Optimal Control Problems for Differential Inclusions

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16th French-German-Polish Conference on Optimization

Kraków, Poland

September 23-27, 2013
Outline

1. Three basic issues in optimal control
   - Existence of solutions
   - Necessary conditions
   - Dynamic programming

2. Semiconcavity results for nonparameterized control systems
   - Semiconcavity for Mayer problem
   - Semiconcavity for the minimum time problem
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Mayer problem for parameterized control systems

\[ t \leq T \quad x \in \mathbb{R}^N \quad y_{t,x,\alpha}(\cdot) \text{ solution} \]

\[
\begin{cases}
\dot{y}(s) = f(y(s), \alpha(s)) & s \in (t, T) \\
y(t) = x
\end{cases}
\]

where

- \( A \) compact
- \( \alpha : [t, T] \rightarrow A \) measurable
- \( f \in C(\mathbb{R}^N \times A; \mathbb{R}^N) \) continuous
  - \(|f(x, a)| \leq C_0(1 + |x|)\)
  - \(|f(x, a) - f(y, a)| \leq C_1|x - y|\)

given \( \phi \in \text{Lip}(\mathbb{R}^N) \) want to minimize

\[ \alpha \mapsto \phi(y_{t,x,\alpha}(T)) \]
Associated differential inclusion

the control system

\[
\begin{aligned}
\dot{y}(s) &= f(y(s), \alpha(s)) \quad s \in (t, T) \\
y(t) &= x
\end{aligned}
\]  

\hspace{1cm} (CS)

can be recast as the differential inclusion

\[
\begin{aligned}
\dot{y}(s) &\in F(y(s)) \quad \text{a.e in } (t, T) \\
y(t) &= x
\end{aligned}
\]  

\hspace{1cm} (DI)

where \( F(x) = \{ f(x, a) : a \in A \} \)

Our goal

point out analogies and differences in the way to address

- existence of solutions
- necessary conditions
- dynamic programming method and optimality conditions

when passing from parameterized to nonparameterized control systems
Associated differential inclusion

the control system

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Existence of solutions

\[
\begin{align*}
\min_\alpha \phi(y_{t,x,\alpha}(T)) \quad \text{subject to} \quad & \begin{cases}
\dot{y}(s) = f(y(s), \alpha(s)) \\
y(t) = x
\end{cases} \quad s \in (t, T)
\end{align*}
\]

Theorem

If \( f(x, A) \) is convex then (M) has a solution

- control system is recast as differential inclusion

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\begin{cases}
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y(t) = x
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\end{align*}
\]

where \( F(x) = \{ f(x, a) : a \in A \} \)

- compactness of trajectories of (DI) since \( F(\cdot) \) has compact convex images
Existence of solutions

\[ \min_{\alpha} \phi(y_{t,x,\alpha}(T)) \quad \text{subject to} \quad \begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s \in (t, T) \\ y(t) = x \end{cases} \]  

(MP)

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where \( F(x) = \{ f(x, a) : a \in A \} \)

- compactness of trajectories of (DI) since \( F(\cdot) \) has compact convex images
Set-valued functions

$F : \mathbb{R}^N \to \mathbb{R}^N$ is assumed to satisfy

- $F(x) \neq \emptyset$ convex compact $\forall x$
- $F$ locally Lipschitz with respect to $\text{dist}_H$ (Hausdorff)
- $\exists r > 0$ so that $\max\{|v| : v \in F(x)\} \leq r(1 + |x|)$

the Hausdorff distance of two compact sets $S, S' \subset \mathbb{R}^N$

- semidistance

$$\text{dist}_H^+(S, S') = \inf\{\varepsilon : S \subseteq S' + \varepsilon B\}$$

- distance

$$\text{dist}_H(S, S') = \max\{\text{dist}_H^+(S, S'), \text{dist}_H^+(S', S)\}$$
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  \]
Mayer problem for differential inclusions

given
- \( \phi \in \text{Lip}(\mathbb{R}^N) \) and \( F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N \) with
  - \( F(x) \neq \emptyset \) convex compact \( \forall x \)
  - \( F \) locally Lipschitz with respect to \( \text{dist}_H \) (Hausdorff)
  - \( \exists r > 0 \) so that \( \max\{|v| : v \in F(x)\} \leq r(1 + |x|) \)
- \( T \in \mathbb{R}, \; t \leq T, \; x \in \mathbb{R}^N \)


\[ y(s) \in F(y(s)) \; \text{a.e in} \; (t, T) \]
\[ y(t) = x \]

Theorem

The infimum

\[ \inf \left\{ \phi(y(T)) : y \in \mathcal{Y}_T(t, x) \right\} \]

is attained
Mayer problem for differential inclusions

given

- \( \phi \in \text{Lip}(\mathbb{R}^N) \) and \( F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N \) with
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- \( T \in \mathbb{R}, \ t \leq T, \ x \in \mathbb{R}^N \)

denote by \( \mathcal{Y}_T(t, x) \) all absolutely continuous arcs

\[
\begin{cases} 
\dot{y}(s) \in F(y(s)) & \text{a.e in } (t, T) \\
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\end{cases}
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Theorem

The infimum

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is attained
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Maximum Principle: parameterized case

want to find necessary conditions for a control $\alpha^*$ to be optimal

$$\phi(y_{t,x}, \alpha^*(T)) = \min_{\alpha} \phi(y_{t,x}, \alpha(T))$$

Theorem

- $\partial_x f \in C(\mathbb{R}^N \times A; \mathbb{R}^{N \times N})$ and $\phi \in C^1(\mathbb{R}^N)$
- $\alpha^*$ and $y^* := y_{t,x}, \alpha^*$ optimal pair

let $p^*$ be the solution of the adjoint problem

$$\begin{cases}
\dot{p}(s) = -\partial_x f(y^*(s), \alpha^*(s))^\text{tr} p(s) & (s \in [0, T]) \\
p(T) = -\nabla \phi(y^*(T))
\end{cases}$$

then

$$p^*(s) \cdot f(y^*(s), \alpha^*(s)) = \max_{a \in A} p^*(s) \cdot f(y^*(s), a) \quad (s \in [0, T] \ a.e.)$$
Maximum Principle: parameterized case

want to find necessary conditions for a control \( \alpha^* \) to be optimal

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\phi(y_t, x, \alpha^*(T)) = \min_\alpha \phi(y_t, x, \alpha(T))
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$$p^*(s) \cdot f(y^*(s), \alpha^*(s)) = \max_{a \in A} p^*(s) \cdot f(y^*(s), a) \quad (s \in [0, T] \text{ a.e.})$$
Necessary conditions: nonparameterized case

Let \( \phi : \mathbb{R}^N \to \mathbb{R} \) be Lipschitz and let \( F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N \) satisfy

- \( F(x) \neq \emptyset \) convex compact \( \forall x \)
- \( F \) locally Lipschitz with respect to \( \text{dist}_H \) (Hausdorff)
- \( \exists r > 0 \) so that \( \max\{ |v| : v \in F(x) \} \leq r(1 + |x|) \)

want to find necessary conditions for a trajectory \( y^* \) to be optimal

\[
\phi(y^*(T)) = \min \left\{ \phi(y(T)) : \begin{cases} \dot{y} \in F(y) \\ y(t) = x \end{cases} \right\}
\]

difficulty: existence of smooth parameterizations the Hamiltonian \( H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) associated to \( F \) is defined by

\[
H(x, p) = \max_{v \in F(x)} p \cdot v
\]

there is a one-to-one correspondence between \( H \) and \( F \):

\[
v \in F(x) \iff p \cdot v \leq H(x, p) \quad \forall p \in \mathbb{R}^N
\]
Necessary conditions: nonparameterized case

Let $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ be Lipschitz and let $F : \mathbb{R}^N \Rightarrow \mathbb{R}^N$ satisfy

- $F(x) \neq \emptyset$ convex compact $\forall x$
- $F$ locally Lipschitz with respect to $\text{dist}_H$ (Hausdorff)
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difficulty: existence of smooth parameterizations

The Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ associated to $F$ is defined by

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There is a one-to-one correspondence between $H$ and $F$:

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difficulty: existence of smooth parameterizations the Hamiltonian

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there is a one-to-one correspondence between $H$ and $F$:

$$v \in F(x) \iff p \cdot v \leq H(x, p) \quad \forall p \in \mathbb{R}^N$$
The nonsmooth Maximum Principle

**Theorem**

Let

\[ \phi(y^*(T)) = \min \left\{ \phi(y(T)) : \begin{cases} \dot{y} \in F(y) \\ y(t) = x \end{cases} \right\} \]

Then there exists \( p^* : [t, T] \rightarrow \mathbb{R}^N \) absolutely continuous so that

(a) \( (-\dot{p}^*(s), \dot{y}^*(s)) \in \partial H(y^*(s), p^*(s)) \) for a.e. \( s \in [0, T] \)

(b) \( p^*(T) \in -\partial \phi(y^*(T)) \)

- \( \partial \phi(x) \) denotes the Clarke subgradient of \( \phi \) at \( x \), that is,
  \[ \partial \phi(x) = \text{co} \left\{ p \in \mathbb{R}^N : p = \lim_{n} \nabla \phi(x_n) \right\} \]

- Condition (b) above encodes maximum principle
  \[ p^*(s) \cdot \dot{y}^*(s) = H(y^*(s), p^*(s)) = \max_{v \in F(y(s))} p^*(s) \cdot v \]
The nonsmooth Maximum Principle

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Dynamic Programming method

Mayer problem

\[
\min_{\alpha} \phi(y_{t}, x, \alpha(T)) \quad \text{subject to} \quad \begin{cases} 
\dot{y}(s) = f(y(s), \alpha(s)) & s \in (t, T) \\
y(t) = x 
\end{cases}
\]  

(MP)

- define the value function of (MP)

\[
V(t, x) = \min_{\alpha} \phi(y_{t}, x, \alpha(T)) \quad (t, x) \in (-\infty, T] \times \mathbb{R}^N
\]

- consider the Hamilton-Jacobi equation

\[
\begin{cases} 
-u_t(t, x) + H(x, -\nabla u(t, x)) = 0 & (t, x) \in (-\infty, T) \times \mathbb{R}^N \\
u(T, x) = \phi(x) & x \in \mathbb{R}^N
\end{cases}
\]  

(HJ)

- characterize \( V \) via (HJ)

- recover optimality conditions from the properties of \( V \) (feedback)
Dynamic Programming method

Mayer problem

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- define the value function of \((MP)\):

\[ V(t, x) = \min_{\alpha} \phi(y_{t,x}, \alpha(T)) \quad (t, x) \in (-\infty, T] \times \mathbb{R}^N \]

- consider the Hamilton-Jacobi equation

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- characterize \( V \) via \((HJ)\)

- recover optimality conditions from the properties of \( V \) (feedback)
Dynamic Programming method

Mayer problem

\[ \min_{\alpha} \phi(y_t, x, \alpha(T)) \quad \text{subject to} \quad \begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s \in (t, T) \\ y(t) = x \end{cases} \] (MP)

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\[ V(t, x) = \min_{\alpha} \phi(y_t, x, \alpha(T)) \quad (t, x) \in (-\infty, T] \times \mathbb{R}^N \]

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- characterize \( V \) via (HJ)

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- define the value function of (MP)
  \[ V(t, x) = \min_{\alpha} \phi(y(t, x, \alpha(T)) \quad (t, x) \in (-\infty, T] \times \mathbb{R}^N \]

- consider the Hamilton-Jacobi equation
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- characterize \( V \) via (HJ)
  - recover optimality conditions from the properties of \( V \) (feedback)
Dynamic Programming method

Mayer problem

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\min_{\alpha} \phi(y_{t,x},\alpha(T)) \quad \text{subject to} \quad \left\{ \begin{array}{l}
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y(t) = x
\end{array} \right.
\]

- define the value function of (\(MP\))

\[
V(t, x) = \min_{\alpha} \phi(y_{t,x},\alpha(T)) \quad (t, x) \in (-\infty, T] \times \mathbb{R}^N
\]

- consider the Hamilton-Jacobi equation

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\begin{cases}
-u_t(t, x) + H(x, -\nabla u(t, x)) = 0 & (t, x) \in (-\infty, T) \times \mathbb{R}^N \\
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\end{cases}
\]

- characterize \(V\) via (\(HJ\))

- recover optimality conditions from the properties of \(V\) (feedback)
Weak solutions to Hamilton-Jacobi equations

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(HJ)

- has no global smooth solution
- may have infinitely many Lipschitz solutions satisfying (HJ) a.e.
  - Dacorogna and Marcellini (1999)
- has a unique viscosity solution
- the viscosity solution is the unique semiconcave $u$ satisfying (HJ) a.e.
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Semiconcave functions

Definition

We say that a continuous function \( u : \mathbb{R}^N \rightarrow \mathbb{R} \) is (linearly) semiconcave if there exists a constant \( K > 0 \) (a semiconcavity constant for \( u \)) such that

\[
    u(x + z) + u(x - z) - 2u(x) \leq K|z|^2
\]

for all \( x, z \in \mathbb{R}^N \)

- \( u \) is semiconcave with semiconcavity \( K \) if any only if the function

\[
    x \mapsto u(x) - \frac{K}{2}|x|^2
\]

is concave

- \( v \) is semiconvex with semiconvexity constant \( K \) if \(-v\) is semiconcave with semiconcavity constant \( K \)
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P. Cannarsa (Rome Tor Vergata)
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For more on semiconcave functions see

- Control theory
- Nonsmooth and variational analysis
  Rockafellar (1982)
- Differential geometry
- Monographs
  C – Sinestrari (Birkhäuser, 2004)
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Semiconcavity of $V$

Value function

$$V(t, x) = \inf_{\alpha} \phi(y_{t,x,\alpha}(T)) \quad \text{subject to} \quad \begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s \in (t, T) \\ y(t) = x \end{cases}$$

Theorem (C – Frankowska, 1991)

Assume

- $\phi$ semiconcave
- $\|f_x(x, a) - f_x(y, a)\| \leq C_2|x - y|$ for all $x, y \in \mathbb{R}^N$, $a \in A$

Then $V$ is (linearly) semiconcave on $(-\infty, T] \times \mathbb{R}^N$

the proof uses the fact that $f$ is a smooth parameterization of the process
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\[ V(t, x) = \inf_{\alpha} \phi(y_t, x, \alpha(T)) \] subject to \[
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Sketch of the proof

\[ x, z \in \mathbb{R}^N \]

- \( \alpha \) optimal at \( x \)
- \( y(\cdot) = y_{t,x,\alpha}(\cdot), \ y_\pm(\cdot) = y_{t,x\pm z,\alpha}(\cdot) \)

by semiconcavity of \( \phi \) and smoothness of flow

\[
V(t, x + z) + V(t, x - z) - 2V(t, x) \\
\leq \phi(y_+(T)) + \phi(y_-(T)) - 2\phi(y(T)) \\
= \phi(y_+(T)) + \phi(y_-(T)) - 2\phi\left(\frac{y_+(T) + y_-(T)}{2}\right) \\
\leq c|y_+(T) - y_-(T)|^2 \leq c|z|^2 \\
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\[
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Outline

1. Three basic issues in optimal control
   - Existence of solutions
   - Necessary conditions
   - Dynamic programming

2. Semiconcavity results for nonparameterized control systems
   - Semiconcavity for Mayer problem
   - Semiconcavity for the minimum time problem
a semiconcavity result

Value function

\[ V(t, x) = \min \left\{ \phi(y(T)) \mid \begin{array}{l} \dot{y} \in F(y) \\ y(t) = x \end{array} \right\} \]

- \( \phi : \mathbb{R}^N \to \mathbb{R} \) semiconcave
- \( F(x) \neq \emptyset \) convex compact dist-\( H \)-Lipschitz and \( \max\{|v| : v \in F(x)\} \leq r(1 + |x|) \)
- \( H(x, p) = \max_{v \in F(x)} p \cdot v \)

Theorem (C – Wolenski)

Assume

(H1) \( x \mapsto H(x, p) \) semiconvex with constant \( c|p| \)
(H2) \( \nabla_p H(x, p) \) Lipschitz in \( x \) uniformly for \( |p| = 1 \)

Then \( V \) is semiconcave on \( (-\infty, T] \times \mathbb{R}^N \)
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proof I: construction of perturbed trajectories

- Lipschitz regularity of $V$ known
- fix $x, z \in \mathbb{R}^N$, want to show: $V(t, x + z) + V(t, x - z) - 2V(t, x) \leq c|z|^2$
- let $y$ be optimal and invoke maximum principle to obtain
  $$\begin{cases}
  \dot{y}(s) = \nabla_p H(y(s), p(s)), & y(t) = x \\
  -p(s) \in \partial_x H(y(s), p(s)), & -p(T) \in \partial^+ \phi(y(T))
  \end{cases}$$

- supposing $p(s) \neq 0$ define $y_\pm(\cdot)$ by
  $$\begin{cases}
  \dot{y}_\pm(s) = \nabla_p H(y_\pm(s), p(s)) \\
  y_\pm(t) = x \pm z
  \end{cases}$$
  and observe
  - $|y_+(s) - y_-(s)| \leq c|z|$ and $|y_\pm(s) - y(s)| \leq c|z|$
  - $p \cdot \dot{y} = H(y, p)$ and $p \cdot \dot{y}_\pm = H(y_\pm, p)$
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proof II: use of transversality condition

\[ V(t, x + z) + V(t, x - z) - 2V(t, x) \]
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\[ = \phi(y_+(T)) + \phi(y_-(T)) - 2\phi\left(\frac{y_+(T) + y_-(T)}{2}\right) \]
\[ \leq c|y_+(T) - y_-(T)|^2 \leq c|z|^2 \]
\[ + 2\left[ \phi\left(\frac{y_+(T) + y_-(T)}{2}\right) - \phi(y(T)) \right] \]

since \(-p(T) \in \partial^+ \phi(y(T))\) one has

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\[ \leq -p(T) \cdot \left[ \frac{y_+(T) + y_-(T)}{2} - y(T) \right] + c\left| \frac{y_+(T) + y_-(T)}{2} - y(T) \right|^2 \]
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\[ V(t, x + z) + V(t, x - z) - 2V(t, x) \]

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\[ \leq c|z|^2 \]
proof III: use of maximum principle

\[ -p(T) \cdot \left[ \frac{y_+(T) + y_-(T)}{2} - y(T) \right] = \frac{1}{2} \int_t^T \left[ -\dot{p} \cdot (y_+ + y_- - 2y) + p \cdot (2\dot{y} - \dot{y}_+ - \dot{y}_-) \right] ds \]

\[ \frac{1}{2} \int_t^T p \cdot (2\dot{y} - \dot{y}_+ - \dot{y}_-) \, ds \]

\[ = \int_t^T \left[ H(y, p) - \frac{H(y_+, p) + H(y_-, p)}{2} \right] ds \]

\[ = \int_t^T \left[ H(y, p) - H\left( \frac{y_+ + y_-}{2}, p \right) \right] ds \]

\[ + \int_t^T \left[ H\left( \frac{y_+ + y_-}{2}, p \right) - \frac{H(y_+, p) + H(y_-, p)}{2} \right] ds \]

\[ \leq c|p||y_+ - y_-|^2 \leq c|z|^2 \]

\[ \int_t^T \left[ H(y, p) - H\left( \frac{y_+ + y_-}{2}, p \right) \right] ds \]

\[ \leq \int_t^T \left[ \dot{p} \cdot \left( \frac{y_+ + y_-}{2} - y \right) + c|p| \left| \frac{y_+ + y_-}{2} - y \right|^2 \right] ds \]

\[ \leq c|z|^2 \]
proof III: use of maximum principle

\[-p(T) \cdot \left[ \frac{y+(T)+y-(T)}{2} - y(T) \right] \]

\[= \frac{1}{2} \int_t^T \left[ -\dot{p} \cdot (y_+ + y_- - 2y) + p \cdot (2\dot{y} - \dot{y}_+ - \dot{y}_-) \right] \, ds \]

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\[\leq c|p||y_+ - y_-|^2 \leq c|z|^2 \]

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\[+ \int_t^T \left[ H\left(\frac{y_+ + y_-}{2}, p\right) - \frac{H(y_+, p) + H(y_-, p)}{2} \right] ds \leq c|p||y_+ - y_-|^2 \leq c|z|^2 \]

\[\int_t^T \left[ H(y, p) - H\left(\frac{y_+ + y_-}{2}, p\right) \right] ds \leq \int_t^T \left[ -\dot{p} \cdot \left( \frac{y_+ + y_-}{2} - y \right) + c|p| \left| \frac{y_+ + y_-}{2} - y \right|^2 \right] ds \leq c|z|^2 \]
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\[\leq c|z|^2 \]
Outline

1. Three basic issues in optimal control
   - Existence of solutions
   - Necessary conditions
   - Dynamic programming

2. Semiconcavity results for nonparameterized control systems
   - Semiconcavity for Mayer problem
   - Semiconcavity for the minimum time problem
Glossary

\( \mathcal{Y}(x) \)  admissible trajectories at \( x \): solutions to
\[
\begin{aligned}
\dot{y}(t) &\in F(y(t)) \\
y(0) &= x
\end{aligned}
\]

also of interest parameterized case: \( F(x) = f(x, A) \)

\( S \)  target: nonempty closed subset of \( \mathbb{R}^N \)

\( \mathcal{T}(\cdot) \)  minimum time function:
\[
\mathcal{T}(x) = \inf_{y \in \mathcal{Y}(x)} \{ t \geq 0 : y(t) \in S \}
\]

\( C, C_t \)  controllable sets:
\[
\begin{aligned}
C &= \{ x \in \mathbb{R}^n | \mathcal{T}(x) < \infty \} \\
C_t &= \{ x \in \mathbb{R}^n | \mathcal{T}(x) \leq t \} \quad \text{(in time } t \geq 0)\end{aligned}
\]

(HJ)  Hamilton-Jacobi equation:
\[
\begin{cases}
H(x, -\nabla \mathcal{T}) = 1 & \text{in } C \setminus S \\
\mathcal{T}(x) = 0 & x \in \partial S \\
(\mathcal{T}(x) \to +\infty) & x \to \partial C
\end{cases}
\]
Petrov condition & Lipschitz continuity of $\mathcal{T}(\cdot)$

inward pointing condition

$F$ satisfies the **inward pointing (or Petrov) condition** on $S$ if $\exists r > 0$ such that

$$H(x, -\nu) = \max_{\nu \in F(x)} -\nu \cdot \nu \geq r|\nu|$$  \hspace{1cm} (PC)

for any $x \in \partial S$ and any proximal normal $\nu$ to $S$ at $x$

under $(PC)$:

- $\mathcal{C}$ is an **open neighborhood** of $S$
- $\mathcal{T}(\cdot)$ is **locally Lipschitz** on $\mathcal{C}$

**Inner ball property & semiconcavity of \( T(\cdot) \)**

**Inner ball property**

\( S \) has the **inner ball (or interior sphere) property** if, for some \( r > 0 \),

\[
\forall x \in \partial S \; \exists y \in S \; \text{such that} \; x \in B_r(y) \subset S \quad \text{(IB)}
\]

under \((PC)\) and \((IB)\):

\( T(\cdot) \) **locally semiconcave** on \( C \setminus \text{int}(S) \)

- **C-Sinestrari (1995):**
  
  (a) \( F(x) = f(x, A) \)
  
  (b) \( \| f_x(x, a) - f_x(y, a) \| \leq C_2 |x - y| \)

- **C-Marino, Wolenski (2012):**
  
  (a') \( x \mapsto H(x, p) \) semiconvex
  
  (b') \( x \mapsto \nabla_p H(x, p) \) Lipschitz
Our goal
To obtain semiconcavity removing the inner ball property (from $S$)

Theorem (C-Frankowska, 2006)
Assume ($PC$) and suppose

- $F(x) = f(x, A)$ has the inner ball property $\forall x$ near $\partial S$
- $\|f_x(x, a) - f_x(y, a)\| \leq C_2 |x - y|$
- $x \mapsto \nabla_p H(x, p)$ Lipschitz

Then $T(\cdot)$ is locally semiconcave on $\mathcal{C} \setminus S$

C-Khai T. Nguyen (2011): above conclusion also true under the following

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Maximum principle & dual arc inclusion

**Theorem**

Let $x \notin S$ and

- $y \in \mathcal{Y}(x)$ time-optimal, $T = T(x)$
- $\nu$ inner normal to $S$ at $y(T)$ with $|\nu| = 1$ and $r := H(y(T), \nu) > 0$

Then $\exists p : [0, T] \rightarrow \mathbb{R}^N$ such that

$$
\begin{cases}
\dot{y} = \nabla_p H(y, p) \\
-\dot{p} \in \partial_x H(y, p), \quad p(T) = \frac{1}{r} \nu
\end{cases}
$$

and $-p(t) \in \partial^+ T(y(t)) \quad \forall t \in [0, T)$

- C-Frankowska, Sinestrari (2000): $F(x) = f(x, A)$
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The constancy of $H$

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Then $H(y(t), p(t)) \equiv 1$ for all $t \in (0, T)$

Proof based on the fact that for any $x \notin S$ and $-p \in \partial^+ T(x)$:

- $H(x, p) \leq 1$ (subsolution property), and
- if $x$ is an optimal point (interior to optimal arc), then $H(x, p) = 1$
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Let $x \notin S$ and

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  \end{cases}
  \quad \text{and} \quad -p(t) \in \partial^+ T(y(t)) \quad \forall t \in [0, T)
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Theorem

Assume

- $T(\cdot)$ locally semiconcave on $C \setminus \text{int}(S)$
- $x \notin S$ and $y \in \mathcal{Y}(x)$ with $y(T) \in S$
- $p : [0, T] \to \mathbb{R}^N$ measurable such that
  \[
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  $t \in [0, T]$ a.e.

Then $T = T(x)$ and $y$ is time-optimal
Optimality conditions

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Beyond semiconcavity

Why go beyond semiconcavity? For several reasons including:

- treatment of state constraints
- relaxing controllability assumptions for time optimal control
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Removing Petrov condition

Roughly speaking, semiconcavity of $\mathcal{T}$ is equivalent to:

\begin{itemize}
  \item \textbf{(PC)} inward pointing condition on $\partial S$
  \begin{equation*}
    \max_{\nu \in F(x)} -\nu \cdot \nu \geq r|\nu| \quad (x \in \partial S, \nu \perp S)
  \end{equation*}
  \item \textbf{(PS)} exterior ball property of the hypograph
  \begin{equation*}
    \text{hypo}(\mathcal{T}) = \{(x, \tau) \in C \times \mathbb{R} : \tau \leq \mathcal{T}(x)\}
  \end{equation*}
\end{itemize}

Removing (PC) retaining (PS) possible under mild controllability assumptions:

- C-Khai T. Nguyen (2011)
Removing Petrov condition

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\text{hypo}(T) = \{(x, \tau) \in C \times \mathbb{R} : \tau \leq T(x)\}
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Conclusions

Passing from parameterized to non parameterized models has +’s and −’s:

- **advantages:**
  - allows to treat systems with no smooth parameterization
  - preserves most of the basic results

- **disadvantages:**
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Merci

Danke

Dziękuuję