EAM2 Lecture Notes

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Contents

Notation

- $\mathbb{R} = (-\infty, \infty)$ stands for the real line, $\mathbb{R}_+$ for $[0, \infty)$, and $\mathbb{R}_+^*$ for $(0, \infty)$.
- $\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, \dots\}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\} = \{\pm 1, \pm 2, \dots\}$.
- For any $\lambda \in \mathbb{C}$, $\Re \lambda$ and $\Im \lambda$ denote the real and imaginary parts of λ, respectively.
- $|\cdot|$ stands for the norm of a Banach space X, as well as for the absolute value of a real number or the modulus of a complex number.
- $\mathcal{L}(X)$ is the Banach space of all bounded linear operators $\Lambda : X \to X$ equipped with norm $\|\Lambda\| = \sup_{|x| \leqslant 1} |\Lambda x|$.
- For any metric space (X, d), $\mathcal{C}_b(X)$ denotes the Banach space of all bounded uniformly continuous functions $f : X \to \mathbb{R}$ with norm
$$\|f\|_{\infty, X} = \sup_{x \in X} |f(x)|.$$
- Given a Banach space $(X, |\cdot|)$ and a closed interval $I \subseteq \mathbb{R}$ (bounded or unbounded), we denote by $\mathcal{C}_b(I; X)$ the Banach space of all bounded uniformly continuous functions $f : I \to X$ with norm
$$\|f\|_{\infty, I} = \sup_{s \in I} |f(s)|.$$
- $\Pi_\omega = \left\{\lambda \in \mathbb{C} \ : \ \Re \lambda > \omega\right\}$ for any $\omega \in \mathbb{R}$.

1 Semigroups of bounded linear operators

1.1 Uniformly continuous semigroups

Let $(X, |\cdot|)$ be a (real or complex) Banach space. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators $\Lambda : X \to X$ with norm

$$\|\Lambda\| = \sup_{|x|\leqslant 1} |\Lambda x|.$$

Definition 1 *A semigroup of bounded linear operators on X is a map $S : [0,\infty) \to \mathcal{L}(X)$ with the following properties:*

(*a*) $S(0) = I$,

(*b*) $S(t+s) = S(t)S(s)$ *for all* $t, s \geqslant 0$.

Equivalent notations are $S(\cdot)$, $\{S(t)\}_{t\geqslant 0}$, and even the simpler form $S(t)$.

Definition 2 *The infinitesimal generator of a semigroup of bounded linear operators $S(t)$ is the map $A : D(A) \subset X \to X$ defined by*

$$\begin{cases} D(A) = \left\{x \in X \ : \ \exists \lim_{t\downarrow 0} \frac{S(t)x - x}{t}\right\} \\ Ax = \lim_{t\downarrow 0} \frac{S(t)x-x}{t} \qquad\qquad \forall x \in D(A) \end{cases} \tag{1.1.1}$$

Exercise 1 Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a semigroup of bounded linear operators $S(t)$. Prove that

(*a*) $D(A)$ is a subspace of X,

(*b*) A is a linear operator.

Definition 3 *A semigroup $S(t)$ of bounded linear operators on X is* uniformly continuous *if*

$$\lim_{t\downarrow 0} \|S(t) - I\| = 0.$$

Exercise 2 Let $S(t)$ be a uniformly continuous semigroup of bounded linear operators. Prove that for all $\tau \geqslant 0$ there exists $M_\tau \geqslant 0$ such that

$$\|S(t)\| \leqslant M_\tau \qquad \forall t \in [0, \tau].$$

Remark 1 A semigroup $S(t)$ is uniformly continuous if and only if

$$\lim_{s\to t} \|S(s) - S(t)\| = 0 \qquad \forall t \geqslant 0.$$

Example 1 let $A \in \mathcal{L}(X)$. Then

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x$$

is a uniformly continuous semigroup of bounded linear operators on X. Moreover, A is the infinitesimal generatorof e^{tA}. Indeed, the proof of the following properties is left as an exercise.

(*a*) $e^{tA} \in \mathcal{L}(X)$ because $\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x$ converges for all $t \geqslant 0$.

(*b*) $e^{(t+s)A} = e^{tA} e^{sA}$ for all $s, t \geqslant 0$.

(*c*) $\|e^{tA} - I\| = \|\sum_{n=1}^{\infty} \frac{t^n}{n!} A^n\| \leqslant t\|A\| e^{t\|A\|}$ for all $t \geqslant 0$.

(*d*) $\|\frac{e^{tA}-I}{t} - A\| = \|\sum_{n=2}^{\infty} \frac{t^{n-1}}{n!} A^n\| \leqslant t\|A\|^2 e^{t\|A\|}$ for all $t \geqslant 0$.

Theorem 1 *For any linear operator $A : D(A) \subset X \to X$ the following properties are equivalent:*

(*a*) *A is the infinitesimal generator of a uniformly continuous semigroup,*

(*b*) *$A \in \mathcal{L}(X)$.*

Proposition 1 *Let $S(t)$ and $T(t)$ be uniformly continuous semigroups of bounded linear operators on X and let $A \in \mathcal{L}(X)$. If*

$$\lim_{t \downarrow 0} \frac{S(t) - I}{t} = A = \lim_{t \downarrow 0} \frac{T(t) - I}{t},$$

then $S(t) = T(t)$ for all $t \geqslant 0$.

Let $T > 0$. For any $A \in \mathcal{L}(X)$, a solution of the Cauchy problem

$$\begin{cases} y'(t) = Ay(t) \quad (t \in [0, T]) \\ y(0) = x \in X \end{cases} \tag{1.1.2}$$

is a function $y \in \mathcal{C}^1([0, T]; X)$ which satisfies (1.1.2) pointwise.

Proposition 2 *Problem* (1.1.2) *has a unique solution given by $y(t) = e^{tA}x$.*

Example 2 Consider the integral equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \int_0^1 k(x, y) u(t, y)\, dy \quad (t \in [0, T]) \\ u(0, x) = u_0(x) \end{cases} \tag{1.1.3}$$

where $k \in L^2\big([0,1]\times[0,1]\big)$ and $u_0 \in L^2(0,1)$. Problem (1.1.3) can be seen as an abstract Cauchy problem of the form

$$\begin{cases} u'(t) = Ku(t) & (t \in [0,T]) \\ u(0) = u_0 \in X \end{cases} \tag{1.1.4}$$

where $X = L^2(0,1)$ and

$$Ku(x) = \int_0^1 k(x,y)u(t,y)\,dy \qquad \forall x \in X$$

is a bounded linear operator on X. Then Proposition 2 insures that (1.1.4) has a unique solution $u \in \mathcal{C}^1([0,T];X)$ given by $u(t) = e^{tK}u_0$.

1.2 Strongly continuous semigroups

Example 3 Let $\mathcal{C}_b(\mathbb{R})$ be the Banach space of all bounded uniformly continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the uniform norm

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|.$$

For any $t \in \mathbb{R}_+$ define

$$\big(S(t)f\big)(x) = f(x+t) \qquad \forall f \in \mathcal{C}_b(\mathbb{R}).$$

The reader is invited to check that:

1. $S(t)$ is a semigroup of bounded linear operators on $\mathcal{C}_b(\mathbb{R})$,

2. $S(t)$ fails to be uniformly continuous,

3. for all $f \in \mathcal{C}_b(\mathbb{R})$ we have that $\|S(t)f - f\|_\infty \to 0$ as $t \downarrow 0$.

Definition 4 *A semigroup $S(t)$ of bounded linear operators on X is called* strongly continuous *(or of class $\mathcal{C}_0$) if*

$$\lim_{t\downarrow 0} S(t)x = x \qquad \forall x \in X. \tag{1.2.1}$$

Theorem 2 *Let $S(t)$ be a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Then there exist $\omega \geqslant 0$ and $M \geqslant 1$ such that*

$$\|S(t)\| \leqslant Me^{\omega t} \qquad \forall t \geqslant 0. \tag{1.2.2}$$

When $\omega = 0$ in (1.2.2), $S(t)$ is called *uniformly bounded*. If, in addition, $M = 1$, we say that $S(t)$ is a *contraction semigroup*.

Proof. We first prove the following:

$$\exists \tau > 0 \quad \text{and} \quad M \geqslant 1 \quad \text{such that} \quad \|S(t)\| \leqslant M \quad \forall t \in [0, \tau]. \tag{1.2.3}$$

We argue by contradiction assuming there exists a sequence $t_n \downarrow 0$ such that $\|S(t_n)\| \geqslant n$ for all $n \geqslant 1$. Then, the principle of uniform boundedness implies that, for some $x \in X$, $\|S(t_n)x\| \to \infty$ as $n \to \infty$, in contrast with (1.2.1).

Now, given $t \in \mathbb{R}_+$, let $n \in \mathbb{N}$ and $\delta \in [0, \tau[$ be such that

$$t = n\tau + \delta.$$

Then, in view of (1.2.3),

$$\|S(t)\| = \|S(\delta)S(\tau)^n\| \leqslant M \cdot M^n = M \cdot (M^{1/\tau})^{n\tau} \leqslant M \cdot (M^{1/\tau})^t$$

which yields (1.2.2) with $\omega = \frac{\log M}{\tau}$. □

Corollary 1 *Let $S(t)$ be a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Then for every $x \in X$ the map $t \mapsto S(t)x$ is continuous from $\mathbb{R}_+$ into X.*

1.3 The infinitesimal generator of a $\mathcal{C}_0$-semigroup

Theorem 3 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X, denoted by $S(t)$. Then the following properties hold true.*

(a) *For all $t \geqslant 0$*

$$\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} S(s)x\, ds = S(t)x \qquad \forall x \in X.$$

(b) *For all $t \geqslant 0$ and $x \in X$*

$$\int_0^t S(s)x\, ds \in D(A) \quad \textit{and} \quad A\Big(\int_0^t S(s)x\, ds\Big) = S(t)x - x.$$

(c) *$D(A)$ is dense in X.*

(d) *For all $x \in D(A)$ and $t \geqslant 0$ we have that $S(t)x \in D(A)$, $t \mapsto S(t)x$ is continuously differentiable, and*

$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax.$$

(e) *For all* $x \in D(A)$ *and all* $0 \leqslant s \leqslant t$ *we have that*

$$S(t)x - S(s)x = \int_s^t S(\tau)Ax\, d\tau = \int_s^t AS(\tau)x\, d\tau.$$

Proof. All integrals are to be understood in the Riemann sense.

(a) This point is an immediate consequence of the strong continuity of S.

(b) For any $t \geqslant h > 0$ we have that

$$\begin{aligned}\frac{S(h)-I}{h}\Big(\int_0^t S(s)x\, ds\Big) &= \frac{1}{h}\int_0^t (S(h+s)-S(s))x\, ds\\ &= \frac{1}{h}\Big(\int_h^{t+h} S(s)x\, ds - \int_0^t S(s)x\, ds\Big)\\ &= \frac{1}{h}\Big(\int_t^{t+h} S(s)x\, ds - \int_0^h S(s)x\, ds\Big).\end{aligned}$$

Therefore, by (a),

$$\lim_{h\downarrow 0}\frac{S(h)-I}{h}\Big(\int_0^t S(s)x\, ds\Big) = S(t)x - x$$

which proves (b).

(c) This point follows from (a) and (b).

(d) For all $x \in D(A)$, $t \geqslant 0$, and $h > 0$ we have that

$$\frac{S(h)-I}{h}\, S(t)x = S(t)\,\frac{S(h)-I}{h}\, x \to S(t)Ax \quad \text{as} \quad h \downarrow 0.$$

Therefore $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax = \frac{d^+}{dt}\, S(t)x$. In order to prove the existence of the left derivative, observe that for all $0 < h < t$

$$\frac{S(t-h)x - S(t)x}{-h} = S(t-h)\,\frac{S(h)-I}{h}\, x.$$

Moreover, by (1.2.2),

$$\begin{aligned}&\Big|S(t-h)\,\frac{S(h)-I}{h}\, x - S(t)Ax\Big|\\ &\qquad\leqslant \Big|S(t-h)\Big|\cdot\Big|\frac{S(h)-I}{h}\, x - S(h)Ax\Big|\\ &\qquad\qquad\leqslant Me^{\omega t}\Big|\frac{S(h)-I}{h}\, x - S(h)Ax\Big| \longrightarrow 0 \text{ as } h \downarrow 0.\end{aligned}$$

Therefore

$$\frac{S(t-h)x - S(t)x}{-h} \longrightarrow S(t)Ax = AS(t)x \quad \text{as} \quad h \downarrow 0,$$

showing that the left and right derivatives coincide.

(e) This point follows from (d).

The proof is complete. $\square$

1.4 Closedness of A

We recall that $X \times X$ is a Banach space with norm

$$\|(x,y)\| = |x| + |y| \qquad \forall (x,y) \in X \times X.$$

Definition 5 *An operator $A : D(A) \subset X \to X$ is said to be closed if its graph*

$$graph(A) := \big\{(x,y) \in X \times X \ : \ x \in D(A)\,,\ y = Ax\big\}$$

is a closed subset of $X \times X$.

Exercise 3 Prove that $A : D(A) \subset X \to X$ is closed if and only if for any sequence $\{x_n\} \subset D(A)$

$$\begin{cases} x_n \to x \\ Ax_n \to y \end{cases} \qquad \Longrightarrow \qquad x \in D(A) \quad \text{and} \quad Ax = y. \tag{1.4.1}$$

Proposition 3 *The infinitesimal generator of a $\mathcal{C}_0$-semigroup $S(t)$ is a closed operator.*

Proof. Let $A : D(A) \subset X \to X$ be the infinitesimal generator of $S(t)$ and let $\{x_n\} \subset D(A)$ be as in (1.4.1). By Theorem 3$-(d)$ we have that, for all $t \geqslant 0$,

$$S(t)x_n - x_n = \int_0^t S(s)Ax_n dx.$$

Hence, taking the limit as $n \to \infty$ and dividing by t, we obtain

$$\frac{S(t)x - x}{t} = \frac{1}{t}\int_0^t S(s)y dx.$$

Passing to the limit as $t \downarrow 0$, we conclude that $Ax = y$. $\square$

Remark 2 From Proposition 3 it follows that the domain $D(A)$ of the infinitesimal generator of a $\mathcal{C}_0$-semigroup is a Banach space with the *graph norm*

$$|x|_{D(A)} = |x| + |Ax| \qquad \forall x \in D(A).$$

Proposition 4 (Cauchy problem) *Let S be a $\mathcal{C}_0$-semigroup of bounded linear operators on X and $A : D(A) \subset X \to X$ be its infinitesimal generator. Then for every $x \in D(A)$ the Cauchy problem*

$$\begin{cases} y'(t) = Ay(t) \\ y(0) = x \end{cases} \tag{1.4.2}$$

has a unique solution $y \in \mathcal{C}^1([0,\infty); X) \cap \mathcal{C}([0,\infty); D(A))$[1] *given by*

$$y(t) = S(t)x \qquad \forall t \geqslant 0.$$

Proof. The fact that $y(t) = S(t)x$ satisfies (1.4.2) is point (d) of Theorem 3. Let us show that this is the unique solution of the problem. Let $z \in \mathcal{C}^1([0,\infty); X) \cap \mathcal{C}([0,\infty); D(A))$ be a solution of (1.4.2), fix $t > 0$, and set

$$u(s) = S(t-s)z(s)\,, \quad \forall s \in [0,t].$$

Then

$$u'(s) = -AS(t-s)z(s) + S(t-s)Az(s) = 0\,, \quad \forall s \in [0,t].$$

Therefore, $z(t) = u(t) = u(0) = y(t)$. □

Exercise 4 Let $S(t)$ and $T(t)$ be $\mathcal{C}_0$-semigroups with generator $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$, respectively. Show that

$$A = B \quad \Longrightarrow \quad S(t) = T(t) \quad \forall t \geqslant 0.$$

Exercise 5 Find the infinitesimal generator of the $\mathcal{C}_0$-semigroup of left translations discussed in Exampe 3.

Example 4 (Transport equation in $C_b(\mathbb{R})$) Returning to the left-translation semigroup on $C_b(\mathbb{R})$ of Example 3, by Proposition 4 and Exercise 5 we conclude that for each $f \in C_b^1(\mathbb{R})$ the unique solution of the problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial u}{\partial x}(t,x) & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ u(0,x) = f(x) & x \in \mathbb{R} \end{cases}$$

is given by $u(t,x) = f(x+t)$.

Exercise 6 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$. Define

$$\begin{cases} D(A) = H^2(\Omega) \cap H_0^1(\Omega) \\ Au = \Delta u & \forall u \in D(A). \end{cases}$$

Prove that A is a closed operator.

[1] Here $D(A)$ is ragarded as a Banach space with the graph norm.

1.5 Spectral properties of closed operators

Let $A : D(A) \subset X \to X$ be a closed operator on a complex Banach space X.

Definition 6 *The* resolvent set *of A, $\rho(A)$, is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A : D(A) \to X$ is bijective. The set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called the* spectrum *of A. For any $\lambda \in \rho(A)$ the linear operator*

$$R(\lambda, A) := (\lambda I - A)^{-1} : X \to X$$

is called the resolvent of A at λ.

Example 5 On $X = \mathcal{C}([0,1])$ with the uniform norm consider the linear operator $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) = \mathcal{C}^1([0,1]) \\ Af = f', \quad \forall f \in D(A) \end{cases}$$

is closed (*Exercise*). Then $\sigma(A) = \mathbb{C}$ because for any $\lambda \in \mathbb{C}$ the function $f_\lambda(x) = e^{\lambda x}$ satisfies

$$\lambda f_\lambda(x) - f'_\lambda(x) = 0 \qquad \forall x \in [0,1].$$

On the other hand, for the closed operator A_0 defined by

$$\begin{cases} D(A_0) = \big\{ f \in \mathcal{C}^1([0,1]) \ : \ f(0) = 0 \big\} \\ A_0 f = f', \quad \forall f \in D(A_0), \end{cases}$$

we have that $\sigma(A_0) = \varnothing$. Indeed, for any $g \in X$ the problem

$$\begin{cases} \lambda f(x) - f'(x) = g(x) \quad x \in [0,1] \\ f(0) = 0 \end{cases}$$

admits the unique solution

$$f(x) = -\int_0^x e^{\lambda(x-s)} g(s)\, dx \quad (x \in [0,1])$$

which belongs to $D(A_0)$.

Proposition 5 *Any closed operator $A : D(A) \subset X \to X$ on a complex Banach space X has the following properties.*

(a) $R(\lambda, A) \in \mathcal{L}(X)$ for any $\lambda \in \rho(A)$.

(b) For any $\lambda \in \rho(A)$

$$AR(\lambda, A) = \lambda R(\lambda, A) - I. \tag{1.5.1}$$

(c) The resolvent identity *holds:*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \qquad \forall \lambda, \mu \in \rho(A). \quad (1.5.2)$$

(d) For any $\lambda, \mu \in \rho(A)$

$$R(\lambda, A)R(\mu, A) = R(\mu, A)R(\lambda, A). \quad (1.5.3)$$

Proof. Let $\lambda, \mu \in \rho(A)$.

(a) Since A is closed, so is $\lambda I - A$ and aslo $R(\lambda, A) = (\lambda I - A)^{-1}$. So, $R(\lambda, A) \in \mathcal{L}(X)$ by the closed graph theorem.

(b) This point follows from the definition of $R(\lambda, A)$.

(c) By (1.5.1) we have that

$$[\lambda R(\lambda, A) - AR(\lambda, A)]R(\mu, A) = R(\mu, A)$$

and

$$R(\lambda, A)[\mu R(\mu, A) - AR(\mu, A)] = R(\lambda, A).$$

Since $AR(\lambda, A) = R(\lambda, A)A$ on $D(A)$, (1.5.2) follows.

(d) Apply (1.5.2) to compute

$$\begin{aligned} R(\lambda, A) - R(\mu, A) &= (\mu - \lambda)R(\lambda, A)R(\mu, A) \\ R(\mu, A) - R(\lambda, A) &= (\lambda - \mu)R(\mu, A)R(\lambda, A). \end{aligned}$$

Adding the above identities side by side yields the conclusion.

The proof is complete. □

Theorem 4 *Let $\lambda_0 \in \rho(A)$. Then, for any $\lambda \in \mathbb{C}$ such that*

$$|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0, A)\|}, \quad (1.5.4)$$

the resolvent $R(\lambda, A)$ is given by the (Neumann) series

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}. \quad (1.5.5)$$

Consequently, the resolvent set $\rho(A)$ is open in $\mathbb{C}$, $\lambda \mapsto R(\lambda, A)$ is analytic on $\rho(A)$, and for any $\lambda \in \rho(A)$

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n\, n!\, R(\lambda, A)^{n+1} \qquad \forall n \in \mathbb{N}. \quad (1.5.6)$$

Proof. For all $\lambda \in \mathbb{C}$ and $\lambda_0 \in \rho(A)$ we have that

$$\lambda I - A = \lambda_0 I - A + (\lambda - \lambda_0)I = [I - (\lambda_0 - \lambda)R(\lambda_0, A)](\lambda_0 I - A).$$

This operator is bijective if and only if $[I - (\lambda_0 - \lambda)R(\lambda_0, A)]$ is invertible, which is the case if λ satisfies (1.5.4). Then

$$R(\lambda, A) = R(\lambda_0, A)[I - (\lambda_0 - \lambda)R(\lambda_0, A)]^{-1} = \sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}.$$

The analyticity of $R(\lambda, A)$ and (1.5.6) follows from (1.5.5). $\square$

1.6 Integral representation of $R(\lambda, A)$

Theorem 5 (Integral representation) *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X, $S(t)$, such that*

$$\|S(t)\| \leqslant M e^{\omega t} \qquad \forall t \geqslant 0 \tag{1.6.1}$$

for some constants $M \geqslant 1$ and $\omega \in \mathbb{R}$. Then $\rho(A)$ contains the half-plane

$$\Pi_\omega = \big\{\lambda \in \mathbb{C} \ : \ \Re\,\lambda > \omega\big\} \tag{1.6.2}$$

and

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t)x\,dt \qquad \forall x \in X\,,\ \forall \lambda \in \Pi_\omega. \tag{1.6.3}$$

Proof. We must prove that, given any $\lambda \in \Pi_\omega$ and $x \in X$, the equation

$$\lambda u - Au = x \tag{1.6.4}$$

has a unique solution given by (1.6.3).
<u>Existence</u>: observe that $u := \int_0^\infty e^{-\lambda t} S(t)x\,dt \in X$ because $\Re\,\lambda > \omega$. Moreover, for all $h > 0$,

$$\begin{aligned}
\frac{S(h)u - u}{h} &= \frac{1}{h}\Big\{\int_0^\infty e^{-\lambda t} S(t+h)x\,dt - \int_0^\infty e^{-\lambda t} S(t)x\,dt\Big\} \\
&= \frac{1}{h}\Big\{e^{\lambda h}\int_h^\infty e^{-\lambda t} S(t)x\,dt - \int_0^\infty e^{-\lambda t} S(t)x\,dt\Big\} \\
&= \frac{e^{\lambda h} - 1}{h}\,u - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} S(t)x\,dt.
\end{aligned}$$

So

$$\lim_{h \downarrow 0} \frac{S(h)u - u}{h} = \lambda u - x$$

which in turn yields that $u \in D(A)$ and (1.6.4) holds true.

Uniqueness: let $u \in D(A)$ be a solution of (1.6.4). Then

$$\int_0^\infty e^{-\lambda t} S(t)(\lambda u - Au)\, dt = \lambda \int_0^\infty e^{-\lambda t} S(t) u\, dt - \int_0^\infty e^{-\lambda t} \frac{d}{dt} S(t) u\, dt = u$$

which implies that u is given by (1.6.3). $\square$

Proposition 6 *Let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be closed linear operators in X and suppose $B \subset A$, that is,*

$$D(B) \subset D(A) \quad \textit{and} \quad Ax = Bx \quad \forall x \in D(B).$$

If $\rho(A) \cap \rho(B) \neq \varnothing$, then $A = B$.

Proof. It suffices to show that $D(A) \subset D(B)$. Let $x \in D(A)$, $\lambda \in \rho(A) \cap \rho(B)$, and set

$$y = \lambda x - Ax \quad \text{and} \quad z = R(\lambda, B) y.$$

Then $z \in D(B)$ and $\lambda z - Bz = \lambda x - Ax$. Since $B \subset A$, $\lambda z - Bz = \lambda z - Az$. Thus, $(\lambda I - A)(x - z) = 0$. So, $x = z \in D(B)$. $\square$

Example 6 (Right-translation semigroup on $\mathbb{R}_+$) On the real Banach space

$$X = \{ f \in \mathcal{C}_b(\mathbb{R}_+) \ : \ f(0) = 0 \}$$

with the uniform norm, consider the right-translation semigroup

$$\big(S(t) f\big)(x) = \begin{cases} f(x - t) & x > t \\ 0 & x \in [0, t] \end{cases} \qquad \forall x, t \geqslant 0.$$

It is easy to check that S is a $\mathcal{C}_0$-semigroup on X with $\|S(t)\| = 1$ for all $t \geqslant 0$. In order to characterize its infinitesimal generator A, let us consider the operator $B : D(B) \subset X \to X$ defined by

$$\begin{cases} D(B) = \big\{ f \in X \ : \ f' \in X \big\} \\ Bf = -f', \quad \forall f \in D(B). \end{cases}$$

We claim that:

(i) $B \subset A$

Proof. Let $f \in D(B)$. Then, for all $x, t \geqslant 0$ we have

$$\frac{\big(S(t) f\big)(x) - f(x)}{t} = \begin{cases} -\frac{f(x)}{t} = -f'(x_t), & 0 \leqslant x \leqslant t \\ \frac{f(x-t) - f(x)}{t} = -f'(x_t) & x \geqslant t \end{cases}$$

with $0 \leqslant x - x_t \leqslant t$. Therefore

$$\sup_{x \geqslant 0} \left| \frac{\big(S(t)f\big)(x) - f(x)}{t} + f'(x) \right| \leqslant \sup_{|x-y| \leqslant t} |f'(x) - f'(y)| \to 0 \quad \text{as} \quad t \downarrow 0$$

because f' is uniformly continuous. □

(*ii*) $1 \in \rho(B)$

Proof. For any $g \in X$ the unique solution f of the problem

$$\begin{cases} f \in D(B) \\ f(x) + f'(x) = g(x) \quad \forall x \geqslant 0 \end{cases}$$

is given by

$$f(x) = \int_0^x e^{s-x} g(s)\, ds \qquad (x \geqslant 0).$$ □

Since $1 \in \rho(A)$ by Proposition 5, Proposition 6 yields that $A = B$.

1.7 Asymptotic behaviour of $\mathcal{C}_0$-semigroups

Let $S(t)$ be a $\mathcal{C}_0$-semigroup of bounded linear operators on X.

Definition 7 *The number*

$$\omega_0(S) = \inf_{t>0} \frac{\log \|S(t)\|}{t} \tag{1.7.1}$$

is called the type *or* growth bound *of $S(t)$.*

Proposition 7 *The growth bound of S satisfies*

$$\omega_0(S) = \lim_{t\to\infty} \frac{\log \|S(t)\|}{t} < \infty. \tag{1.7.2}$$

Moreover, for any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$\|S(t)\| \leqslant M_\varepsilon e^{(\omega_0(S)+\varepsilon)t} \qquad \forall t \geqslant 0. \tag{1.7.3}$$

Proof. The fact that $\omega_0(S) < \infty$ is a direct consequence of (1.7.1). In order to prove (1.7.2) it suffices to show that

$$\limsup_{t\to\infty} \frac{\log \|S(t)\|}{t} \leqslant \omega_0(S). \tag{1.7.4}$$

For any $\varepsilon > 0$ let $t_\varepsilon > 0$ be such that

$$\frac{\log \|S(t_\varepsilon)\|}{t_\varepsilon} < \omega_0(S) + \varepsilon. \tag{1.7.5}$$

Let us write any $t \geqslant t_\varepsilon$ as $t = nt_\varepsilon + \delta$ with $n0n(\varepsilon) \in \mathbb{N}$ and $\delta = \delta(\varepsilon) \in [0, t_\varepsilon[$. Then, by (1.2.2) and (1.7.5),

$$\|S(t)\| \leqslant \|S(\delta)\| \, \|S(t_\varepsilon)\|^n \leqslant Me^{\omega\delta} \, e^{nt_\varepsilon(\omega_0(S)+\varepsilon)} = Me^{(\omega-\omega_0(S)-\varepsilon)\delta} e^{(\omega_0(S)+\varepsilon)t}$$

which proves (1.7.3) with $M_\varepsilon = Me^{(\omega-\omega_0(S)-\varepsilon)\delta}$. Moreover, taking the logarithm of both sides of the above inequality we get

$$\frac{\log \|S(t)\|}{t} \leqslant \omega_0(S) + \varepsilon + \frac{\log M + (\omega - \omega_0(S) - \varepsilon)\delta}{t}$$

and (1.7.4) follows as $t \to \infty$. □

Definition 8 *For any operator $A : D(A) \subset X \to X$ we define the* spectral bound *of A as*

$$s(A) = \sup\{\, \Re\lambda \;:\; \lambda \in \sigma(A)\,\}.$$

Corollary 2 *Let $S(t)$ be a $\mathcal{C}_0$-semigroup on X with infinitesimal generator A. Then*

$$-\infty \leqslant s(A) \leqslant \omega_0(S) < +\infty.$$

Proof. By combining Theorem 5 and (1.7.3) we conclude that

$$\Pi_{\omega_0(S)+\varepsilon} \subset \rho(A) \qquad \forall \varepsilon > 0.$$

Therefore, $s(A) \leqslant \omega_0(S) + \varepsilon$ for all $\varepsilon > 0$. The conclusion follows. □

Example 7 For fixed $T > 0$ and $p \geqslant 1$ let $X = L^p(0,T)$ and

$$\big(S(t)f\big)(x) = \begin{cases} f(x-t) & x \in [t,T] \\ 0 & x \in [0,t) \end{cases} \qquad \forall x \in [0,T]\,, \, \forall t \geqslant 0.$$

Then S is a $\mathcal{C}_0$-semigroup of bounded linear operators on X which satisfies $\|S(t)\| \leqslant 1$ for all $t \geqslant 0$. Moreover, observe that S is *nilpotent*, that is, we have $S(t) \equiv 0$, $\forall t \geqslant T$. Deduce that $\omega_0(S) = -\infty$. So, the spectral bound of the infinitesimal generator of $S(t)$ also equals $-\infty$.

Example 8 ($-\infty < s(A) = \omega_0(S)$) In the Banach space

$$X = \mathcal{C}_b(\mathbb{R}_+; \mathbb{C}),$$

with the uniform norm, the left-translation semigroup

$$\big(S(t)f\big)(x) = f(x+t) \qquad \forall x, t \geqslant 0$$

is a $\mathcal{C}_0$-semigroup of contractions on X which satisfies $\|S(t)\| = 1$ (*Exercise*). Therefore

$$\omega_0(S) = 0.$$

The infinitesimal generator of $S(t)$ is given by

$$\begin{cases} D(A) = \mathcal{C}_b^1(\mathbb{R}_+;\mathbb{C}) \\ Af = f' & \forall f \in D(A). \end{cases}$$

By Theorem 5 we have that

$$\rho(A) \supset \{\lambda \in \mathbb{C} \ : \ \Re\lambda > 0\}.$$

We claim that

$$\sigma(A) \supset \{\lambda \in \mathbb{C} \ : \ \Re\lambda \leqslant 0\}.$$

Indeed, for any $\lambda \in \mathbb{C}$ the function $f_\lambda(x) := e^{\lambda x}$ satisfies $\lambda f - f' = 0$. Moreover, $f_\lambda \in D(A)$ for $\Re\lambda \leqslant 0$. Therefore

$$s(A) = 0.$$

Example 9 **($s(A) < \omega_0(S)$)** Let us denote by $\mathcal{C}_0(\mathbb{R}_+;\mathbb{C})$ the Banach space of all continuous functions $f : \mathbb{R}_+ \to \mathbb{C}$ such that

$$\lim_{x\to\infty} f(x) = 0$$

with the uniform norm. We define X to be the Banach space (*Exercise*) of all functions $f \in \mathcal{C}_0(\mathbb{R}_+;\mathbb{C})$ such that

$$\|f\| := \sup_{x\in\mathbb{R}_+} |f(x)| + \int_0^\infty |f(x)|e^x dx < \infty.$$

Once again, the left-translation semigroup

$$\big(S(t)f\big)(x) = f(x+t) \qquad \forall x, t \geqslant 0$$

is a $\mathcal{C}_0$-semigroup of contractions on X. Indeed, for all $t \geqslant 0$

$$\begin{aligned} \|S(t)f\| &= \sup_{x\in\mathbb{R}_+} |f(x+t)| + \int_0^\infty |f(x+t)|e^x dx \\ &\leqslant \sup_{x\in\mathbb{R}_+} |f(x)| + e^{-t}\int_0^\infty |f(x)|e^x dx. \end{aligned}$$

Moreover, $\|S(t)\| = 1$ (*Exercise*). Therefore

$$\omega_0(S) = 0.$$

The infinitesimal generator of $S(t)$ is given by

$$\begin{cases} D(A) = \{f \in X \ : \ f' \in X\} \\ Af = f' & \forall f \in D(A). \end{cases}$$

For any $\lambda \in \mathbb{C}$ the function $f_\lambda(x) := e^{\lambda x}$ satisfies $\lambda f - f' = 0$ and $f_\lambda \in D(A)$ for $\Re\lambda < -1$. So,

$$s(A) \geqslant -1. \tag{1.7.6}$$

We claim that

$$\rho(A) \supset \big\{\lambda \in \mathbb{C} \ : \ \Re\lambda > -1\big\}. \tag{1.7.7}$$

Indeed, a direct calculation shows that, for any $g \in X$, the function

$$f(x) = \int_0^\infty e^{-\lambda t}\big(S(t)g\big)(x)dt = \int_0^\infty e^{-\lambda t} g(x+t)dt \qquad (x \geqslant 0)$$

satisfies $\lambda f - f' = g$. Consequently, if we show that $f \in X$, then $f \in D(A)$ follows and so $\lambda \in \rho(A)$. In order to check $f \in X$, observe that, for all $x \geqslant 0$,

$$\begin{aligned} |f(x)| &\leqslant \int_0^\infty \big|e^{-\lambda t} g(x+t)\big|dt \\ &= \int_0^\infty e^{-t\Re\lambda}\big|g(x+t)\big|e^{x+t}e^{-x-t}dt \\ &= e^{-x}\int_0^\infty e^{-t(1+\Re\lambda)}e^{x+t}\big|g(x+t)\big|dt \\ &\leqslant e^{-x}\int_x^\infty e^s\big|g(s)\big|ds \end{aligned} \tag{1.7.8}$$

which insures that $f \in \mathcal{C}_0(\mathbb{R}_+;\mathbb{C})$. Furthermore, by (1.7.8) we compute

$$\begin{aligned} \int_0^\infty |f(x)|e^x dx &\leqslant \int_0^\infty dx \int_0^\infty e^{-t(1+\Re\lambda)}e^{x+t}\big|g(x+t)\big|dt \\ &= \int_0^\infty e^{-t(1+\Re\lambda)}dt \int_0^\infty e^{x+t}\big|g(x+t)\big|dx \\ &\leqslant \int_0^\infty e^{-t(1+\Re\lambda)}dt \int_0^\infty e^{\tau}\big|g(\tau)\big|d\tau < \infty. \end{aligned}$$

From (1.7.6) and (1.7.7) it follows that $s(A) = -1 < 0 = \omega_0(S)$.

Exercise 7 Let $S(t)$ be a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Prove that $\omega_0(S) < 0$ if and only if

$$\lim_{t\to+\infty} \|S(t)\| = 0. \tag{1.7.9}$$

Solution. One only needs to show that (1.7.9) implies that $\omega_0(S) < 0$. Let $t_0 > 0$ be such that $\|S(t_0)\| < 1/e$. For any $t > 0$ let $n \in \mathbb{N}$ be the unique integer such that

$$nt_0 \leqslant t < \big(n+1\big)t_0. \tag{1.7.10}$$

Then

$$\|S(t)\| = \big\|S\big(nt_0\big)S\big(t - nt_0\big)\big\| \leqslant \frac{Me^{\omega(t-nt_0)}}{e^n} \leqslant \frac{Me^{\omega t_0}}{e^n}\,.$$

Therefore, on account of (1.7.9), we conclude that

$$\begin{aligned}\frac{\log\|S(t)\|}{t} &\leqslant \frac{\log\left(Me^{\omega t_0}\right)}{t} - \frac{n}{t}\\ &\leqslant \frac{\log\left(Me^{\omega t_0}\right)}{t} - \Big(\frac{1}{t_0} - \frac{1}{t}\Big) \qquad \forall t > 0.\end{aligned}$$

Taking the limit as $t \to +\infty$ we conclude that $\omega_0(S) < 0$. □

1.8 The Hille-Yosida generation theorem

Theorem 6 *Let $M \geqslant 1$ and $\omega \in \mathbb{R}$. For a linear operator $A : D(A) \subset X \to X$ the following properties are equivalent:*

(a) A is closed, $D(A)$ is dense in X, and

$$\rho(A) \supseteq \Pi_\omega = \left\{\lambda \in \mathbb{C} \ : \ \Re\lambda > \omega\right\} \tag{1.8.1}$$

$$\|R(\lambda, A)^k\| \leqslant \frac{M}{(\Re\lambda - \omega)^k} \quad \forall k \geqslant 1, \forall\lambda \in \Pi_\omega \tag{1.8.2}$$

(b) A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup, $S(t)$, such that

$$\|S(t)\| \leqslant Me^{\omega t} \qquad \forall t \geqslant 0. \tag{1.8.3}$$

$\boxed{\textit{Proof of } (b) \Rightarrow (a)}$ The fact that A is closed, $D(A)$ is dense in X, and (1.8.1) holds true has already been proved, see Theorem 3-(c), Proposition 3, and Theorem 5. In order to prove (1.8.2) observe that, by using (1.6.3) to compute the k-th derivative of the resolvent of A, we obtain

$$\frac{d^k}{d\lambda^k} R(\lambda, A)x = (-1)^k \int_0^\infty t^k e^{-\lambda t} S(t)x\, dt \qquad \forall x \in X\,,\ \forall\lambda \in \Pi_\omega.$$

Therefore,

$$\Big\|\frac{d^k}{d\lambda^k} R(\lambda, A)\Big\| \leqslant M \int_0^\infty t^k e^{-(\Re\lambda - \omega)t}\, dt = \frac{M\, k!}{(\Re\lambda - \omega)^{k+1}}$$

where the integral is easily computed by induction. The conclusion follows recalling (1.5.6). □

Lemma 1 *Let $A : D(A) \subset X \to X$ be as in (a) of Theorem 6. Then:*

(i) For all $x \in X$

$$\lim_{n\to\infty} nR(n, A)x = x. \tag{1.8.4}$$

(ii) *The* Yosida Approximation A_n *of* A*, defined as*

$$A_n = nAR(n, A) \qquad (n \geqslant 1) \tag{1.8.5}$$

is a sequence of bounded operator on X *which satisfies*

$$A_n A_m = A_m A_n \qquad \forall n, m \geqslant 1 \tag{1.8.6}$$

and

$$\lim_{n\to\infty} A_n x = Ax \qquad \forall x \in D(A). \tag{1.8.7}$$

(iii) *For all* $m, n > 2\omega$, $x \in D(A)$, $t \geqslant 0$ *we have that*

$$\begin{aligned} \|e^{tA_n}\| &\leqslant Me^{\frac{n\omega t}{n-\omega}} \leqslant Me^{2\omega t} && (1.8.8)\\ |e^{tA_n}x - e^{tA_m}x| &\leqslant M^2 te^{2\omega t}|A_n x - A_m x|. && (1.8.9)\end{aligned}$$

Consequently, for all $x \in D(A)$ *the sequence* $u_n(t) := e^{tA_n}x$ *is Cauchy in* $\mathcal{C}([0, T]; X)$ *for any* $T > 0$*.*

Proof of (*i*): owing to (1.5.1), for any $x \in D(A)$ we have that

$$|nR(n, A)x - x| = |AR(n, A)x| = |R(n, A)Ax| \leqslant \frac{M|Ax|}{n-\omega} \overset{(n\to\infty)}{\longrightarrow} 0,$$

where we have used (1.8.2) with $k = 1$. Moreover, again by (1.8.2) ,

$$\|nR(n, A)\| \leqslant \frac{Mn}{n-\omega} \leqslant 2M \quad \forall n > 2\omega.$$

The last two inequalities yield the conclusion because $D(A)$ is dense in X. Indeed, let $x \in X$ and fix any $\varepsilon > 0$. Let $x_\varepsilon \in D(A)$ be such that $|x_\varepsilon - x| < \varepsilon$. Then

$$\begin{aligned} |nR(n, A)x - x| &\leqslant |nR(n, A)(x - x_\varepsilon)| + |nR(n, A)x_\varepsilon - x_\varepsilon| + |x_\varepsilon - x| \\ &< (2M + 1)\varepsilon + \frac{M|Ax_\varepsilon|}{n-\omega} \overset{(n\to\infty)}{\longrightarrow} (2M + 1)\varepsilon.\end{aligned}$$

Since ε is arbitrary, (1.8.4) follows.
Proof of (*ii*): observe that $A_n \in \mathcal{L}(X)$ because

$$A_n = n^2 R(n, A) - nI \qquad \forall n \geqslant 1. \tag{1.8.10}$$

Moreover, in view of (1.5.3) we have that

$$\begin{aligned} A_n A_m &= [n^2 R(n, A) - nI]\,[m^2 R(m, A) - mI] \\ &= [m^2 R(m, A) - mI]\,[n^2 R(n, A) - nI] = A_m A_n.\end{aligned}$$

Finally, owing to (1.8.4), for all $x \in D(A)$ we have that

$$A_n x = nAR(n,A)x = nR(n,A)Ax \overset{(n\to\infty)}{\longrightarrow} Ax.$$

Proof of (*iii*): recalling (1.8.10) we have that

$$e^{tA_n} = e^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k} t^k R(n,A)^k}{k!}, \quad \forall t \geqslant 0.$$

Therefore, in view of (1.8.2),

$$\|e^{tA_n}\| \leqslant M e^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k} t^k}{k!(n-\omega)^k} = M e^{\frac{n\omega t}{n-\omega}} \leqslant M e^{2\omega t}$$

for all $t \geqslant 0$ and $n > 2\omega$. This proves (1.8.8).

Next, observe that for any $x \in D(A)$ we have that

$$\begin{cases} (u_n - u_m)'(t) = A_n(u_n - u_m)(t) + (A_n - A_m)u_m(t) & \forall t \geqslant 0 \\ (u_n - u_m)(0) = 0. \end{cases}$$

where we have set $u_n(t) = e^{tA_n}x$. Therefore, for all $t \geqslant 0$ we have that

$$\begin{aligned} e^{tA_n}x - e^{tA_m}x &= \int_0^t e^{(t-s)A_n}(A_n - A_m)e^{sA_m}x\,ds \\ &= \int_0^t e^{(t-s)A_n}e^{sA_m}(A_n - A_m)x\,ds \end{aligned} \qquad (1.8.11)$$

because A_n and $e^{sA_m}x$ commute in view of (1.8.6). Thus, by combining (1.8.11) and (1.8.8) we obtain

$$\begin{aligned} |e^{tA_n}x - e^{tA_m}x| &\leqslant M^2 \int_0^t e^{2\omega(t-s)} e^{2\omega s} |A_n x - A_m x|, ds \\ &\leqslant M^2\, t\, e^{2\omega t}\, |A_n x - A_m x|. \end{aligned}$$

In view of (1.8.7), the last inequality shows that $e^{tA_n}x$ is a Cauchy sequence in $\mathcal{C}([0,T];X)$ for any $T > 0$, thus completing the proof. □

Exercise 8 Use a density argument to prove that $e^{tA_n}x$ is a Cauchy sequence on all compact subsets of $\mathbb{R}_+$ for all $x \in X$.

Proof of (a) $\Rightarrow$ (b) On account of Lemma 1 and Exercise 8, we have that $e^{tA_n}x$ is a Cauchy sequence on all compact subsets of $\mathbb{R}_+$ for all $x \in X$. Consequently, the limit (uniform on all $[0,T] \subset \mathbb{R}_+$)

$$S(t)x = \lim_{n\to\infty} e^{tA_n}x, \quad \forall x \in X, \qquad (1.8.12)$$

defines a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Moreover, passing to the limit as $n \to \infty$ in (1.8.8), we conclude that $\|S(t)\| \leqslant Me^{\omega t}$, $\forall t \geqslant 0$.

Let us identify the infinitesimal generator of $S(t)$. By (1.8.8), for $x \in D(A)$ we have that

$$\begin{aligned}\Big|\frac{d}{dt}\, e^{tA_n}x - S(t)Ax\Big| &\leqslant |e^{tA_n}A_n x - e^{tA_n}Ax| + |e^{tA_n}Ax - S(t)Ax| \\ &\leqslant Me^{2\omega t}|A_n x - Ax| + |e^{tA_n}Ax - S(t)Ax| \stackrel{(n\to\infty)}{\longrightarrow} 0\end{aligned}$$

uniformly on all compact subsets of $\mathbb{R}_+$ by (1.8.12). Therefore, for all $T > 0$ and $x \in D(A)$ we have that

$$\begin{cases} e^{tA_n}x \stackrel{(n\to\infty)}{\longrightarrow} S(t)x \\ \frac{d}{dt}\, e^{tA_n}x \stackrel{(n\to\infty)}{\longrightarrow} S(t)Ax \end{cases} \qquad \text{uniformly on } [0,T].$$

This implies that

$$S'(t)x = S(t)Ax, \quad \forall x \in D(A)\,,\ \forall t \geqslant 0. \tag{1.8.13}$$

Now, let $B : D(B) \subset X \to X$ be the infinitesimal generator of $S(t)$. Then $A \subset B$ in view of (1.8.13). Moreover, $\Pi_\omega \subset \rho(A)$ by assumption (a) and $\Pi_\omega \subset \rho(B)$ by Proposition 5. So, on account of Proposition 6, $A = B$. $\square$

Remark 3 The above proof shows that condition (a) in Theorem 6 can be relaxed as follows:

(a') *A is closed, $D(A)$ is dense in X, and*

$$\rho(A) \supseteq]\omega, \infty[\tag{1.8.14}$$

$$\|R(n,A)^k\| \leqslant \frac{M}{(n-\omega)^k} \quad \forall k \geqslant 1, \forall n > \omega. \tag{1.8.15}$$

Remark 4 When $M = 1$, the countably many bounds in condition (a) follow from (1.8.2) for $k = 1$, that is,

$$\|R(\lambda, A)\| \leqslant \frac{1}{\Re\lambda - \omega} \qquad \forall k \geqslant 1\,,\ \forall \lambda \in \Pi_\omega.$$

Example 10 (Second order parabolic equations in $L^2(\Omega)$) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$. Define

$$\begin{cases} D(A) = H^2(\Omega) \cap H^1_0(\Omega) \\ Au = \sum_{i,j=1}^n D_j(a_{ij}D_j)u + \sum_{i=1}^n b_i D_i u + cu \quad \forall u \in D(A). \end{cases}$$

where

(H1) $a_{ij} \in \mathcal{C}^1(\overline{\Omega})$ satisfies $a_{ij} = a_{ji}$ for all $i, j = 1, \dots, n$ and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_i \geqslant \theta|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n,\ x \in \Omega$$

(H2) $b_i \in L^\infty(\Omega)$ for all $i = 1, \dots, n$ and $c \in L^\infty(\Omega)$.

In order to apply the Hille-Yosida theorem to show that A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup $S(t)$ on $L^2(\Omega)$, one can check that the following assumptions are satisfied.

1. $D(A)$ is dense in $L^2(\Omega)$.

   [This is a known property of Sobolev spaces.]

2. A is a closed operator.

   Proof. Let $u_k \in D(A)$ be such that

   $$u_k \overset{k\to\infty}{\longrightarrow} u \quad \text{and} \quad Au_k \overset{k\to\infty}{\longrightarrow} f.$$

   Then, for all $h, k \geqslant 1$ we have that $v_{hk} := u_h - u_k$ satisfies

   $$\begin{cases} \sum_{i,j=1}^{n} D_j(a_{ij}D_j)v_{hk} + \sum_{i=1}^{n} b_i D_i v_{hk} + c v_{hk} =: f_{hk} & \text{in } \Omega \\ v_{hk} = 0 & \text{on } \partial\Omega. \end{cases}$$

   So, elliptic regularity insures that

   $$\|v_{hk}\|_{2,\Omega} \leqslant C\big(\|f_{hk}\|_{0,\Omega} + \|v_{hk}\|_{0,\Omega}\big)$$

   for some constant $C > 0$. The above inequality implies that $\{u_k\}$ is a Cauchy sequence in $D(A)$ and this yields $f = Au$. $\square$

3. $\exists \omega \in \mathbb{R}$ such that $\rho(A) \supset]\omega, \infty[$.

   [This follows from elliptic theory.]

4. $\|R(\lambda, A)\| \leqslant \frac{1}{\lambda - \omega}$ for all $k \geqslant 1$ and $\lambda > \omega$.

   [This follows from elliptic theory.]

Then, for any $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, the function $u(t,x) = \big(S(t)u_0\big)(x)$ is the unique solution of the initial-boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} D_j(a_{ij}D_j)u + \sum_{i=1}^{n} b_i D_i u + cu & \text{in }]0,\infty[\times\Omega \\ u = 0 & \text{on }]0,\infty[\times\partial\Omega \\ u(0,x) = u_0(x) & x \in \Omega. \end{cases}$$

in the class

$$C^1\big([0,\infty); L^2(\Omega)\big) \cap \mathcal{C}\big([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)\big).$$

Exercise 9 Let $S(t)$ be the $\mathcal{C}_0$-semigroup on $L^2(\Omega)$ associated with the initial-boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in }]0,\infty[\times\Omega \\ u = 0 & \text{on }]0,\infty[\times\partial\Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases} \tag{1.8.16}$$

Show that $\omega_0(S) < 0$.

Solution. We know from Example 10 that the infinitesimal generator of $S(t)$ is the operator A defined by

$$\begin{cases} D(A) = H^2(\Omega) \cap H_0^1(\Omega) \\ Au = \Delta u & \forall u \in D(A). \end{cases}$$

For $u_0 \in D(A)$, let $u(t,x) = \big(S(t)u_0\big)(x)$. Then u satisfies (1.8.16). So

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega |u(t,x)|^2 dx\Big) = -\frac{1}{2}\int_\Omega |Du(t,x)|^2 dx \qquad \forall t > 0.$$

Moreover, by Poincaré's inequality we have that

$$\int_\Omega |u(t,x)|^2 dx \leqslant c(\Omega)\int_\Omega |Du(t,x)|^2 dx.$$

Therefore,

$$\frac{d}{dt}\,|u(t)|^2 \leqslant -\frac{2}{c(\Omega)}\,|u(t)|^2$$

which ensures, by Gronwall's lemma, that

$$|u(t)| \leqslant e^{-t/c(\Omega)}|u_0| \qquad \forall t > 0.$$

By a density argument, one concludes that the above inequality holds true for any $x \in L^2(\Omega)$, so that $\omega_o(S) \leqslant -1/c(\Omega)$. □

1.9 Additional exercises for Chapter 1

Exercise 10 Let X be a Banach space and let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Prove that, for every $n \geqslant 1$,

$$D(A^n) := \{x \in D(A^{n-1}) \ : \ Ax \in D(A)\}$$

is dense in X.

Solution. For $n = 1$ the conclusion follows from Theorem 3. Let the conclusion be true for some $n \geqslant 1$ and fix any $y \in X$. Then, for any $\varepsilon > 0$ there exists $x_\varepsilon \in D(A^n)$ such that $|x_\varepsilon - y| < \varepsilon$. Moreover,

$$A^n\Big(\frac{1}{t}\int_0^t S(s)x_\varepsilon\, ds\Big) = \frac{1}{t}\int_0^t S(s)A^n x_\varepsilon\, ds$$

Since

$$\frac{1}{t}\int_0^t S(s)x_\varepsilon\, ds \in D(A) \qquad \forall t > 0$$

we conclude that

$$\frac{1}{t}\int_0^t S(s)x_\varepsilon\, ds \in D(A^{n+1}) \qquad \forall t > 0.$$

Moreover, there exists $t_\varepsilon > 0$ such that

$$\Big|\frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} S(s)x_\varepsilon\, ds - y\Big| \leqslant \Big|\frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} S(s)x_\varepsilon\, ds - x_\varepsilon\Big| + |x_\varepsilon - y| < 2\varepsilon. \qquad \square$$

Exercise 11 Given a uniformly bounded $\mathcal{C}_0$-semigroup, $\|S(t)\| \leqslant M$, define

$$|x|_S = \sup_{t\geqslant 0} |S(t)x|\,, \quad \forall x \in X. \tag{1.9.1}$$

Show that:

1. $|\cdot|_S$ is a norm on X,
2. $|x| \leqslant |x|_S \leqslant M|x|$ for all $x \in X$, and
3. S is a contraction semigroup with respect to $|\cdot|_S$.

Exercise 12 Let S be $\mathcal{C}_0$-semigroup of bounded linear operators on X and let $K \subset X$ be compact. Prove that for every $t_0 \geqslant 0$

$$\lim_{t\to t_0} \sup_{x\in K} \big|S(t)x - S(t_0)x\big| = 0\,. \tag{1.9.2}$$

Solution. We may assume $S \in \mathcal{G}(M,0)$ for some $M >$) without loss of generality. Let $t_0 > 0$ and fix any $\varepsilon > 0$. Since K is totally bounded, there exist $x_1, \dots, x_{N_\varepsilon} \in X$ such that

$$K \subset \bigcup_{n=1}^{N_\varepsilon} B\Big(x_n, \frac{\varepsilon}{M}\Big).$$

Moreover, there exists $\tau > 0$ such that

$$|t - t_0| < \tau \quad \Longrightarrow \quad \big|S(t)x_n - S(t_0)x_n\big| < \varepsilon \qquad \forall n = 1, \dots, N_\varepsilon.$$

Thus, for all $|t - t_0| < \tau$ we have that, if $x \in K$ is such that $x \in B\big(x_n, \frac{\varepsilon}{M}\big)$, then

$$\begin{aligned} &\big|S(t)x - S(t_0)x\big| \\ &\quad \leqslant \ \big|S(t)x - S(t)x_n\big| + \big|S(t)x_n - S(t_0)x_n\big| + \big|S(t_0)x_n - S(t_0)x\big| \\ &\quad \leqslant \ 2M|x - x_n| + \varepsilon < 3\varepsilon. \end{aligned}$$

So, the limit of $\big|S(t)x - S(t_0)x\big|$ as $t \to t_0$ is uniform on K. □

Exercise 13 Prove that if $A : D(A) \subset X \to X$ is a closed operator and $B \in \mathcal{L}(X)$, then $A + B : D(A) \subset X \to X$ is also closed.

Exercise 14 Let $A : D(A) \subset X \to X$ be a closed operator satisfying (1.8.2) but suppose $D(A)$ fails to be dense in X. In the Banach space $Y := \overline{D(A)}$, define the operator B, called the *part of A in Y*, by

$$\begin{cases} D(B) = \big\{x \in D(A) \ : \ Ax \in Y\big\} \\ Bx = Ax \qquad \forall x \in D(B). \end{cases}$$

Prove that B is the infinitesimal generator of a $\mathcal{C}_0$-semigroup on Y.

Solution. $R(\lambda, A)(Y) \subset D(B)$ for all $\lambda \in \mathbb{C}$ such that $\Re\lambda > \omega$. Indeed, owing to (1.5.1) for all $x \in D(A)$ we have that

$$\lim_{n\to\infty} nR(n,A)x = \lim_{n\to\infty} \big\{R(n,A)Ax + x\big\} = x. \tag{1.9.3}$$

Since $\|nR(n,A)\|$ is bounded, (1.9.3) holds true for all $x \in Y$. Hence, $D(B)$ is dense in Y. Consequently, B satisfies in Y all the assumptions of Theorem 6. □

Exercise 15 For any fixed $p \geqslant 1$, let $X = L^p(\mathbb{R})$ and define, $\forall f \in X$,

$$\big(S(t)f\big)(x) = f(x+t) \quad \forall x \in \mathbb{R}\,, \forall t \geqslant 0. \tag{1.9.4}$$

Prove that S is $\mathcal{C}_0$-semigroup which fails to be uniformly continuous.

(Observe that (1.9.4) makes sense for $t < 0$ as well. On the other hand, if one takes $X = L^p(\mathbb{R}_+)$, then (1.9.4) makes sense only for $t \geqslant 0$.)

Solution. Suppose S is uniformly continuous and let $\tau > 0$ be such that $\|S(t) - I\| < 1/2$ for all $t \in [0, \tau]$. Then by taking $f_n(x) = n^{1/p}\chi_{[0,1/n]}(x)$ for $p < \infty$ and $n > 1/\tau$ we have that $|f_n| = 1$ and

$$|S(\tau)f_n - f_n| = \Big(\int_{\mathbb{R}} n|\chi_{[0,1/n]}(x+\tau) - \chi_{[0,1/n]}(x)|^p dx\Big)^{\frac{1}{p}} = 2^{1/p}.$$

Exercise 16 Denoting by $|f|_p$ the norm of f in $L^p(\mathbb{R})$ and by $W^{1,p}(\mathbb{R})$ the Banach space of all locally absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$|f|_{1,p} := |f|_p + |f'|_p < \infty, \tag{1.9.5}$$

show that the infinitesimal generator of the left-translation semigroup $S(t)$ on $L^p(\mathbb{R})$ defined in (1.9.4) is given by

$$\begin{cases} D(A) = W^{1,p}(\mathbb{R}) \\ Af(x) = f'(x) \quad (x \in \mathbb{R} \text{ a.e.}) \quad \forall f \in D(A). \end{cases} \tag{1.9.6}$$

Exercise 17 Let $p \geqslant 2$. On $X = L^p(0, \pi)$ consider the operator defined by

$$\begin{cases} D(A) = W^{2,p}(0,\pi) \cap W_0^{1,p}(0,\pi) \\ Af(x) = f''(x) \qquad\qquad x \in (0,\pi) \text{ a.e.} \end{cases} \tag{1.9.7}$$

where

$$W_0^{1,p}(0,\pi) = \big\{f \in W^{1,p}(0,\pi) \ : \ f(0) = 0 = f(\pi)\big\}.$$

Since $\mathcal{C}_c^\infty(0,\pi) \subset D(A)$, we have that $D(A)$ is dense in X. Show that A is closed and satisfies condition (a') of Remark 3 with $M = 1$ and $\omega = 0$. Theorem 6 will imply that A generates a $\mathcal{C}_0$-semigroup of contractions on X.

Solution. Step 1: $\sigma(A) = \{-n^2 \ : \ n \in \mathbb{N}\}$.

Fix any $g \in X$. We will show that, for all $\lambda \neq n^2 (n \geqslant 1)$, the Sturm-Liouville system

$$\begin{cases} \lambda f(x) - f''(x) = g(x), \quad 0 < x < \pi \\ f(0) = 0 = f(\pi) \end{cases} \tag{1.9.8}$$

admits a unique solution $f \in D(A)$. Denoting by

$$g(x) = \sum_{n=1}^{\infty} g_n \sin(nx) \qquad (x \in [0,\pi])$$

the Fourier series of g, we seek a candidate solution f of the form

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(nx) \qquad (x \in [0,\pi]).$$

In order to satisfy (1.9.8) one must have

$$(\lambda+n^2)f_n = g_n \qquad \forall n \geqslant 1.$$

So, for any $\lambda \neq -n^2$, (1.9.8) has a unique solution given by

$$f(x) = \sum_{n=1}^{\infty} \frac{g_n}{\lambda+n^2}\sin(nx) \qquad (x\in[0,\pi]).$$

From the above representation it follows that $f \in H^2(0,\pi)\cap H_0^1(0,\pi)$. In fact, returning to the equation in (1.9.8) one concludes that $f \in D(A)$.

Step 2: resolvent estimate.
By multiplying both members of the equation in (1.9.8) by $|f|^{p-2}f$ and integrating over $(0,\pi)$ one obtains, for all $\lambda > 0$,

$$\lambda\int_0^\pi |f(x)|^p dx + (p-1)\int_0^p |f(x)|^{p-2}|f'(x)|^2 dx = \int_0^\pi g(x)|f(x)|^{p-2}f(x)\,dx$$

which yields

$$|f|_p \leqslant \frac{1}{\lambda}\,|g|_p \qquad \forall\lambda>0.$$

Step 3: conclusion.
By Proposition 4 we conclude that for each $f\in W^{2,p}(0,\pi)\cap W_0^{1,p}(0,\pi)$ the unique solution of

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) & (t,x)\in\mathbb{R}_+\times(0,\pi)\\ u(t,0)=0=u(t,\pi) & t\geqslant 0\\ u(0,x)=f(x) & x\in(0,\pi)\end{cases}$$

is given by $u(t,x) = (S(t)f)(x)$. □

Exercise 18 Let $S(t)$ be the $\mathcal{C}_0$-semigroup generated by operator A in (1.9.7). Prove that, for any $f\in L^p(0,\pi)$,

$$(S(t)f)(x) = \int_0^\pi K(t,x,y)f(y)\,dy\,, \quad \forall t\geqslant 0,\ x\in(0,\pi)\text{ a.e.}$$

where

$$K(t,x,y) = \frac{2}{\pi}\sum_{k=1}^{\infty} e^{-k^2t}\sin(kx)\sin(ky).$$

Exercise 19 Let $f\in W^{2,p}(\mathbb{R})$ with $p\geqslant 2$. Solve the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) & (t,x)\in\mathbb{R}_+\times\mathbb{R}\\ u(0,x)=f(x) & x\in\mathbb{R}.\end{cases}$$

Solution. The operator defined by

$$\begin{cases} D(A) = W^{2,p}(\mathbb{R}) \\ Af(x) = f''(x) \qquad x \in \mathbb{R} \text{ a.e.} \end{cases}$$

is densely defined and closed. Let us begin by studying the problem

$$\begin{cases} f \in D(A) \\ \lambda f - f'' = g \in X \end{cases} \tag{1.9.9}$$

in the special case $p = 2$. Taking the Fourier transform of both members of the above equation we find

$$(\lambda + \xi^2)\widehat{f}(\xi) = \widehat{g}(\xi) \quad \forall \xi \in \mathbb{R}.$$

So, for any $\lambda > 0$ we have that the solution to problem (1.9.9) is given by

$$f(x) = (g * \phi_\lambda)(x) \quad \text{with} \quad \phi_\lambda(x) = \frac{e^{-\sqrt{\lambda}\,|x|}}{2\sqrt{\lambda}}\,,$$

that is,

$$f(x) = \frac{1}{2\sqrt{\lambda}}\Big\{\int_{-\infty}^{x} g(y)e^{-\sqrt{\lambda}\,(x-y)}dy + \int_{x}^{\infty} g(y)e^{-\sqrt{\lambda}\,(y-x)}dy\Big\}.$$

Moreover, the above representation formula holds true for any $p \geqslant 2$. We have thus proved that $(0,\infty) \subset \rho(A)$. Finally, by multiplying both members of the equation in (1.9.8) by $|f|^{p-2}f$ and integrating over $\mathbb{R}$ we obtain as in Exercise 17

$$\lambda\int_{-\infty}^{\infty}|f|^p dx + (p-1)\int_{-\infty}^{\infty}|f|^{p-2}|f'|^2 dx = \int_{-\infty}^{\infty} g|f|^{p-2}f\,dx$$

which yields

$$|f|_p \leqslant \frac{1}{\lambda}\,|g|_p.$$

Therefore, A satisfies condition (a') of Remark 3 and generates a $\mathcal{C}_0$-semigroup of bounded linear operators on X which gives the solution of our problem. □

Exercise 20 On $X = \{f \in \mathcal{C}([0,\pi]) \ : \ f(0) = 0 = f(\pi)\}$ with the uniform norm, consider the linear operator $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) = \big\{f \in \mathcal{C}^2([0,1]) \ : \ f(0) = f(\pi) = 0 = f''(0) = f''(\pi)\big\} \\ Af = f'', \quad \forall f \in D(A). \end{cases}$$

Show that A generates a $\mathcal{C}_0$-semigroup of contractions on X and derive the initial-boundary value problem which is solved by such semigroup.

Solution. We only prove that $\|R(\lambda, A)\| \leqslant 1/\lambda$ for all $\lambda > 0$. Fix any $g \in X$ and let $f = R(\lambda, A)g$. Let $x_0 \in [0, \pi]$ be such that $|f(x_0)| = |f|_\infty$. If $f(x_0) > 0$, then $x_0 \in (0, \pi)$ is a maximum point of f. So, $f''(x_0) \leqslant 0$ and we have that

$$\lambda |f|_\infty = \lambda f(x_0) \leqslant \lambda f(x_0) - f''(x_0) = g(x_0) \leqslant |g|_\infty.$$

On the other hand, if $f(x_0) < 0$, then $x_0 \in (0, \pi)$ once again and x_0 is a minimum point of f. Thus, $f''(x_0) \geqslant 0$ and

$$\lambda |f|_\infty = -\lambda f(x_0) \leqslant -\lambda f(x_0) + f''(x_0) = -g(x_0) \leqslant |g|_\infty.$$

In any case, we have that $\lambda |f|_\infty \leqslant |g|_\infty$. $\square$

Exercise 21 Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a uniformly bounded semigroup $\|S(t)\| \leqslant M$. Prove the *Laundau-Kolmogorov inequality:*

$$|Ax|^2 \leqslant 4M^2 \, |x| \, |A^2 x| \qquad \forall x \in D(A^2), \tag{1.9.10}$$

where

$$\begin{cases} D(A^2) = \{x \in D(A) \; : \; Ax \in D(A)\} \\ A^2 x = A(Ax) \, , \quad \forall x \in D(A^2). \end{cases} \tag{1.9.11}$$

Solution. Assume $M = 1$. For any $x \in D(A^2)$ and all $t \geqslant 0$ we have

$$\begin{aligned} \int_0^t (t-s) S(s) A^2 x \, ds &= \big[(t-s)S(s)Ax\big]_{s=0}^{s=t} + \int_0^t S(s) Ax \, ds \\ &= -tAx + \big[S(s)x\big]_{s=0}^{s=t} = -tAx + S(t)x - x. \end{aligned}$$

Therefore, for all $t > 0$,

$$\begin{aligned} |Ax| &\leqslant \frac{1}{t} \, |S(t)x - x| + \frac{1}{t} \int_0^t (t-s) |S(s) A^2 x| ds \\ &\leqslant \frac{2}{t} \, |x| + \frac{t}{2} \, |A^2 x|. \end{aligned} \tag{1.9.12}$$

If $A^2 x = 0$, then the above inequality yields $Ax = 0$ by letting $t \to \infty$. So, (1.9.10) is true in this case. On the other hand, for $A^2 x \neq 0$ the function of t on the right-hand side of (1.9.12) attains its minimum at

$$t_0 = \frac{2|x|^{1/2}}{|A^2 x|^{1/2}}.$$

By taking $t = t_0$ in (1.9.12) we obtain (1.9.10) once again. $\square$

Exercise 22 Use the Landau-Kolmogorov inequality to deduce the interpolation inequality

$$|f'|_p \leqslant 2 \sqrt{|f|_p \, |f''|_p} \qquad \forall f \in W^{2,p}(\mathbb{R}).$$

2 Special classes of semigroups

2.1 Dissipative operators

Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$.

Definition 9 *We say that an operator $A : D(A) \subset H \to H$ is* dissipative *if*

$$\Re \langle Ax, x \rangle \leqslant 0 \qquad \forall x \in D(A). \tag{2.1.1}$$

Example 11 In $H = L^2(\mathbb{R}_+; \mathbb{C})$ consider the operator

$$\begin{cases} D(A) = H^1(\mathbb{R}_+; \mathbb{C}) \\ Af(x) = f'(x) \qquad\qquad x \in \mathbb{R}_+ \ \text{ a.e.} \end{cases}$$

Then

$$2\Re \langle Af, f \rangle = 2\Re \Big(\int_0^\infty f'(x)\overline{f(x)}\, dx \Big) = \int_0^\infty \frac{d}{dx} |f(x)|^2 \, dx = -|f(0)|^2 \leqslant 0.$$

So, A is dissipative.

Proposition 8 *An operator $A : D(A) \subset H \to H$ is dissipative if and only if*

$$|(\lambda I - A)x| \geqslant \lambda |x| \qquad \forall x \in D(A) \quad \textit{and} \quad \forall \lambda > 0. \tag{2.1.2}$$

Proof. Let A be dissipative. Then for every $\lambda > 0$

$$|(\lambda I - A)x|^2 = \lambda^2 |x|^2 - 2\lambda \Re \langle Ax, x \rangle + |Ax|^2 \geqslant \lambda^2 |x|^2 \qquad \forall x \in D(A).$$

Conversely, suppose A satisfies (2.1.2). Then for every $\lambda > 0$ and $x \in D(A)$

$$\lambda^2 |x|^2 - 2\lambda \Re \langle Ax, x \rangle + |Ax|^2 = |(\lambda I - A)x|^2 \geqslant \lambda^2 |x|^2$$

So, $2\lambda \Re \langle Ax, x \rangle \leqslant |Ax|^2$ which in turn yields (2.1.1) as $\lambda \to \infty$. □

The above characterization can be used to extend the notion of dissipative operators to a Banach space X.

Definition 10 *We say that an operator $A : D(A) \subset X \to X$ is* dissipative *if*

$$|(\lambda I - A)x| \geqslant \lambda |x| \qquad \forall x \in D(A) \quad \textit{and} \quad \forall \lambda > 0. \tag{2.1.3}$$

Remark 5 It follows from (2.1.3) that, if A is dissipative then

$$\lambda I - A : D(A) \to X$$

is one-to-one for all $\lambda > 0$.

Proposition 9 *Let* $A : D(A) \subset X \to X$ *be dissipative. If*

$$\exists \lambda_0 > 0 \quad \textit{such that} \quad (\lambda_0 I - A)D(A) = X, \tag{2.1.4}$$

then the following properties hold:

(a) $\lambda_0 \in \rho(A)$ *and* $\|R(\lambda_0, A)\| \leqslant 1/\lambda_0$*,*

(b) A *is closed,*

(c) $(\lambda I - A)D(A) = X$ *and* $\|R(\lambda, A)\| \leqslant 1/\lambda$ *for all* $\lambda > 0$*.*

We observe that point *(a)* follows from Remark 5 and inequality (2.1.3). As for point *(b)*, we note that, since $R(\lambda_0, A)$ is closed, $\lambda_0 I - A$ is also closed, and therefore A is closed.

Proof of (c). By point *(a)* the set

$$\Lambda = \big\{\lambda \in]0, \infty[\,: \; (\lambda I - A)D(A) = X\big\}$$

is contained in $\rho(A)$ which is open in $\mathbb{C}$. This implies that Λ is also open. Let us show that Λ is closed: let $\Lambda \ni \lambda_n \to \lambda > 0$ and fix any $y \in X$. There exists $x_n \in D(A)$ such that

$$\lambda_n x_n - A x_n = y. \tag{2.1.5}$$

From (2.1.2) it follows that $|x_n| \leqslant |y|/\lambda_n \leqslant C$ for some $C > 0$. Again by (2.1.2),

$$\begin{aligned} \lambda_m |x_n - x_m| &\leqslant |\lambda_m (x_n - x_m) - A(x_n - x_m)| \\ &\leqslant |\lambda_m - \lambda_n|\,|x_n| + |\lambda_n x_n - A x_n - (\lambda_m x_m - A x_m)| \\ &\leqslant C|\lambda_m - \lambda_n|. \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence. Let $x_n \to x$. Then $Ax_n \to \lambda x - y$ by (2.1.5). Since A is closed by point *(b)*, $x \in D(A)$ and $\lambda x - Ax = y$. This show that $\lambda I - A$ is surjective and implies that $\lambda \in \Lambda$. Thus, Λ is both open and closed in $(0, \infty)$. Moreover, $\Lambda \neq \varnothing$ because $\lambda_0 \in \Lambda$. So, $\Lambda = (0, \infty)$. The inequality $\|R(\lambda, A)\| \leqslant 1/\lambda$ is a consequence of dissipativity. □

Definition 11 *A dissipative operator* $A : D(A) \subset X \to X$ *is called* maximal dissipative *if* (2.1.4) *holds true.*

Theorem 7 *Let* X *be a reflexive Banach space. If* $A : D(A) \subset X \to X$ *is a maximal dissipative operator, then* $D(A)$ *is dense in* X*.*

We give the proof assuming that X is a Hilbert space. The case of a reflexive Banach space is treated in exercises 24 to 27.

Proof. Let $z \in X$ be such that $\langle z, x\rangle = 0$ for all $x \in D(A)$. We will show that $z = 0$, or

$$\langle z, y\rangle = 0 \qquad \forall y \in X.$$

Since $(I - A)$ is surjective, the above is equivalent to

$$0 = \langle z, x - Ax\rangle \qquad \forall x \in D(A).$$

Finally, what we need to prove is that

$$\langle z, x\rangle = 0 \qquad \forall x \in D(A) \quad \Longrightarrow \quad \langle z, Ax\rangle = 0 \qquad \forall x \in D(A). \tag{2.1.6}$$

Let $x \in D(A)$. Since $nI - A$ is onto, there exists a sequence $\{x_n\} \subset D(A)$ such that

$$nx = nx_n - Ax_n \qquad \forall n \geqslant 1. \tag{2.1.7}$$

Since $Ax_n = n(x_n - x) \in D(A)$, we have that $x_n \in D(A^2)$ and

$$Ax = Ax_n - \frac{1}{n} A^2 x_n \quad \text{or} \quad Ax_n = \Big(I - \frac{1}{n} A\Big)^{-1} Ax.$$

Since $\|(I - \frac{1}{n} A)^{-1}\| \leqslant 1$ by (2.1.2), the above identity yields $|Ax_n| \leqslant |Ax|$. So, by (2.1.7) we obtain

$$|x_n - x| \leqslant \frac{1}{n} |Ax|.$$

Therefore, $x_n \to x$. Moreover, since $\{Ax_n\}$ is bounded, there is a subsequence Ax_{n_k} such that $Ax_{n_k} \rightharpoonup y$. Since A is closed, we have that $y = Ax$. Therefore,

$$\langle z, Ax\rangle = \lim_{k\to\infty} \langle z, Ax_{n_k}\rangle = \lim_{k\to\infty} n_k \langle z, x_{n_k} - x\rangle$$

and (2.1.6) follows from the vanishing of the rightmost term above. $\square$

Example 12 We now show that the above density may be fail in a general Banach space. On $X = \mathcal{C}([0, 1])$ with the uniform norm consider the linear operator $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) = \big\{u \in \mathcal{C}^1([0, 1]) \;:\; u(0) = 0\big\} \\ Au(x) = -u'(x) & \qquad\qquad \forall x \in [0, 1]. \end{cases}$$

Then, for all $\lambda > 0$ and $f \in X$ we have that the equation $\lambda u - Au = f$ has the unique solution $u \in D(A)$ given by

$$u(x) = \int_0^x e^{\lambda(y-x)} f(y)\, dy \quad (x \in [0, 1])$$

Thereforre, $\lambda I - A$ is onto. Moreover,

$$\lambda |u(x)| \leqslant \int_0^x \lambda e^{\lambda(y-x)} \|f\|_\infty \, dy = (1 - e^{-\lambda x})\|f\|_\infty \leqslant \|\lambda u - Au\|_\infty.$$

So, A is dissipative. On the other hand, $D(A)$ is not dense in X because all functions in $D(A)$ vanish at $x = 0$.

Theorem 8 (Lumer-Phillips 1) *Let $A : D(A) \subset X \to X$ be a densely defined linear operator. Then the following properties are equivalent:*

(a) A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions,

(b) A is maximal dissipative.

$\boxed{\textit{Proof of } (a) \Rightarrow (b)}$ In view of Theorem 5, we have that $]0, \infty[\subset \rho(A)$. So, $(\lambda I - A)D(A) = X$ for all $\lambda > 0$. Moreover, by the Hille-Yosida theorem for all $\lambda > 0$ and $y \in X$ we have that $\lambda |R(\lambda, A)y| \leqslant |y|$ or, setting $x = R(\lambda, A)y$,

$$\lambda |x| \leqslant |(\lambda I - A)x| \qquad \forall x \in D(A).$$

So, A is maximal dissipative. $\square$

$\boxed{\textit{Proof of } (b) \Rightarrow (a)}$ We have that:

(i) $D(A)$ is dense by hypothesis,

(ii) A is closed by Proposition 9-*(b)*,

(iii) $]0, \infty[\subset \rho(A)$ and $\|R(\lambda, A)\| \leqslant 1/\lambda$ for all $\lambda > 0$ by Proposition 9-*(c)*.

The conclusion follows by the Hille-Yosida theorem. $\square$

Example 13 (Wave equation in $L^2(\Omega)$) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$. For any given $f \in H^2(\Omega) \cap H^1_0(\Omega)$ and $g \in H^1_0(\Omega)$, consider the problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u & \text{in }]0,\infty[\times\Omega \\ u = 0 & \text{on }]0,\infty[\times\partial\Omega \\ u(0,x) = f(x)\,,\ \frac{\partial u}{\partial t}(0,x) = g(x) & x \in \Omega \end{cases} \tag{2.1.8}$$

Let H be the Hilbert space $H^1_0(\Omega) \times L^2(\Omega)$ with the scalar product

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \right\rangle = \int_\Omega \big(Du(x) \cdot D\bar{u}(x) + v(x)\bar{v}(x)\big)dx.$$

Define $A : D(A) \subset H \to H$ by

$$\begin{cases} D(A) = \big(H^2(\Omega) \cap H^1_0(\Omega)\big) \times H^1_0(\Omega) \\ A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \end{pmatrix} \end{cases} \tag{2.1.9}$$

We will show that A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions on H by checking that A is maximal dissipative.

Let $\begin{pmatrix} u \\ v \end{pmatrix} \in D(\mathcal{A})$. Then, integrating by parts we obtain

$$\Big\langle A\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\Big\rangle = \int_\Omega \big(Du(x) \cdot Dv(x) + v(x)\Delta u(x)\big)dx = 0\,. \tag{2.1.10}$$

So, A is dissipative.

Now, consider the resolvent equation

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} \in D(\mathcal{A}) \\ \\ (I - A)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \in H \end{cases} \tag{2.1.11}$$

which is equivalent to the system

$$\begin{cases} u \in H^2(\Omega) \cap H_0^1(\Omega), \quad v \in H_0^1(\Omega) \\ u - v = f \in H_0^1(\Omega) \\ v - \Delta u = g \in L^2(\Omega). \end{cases} \tag{2.1.12}$$

Using elliptic theory one can show that the boundary value problem

$$\begin{cases} u \in H^2(\Omega) \cap H_0^1(\Omega), \\ u - \Delta u = f + g \in L^2(\Omega) \end{cases}$$

has a unique solution. Then, taking $v = u - f \in H_0^1(\Omega)$ we obtain the unique solution of problem (2.1.12). So, A is maximal dissipative and therefore A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions, $S(t)$, thanks to Theorem 8.

For any $f \in H^2(\Omega) \cap H_0^1(\Omega)\,,\, g \in H_0^1(\Omega)$, let $u(t)\,(t \in \mathbb{R}_+)$ be the first component of

$$S(t)\begin{pmatrix} f \\ g \end{pmatrix}$$

Then u is the unique solution of problem (2.1.8) in the space

$$\mathcal{C}^2\big(\mathbb{R}_+; L^2(\Omega)\big) \cap \mathcal{C}^1\big(\mathbb{R}_+; H_0^1(\Omega)\big) \cap \mathcal{C}\big(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega)\big).$$

2.2 Strongly continuous groups

Definition 12 *A $\mathcal{C}_0$-group of bounded linear operators on X is is a map $G : \mathbb{R} \to \mathcal{L}(X)$ with the following properties:*

(a) $G(0) = I$ and $G(t + s) = G(t)G(s)$ for all $t, s \in \mathbb{R}$,

(b) for all $x \in X$

$$\lim_{t\to 0} G(t)x = x. \tag{2.2.1}$$

Definition 13 *The infinitesimal generator of a $\mathcal{C}_0$-group of bounded linear operators on X, $G(t)$, is the map $A : D(A) \subset X \to X$ defined by*

$$\begin{cases} D(A) = \left\{ x \in X \ : \ \exists \lim_{t\to 0} \frac{S(t)x-x}{t} \right\} \\ Ax = \lim_{t\to 0} \frac{S(t)x-x}{t} & \forall x \in D(A) \end{cases}$$

Theorem 9 *Let $M \geqslant 1$ and $\omega \geqslant 0$. For a linear operator $A : D(A) \subset X \to X$ the following properties are equivalent:*

(a) *A is the infinitesimal generator of a $\mathcal{C}_0$-group, $G(t)$, such that*

$$\|G(t)\| \leqslant Me^{\omega|t|} \qquad \forall t \in \mathbb{R}. \tag{2.2.2}$$

(b) *A and $-A$ are the infinitesimal generators of $\mathcal{C}_0$-semigroups, $S_+(t)$ and $S_-(t)$) respectively, satisfying*

$$\|S_\pm(t)\| \leqslant Me^{\omega t} \qquad \forall t \geqslant 0. \tag{2.2.3}$$

(c) *A is closed, $D(A)$ is dense in X, and*

$$\rho(A) \supseteq \left\{ \lambda \in \mathbb{C} \ : \ |\Re \lambda| > \omega \right\} \tag{2.2.4}$$

$$\|R(\lambda, A)^k\| \leqslant \frac{M}{(|\Re \lambda| - \omega)^k} \quad \forall k \geqslant 1, \forall |\Re \lambda| > \omega \tag{2.2.5}$$

Remark 6 Let A and $S_\pm(t)$ be as in point *(b)* above. We claim that

(i) $S_+(t)S_-(s) = S_-(s)S_+(t)$ for all $s, t \geqslant 0$,

(ii) $S_+(t)^{-1} = S_-(t)$ for all $t \geqslant 0$.

Indeed, $S_+(t)$ and $S_-(t)$ commute because

$$S_\pm(t) = \lim_{n\to\infty} e^{\pm tA_n},$$

where e^{tA_n} and e^{-tA_m} commute since A_n and A_m do so. Hence, (i) holds true.

Consequently,

$$S(t) := S_+(t)S_-(t) \qquad (t \geqslant 0)$$

is also a $\mathcal{C}_0$-semigroup and, for all $x \in D(A) = D(-A)$, we have that

$$\frac{S(t)x - x}{t} = S_+(t)\,\frac{S_-(t)x - x}{t} + \frac{S_+(t)x - x}{t} \xrightarrow{t\downarrow 0} -Ax + Ax = 0.$$

So, $\frac{d}{dt}\,S(t)x = 0$ for all $t \geqslant 0$. Hence, $S(t)x = x$ for all $t \geqslant 0$ and $x \in D(A)$. By density, $S(t)x = x$ for all $x \in X$, which yields $S_+(t)^{-1} = S_-(t)$.

$\boxed{\textit{Proof of (a)} \Rightarrow \textit{(b)}}$ Define, for all $t \geqslant 0$,

$$S_+(t) = G(t) \quad \text{and} \quad S_-(t) = G(-t).$$

Then it can be checked that $S_\pm(t)$ is $\mathcal{C}_0$-semigroup satisfying (2.2.3). Moreover, observing that

$$\frac{S_-(t)x - x}{t} = \frac{G(-t)x - x}{t} = -G(-t)\frac{G(t)x - x}{t},$$

it is easy to show that $\pm A$ is the infinitesimal generator of $S_\pm(t)$. $\square$

$\boxed{\textit{Proof of (b)} \Rightarrow \textit{(c)}}$ By the Hille-Yosida theorem we conclude that A is closed, $D(A)$ is dense in X, and

$$\rho(A) \supseteq \Pi_\omega = \{\lambda \in \mathbb{C} \ : \ \Re\lambda > \omega\}$$
$$\|R(\lambda, A)^k\| \leqslant \frac{M}{(\Re\lambda - \omega)^k} \quad \forall k \geqslant 1, \ \forall \lambda \in \Pi_\omega.$$

Since

$$(\lambda I + A)^{-1} = -(-\lambda I - A)^{-1}, \tag{2.2.6}$$

we have that $-\rho(A) = \rho(-A) \supseteq \Pi_\omega$, or

$$\rho(A) \supseteq -\Pi_\omega = \{\lambda \in \mathbb{C} \ : \ \Re\lambda < -\omega\},$$

and

$$\|R(\lambda, A)^k\| = \|R(-\lambda, -A)^k\| \leqslant \frac{M}{(-\Re\lambda - \omega)^k} \quad \forall k \geqslant 1, \ \forall \lambda \in -\Pi_\omega. \quad \square$$

$\boxed{\textit{Proof of (c)} \Rightarrow \textit{(a)}}$ Recalling (2.2.6), by the Hille-Yosida theorem it follows that $\pm A$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup, $S_\pm(t)$, satisfying (2.2.3). For all $x \in X$ define

$$G(t)x = \begin{cases} S_+(t)x & (t \geqslant 0) \\ S_-(-t)x & (t < 0). \end{cases}$$

Then, it follows that (2.2.1) and (2.2.2) hold true, and A is the infinitesimal generator of $G(t)$. Let us check that $G(t+s) = G(t)G(s)$ for all $t \geqslant 0$ and all $s \leqslant 0$ such that $t + s \geqslant 0$. We have that

$$G(t)G(s) = S_+(t)S_-(-s) = S_+(t+s)S_+(-s)S_+(-s)^{-1} = G(t+s). \quad \square$$

Corollary 3 *Let $A : D(A) \subset X \to X$ be a densely defined linear operator. If both A and $-A$ are maximal dissipative, the A is the infinitesimal generator of a $\mathcal{C}_0$-group, $G(t)$, which satisfies $\|G(t)\| = 1$ for all $t \in \mathbb{R}$.*

Proof. By the Lumer-Phillips theorem, A and $-A$ are infinitesimal generators of $\mathcal{C}_0$-semigroups of contractions, $S_+(t)$ and $S_-(t)$ respectively. Therefore, Theorem 9 ensures that A is the infinitesimal generator of a $\mathcal{C}_0$-group, $G(t)$. Moreover, $1 = \|S_+(t)S_-(t)\| \leqslant \|S_+(t)\|\,\|S_-(t)\| \leqslant 1$. Hence, $\|G(t)\| = 1$. $\square$

Example 14 (Wave equation continued) We return to the wave equation that was studied in Example 13. We proved that operator A, defined in (2.1.9), is maximal dissipative. We claim that $-A$ is maximal dissipative as well. Indeed, equation (2.1.10) implies that $-A$ is dissipative. Moreover, the resolvent equation for $-A$ takes the form

$$\begin{cases} u \in H^2(\Omega) \cap H_0^1(\Omega), \qquad v \in H_0^1(\Omega) \\ u + v = f \in H_0^1(\Omega) \\ v + \Delta u = g \in L^2(\Omega)\,, \end{cases}$$

which can be uniquely solved arguing exactly as we did for system (2.1.12).

Then, by Corollary 3, A is the infinitesimal generator of a $\mathcal{C}_0$-group, $G(t)$, which satisfies $\|G(t)\| = 1$ for all $t \in \mathbb{R}$. So, for any $f \in H^2(\Omega) \cap H_0^1(\Omega)$, $g \in H_0^1(\Omega)$, the first component $u(t)\,(t \in \mathbb{R}_+)$ of

$$G(t)\begin{pmatrix} f \\ g \end{pmatrix}$$

is the unique solution of problem (2.1.8) in the space

$$\mathcal{C}^2\big(\mathbb{R}; L^2(\Omega)\big) \cap \mathcal{C}^1\big(\mathbb{R}; H_0^1(\Omega)\big) \cap \mathcal{C}\big(\mathbb{R}; H^2(\Omega) \cap H_0^1(\Omega)\big).$$

2.3 The adjoint of a linear operator

In this section, we consider the special case when $\big(X, \langle\cdot,\cdot\rangle\big)$ is a Hilbert space. We denote by $j_X : X^* \to X$ the Riesz isomorphism, which associates with any $\phi \in X^*$ the unique element $j_X(\phi) \in X$ such that

$$\phi(x) = \langle x, j_X(\phi)\rangle \qquad \forall x \in X.$$

Let $A : D(A) \subset X \to X$ be a densely defined linear operator.

Remark 7 The set

$$D(A^*) = \Big\{ y \in X \;\Big|\; \exists C \geqslant 0 \;:\; x \in D(A) \quad \Longrightarrow \quad |\langle Ax, y\rangle| \leqslant C|x| \Big\} \qquad (2.3.1)$$

is a subspace of X and, for any $y \in D(A^*)$, the linear map $x \mapsto \langle Ax, y\rangle$ can be uniquely extended to a bounded linear functional $\phi_y \in X^*$.

Definition 14 *The adjoint of A is the map $A^* : D(A^*) \subset X \to X$ defined by*

$$A^* y = j_X(\phi_y) \qquad \forall y \in D(A^*)$$

where $D(A^*)$ and ϕ_y are defined in Remark 7.

Proposition 10 *Let $A : D(A) \subset X \to X$ be a densely defined linear operator. Then $A^* : D(A^*) \subset X \to X$ is a closed linear operator satisfying the identity*

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \qquad \forall x \in D(A), \forall y \in D(A^*). \tag{2.3.2}$$

Proof. We only prove that A^* is closed, leaving the remaining item for the reader to check. Let $\{y_n\} \subset D(A^*)$ and $y, z \in X$ be such that

$$\begin{cases} y_n \to y \\ A^* y_n \to z \end{cases} \qquad (n \to \infty)$$

Then $\{A^* y_n\}$ is bounded, say $|A^* y_n| \leqslant C$. So, recalling (2.3.2), we have that

$$|\langle Ax, y_n \rangle| = |\langle x, A^* y_n \rangle| \leqslant C|x| \qquad \forall x \in D(A)$$

This yields

$$|\langle Ax, y \rangle| \leqslant C|x| \qquad \forall x \in D(A)$$

which in turn implies that $y \in D(A^*)$. Moreover

$$\langle Ax, y \rangle = \lim_{n \to \infty} \langle Ax, y_n \rangle = \langle x, z \rangle \qquad \forall x \in D(A).$$

Thus, $\langle x, A^* y - z \rangle = 0$ for all $x \in D(A)$. Since $D(A)$ is dense, $A^* y = z$. □

Theorem 10 (Lumer-Phillips 2) *Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator. If A and A^* are dissipative, then A is the infinitesimal generator of a contraction semigroup on X.*

Proof. In view of Theorem 8 it suffices to show that $]0, \infty[\subset \rho(A)$. For this purpose, since $\lambda I - A$ is one-to-one for any $\lambda > 0$, one just has to check that

$$(\lambda I - A) D(A) = X \qquad \forall \lambda > 0.$$

Step 1: $(\lambda I - A) D(A)$ is dense in X for every $\lambda > 0$.
Let $y \in X$ be such that

$$\langle \lambda x - Ax, y \rangle = 0 \qquad \forall x \in D(A).$$

The identity $\langle Ax, y \rangle = \lambda \langle x, y \rangle$ yields $y \in D(A^*)$ and the fact that

$$\langle x, \lambda y - A^* y \rangle = 0,$$

first for all $x \in D(A)$ and then, by density, for all $x \in X$. So, $\lambda y - A^* y = 0$. Since, being dissipative, $\lambda I - A^*$ is also one-to-one, we conclude that $y = 0$.

Step 2: *$\lambda I - A$ is onto for every $\lambda > 0$.*
Fix any $y \in X$. By Step 1, there exists $\{x_n\} \subset D(A)$ such that

$$\lambda x_n - A x_n =: y_n \to y \quad \text{as} \quad n \to \infty.$$

By (2.1.2) we deduce that, for all $n, m \geqslant 1$,

$$|x_n - x_m| \leqslant \frac{1}{\lambda} |y_n - y_m|$$

which insures that $\{x_n\}$ is a Cauchy sequence in X. Therefore, there exists $x \in X$ such that

$$\begin{cases} x_n \to x \\ A x_n = \lambda x_n - y_n \to \lambda x - y \end{cases} \qquad (n \to \infty)$$

Since A is closed, $x \in D(A)$ and $\lambda x - A x = y$. □

Definition 15 *A densely defined linear operator $A : D(A) \subset X \to X$ is called:*

(a) symmetric *if $A \subset A^*$, that is,*

$$D(A) \subset D(A^*) \quad \textit{and} \quad Ax = A^* x \qquad \forall x \in D(A).$$

(b) self-adjoint *if $A = A^*$.*

Remark 8 Observe that a symmetric operator A is self-adjoint if and only if $D(A) \subseteq D(A^*)$. Moreover, in view of Proposition 10, any self-adjoint operator is closed.

Corollary 4 (Lumer-Phillips 3) *Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator. If A is self-adjoint and dissipative, then A is the infinitesimal generator of a contraction semigroup on X.*

Example 15 In $X = L^2(0,1;\mathbb{C})$, consider the linear operator

$$\begin{cases} D(A) = H_0^1(0,1;\mathbb{C}) \\ Au(x) = i\, u'(x) \qquad x \in [0,1] \ a.e. \end{cases}$$

Then, A is densely defined and symmetric. Indeed, for all $u, v \in D(A)$,

$$\begin{aligned} \langle Au, v \rangle &= i \int_0^1 u'(x) \overline{v(x)}\, dx & (2.3.3) \\ &= \big[i u(x) \overline{v(x)} \big]_{x=0}^{x=1} - i \int_0^1 u(x) \overline{v'(x)}\, dx = \langle u, Av \rangle. \end{aligned}$$

On the other hand, A fails to be self-adjoint because, as we show next,

$$D(A^*) \supseteq H^1(0,1;\mathbb{C}),$$

so that $D(A) \subsetneq D(A^*)$. Indeed, integrating by parts as in (2.3.3), for all $v \in H^1(0,1;\mathbb{C})$ and $u \in H_0^1(0,1;\mathbb{C})$ we have that

$$\big|\langle Au, v\rangle\big| = \Big| - i\int_0^1 u(x)\overline{v'(x)}\,dx\Big| \leqslant |u|_2|v'|_2. \qquad \square$$

Proposition 11 *Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator such that $\rho(A) \cap \mathbb{R} \neq \varnothing$. If A is symmetric, then A is self-adjoint.*

Proof. We prove that $D(A^*) \subset D(A)$ in two steps. Fix any $\lambda \in \rho(A) \cap \mathbb{R}$.

Step 1: $\boxed{R(\lambda, A) = R(\lambda, A)^*}$
Since $R(\lambda, A) \in \mathcal{L}(X)$, in view of Exercise 23 it suffices to show that

$$\langle R(\lambda, A)x, y\rangle = \langle x, R(\lambda, A)y\rangle \qquad \forall x, y \in X.$$

Fix any $x, y \in X$ and set

$$u = R(\lambda, A)x \quad \text{and} \quad v = R(\lambda, A)y$$

so that $u, v \in D(A)$ and

$$\lambda u - Au = x \quad \text{and} \quad \lambda v - Av = y.$$

Since A is symmetric, we have that

$$\langle R(\lambda, A)x, y\rangle = \langle u, y\rangle = \langle u, \lambda v - Av\rangle = \langle \lambda u - Au, v\rangle = \langle x, R(\lambda, A)y\rangle.$$

Step 2: $\boxed{D(A^*) \subset D(A)}$
Let $u \in D(A^*)$ and set $x = \lambda u - A^*u$. Observe that, for all $v \in D(A)$,

$$\langle x, v\rangle = \langle \lambda u - A^*u, v\rangle = \langle u, \lambda v - Av\rangle.$$

Now, take any $y \in X$ and let $v = R(\lambda, A)y$. Then the above identity yields

$$\langle x, R(\lambda, A)y\rangle = \langle u, y\rangle \qquad \forall y \in X.$$

So, by Step 1 we conclude that $u = R(\lambda, A)^*x = R(\lambda, A)x \in D(A)$. $\square$

Example 16 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$. Define

$$\begin{cases} D(A) = H^2 \cap H_0^1(\Omega;\mathbb{C}) \\ Au(x) = \Delta u(x) - V(x)u(x) \quad x \in \Omega \text{ a.e.} \end{cases} \tag{2.3.4}$$

where we assume $V \in L^\infty(\Omega)$. Let us check that A is self-adjoint in $L^2(\Omega;\mathbb{C})$. Indeed, integration by parts insures that A is symmetric. So, by Proposition 11, it suffices to check that $\rho(A)\cap\mathbb{R} \neq \varnothing$. We claim that, for $\lambda \in \mathbb{R}$ large enough, for any $h \in L^2(\Omega;\mathbb{C})$ the problem

$$\begin{cases} w \in H^2 \cap H_0^1(\Omega;\mathbb{C}) \\ (\lambda+V)w - \Delta w = h \quad x \in \Omega \end{cases} \tag{2.3.5}$$

has a unique solution. Equivalently, by setting $f = \Re h$, $g = \Im h \in L^2(\Omega)$ and $u = \Re w$, $v = \Im w$, we have to prove solvability for the boundary value problems

$$\begin{cases} u \in H^2 \cap H_0^1(\Omega) \\ (\lambda+V)u - \Delta u = f \quad x \in \Omega \end{cases} \quad \text{and} \quad \begin{cases} v \in H^2 \cap H_0^1(\Omega) \\ (\lambda+V)v - \Delta v = g \quad x \in \Omega. \end{cases}$$

The latter is a well-established fact in elliptic theory.

The following property of self-adjoint operators is very useful. We recall that an operator $U \in \mathcal{L}(X)$ is unitary if $UU^* = U^*U = I$.

Theorem 11 (Stone) *Let X be a complex Hilbert space. For any densely defined linear operator $A : D(A) \subset X \to X$ the following properties are equivalent:*

(a) A is self-adjoint,

(b) iA is the infinitesimal generator of a $\mathcal{C}_0$-group of unitary operators.

$\boxed{\textit{Proof of } (a) \Rightarrow (b)}$ Since A is self-adjoint, A is closed and we have that

$$\langle Ax, x\rangle = \langle x, A^*x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle} \qquad \forall x \in D(A).$$

Thus, $\langle Ax, x\rangle$ is real so that

$$\Re\langle iAx, x\rangle = 0 \qquad \forall x \in D(A).$$

The above identity implies that $\pm iA$ is dissipative. Since

$$\langle iAx, y\rangle = i\langle x, Ay\rangle = \langle x, -iAy\rangle \qquad \forall x, y \in D(A),$$

we have that $(iA)^* = -iA$. So, by Theorem 10 we deduce that $\pm iA$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions. Then, by Theorem 9, iA generates a $\mathcal{C}_0$ group $G(t)$. Such a group is unitary because

$$1 = \|G(t)G(-t)\| = \|G(t)G(t)^*\| \leqslant 1. \qquad \square$$

$\boxed{\textit{Proof of } (b) \Rightarrow (a)}$ Let iA be the infinitesimal generator of a $\mathcal{C}_0$-group of unitary operators on X, say $G(t)$. Then, for all $x \in D(A)$, we have that

$$\begin{aligned} iAx &= \lim_{t\to 0} \frac{G(t)x - x}{t} = -\lim_{t\to 0} \frac{G(-t)x - x}{t} = -\lim_{t\to 0} \frac{G(t)^* x - x}{t} = \\ &= -\lim_{t\to 0} \Big(\frac{G(t) - I}{t}\Big)^* x = -(iA)^* x = iA^* x. \end{aligned}$$

Thus, $x \in D(A^*)$ and $Ax = A^* x$. By running the above computation backwards, we conclude that $D(A^*) \subseteq D(A)$. Therefore, A is self-adjoint. □

Example 17 (Schrödinger equation in a bounded domain) Let us consider the initial-boundary value problem

$$\begin{cases} \frac{1}{i}\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - V(x)u(t,x) & (t,x) \in \mathbb{R} \times \Omega \\ u(t,x) = 0 & t \in \mathbb{R},\ x \in \partial\Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases} \tag{2.3.6}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class $\mathcal{C}^2$ and $V \in L^\infty(\Omega)$. In Example 16, we have already checked that the operator A, defined in (2.3.4), is self-adjoint on $L^2(\Omega;\mathbb{C})$. Therefore, by Theorem 11 we conclude that, for any $u_0 \in H^2 \cap H_0^1(\Omega;\mathbb{C})$, problem (2.3.6) has a unique solution

$$u \in \mathcal{C}^1\big(\mathbb{R}; L^2(\Omega;\mathbb{C})\big) \cap \mathcal{C}\big(\mathbb{R}; H^2 \cap H_0^1(\Omega;\mathbb{C})\big).$$ □

2.4 Additional exercises for Chapter 2

Exercise 23 Prove that, if $A \in \mathcal{L}(X)$, then A^* is also bounded.

Exercise 24 We recall that the duality set of a point $x \in X$ is defined as

$$\Phi(x) = \big\{\phi \in X^* \ : \ \langle x, \phi\rangle = |x|^2 = \|\phi\|^2\big\}. \tag{2.4.1}$$

Observe that the Hahn-Banach theorem ensures $\Phi(x) \neq \varnothing$.

We also recall that, for all $x \in X$,

$$\partial|x| = \big\{\phi \in X^* \ : \ |x+h| - |x| \geqslant \langle h, \phi\rangle\,,\ \forall x, h \in X\big\}. \tag{2.4.2}$$

Prove that

$$\Phi(x) = x\partial|x| = \big\{\psi \in X^* \ : \ \psi = |x|\phi\,,\ \phi \in \partial|x|\big\}.$$

Exercise 25 Prove that, for any operator $A : D(A) \subset X \to X$ the following properties are equivalent:

(a) A is dissipative,

(b) for all $x \in D(A)$ there exists $\phi \in \Phi(x)$ such that $\Re\langle Ax, \phi\rangle \leqslant 0$.

Exercise 26 Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions. Prove that, for all $x \in D(A)$,

$$\Re\langle Ax, \phi\rangle \leqslant 0 \qquad \forall \phi \in \Phi(x).$$

Exercise 27 Mimic the proof of Theorem 7 in the case of a Hilbert space to treat the general case of a reflexive Banach space.

3 The inhomogeneous Cauchy problem

In this chapter, we assume that $\big(X, \langle\cdot,\cdot\rangle\big)$ is a separable Hilbert space and denote by $\{e_j\}_{j\in\mathbb{N}}$ a complete orthonormal system in X.

We study the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t) \\ u(0) = x, \end{cases} \tag{3.0.1}$$

where $f \in L^2(0,T;X)$ and $A : D(A) \subset X \to X$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup on X, $S(t)$, which satisfies the growth condition (1.8.3). For the extension of this theory to a general Banach space, we refer the reader to the classic monograph by Pazy [3] or the more recent text [2].

3.1 Notions of solution

Definition 16 *Let $x \in X$ and $f \in L^2(0,T;X)$.*

(I) We say that $u \in H^1(0,T;X) \cap L^2(0,T;D(A))$ is a strict solution *of* (3.0.1) *if $u(0) = x$ and*

$$u'(t) = Au(t) + f(t) \qquad (t \in [0,T] \ \text{a.e.})$$

(II) We say that $u \in \mathcal{C}\big([0,T];X\big)$ is a strong solution *of* (3.0.1) *if there exists a sequence $u_n \in H^1(0,T;X) \cap L^2(0,T;D(A))$ such that*

$$\begin{cases} u_n \to u & \text{in } C\big([0,T];X\big) \\ u_n' - Au_n \to f & \text{in } L^2(0,T;X) \qquad (n\to\infty) \\ u_n(0) \to x & \text{in } X \end{cases} \tag{3.1.1}$$

3.2 Well posedness in $L^2(0,T;H)$

Theorem 12 (Existence and uniqueness of strong solutions) *For any $x \in X$ and $f \in L^2(0,T;X)$ there exists a unique strong solution u of* (3.0.1)*, which is given by the variation-of-constants formula*

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds \tag{3.2.1}$$

Moreover, $u_n := nR(n,A)u$ satisfies

$$u_n \in H^1(0,T;X) \cap L^2(0,T;D(A)) \quad \text{and} \quad u_n \overset{(n\to\infty)}{\longrightarrow} u \ \text{ in } \mathcal{C}\big([0,T];X\big).$$

Observe that u in (3.2.1) is well defined in view of Proposition 17.

Proof. Step 1: existence. Let u be given by (3.2.1) and define

$$\begin{cases} u_n(t) = nR(n,A)u(t) \\ f_n(t) = nR(n,A)f(t) \\ x_n = nR(n,A)x \end{cases} \qquad \forall n \in \mathbb{N}\,,\ n > \omega$$

where $\omega \geqslant 0$ is such that (1.8.3) holds true. Then

$$u_n(t) = S(t)x_n + \int_0^t S(t-s)f_n(s)\,ds \quad (t \in [0,T]). \tag{3.2.2}$$

Since $x_n \in D(A)$ and $f_n \in L^2(0,T;D(A))$, by propositions 16 and 17 we conclude that

$$u_n \in H^1(0,T;X) \cap L^2(0,T;D(A)) \quad \text{and} \quad \begin{cases} u_n' - Au_n = f_n \\ u_n(0) = x_n. \end{cases}$$

Moreover, invoking Lemma 1 we conclude that $x_n \to x$ as $n \to \infty$ while

$$f_n(t) \overset{(n\to\infty)}{\longrightarrow} f(t) \quad \text{and} \quad |f_n(t)| \leqslant \frac{Mn}{n-\omega}\,|f(t)| \quad (\text{a.e. } t \in [0,T])$$

Therefore, $f_n \overset{(n\to\infty)}{\longrightarrow} f$ in $L^2(0,T;X)$.

Finally,we have that

$$\sup_{t\in[0,T]} |u_n(t) - u(t)| \leqslant Me^{\omega T}\Big(|x_n - x| + \int_0^T |f_n(s) - f(s)|\,ds\Big) \overset{(n\to\infty)}{\longrightarrow} 0.$$

So, u is a strong solution of (3.0.1).

Step 1: uniqueness.
Let v be a strong solution of (3.0.1) and let $\{v_n\}$ be a sequence satisfying (3.1.1). Setting $f_n = v_n' - Av_n$, for any fixed $t \in]0,T]$ we have that

$$\frac{d}{ds}\big(S(t-s)v_n(s)\big) = S(t-s)f_n(s) \qquad (\text{a.e. } s \in [0,t]).$$

By integrating over $[0,t]$ we deduce that v_n satisfies (3.2.2). Then, passing to the limit as $n \to \infty$ we conclude that v is given by (3.2.1). $\square$

The following result provides a useful approximation of strong solutions.

Proposition 12 *Let $\{x_n\} \subset X$ and $\{f_n\} \subset L^2(0,T;X)$ be such that*

$$x_n \overset{X}{\longrightarrow} x \quad \textit{and} \quad f_n \overset{L^2(0,T;X)}{\longrightarrow} f \quad (n \to \infty).$$

Let u_n satisfy

$$\begin{cases} u_n'(t) = A_n u_n(t) + f_n(t), & t \in (0,T) \\ u_n(0) = x_n \end{cases} \tag{3.2.3}$$

where $A_n = n^2 R(n,A) - n$ $(n > \omega)$ is the Yosida approximation of A. Then $\{u_n\}_n$ is bounded in $\mathcal{C}\big([0,T];X\big)$ and

$$u_n(t) \overset{(n\to\infty)}{\longrightarrow} u(t) \quad \forall t \in [0,T],$$

where u is the strong solution of (3.0.1).

Proof. Since $A_n \in \mathcal{L}(X)$ we have that

$$u_n(t) = e^{tA_n} x_n + \int_0^t e^{(t-s)A_n} f_n(s)\, ds \quad (t \in [0,T]).$$

Thus, recalling (1.8.8) and (1.8.12), we obtain

$$|e^{tA_n} x_n - S(t)x| \leqslant M e^{2\omega t} |x_n - x| + |e^{tA_n} x - S(t)x| \overset{n\to\infty}{\longrightarrow} 0$$

uniformly on $[0,T]$. Moreover,

$$\begin{aligned} &\Big| \int_0^t \Big(e^{(t-s)A_n} f_n(s) - S(t-s) f(s) \Big) ds \Big| \\ &\qquad\qquad \leqslant M \int_0^t e^{2\omega(t-s)} |f_n(s) - f(s)| ds \overset{\mathcal{C}([0,T];X)}{\longrightarrow} 0. \\ &\qquad\qquad\qquad\qquad + \int_0^t |e^{(t-s)A_n} f(s) - S(t-s) f(s)| ds. \end{aligned}$$

By Lebesgue's dominated convergence theorem, for any $t \in [0,T]$ we have that

$$\lim_{n\to\infty} \int_0^t |e^{(t-s)A_n} f(s) - S(t-s) f(s)| ds = 0.$$

The conclusion follows. □

3.3 Regularity

Our first result guarantees that the strong solution of (3.0.1) is strict when f has better "space regularity".

Theorem 13 *Let $x \in D(A)$ and let $f \in L^2(0,T;D(A))$. Then the strong solution u of problem* (3.0.1) *is strict.*

Proof. Let u be the strong solution of problem (3.0.1) and let u_n be the solution of (3.2.3) with $f_n \equiv f$. Then

$$v_n(t) := A_n u_n(t) \quad (t \in [0,T])$$

satisfies

$$\begin{cases} v_n'(t) = A_n v_n(t) + A_n f(t), & t \in (0,T) \\ v_n(0) = A_n x \end{cases}$$

where

$$A_n x \overset{(n\to\infty)}{\longrightarrow} Ax \quad \text{and} \quad A_n f \overset{(n\to\infty)}{\longrightarrow} Af \quad \text{in } L^2(0,T;X).$$

So, Proposition 12 ensures that v_n is bounded in $\mathcal{C}\big([0,T];X\big)$ and converges point-wise to the strong solution of

$$\begin{cases} v'(t) = Av(t) + Af(t), & t \in (0,T) \\ v(0) = Ax \end{cases}$$

which is given by

$$v(t) = S(t)Ax + \int_0^t S(t-s)Af(s)\,ds = Au(t) \quad (t \in [0,T] \text{ a.e.})$$

Moreover, owing to Proposition 16 we have that $v = Au$. This shows that $u \in \mathcal{C}([0,T];D(A))$. Furthermore,

$$u_n' = A_n u_n + f = v_n + f \overset{L^2(0,T;;X)}{\longrightarrow} Au + f \quad (n \to \infty)$$

because v_n is bounded in $\mathcal{C}\big([0,T];X\big)$ and converges point-wise. Therefore, $u \in H^1(0,T;X)$ and $u'(t) = Au(t) + f(t)$ for a.e. $t \in [0,T]$. □

We will now show a similar result if f has better "time regularity". In this case, one can prove that strong solutions are classical in the following sense. Let $x \in D(A)$ and let $f \in \mathcal{C}\big([0,T];X\big)$.

Definition 17 *We say that* $u \in \mathcal{C}^1\big([0,T];X\big) \cap \mathcal{C}\big([0,T];D(A)\big)$ *is a* classical solution *of* (3.0.1) *if* $u(0) = x$ *and*

$$u'(t) = Au(t) + f(t) \qquad \forall t \in [0,T].$$

Theorem 14 *Let* $x \in D(A)$ *and let* $f \in H^1(0,T;X)$. *Then the strong solution* u *of problem* (3.0.1) *is classical.*

We begin by studying the case of $x = 0$.

Lemma 2 *For any $f \in H^1(0,T;X)$ define*

$$F_A(t) = \int_a^t S(t-s)f(s)\,ds \quad (t \in [0,T]). \tag{3.3.1}$$

Then $F_A \in \mathcal{C}^1\big([0,T];X\big) \cap \mathcal{C}\big([0,T];D(A)\big)$ and

$$F_A'(t) = AF_A(t) + f(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)ds \quad (t \in [0,T]).$$

Proof. Since F_A can be rewritten as

$$F_A(t) = \int_0^t S(s)f(t-s)ds \quad (t \in [0,T]),$$

by differentiating the integral we conclude that

$$F_A'(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)ds \qquad \forall t \in [0,T].$$

In view of Proposition 17, this implies that $F_A \in \mathcal{C}^1\big([0,T];X\big)$.

Moreover, returning to (3.3.1), for all $t \in [0,T]$ we also have that

$$\begin{aligned} F_A'(t) &= \lim_{h\downarrow 0} \frac{1}{h}\Big\{ \int_0^{t+h} S(t+h-s)f(s)\,ds - \int_0^t S(t-s)f(s)\,ds \Big\} \\ &= \lim_{h\downarrow 0} \Big\{ \frac{S(h)-I}{h} \int_0^t S(t-s)f(s)\,ds + \frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,ds \Big\}. \end{aligned}$$

Since

$$\lim_{h\downarrow 0} \frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,ds = f(t),$$

the above identity implies that $F_A(t) \in D(A)$ and

$$F_A(t) = F_A'(t) - f(t) \qquad \forall t \in [0,T].$$

Consequently, $F_A \in \mathcal{C}\big([0,T];D(A)\big)$ and the proof is complete. □

Proof of Theorem 14. Let u be the strong solution of problem (3.0.1). Then

$$u(t) = S(t)x + F_A(t) \qquad \forall t \in [0,T],$$

where F_A is defined in (3.3.1). The conclusion follows from Theorem 3 and Lemma 2. □

Example 18 In general, the strong solution of (3.0.1) fails to be classical, or even strict, assuming just $f \in \mathcal{C}([0,T];X)$. Indeed, let $y \in X \setminus D(A)$ and take $f(t) = S(t)y$. Then the strong solution of (3.0.1) with $x = 0$ is given by

$$u(t) = tS(t)y \qquad \forall t \geqslant 0$$

which fails to be differentiable for $t > 0$.

3.4 Maximal regularity for dissipative operators

For special classes of generators the strong solution of (3.0.1) enjoys additional regularity properties, as we show in this section.

Theorem 15 *Let $A : D(A) \subset X \to X$ be a densely defined self-adjoint dissipative operator and let $f \in L^2(0,T;X)$. Define*

$$F_A(t) = \int_a^t S(t-s)f(s)\,ds \quad (t \in [0,T]).$$

Then F_A is the strict solution of the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t) \\ u(0) = 0. \end{cases} \tag{3.4.1}$$

Moreover, $t \mapsto \langle AF_A(t), F_A(t)\rangle$ is absolutely continuous on $[0,T]$,

$$\frac{d}{dt}\langle AF_A(t), F_A(t)\rangle = 2\Re\langle F_A'(t), AF_A(t)\rangle \qquad (a.e.\ t \in [0,T]), \tag{3.4.2}$$

and

$$\|AF_A\|_2 \leqslant \|f\|_2. \tag{3.4.3}$$

Lemma 3 *Let $A : D(A) \subset X \to X$ be a densely defined self-adjoint dissipative operator and let $v \in H^1(0,T;X) \cap L^2(0,T;D(A))$ be such that $v(0) = 0$. Then $t \mapsto \langle Av(t), v(t)\rangle$ is absolutely continuous on $[0,T]$ and*

$$\frac{d}{dt}\langle Av(t), v(t)\rangle = 2\Re\langle v'(t), Av(t)\rangle \qquad (a.e.\ t \in [0,T]). \tag{3.4.4}$$

Proof. Define $v_n(t) = \langle A_n v(t), v(t)\rangle$ $(t \in [0,T])$, where $A_n = nAR(n,A)$ is the Yosida approximation of A. Then v_n is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\langle A_n v(t), v(t)\rangle = 2\Re\langle v'(t), A_n v(t)\rangle \qquad (\text{a.e. } t \in [0,T])$$

or

$$\langle A_n v(t), v(t)\rangle = 2\Re\int_0^t \langle v'(s), A_n v(s)\rangle ds \qquad \forall t \in [0,T]. \tag{3.4.5}$$

Now, since for a.e. $t \in [0,T]$

$$\begin{array}{rcl} A_n v(t) & = & nR(n,A)Av(t) \stackrel{(n\to\infty)}{\longrightarrow} Av(t) \\ |A_n v(t)| & \leqslant & |Av(t)|, \end{array}$$

we can pass to the limit as $n \to \infty$ in (3.4.5) to obtain

$$\langle Av(t), v(t)\rangle = 2\Re\int_0^t \langle v'(s), Av(s)\rangle ds \qquad \forall t \in [0,T].$$

So, $t \mapsto \langle Av(t), v(t)\rangle$ is absolutely continuous on $[0,T]$ and satisfies (3.4.4). $\square$

Proof of Theorem 15. Define

$$f_n(t) = nR(n,A)f(t) \quad \text{and} \quad F_n(t) = nR(n,A)F_A(t) \qquad \forall t \in [0,T]$$

and observe that

$$F_n(t) = \int_a^t S(t-s)f_n(s)\,ds \quad (t \in [0,T]).$$

Owing to Theorem 13, we have that $F_n \in H^1(0,T;X)\cap L^2(0,T;D(A))$ satisfies $F_n(0)=0$ and

$$F_n'(t) = AF_n(t) + f_n(t) \qquad (\text{a.e. } t \in [0,T]). \tag{3.4.6}$$

Moreover, by (3.4.2) we have that

$$2\int_0^t \Re\langle F_n'(s), AF_n(s)\rangle ds = \langle AF_n(t), F_n(t)\rangle \leqslant 0 \qquad \forall t \in [0,T]$$

because A is dissipative. Therefore, by multiplying each member of (3.4.6) by $2AF_n(t)$, taking real parts, and integrating over $[0,T]$ we obtain

$$\begin{aligned} 2\int_0^T |AF_n(t)|^2 dt &\leqslant -2\int_0^T \Re\langle f_n(t), AF_n(t)\rangle dt \\ &\leqslant \int_0^T \Big(|f_n(t)|^2 + |AF_n(t)|^2\Big)dt. \end{aligned}$$

Hence

$$\int_0^T |AF_n(t)|^2 dt \leqslant \int_0^T |f_n(t)|^2 dt \leqslant \int_0^T |f(t)|^2 dt.$$

Thus, $\{F_n\}_n$ is bounded in $H^1(0,T;X)\cap L^2(0,T;D(A))$. Therefore, there exists a subsequence $\{F_{n_k}\}_k$ and a function F_∞ such that

$$F_{n_k} \overset{(n\to\infty)}{\rightharpoonup} F_\infty \ \text{ in } \ H^1(0,T;X)\cap L^2(0,T;D(A)).$$

Recalling that $F_{n_k} \overset{(n\to\infty)}{\longrightarrow} F$ in $\mathcal{C}\big([0,T];X\big)$ by Theorem 12, we conclude that $F \in H^1(0,T;X)\cap L^2(0,T;D(A))$.

Now, fix any $g \in L^2(0,T;X)$. Then, taking the product of each member of (3.4.6) with g we have that

$$\int_0^T \langle F_n'(t), g(t)\rangle\,dt = \int_0^T \langle AF_n(t) + f_n(t), g(t)\rangle\,dt.$$

So, in the limit as $n \to \infty$,

$$\int_0^T \langle F'(t) - AF(t) - f(t), g(t)\rangle\,dt = 0 \qquad \forall g \in L^2(0,T;X)$$

which in turn yields $F'(t) = AF(t) + f(t)$ for a.e. $t \in [0, T]$. $\square$

Since the strong solution of (3.0.1) is given by (3.2.1), by Theorem 15 we obtain the following.

Corollary 5 *Let $A : D(A) \subset X \to X$ be a densely defined self-adjoint dissipative operator and let $x \in D(A)$. Then, for any $f \in L^2(0, T; X)$ the strong solution of* (3.0.1) *is strict.*

Remark 9 The above result ca be refined by introducing an intermediate subspace between X and $D(A)$, namely the interpolation space $[X, D(A)]_{1/2}$, which is such that $t \mapsto S(t)x$ belongs to $H^1(0, T; X) \cap L^2(0, T; D(A))$ for any $x \in [X, D(A)]_{1/2}$. The reader is referred to [1] for such an extension.

4 Appendix A: Riemann integral on $\mathcal{C}([a,b];X)$

We recall the construction of the Riemann integral for a continuous function $f:[a,b]\to X$, where X is a Banach space and $-\infty<a<b<\infty$.

Let us consider the family of partitions of $[a,b]$

$$\Pi(a,b)=\Big\{\pi=\{t_i\}_{i=0}^n\ :\ n\geqslant 1,\ a=t_0<t_1<\cdots<t_n=b\Big\}$$

and define

$$diam(\pi)=\max_{1\leqslant i\leqslant n}(t_1-t_{i-1})\qquad(\pi\in\Pi(a,b)).$$

For any $\pi\in\Pi(a,b),\pi=\{t_i\}_{i=0}^n$, we set

$$\Sigma(\pi)=\Big\{\sigma=(s_1,\dots,s_n)\ :\ s_i\in[t_{i-1},t_i],\ 1\leqslant i\leqslant n=b\Big\}.$$

Finally, for any $\pi\in\Pi(a,b),\pi=\{t_i\}_{i=0}^n$, and $\sigma\in\Sigma(\pi),\sigma=(s_1,\dots,s_n)$, we define

$$S_\pi^\sigma(f)=\sum_{i=1}^n f(s_i)(t_1-t_{i-1}).$$

Theorem 16 *The limit*

$$\lim_{diam(\pi)\downarrow 0}S_\pi^\sigma(f)=:\int_a^b f(t)dt$$

exists uniformly for $\sigma\in\Sigma(\pi)$.

Lemma 4 *For any $\varepsilon>0$ there exists $\delta>0$ such that for all $\pi,\pi'\in\Pi(a,b)$ with $\pi\subseteq\pi'$ we have that*

$$diam(\pi)<\delta\quad\Longrightarrow\quad\big|S_\pi^\sigma(f)-S_{\pi'}^{\sigma'}(f)\big|<\varepsilon$$

for all $\sigma\in\Sigma(\pi)$ and $\sigma'\in\Sigma(\pi')$.

Proof. Since f is uniformly continuous, for any $\varepsilon>0$ there exists $\delta>0$ such that for all $t,s\in[a,b]$

$$|t-s|<\delta\quad\Longrightarrow\quad|f(t)-f(s)|<\frac{\varepsilon}{b-a}\,. \tag{4.0.1}$$

Let

$$\begin{cases}\pi=\{t_i\}_{i=0}^n\,, & \sigma=(s_1,\dots,s_n)\\ \pi'=\{t'_j\}_{j=0}^m\,, & \sigma'=(s'_1,\dots,s'_m)\end{cases}$$

be such that $\pi\subseteq\pi'$ and $diam(\pi)<\delta$. Then there exist positive integers

$$0=j_0<j_1<\cdots<j_n=m$$

such that $t'_{j_i} = t_i$ for all $i = 0, \dots, n$. For any such i, it holds that

$$t_1 - t_{i-1} = t'_{j_i} - t'_{j_{i-1}} = \sum_{j=j_{i-1}+1}^{j_i} (t'_j - t'_{j-i}).$$

Then

$$\begin{aligned} S_\pi^\sigma(f) - S_{\pi'}^{\sigma'}(f) &= \sum_{i=1}^{n} f(s_i)(t_1 - t_{i-1}) - \sum_{j=1}^{m} f(s'_j)(t'_j - t'_{j-1}) \\ &= \sum_{i=1}^{n} \sum_{j=j_{i-1}+1}^{j_i} \big(f(s_i) - f(s'_j)\big)(t'_j - t'_{j-1}) \end{aligned}$$

Since for all $i = 1, \dots, n$ we have that

$$s_i,\, s'_j \in [t_{i-1}, t_i] \qquad \forall j_{i-1} + 1 \leqslant j \leqslant j_i,$$

from (4.0.1) it follows that

$$\begin{aligned} \big|S_\pi^\sigma(f) - S_{\pi'}^{\sigma'}(f)\big| &\leqslant \sum_{i=1}^{n} \sum_{j=j_{i-1}+1}^{j_i} \big|f(s_i) - f(s'_j)\big|(t'_j - t'_{j-1}) \\ &\leqslant \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (t_1 - t_{i-1}) = \varepsilon. \end{aligned}$$

The proof is complete. $\square$

Proof of Theorem 16. For any given $\varepsilon > 0$ let δ be as in Lemma 4. Let $\pi, \pi' \in \Pi(a, b)$ be such that $diam(\pi) < \delta$ and $diam(\pi') < \delta$. Finally, let $\sigma \in \Sigma(\pi)$ and $\sigma' \in \Sigma(\pi')$. Define $\pi'' = \pi \cup \pi'$ and fix any $\sigma'' \in \Sigma(\pi'')$. Then

$$\big|S_\pi^\sigma(f) - S_{\pi'}^{\sigma'}(f)\big| \leqslant \big|S_\pi^\sigma(f) - S_{\pi''}^{\sigma''}(f)\big| + \big|S_{\pi''}^{\sigma''}(f) - S_{\pi'}^{\sigma'}(f)\big| < 2\varepsilon.$$

This completes the proof since ε is arbitrary. $\square$

Proposition 13 *For any $f, g \in \mathcal{C}([a, b]; X)$ and $\lambda \in \mathbb{C}$ we have that*

$$\begin{aligned} \int_a^b \big(f(t) + g(t)\big)dt &= \int_a^b f(t)dt + \int_a^b g(t)dt \\ \int_a^b \lambda f(t)dt &= \lambda \int_a^b f(t)dt \\ \Big|\int_a^b f(t)dt\Big| &\leqslant \int_a^b |f(t)|dt. \end{aligned}$$

Moreover, for any $\Lambda \in \mathcal{L}(X)$ *we have that*

$$\Lambda\Big(\int_a^b f(t)dt\Big) = \int_a^b \Lambda f(t)dt.$$

Furthermore, if $A : D(A) \subset X \to X$ *is a closed operator and* $f \in \mathcal{C}([a,b]; D(A))$*, then*

$$A\Big(\int_a^b f(t)dt\Big) = \int_a^b Af(t)dt.$$

5 Appendix B: Lebesgue integral on $L^2(a, b; H)$

Let $\big(H, \langle\cdot,\cdot\rangle\big)$ be a separable Hilbert space and let $\{e_j\}_{j\in\mathbb{N}}$ be a complete orthonormal system in H.

5.1 The Hilbert space $L^2(a, b; H)$

Definition 18 *A function $f : [a, b] \to H$ is said to be Borel (resp. Lebesgue) measurable if so is the scalar function $t \mapsto \langle f(t), x\rangle$ for every $x \in H$.*

Remark 10 Let $f : [a, b] \to H$.

1. Since, for any $x \in H$,

$$\langle f(t), x\rangle = \sum_{j=1}^{\infty} \langle f(t), e_j\rangle\, \overline{\langle x, e_j\rangle} \qquad (t \in [a, b]),$$

   we conclude that f is Borel (resp. Lebesgue) measurable if and only if so is the scalar function $t \mapsto \langle f(t), e_j\rangle$ for every $j \in \mathbb{N}$.

2. Since

$$|f(t)|^2 = \sum_{j=1}^{\infty} \big|\langle f(t), e_j\rangle\big|^2 \qquad (t \in [a, b]),$$

   we have that, if f is Borel (resp. Lebesgue) measurable, then so is the scalar function $t \mapsto \|f(t)\|$.

Definition 19 *We denote by $L^2(a, b; H)$ the space of all Lebesgue measurable functions $f : [a, b] \to H$ such that*

$$\|f\|_2 := \Big(\int_a^b |f(t)|^2 dt\Big)^{\frac{1}{2}} < \infty\,,$$

where two functions f and g are identified if $f(t) = g(t)$ for a.e. $t \in [a, b]$.

Proposition 14 *$L^2(a, b; H)$ is a Hilbert space with the hermitian product*

$$(f|g)_0 = \int_a^b \langle f(t), g(t)\rangle dt \qquad (f, g \in L^2(a, b; H)).$$

Proof. We only prove completeness. □

Remark 11 For any $f \in L^2(a, b; H)$ we have that

$$\sum_{j=1}^{\infty} \Big|\int_a^b \langle f(t), e_j\rangle dt\Big|^2 \leqslant (b-a) \sum_{j=1}^{\infty} \int_a^b \big|\langle f(t), e_j\rangle\big|^2 dt < \infty.$$

Therefore

$$\sum_{j=1}^{\infty} e_j \int_a^b \langle f(t), e_j \rangle dt \in H.$$

Definition 20 *For any $f \in L^2(a,b;H)$ we define*

$$\int_a^b f(t)dt = \sum_{j=1}^{\infty} e_j \int_a^b \langle f(t), e_j \rangle dt\,.$$

Proposition 15 *For any $f \in L^2(a,b;H)$ the following properties hold true.*

(a) For any $x \in H$ we have that

$$\Big\langle x, \int_a^b f(t)dt \Big\rangle = \int_a^b \langle x, f(t) \rangle dt$$

(b)
$$\Big| \int_a^b f(t)dt \Big| \leqslant \int_a^b |f(t)| dt$$

(c) For any $\Lambda \in \mathcal{L}(H)$ we have that

$$\Lambda\Big(\int_a^b f(t)dt \Big) = \int_a^b \Lambda f(t)dt\,.$$

Proposition 16 *Let $A : D(A) \subset H \to H$ be a closed linear operator. Then for any $f \in L^2(a,b;D(A))$ we have that*

$$\int_a^b f(t)dt \in D(A) \quad \text{and} \quad A\Big(\int_a^b f(t)dt \Big) = \int_a^b Af(t)dt\,.$$

Proposition 17 *Let $A : D(A) \subset H \to H$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup on H, $S(t)$, which satisfies the growth condition* (1.8.3). *Then, for any $f \in L^2(a,b;H)$,*

(a) for any $t \in [a,b]$ the function $s \mapsto S(t-s)f(s)$ belongs to $L^2(a,t;H)$, and

(b) the function

$$F_A(t) = \int_a^t S(t-s)f(s)\,ds \qquad (t \in [a,b])$$

belongs to $\mathcal{C}\big([a,b];H\big)$.

Proof. In order to check measurability for $s \mapsto S(t-s)f(s)$ it suffices to observe that, for all $x \in H$ and a.e. $s \in [0,t]$,

$$\langle S(t-s)f(s), x\rangle = \langle f(s), S(t-s)^* x\rangle = \sum_{j=1}^{\infty} \langle f(s), e_j\rangle \, \overline{\langle S(t-s)^* x, e_j\rangle}.$$

Since $s \mapsto \langle S(t-s)^* x, e_j\rangle$ is continuous and $s \mapsto \langle f(s), e_j\rangle$ is measurable for all $j \in \mathbb{N}$, the measurability of $s \mapsto S(t-s)f(s)$ follows. Moreover, by (1.8.3) we have that

$$|S(t-s)f(s)| \leqslant M e^{\omega(t-s)} |f(s) \qquad (s \in [a,t] \text{ a.e.}),$$

which completes the proof of *(a)*.

In order to prove point *(b)*, fix $t \in]a,b[$ and let $t_n \to t$. Fix $\delta \in]0, t-a[$ and let $n_\delta \in \mathbb{N}$ be such that $t_n > t - \delta$ for all $n \geqslant n_\delta$. Then we have that

$$\begin{aligned} &\big|F_A(t_n) - F_A(t)\big| \\ &\leqslant \int_a^{t-\delta} \big|\big[S(t_n - s)f(s) - S(t-s)\big]f(s)\big|\, ds \\ &\quad + \int_{t-\delta}^{t_n} \big|S(t_n - s)f(s)\big|\, ds + \int_{t-\delta}^{t} \big|S(t-s)f(s)\big|\, ds. \end{aligned}$$

To complete the proof it suffices to observe that

$$\lim_{n\to\infty} \int_a^{t-\delta} \big|\big[S(t_n - s)f(s) - S(t-s)\big]f(s)\big|\, ds = 0$$

by the dominated convergence theorem, while the remaining terms on the right-hand side of the above inequality are small with δ. □

5.2 The Sobolev space $H^1(a,b;H)$

Definition 21 *We define $H^1(a,b;H)$ to be the subspace of $L^2(a,b;H)$ which consists of all (equivalence classes of) functions $u \in L^2(a,b;H)$ such that*

$$u(t) - u(a) = \int_a^t f(s)ds \qquad t \in [a,b] \text{ a.e.} \tag{5.2.1}$$

for some $f \in L^2(a,b;H)$.

Remark 12 The proof of the following facts is left to the reader.

1. The function f in (5.2.1) is uniquely determined up to sets of measure zero. We call such a function the *weak derivative* of u and set $u' = f$.

2. $H^1(a,b;H)$ is a Hilbert space with the scalar product

$$(u|v)_1 = \int_a^b \left[\langle u(t), v(t)\rangle + \langle u'(t), v'(t)\rangle\right] dt \qquad (u, v \in H^1(a,b;H)).$$

3. All the elements of $H^1(a,b;H)$ have an absolutely continuous representative.

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