbb{E}[u(X_t, t)\mathbf{1}_{X_t \in B_{2\rho}}] - M\mathbb{P}[|X_t - \bar{x}| > \rho] \geq \mathbb{E}[u(X_t, t)\mathbf{1}_{X_t \in B_{2\rho}}] - C(\rho)(t - \bar{t})^{p-1}. \end{aligned}$$

First, observe that, owing to the space regularity proved above,

$$\begin{aligned} \mathbb{E}[u(X_t, t)\mathbf{1}_{X_t \in B_{2\rho}}] &\geq \mathbb{E}[u(\bar{x}, t)\mathbf{1}_{X_t \in B_{2\rho}}] - C(\rho) \mathbb{E}\left[|X_t - \bar{x}|^{\frac{\theta-p}{\theta-1}}\right] \\ &\geq u(\bar{x}, t) - C(\rho) \left\{ \mathbb{E}\left[|X_t - \bar{x}|^{\frac{\theta-p}{\theta-1}}\right] + (t - \bar{t})^{p-1} \right\}. \end{aligned}$$

Second, by Lemma 2.1,

$$\mathbb{E}\left[|X_t - \bar{x}|^{\frac{\theta-p}{\theta-1}}\right] \leq C \mathbb{E}\left[\left|\int_{\bar{t}}^t \zeta_s ds\right|^{\frac{\theta-p}{\theta-1}}\right] + C(t - \bar{t})^{\frac{\theta-p}{2(\theta-1)}}.$$

Next, by Hölder's inequality and (6.27),

$$\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s| ds\right)^{\frac{\theta-p}{\theta-1}}\right] \leq C(\rho)(t - \bar{t})^{\frac{\theta-p}{\theta}}.$$

Now, combine all the above inequalities to obtain

$$u(\bar{x}, \bar{t}) \geq u(\bar{x}, t) - C(\rho) \left\{ (t - \bar{t})^{p-1} + (t - \bar{t})^{\frac{\theta-p}{2(\theta-1)}} + (t - \bar{t})^{\frac{\theta-p}{\theta}} \right\} - \bar{\eta}_+(t - \bar{t}).$$

Hence, recalling (6.28) and the fact that  $\frac{\theta-p}{2(\theta-1)} \geq \frac{\theta-p}{\theta}$ , we get

$$u(\bar{x}, \bar{t}) \geq u(\bar{x}, t) - C(\rho)(t - \bar{t})^{\frac{\theta-p}{\theta}}.$$

In order to show the reverse inequality, we just need to invoke Lemma 6.3: indeed, taking  $y = \bar{x}$  we obtain

$$u(\bar{x}, \bar{t}) \leq u(\bar{x}, t) + C(t - \bar{t})^{1-\frac{p}{2}} + \eta_-(t - \bar{t}).$$

The desired result follows since  $1 - \frac{p}{2} > \frac{\theta-p}{\theta}$ .  $\square$

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