Optimal control, Hamilton-Jacobi equations and singularities in euclidean and riemaniann spaces

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Outline

1. Introduction to optimal control and Hamilton-Jacobi equations
   - Examples and problem set-up
   - Existence of solutions
   - Necessary conditions
   - Dynamic Programming
   - Solutions to Hamilton-Jacobi equations

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action functional

Example

$x(t) \in \mathbb{R}^3$: a particle moving from time t_1 to time t_2 between two points A and B and subject to a conservative force

$$F(x(t)) = -\nabla V(x(t))$$

among all the (admissible) trajectories, we want to find the one that minimizes the "action", i.e. the functional

$$J(x) = \int_{t_1}^{t_2} \left[\frac{1}{2} m |x'(t)|^2 - V(x(t))\right] dt,$$

where m is the mass of the particle and $\frac{1}{2} m|x'(t)|^2$ is its kinetic energy

want to find the trajectory that goes from A to B in time $t_2 - t_1$ with "minimal dissipation of energy"

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minimal surfaces of revolution

Example

Consider in the space $\mathbb{R}^3$ the circles

$$\begin{cases} y^2 + z^2 = A^2 \\ x = a. \end{cases}, \qquad \begin{cases} y^2 + z^2 = B^2 \\ x = b. \end{cases},$$

where $a \neq b$. Consider any regular curve in the xz-plane $\xi : [a, b] \to \mathbb{R}^3$, $\xi(x) = (x, 0, \alpha(x))$ such that $\alpha(a) = A$ and $\alpha(b) = B$ and the surface of revolution generated by ξ.

We want to minimize the area of the resulting surface among all the regular functions ξ defined above. But the area of any such a surface S is given by

$$\mathrm{Area}(S) = 2\pi \int_a^b \alpha(x)\sqrt{1 + \alpha'(x)^2}\, dx =: J(\alpha),$$

so that the problem deals with the minimization of the functional J over the class of regular functions α such that $\alpha(a) = A$ and $\alpha(b) = B$.

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boat in stream

Example

We want to model the problem of a boat leaving from the shore to enter in a wide basin where the current is parallel to the shore, increasing with the distance from it. We assume that the engine can move the boat in any direction, but its power is limited. For a fixed time $T > 0$, we want to evaluate the furthest point the boat can reach from the starting point, measured along shore. The mathematical model can be formulated as follows: we fix the (x_1, x_2) Cartesian axes in such a way that the starting point is $(0, 0)$ and the x_1–axis coincides with the (starting) shore. Hence, we want to solve the Mayer problem

$$\min \ -x_1(T),$$

where $(x_1, x_2) : [0, T] \to \mathbb{R}^2$ is a solution of the system

$$\begin{cases} x_1'(t) = x_2(t) + \alpha_1(t) \qquad t \in (0, T) \\ x_2'(t) = \alpha_2(t) \qquad t \in (0, T) \\ x_1(0) = 0 \\ x_2(0) = 0, \end{cases}$$

and the controls (α_1, α_2), which represent the engine power, vary in the unit ball of $\mathbb{R}^2$, i.e. $a_1^2 + a_2^2 \leqslant 1$.

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soft landing with minimal fuel consumption

Denote by $x(t)$ the height at time t, $y(t)$ the instantaneous velocity, and $z(t)$ the total mass of the vehicle. If we call $\alpha(t)$ the instantaneous upwards thrust and suppose the rate of decrease of mass is proportional to α, we obtain the following system

$$\begin{cases} x'(t) = y(t), \\ y'(t) = -g + \dfrac{\alpha(t)}{z(t)}, \\ z'(t) = -K\alpha(t), \end{cases}$$

where $K > 0$ and g is the gravity acceleration. At time 0 we have the initial conditions

$$x(0) = x_0, \qquad y(0) = y_0, \qquad z(0) = z_0. \tag{1}$$

In addition we suppose that the thrust cannot exceed some fixed value, say $0 \leqslant \alpha(t) \leqslant R$ for some $R > 0$. The vehicle will land softly at time $T \geq 0$ if

$$x(T) = 0, \qquad y(T) = 0.$$

The problem of soft landing is then to minimize the amount of fuel consumed from time 0 to time T, that is $z_0 - z(T)$. The problem actually includes two state constraints, namely

$$x(t) \geqslant 0 \quad \text{and} \quad z(t) \geqslant m_0,$$

where m_0 is the mass of the vehicle with empty fuel tanks.

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glossary

- a control process in $\mathbb{R}^n$: (f, A) with
  - $A \subset \mathbb{R}^m$ closed set
  - $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ continuous such that

$$\begin{cases} |f(x,a)| \leqslant K_0\big(|x| + |a|\big) \\ \big|f(x,a) - f(y,a)\big| \leqslant K_1|x-y| \end{cases} \qquad \forall x, y \in \mathbb{R}^n,\ a \in A$$

    for some $K_0, K_1 \geqslant 0$
- a control α at $t_0 \in \mathbb{R}$: a measurable map

$$\alpha : [t_0, \infty) \to A \quad \text{such that} \quad \int_{t_0}^{T} |\alpha(t)|dt < \infty \quad \forall T \geqslant t_0$$

- state equation (SE): given $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, $\alpha \in L^1_{\mathrm{loc}}(t_0, \infty; A)$

$$\begin{cases} \dot{y}(t) = f\big(y(t), \alpha(t)\big) & (t \geqslant t_0) \\ y(t_0) = x_0 \end{cases}$$

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trajectories

- given $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and $\alpha \in L^1_{\mathrm{loc}}(t_0, \infty; A)$

$$\exists! y(\cdot; t_0, x_0, \alpha) \quad \text{such that} \quad \begin{cases} \dot y(t) = f(y(t), \alpha(t)) & (t \geqslant t_0) \\ y(t_0) = x_0 \end{cases}$$

- moreover $\quad \forall T \geqslant 0 \ \exists C_T \geqslant 0 \quad$ such that

$$\big|y(t; t_0, x_0, \alpha) - y(t; t_0, x_1, \alpha)\big| \leqslant C_T \big|x_0 - x_1\big| \qquad \forall t \in [t_0, T]$$

for all $x_0, x_1 \in \mathbb{R}^n$ and $\alpha \in L^1_{\mathrm{loc}}(t_0, \infty; A)$

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remarks

- nonautonomous control processes

$$\begin{cases} \dot{y}(t) = f(t, y(t), \alpha(t)) & (t \geqslant t_0) \\ y(t_0) = x_0 \end{cases}$$

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minimum time problem

given

- (f, A) control process in $\mathbb{R}^n$, α control at $t_0 = 0$

$$y(\cdot; x, \alpha) \quad \text{solution of} \quad \begin{cases} \dot{y}(t) = f\big(y(t), \alpha(t)\big) & (t \geqslant 0) \\ y(0) = x \end{cases}$$

- target $S \subset \mathbb{R}^n$ nonempty closed set

define

- transition time $\quad \tau(x, \alpha) = \inf\{t \geqslant 0 \mid y(t; x, \alpha) \in S\}$
  (observe $\tau(x, \alpha) \in [0, \infty]$)
- controllable set $\quad \mathcal{C} = \{x \in \mathbb{R}^n \mid \exists \alpha \ : \ \tau(x, \alpha) < \infty\}$
- minimum time function $\quad T(x) = \inf_\alpha \tau(x, \alpha) \quad x \in \mathcal{C}$

Exercise

A compact $\implies$ $T(x) > 0 \quad \forall x \in \mathcal{C} \setminus S$

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formulation of Mayer problem

given

- (f, A) control process in $\mathbb{R}^n$
- target $\Sigma \subset \mathbb{R}_t \times \mathbb{R}^n_x$ nonempty closed
- constraint set $\mathcal{K} \subset \mathbb{R}_t \times \mathbb{R}^n_x$ nonempty closed with $\Sigma \subset \mathcal{K}$
- cost $\phi : \Sigma \to \mathbb{R}$ lower semicontinuous

define $\alpha \in L^1_{\mathrm{loc}}(t_0, \infty; A)$ admissible at $(t_0, x_0) \in \mathcal{K}$ if

$$\exists T_\alpha \geqslant 0 \ : \ \begin{cases} (t, y(t; t_0, x_0, \alpha)) \in \mathcal{K} & \forall t \in [t_0, T_\alpha] \\ (T_\alpha, y(T_\alpha; t_0, x_0, \alpha)) \in \Sigma \end{cases}$$

denote by $\mathcal{A}(t_0, x_0)$ all controls that are admissible at (t_0, x_0)

Problem (Mayer)

to minimize $J[\alpha] = \phi(T_\alpha, y(T_\alpha; t_0, x_0, \alpha))$ *over all* $\alpha \in \mathcal{A}(t_0, x_0)$

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Bolza problem

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to minimize

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Outline

1. Introduction to optimal control and Hamilton-Jacobi equations
   - Examples and problem set-up
   - Existence of solutions
   - Necessary conditions
   - Dynamic Programming
   - Solutions to Hamilton-Jacobi equations

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the assumption that Σ be compact can be replaced by

$$\{t \in \mathbb{R} \mid \exists x \in \mathbb{R}^n \ : \ (t,x) \in \Sigma\} \quad \text{bounded above} \qquad (*)$$

or by

$$\lim_{t \to +\infty} \inf_{x \in \Sigma_t} \phi(t,x) = +\infty \qquad (**)$$

where $\Sigma_t = \{x \in \mathbb{R}^n \mid (t,x) \in \Sigma\}$

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deduce existence of solutions to the minimum time problem $\forall x \in \mathcal{C}$ if

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- (f, A) control process in $\mathbb{R}^n$, $T > 0$ horizon

$$y(\cdot; x, \alpha) \quad \text{solution of} \quad \begin{cases} \dot{y}(t) = f(y(t), \alpha(t)) & t \in [0, T] \\ y(0) = x \end{cases}$$

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Problem (Mayer with fixed horizon)

to minimize $\varphi(y(T; x, \alpha))$ *over all* $\alpha \in L^1(0, T; A)$

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generalize to target and state constraints

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then $\displaystyle \min_{\alpha\in\mathcal{A}(t_0,x_0)} \int_{t_0}^{T_\alpha} L\big(y(t; t_0, x_0, \alpha), \alpha(t)\big) dt + \phi\big(T_\alpha, y(T_\alpha; t_0, x_0, \alpha)\big)$ *is attained*

Outline

1. Introduction to optimal control and Hamilton-Jacobi equations
   - Examples and problem set-up
   - Existence of solutions
   - Necessary conditions
   - Dynamic Programming
   - Solutions to Hamilton-Jacobi equations

Mayer problem with fixed horizon

- (f, A) control process in $\mathbb{R}^n$, $T > 0$ horizon

$$y(\cdot; x, \alpha) \quad \text{solution of} \quad \begin{cases} \dot{y}(t) = f(y(t), \alpha(t)) & t \in [0, T] \\ y(0) = x \end{cases}$$

- $\varphi : S \to \mathbb{R}$ lower semicontinuous cost

Definition

- $\alpha^* \in L^1(0, T; A)$ *optimal control at x*

$$\varphi(y(T; x, \alpha^*)) = \min_{\alpha \in L^1(0,T;A)} \varphi(y(T; x, \alpha))$$

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to find necessary conditions for a control α^* to be optimal

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Theorem

- A compact
- $\partial_x f \in C\big(\mathbb{R}^n \times A; \mathbb{R}^{n\times n}\big)$ and $\varphi \in C^1(\mathbb{R}^n)$
- $\alpha^* \in L^1(0,T;A)$ and $y^*(\cdot) := y(\cdot; x, \alpha^*)$ optimal pair

$$\varphi\big(y^*(T)\big) = \min_{\alpha \in L^1(0,T;A)} \varphi\big(y(T;x,\alpha)\big)$$

- let p^* be the solution of the adjoint problem

$$\begin{cases} \dot p(s) = -\partial_x f\big(y^*(s), \alpha^*(s)\big)^{tr} p(s) & (s \in [0,T]) \\ p(T) = \nabla\varphi\big(y^*(T)\big) \end{cases}$$

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remarks

1. define hamiltonian $H(x,p) = \max_{a\in A} -p\cdot f(x,a)$ then

$$-p^*(s)\cdot f(y^*(s),\alpha^*(s)) = H(y^*(s),p^*(s)) \quad (s\in[0,T] \text{ a.e.})$$

2. bang-bang principle: let A be convex and

$$f(x,a) = \sum_{i=1}^{m} a_i f^i(x) + f^0(x) \quad \text{with} \quad f^i \in C^1(\mathbb{R}^n;\mathbb{R}^n)\ (i=0,\dots,m)$$

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example

Exercise (optimal bee hive policy)

given $T > 0\,,\quad \nu > 0\,,\quad \lambda > \mu \geqslant 0\,,\quad x_0 > 0\,,\quad y_0 \geqslant 0$

find $\boxed{\max_\alpha y(T;\alpha)}$ *subject to*

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$$-p^*(s) \cdot f\big(y^*(s), \alpha^*(s)\big) = H\big(y^*(s), p^*(s)\big) \quad \big(s \in [0, T^*] \text{ a.e.}\big)$$

and

$$\partial_t \phi\big(T^*, y^*(T^*)\big) + H\big(y^*(T^*), p^*(T^*)\big) = 0$$

PMP for Mayer problem with terminal constraints

Theorem

- A compact convex
- $f(x,a) = \sum_{i=1}^m a_i f^i(x) + f^0(x)$ with $f^i \in C_b^1(\mathbb{R}^n;\mathbb{R}^n)$ $(i = 0,\dots,m)$
- $\varphi_0 \in C^1(\mathbb{R}^n)$ & $\varphi = (\varphi_1,\dots,\varphi_d) \in C^1(\mathbb{R}^n;\mathbb{R}^d)$
- $\{\alpha^*, y^*\}$ optimal pair

$$\min\Big\{\varphi_0(y(T;x,\alpha)) \;\Big|\; \alpha : [0,T] \to A \;:\; \varphi(y(T;x,\alpha)) = 0\Big\}$$

then $\exists(\lambda_0,\lambda) \in [0,\infty) \times \mathbb{R}^d$ *with* $\lambda_0^2 + |\lambda|^2 = 1$ *such that the solution* p^* *of the adjoint problem*

$$\begin{cases} -\dot p(s) = \sum_{i=1}^m \alpha_i^*(s) Df^i\big(y^*(s)\big)^{tr} p(s) + Df^0\big(y^*(s)\big)^{tr} p(s) \\ p(T) = \sum_{j=0}^d \lambda_j \nabla\varphi_j\big(y^*(T)\big) \end{cases}$$

satisfies

$$\sum_{i=1}^m \alpha_i^*(s)\, f^i\big(y^*(s)\big)\cdot p^*(s) = \min_{a\in A} \sum_{i=1}^m a_i\, f^i\big(y^*(s)\big)\cdot p^*(s) \quad (s \in [0,T^*]\ a.e.)$$

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PMP for Bolza problem with fixed horizon

Theorem

- *A compact*
- $\partial_x f \in C\big(\mathbb{R}^n \times A; \mathbb{R}^{n\times n}\big)$, $\partial_x L \in C\big(\mathbb{R}^n \times A; \mathbb{R}^n\big)$, *and* $\varphi \in C^1(\mathbb{R}^n)$
- $\alpha^* : [0,T] \to A$ *optimal* $J[\alpha^*] = \min_{\alpha^*:[0,T]\to A} J[\alpha]$ *with*

$$J[\alpha] = \int_0^T L\big(y(t;x,\alpha),\alpha(t)\big)\,dt + \varphi\big(y(T;x,\alpha)\big)$$

and $y^*(\cdot) := y(\cdot;x,\alpha^*)$

- *let* p^* *be the solution of the adjoint problem*

$$\begin{cases} \dot p(s) = -\partial_x f\big(y^*(s),\alpha^*(s)\big)^{tr} p(s) - \partial_x L\big(y^*(s),\alpha^*(s)\big) & (s\in[0,T]) \\ p(T) = \nabla\varphi\big(y^*(T)\big) \end{cases}$$

$$\Rightarrow \quad p^*(s)\cdot f\big(y^*(s),\alpha^*(s)\big) + L\big(y^*(s),\alpha^*(s)\big)$$
$$= \min_{a\in A}\Big[p^*(s)\cdot f\big(y^*(s),a\big) + L\big(y^*(s),a\big)\Big] \quad (s\in[0,T]\ a.e.)$$

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Outline

1. Introduction to optimal control and Hamilton-Jacobi equations
   - Examples and problem set-up
   - Existence of solutions
   - Necessary conditions
   - Dynamic Programming
   - Solutions to Hamilton-Jacobi equations

an enlightening example

to compute
$$\int_0^\infty \frac{\sin x}{x}\,dx$$
define
$$v(t) = \int_0^\infty e^{-tx}\frac{\sin x}{x}\,dx \qquad (t \geqslant 0)$$
then
$$-\int_0^\infty x e^{-tx}\frac{\sin x}{x}\,dx = \boxed{v'(t) = -\frac{1}{1+t^2}}$$
thus
$$v(t) = k - \arctan t$$
with
$$0 = \lim_{t\to\infty} v(t) = k - \frac{\pi}{2}$$
therefore
$$\int_0^\infty \frac{\sin x}{x}\,dx = v(0) = \frac{\pi}{2}$$

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value function of Mayer problem

- (f, A) control process, $T > 0$, $(t, x) \in [0, T] \times \mathbb{R}^n$

$$y(\cdot; t, x, \alpha) \quad \text{solution of} \quad \begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s \in [t, T] \\ y(t) = x \end{cases}$$

- A compact
- $\varphi : \mathbb{R}^n \to \mathbb{R}$ lower semicontinuous

Mayer Problem $\boxed{\text{minimize} \quad \varphi(y(T; 0, x, \alpha)) \quad \text{over} \quad \alpha : [0, T] \to A}$

Definition

value function

$$v(t, x) = \inf_{\alpha:[t,T]\to A} \varphi(y(T; t, x, \alpha)) \qquad ((t, x) \in [0, T] \times \mathbb{R}^n)$$

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Mayer Problem: minimize $\varphi(y(T; 0, x, \alpha))$ over $\alpha : [0, T] \to A$

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dynamic programming principle

$$v(t,x) = \inf_{\alpha:[t,T]\to A} \varphi(y(T;t,x,\alpha)) \qquad ((t,x)\in[0,T]\times\mathbb{R}^n)$$

Theorem

1. *for any* $(t,x)\in[0,T]\times\mathbb{R}^n$ *and any* $s\in[t,T]$
$$v(t,x) = \inf_{\alpha:[t,s]\to A} v(s,y(s;t,x,\alpha))$$
2. α *optimal at* (t,x) $\Longleftrightarrow$ $\begin{cases} v(t,x) = v(s,y(s;t,x,\alpha)) \\ \forall s\in[t,T] \end{cases}$

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regularity of the value function

$$v(t,x) = \inf_{\alpha:[t,T]\to A} \varphi(y(T;t,x,\alpha)) \qquad ((t,x) \in [0,T] \times \mathbb{R}^n)$$

Theorem

- (f, A) *control process,* $T > 0$
- A *compact*
- $\varphi : \mathbb{R}^n \to \mathbb{R}$ *locally Lipschitz*

then v *is locally Lipschitz in* $[0,T] \times \mathbb{R}^n$

Rademacher $\Rightarrow$ v differentiable $[0,T] \times \mathbb{R}^n$ a.e.

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the Hamilton-Jacobi equation

$$v(t,x) = \inf_{\alpha:[t,T]\to A} \varphi(y(T;t,x,\alpha)) \qquad ((t,x)\in[0,T]\times\mathbb{R}^n)$$

$$H(x,p) = \max_{a\in A} -p\cdot f(x,a) \qquad ((x,p)\in R^n\times\mathbb{R}^n)$$

Theorem

- A *compact*
- $\varphi:\mathbb{R}^n\to\mathbb{R}$ *locally Lipschitz*

then v *satisfies*

$$\begin{cases} -\partial_t v(t,x) + H(x,\partial_x v(t,x)) = 0 & (t,x)\in(0,T)\times\mathbb{R}^n \text{ a.e.} \\ v(T,x) = \varphi(x) & x\in\mathbb{R}^n \end{cases} \qquad (HJ)$$

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verification theorems

$$\begin{cases} -\partial_t u(t,x) + H(x, \partial_x u(t,x)) = 0 & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(T,x) = \varphi(x) & x \in \mathbb{R}^n \end{cases} \qquad (HJ)$$

Theorem

$u \in C([0,T] \times \mathbb{R}^n) \cap C^1((0,T) \times \mathbb{R}^n)$ solution of (HJ) then

1. $u \le v$
2. *if* $\exists\, \alpha^* : [t,T] \to A$ *such that for a.e.* $s \in [t,T]$

$$-\partial_x u(s, y^*(s)) \cdot f(y^*(s), \alpha^*(s)) = H\Big(y^*(s), \partial_x u(s, y^*(s))\Big)$$

then α^* *optimal at* (t,x) *and* $u(t,x) = v(t,x)$

verification theorems

$$\begin{cases} -\partial_t u(t,x) + H\big(x, \partial_x u(t,x)\big) = 0 & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(T,x) = \varphi(x) & x \in \mathbb{R}^n \end{cases} \qquad (HJ)$$

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then α^* *optimal at* (t,x) *and* $u(t,x) = v(t,x)$

verification theorems

$$\begin{cases} -\partial_t u(t,x) + H\big(x, \partial_x u(t,x)\big) = 0 & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(T,x) = \varphi(x) & x \in \mathbb{R}^n \end{cases} \qquad (HJ)$$

Theorem

$u \in C\big([0,T] \times \mathbb{R}^n\big) \cap C^1\big((0,T) \times \mathbb{R}^n\big)$ *solution of* (HJ)

then

1. $u \leq v$
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the Graal of dynamic programming

want to $\boxed{\text{minimize} \quad \varphi(y(T;t,x,\alpha)) \quad \text{over} \quad \alpha : [t,T] \to A}$

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2. construct a 'nice' map (*feedback*) $a : (0,T) \times \mathbb{R}^n \to A$ such that

$$-\partial_x u(t,x) \cdot f(x, a(t,x)) = H(x, \partial_x u(t,x))$$

3. solve the *closed loop* system

$$\begin{cases} \dot{y}(s) = f(y(s), a(s, y(s))) & s \in [t,T] \\ y(t) = x \end{cases}$$

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dynamic programming for Bolza problem

- value function

$$v(t,x) = \inf_{\alpha:[t,T]\to A} \left\{ \int_t^T L(y(s;t,x,\alpha),\alpha(t))\,dt + \varphi(y(T;t,x,\alpha)) \right\}$$

- dynamic programming principle: $\forall (t,x) \in [0,T]\times\mathbb{R}^n$ and $\forall s \in [t,T]$

$$v(t,x) = \inf_{\alpha:[t,s]\to A} \left\{ \int_t^s L(y(r;t,x,\alpha),\alpha(r))\,dr + v(s,y(s;t,x,\alpha)) \right\}$$

- Hamilton-Jacobi equation

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Linear Quadratic Regulator

$$v(t,x) = \min_{\alpha:[t,T]\to\mathbb{R}^n} \left\{ \int_t^T [Py(s)\cdot y(s) + Q\alpha(s)\cdot\alpha(s)]\, ds + Dy(T)\cdot y(T) \right\},$$

subject to

$$\begin{cases} y'(s) = My(s) + N\alpha(s), & s\in(t,T) \\ y(t) = x \end{cases}$$

where

- $P \in \mathbb{R}^{n\times n},\ P = P^{tr} \geqslant 0$
- $D \in \mathbb{R}^{n\times n},\ D = D^{tr} > 0$
- $Q \in \mathbb{R}^{m\times m},\ Q = Q^{tr} > 0$
- $M \in \mathbb{R}^{n\times n},\ N \in \mathbb{R}^{n\times n}$

$$\arg\min_{a\in\mathbb{R}^M} \left\{ p\cdot(Mx + Na) + Px\cdot x + Qa\cdot a \right\} = -\frac{1}{2} Q^{-1} N^{tr} p$$

Hamilton–Jacobi equation

$$\begin{cases} -\partial_t u - Mx\cdot\partial_x u - Px\cdot x + \frac{1}{4} NQ^{-1}N^{tr}\partial_x u\cdot\partial_x u = 0, & (0,T)\times\mathbb{R}^n \\ u(T,x) = Dx\cdot x \end{cases}$$

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take $\boxed{u(t,x) = R(t)x\cdot x}$ with $R(t)\in\mathbb{R}^{n\times n}$ and $R(y) = R(t)^{tr} > 0$ to obtain

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(R) allows to construct optimal trajectories for LQR by the closed loop system

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Outline

1. Introduction to optimal control and Hamilton-Jacobi equations
   - Examples and problem set-up
   - Existence of solutions
   - Necessary conditions
   - Dynamic Programming
   - Solutions to Hamilton-Jacobi equations

characteristics

$u \in C([0,T]\times\mathbb{R}^n) \cap C^2((0,T)\times\mathbb{R}^n)$ solution of

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with H and φ of class C^2

- *characteristic* starting from $z \in \mathbb{R}^n$ the solution $X(\cdot; z)$

$$\dot{X}(t) = \partial_p H(X(t), \partial_x u(t, X(t))), \qquad X(0) = z$$

- $U(t;z) = u(t, X(t;z))$ and $P(t;z) = \partial_x u(t, X(t;z))$ satisfy

$$\dot{U} = -H(X,P) + P\cdot\partial_p H(X,P) \quad \text{and} \quad \dot{P} = -\partial_x H(X,P)$$

Remark

$$(X,P) \quad \textit{solves} \quad \begin{cases} \dot{X} = \partial_p H(X,P), & X(0) = z \\ \dot{P} = -\partial_x H(X,P), & P(0) = \nabla\varphi(z) \end{cases}$$

and

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$u \in C\big([0,T]\times\mathbb{R}^n\big)\cap C^2\big((0,T)\times\mathbb{R}^n\big)$ solution of

$$\begin{cases} -\partial_t u(t,x) + H\big(x,\partial_x u(t,x)\big) = 0 & (t,x)\in(0,T)\times\mathbb{R}^n \\ u(T,x) = \varphi(x) & x\in\mathbb{R}^n \end{cases} \qquad (HJ)$$

with H and φ of class C^2

- *characteristic* starting from $z\in\mathbb{R}^n$ the solution $X(\cdot;z)$

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let $X(t;z)$, $P(t;z)$ denote the solution of problem (2) and let $U(t;z)$ be defined by (3) suppose there exists $T^* > 0$ such that

- *the maximal solution to (2) is defined at least up to T^* for all $z \in \mathbb{R}^n$*
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characteristics for nonlinear pdes's

- $\Omega \subset \mathbb{R}^n$ open domain
- $\Gamma \subset \Omega$ $(n-1)$-dimensional surface of class C^2 without boundary
- $H \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, $\varphi \in C^2(\Omega)$

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Remark

1. *Characteristics provide a (unique) local solution of* (P) *for smooth data*
2. (P) *may have no smooth global solution*

$$\begin{cases} |\nabla u(x)| = 1 & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases}$$

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the method of vanishing viscosity

- elliptic regularization

$$\begin{cases} u_\epsilon \in C^2(\Omega) \cap C(\overline{\Omega}) \\ -\epsilon \Delta u_\epsilon + H(x, u_\epsilon, \nabla u_\epsilon) = 0 & x \in \Omega \\ u_\epsilon(x) = \varphi(x) & x \in \Gamma \end{cases}$$

- suppose

$$\|u_\epsilon\|_\infty + \|\nabla u_\epsilon\|_\infty \leqslant C \qquad \forall \epsilon > 0$$

then, up to a subsequence, $u_\epsilon \to u_0 \in \mathrm{Lip}(\Omega)$ uniformly

- let $\phi \in C^2(\mathbb{R}^n)$ and assume $u_0 - \phi$ has a strict local maximum at $x_0 \in \Omega$
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viscosity solutions

$$H(x, u, \nabla u) = 0 \quad \text{in } \Omega \tag{H}$$

Definition

a function $u \in C(\Omega)$ is a

- *viscosity subsolution of (H) if $\forall \phi \in C^1(\mathbb{R}^n)$*

$$H(x, u(x), \nabla\phi(x)) \leqslant 0 \qquad \forall x \in \arg\max(u - \phi)$$

- *viscosity supersolution of (H) if $\forall \phi \in C^1(\mathbb{R}^n)$*

$$H(x, u(x), \nabla\phi(x)) \geqslant 0 \qquad \forall x \in \arg\min(u - \phi)$$

- *viscosity solution of (H) if it is both a subsolution and a supersolution of (H)*

viscosity solutions

$$H(x, u, \nabla u) = 0 \quad \text{in } \Omega \qquad (H)$$

Definition

a function $u \in C(\Omega)$ is a

- *viscosity subsolution of (H) if $\forall \phi \in C^1(\mathbb{R}^n)$*

$$H(x, u(x), \nabla\phi(x)) \leqslant 0 \qquad \forall x \in \arg\max(u - \phi)$$

- *viscosity supersolution of (H) if $\forall \phi \in C^1(\mathbb{R}^n)$*

$$H(x, u(x), \nabla\phi(x)) \geqslant 0 \qquad \forall x \in \arg\min(u - \phi)$$

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comparison for viscosity solutions

assume $H : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ and $\phi : \Gamma \to \mathbb{R}$ satisfy $\forall R > 0$

$$\forall x, y \in \overline{\Omega}\,,\ \forall p, q \in \mathbb{R}^n \ : \ |p|, |q| \leqslant R \quad \begin{cases} \big|H(x,p) - H(y,p)\big| \leqslant C_R|x-y| \\ \big|H(x,p) - H(x,q)\big| \leqslant C_R|p-q| \end{cases}$$

and

$$\big|\phi(x) - \phi(y)\big| \leqslant C|x-y| \qquad \forall x, y \in \Gamma$$

Theorem

let $\lambda > 0$ and let $u_-, u_+ \in Lip(\overline{\Omega})$ be a subsolution and a supersolution, respectively, of

$$\begin{cases} \lambda u + H(x, \nabla u) = 0 & \text{in } \overline{\Omega} \setminus \Gamma \\ u(x) = \varphi(x) & x \in \Gamma \end{cases}$$

then $\boxed{u_- \leqslant u_+}$

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concluding remarks

- value functions (Mayer, Bolza, minimum time. . .) turn out to be viscosity solutions of their corresponding Hamilton-Jacobi equations
- existence and comparison results for viscosity solutions hold without assuming convexity of H with respect to p (differential games)
- the theory of viscosity solution has had impressive developments in different directions such as
  - discontinuous solutions
  - analysis of singularities
  - second order (degenerate) elliptic equations
  - geometric evolutions
  - numerical analysis
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