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1 Generation of \( C_0 \)-semigroups

1.1 \( C_0 \)-semigroups

Exponential of a bounded operator

Let \((X, |·|)\) be a real or complex Banach space and denote by \(\mathcal{L}(X)\) the Banach algebra of all bounded linear operators \(\Lambda : X \to X\) equipped with the norm

\[
\|\Lambda\| = \sup_{|x| \leq 1} |\Lambda x|.
\]

Let \(T > 0\). For any \(A \in \mathcal{L}(X)\) the Cauchy problem

\[
\begin{aligned}
  y'(t) &= Ay(t) \quad (t \in [0, T]) \\
  y(0) &= x \in X
\end{aligned}
\]

(1.1.1)

can easily be solved by a well-known iteration method. Let us set

\[
y_0(t) = x, \quad y_{n+1}(t) = x + \int_0^t A y_n(s) ds \quad (t \in [0, T]),
\]

where the above integral is understood in the Riemann sense. Then the solution of (1.1.1) is given by

\[
y(t) = \lim_{n \to \infty} y_n(t) = e^{tA}x = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x,
\]

where the series converges uniformly in \(\mathcal{L}(X)\).

Motivated by applications to partial differential equations and other kinds of functional equations, we will extend the theory to problems associated with an unbounded linear operator \(A : D(A) \subset X \to X\).

\( C_0 \)-semigroups

**Definition 1** A \( C_0 \)-semigroup of bounded linear operators on \(X\) is map \(S : [0, \infty) \to \mathcal{L}(X)\) with the following properties:

(a) \(S(0) = I\) and \(S(t+s) = S(t)S(s)\) for all \(t, s \geq 0\), and

(b) for every \(x \in X\) the map \(t \mapsto S(t)x\) is continuous from \([0, \infty)\) to \(X\).

Equivalent notations for the semigroup \(S\) are \(S(\cdot), \{S(t)\}_{t \geq 0}\), and even the simpler form \(S(t)\).
Example 1 For any $A \in \mathcal{L}(X)$ the exponential $S(t) = e^{tA}$ is a $C_0$-semigroup of bounded linear operators on $X$. Moreover, 

(b′) the map $S : [0, \infty) \to \mathcal{L}(X)$ is continuous.

Notice that (b′) is stronger than (b). Moreover, it is known (see, for instance, [4, Theorem I.3.7]) that if $S(\cdot)$ satisfies (a) and (b′), then there exists $A \in \mathcal{L}(X)$ such that $S(t) = e^{tA}$.

Example 2 For a fixed $p \geq 1$ let $X = L^p(\mathbb{R})$ and define, $\forall f \in X$,

$$
(S(t)f)(x) = f(x + t) \quad \forall x \in \mathbb{R}, \forall t \geq 0. 
$$

Then $S$ is $C_0$-semigroup which fails to satisfy (b′). Indeed, suppose $S$ has property (b′) and let $\tau > 0$ be such that $\|S(t) - I\| < 1/2$ for all $t \in [0, \tau]$. Then by taking $f_n(x) = n^{1/p} \chi_{[0,1/n]}(x)$ for $p < \infty$ and $n > 1/\tau$ we have that $|f_n| = 1$ and

$$
|S(\tau)f_n - f_n| = \left( \int_{\mathbb{R}} n^{1/p} \chi_{[0,1/n]}(x + \tau) - \chi_{[0,1/n]}(x) |dx| \right)^{\frac{1}{p}} = 2^{1/p}.
$$

Observe that (1.1.2) makes sense for $t < 0$ as well. In this case we say that $S$ is a $C_0$ group of bounded linear operators on $X$. On the other hand, if one takes $X = L^p(\mathbb{R}_+)$, then (1.1.2) makes sense only for $t \geq 0$.

1.2 The infinitesimal generator of a $C_0$-semigroup

Let $S$ be a $C_0$-semigroup of bounded linear operators on $X$. We are interested in studying the limit

$$
\lim_{h \downarrow 0} \frac{S(h)x - x}{h}
$$

as a function of $x \in X$.

Exercise 1 Show that if $A \in \mathcal{L}(X)$ then

$$
\lim_{h \downarrow 0} \frac{e^{hA}x - x}{h} = Ax \quad \forall x \in X.
$$

Definition 2 The linear operator $A : D(A) \subset X \to X$ defined by

$$
\begin{cases}
D(A) = \{ x \in X : \exists \lim_{h \downarrow 0} \frac{S(h)x - x}{h} \} \\
Ax = \lim_{h \downarrow 0} \frac{S(h)x - x}{h} \quad \forall x \in D(A)
\end{cases}
$$

is called the infinitesimal generator of $S$.

Exercise 2 Check that (1.2.2) defines a linear operator.
Proposition 1 \( D(A) \) is dense in \( X \).

Proof. For any \( x \in X \) define
\[
M_{t,h}x = \frac{1}{h} \int_t^{t+h} S(s)x \, ds \quad (t \geq 0, \; h > 0).
\] (1.2.3)

Then, by continuity, \( \lim_{h \downarrow 0} M_{t,h}x = S(t)x \). Moreover, for any \( t, h > 0 \),
\[
\frac{S(h) - I}{h} M_{0,t}x = \frac{1}{ht} \int_0^t (S(h + s) - S(s))x \, ds
\]
\[
= \frac{1}{ht} \left\{ \int_0^{t+h} S(s)x \, ds - \int_0^t S(s)x \, ds \right\}
\]
\[
= \frac{1}{ht} \left\{ \int_t^{t+h} S(s)x \, ds - \int_0^h S(s)x \, ds \right\} = \frac{1}{t} \{ M_{t,h}x - M_{0,h}x \}.
\]

Therefore
\[
\lim_{h \downarrow 0} \frac{S(h) - I}{h} M_{0,t}x = \frac{S(t)x - x}{t} \quad \forall x \in X.
\]
This yields \( M_{0,t}x \in D(A) \). Since \( \lim_{t \to 0} M_{0,t}x = x \), \( D(A) \) is dense in \( X \). \( \blacksquare \)

Lemma 1 For all \( x \in D(A) \) we have that \( S(t)x \in D(A) \) for every \( t \geq 0 \) and
\[
AS(t)x = S(t)Ax.
\] (1.2.4)

Proof. For all \( h > 0 \) we have that
\[
\frac{S(h) - I}{h} S(t)x = S(t) \frac{S(h) - I}{h} x \to S(t)Ax \quad \text{as} \quad h \downarrow 0.
\]

Therefore \( S(t)x \in D(A) \) and (1.2.4) holds true. \( \blacksquare \)

Remark 1 Fix any \( T \geq 0 \) and observe that for all \( x \in X \) there exists a constant \( N_{T,x} > 0 \) such that \( |S(t)x| \leq N_{T,x} \) for all \( t \in [0,T] \). Then the Uniform Boundedness Principle ensures that, for some constant \( N_T > 0 \),
\[
\|S(t)\| \leq N_T \quad \forall t \in [0,T].
\] (1.2.5)

The following theorem provides a solution to problem (1.1.1) for \( x \in D(A) \).

Theorem 1 For all \( x \in D(A) \) we have that \( t \mapsto S(t)x \) is differentiable for every \( t \geq 0 \) and
\[
\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax.
\]
1.2. THE INFINITESIMAL GENERATOR OF A $C_0$-SEMIGROUP

Proof. Fix any $t \geq 0$. Then Lemma 1 ensures that

\[
\frac{S(t + h)x - S(t)x}{h} = \frac{S(h) - I}{h} \quad \text{as} \quad h \downarrow 0.
\]

Hence, $S(t)x$ has the right derivative at $t$.

In order to complete the proof, let $t > 0$. Then for all $0 < h < t$ we have that

\[
\frac{S(t - h)x - S(t)x}{-h} = S(t - h) \frac{S(h) - I}{h} x.
\]

On the other hand, by (1.2.5),

\[
\left| S(t - h) \frac{S(h) - I}{h} x - S(t)Ax \right| \\
\leq \left| S(t - h) \right| \cdot \left| \frac{S(h) - I}{h} x - S(h)Ax \right| \\
\leq N \left| \frac{S(h) - I}{h} x - S(h)Ax \right| \rightarrow 0 \quad \text{as} \quad h \downarrow 0.
\]

Therefore

\[
\frac{S(t - h)x - S(t)x}{-h} \rightarrow S(t)Ax = AS(t)x \quad \text{as} \quad h \downarrow 0,
\]

showing that the left and right derivatives coincide. \qed

Definition 3 An operator $A : D(A) \subset X \rightarrow X$ is said to be closed if its graph

\[
\text{Graph}(A) = \{ (x, y) : x \in D(A), \ y = Ax \}
\]

is a closed subset of the product space $X \times X$.

Exercise 3 Prove that $A : D(A) \subset X \rightarrow X$ is closed if and only if for any sequence $\{x_n\} \subset D(A)$

\[
\begin{cases}
  x_n \rightarrow x \\
  Ax_n \rightarrow y
\end{cases}
\quad \Rightarrow \quad x \in D(A) \quad \text{and} \quad Ax = y. \quad (1.2.6)
\]

Proposition 2 The infinitesimal generator of a $C_0$-semigroup is a closed operator.

Proof. Let $A$ be the infinitesimal generator of $S$ and let $\{x_n\} \subset D(A)$ be as in (1.2.6). By Theorem 1 we have that, for all $t \geq 0$

\[
S(t)x_n - x_n = \int_0^t S(s)Ax_n dx.
\]

Hence, taking the limit as $n \rightarrow \infty$ and dividing by $t$, we obtain

\[
\frac{S(t)x - x}{t} = \frac{1}{t} \int_0^t S(s)y dx.
\]

Passing to the limit as $t \downarrow 0$, we conclude that $Ax = y$. \qed
Indeed, by H"older's inequality, we have that, for all
\[ |f|_{1,p} := |f|_p + |f'|_p < \infty. \tag{1.2.7} \]

We will show that \( A \) equals the unbounded operator \( B : D(B) \subset X \to X \) defined by
\[
\begin{align*}
\begin{cases}
D(B) = W^{1,p}(\mathbb{R}) \\
Bf(x) = f'(x) \quad (x \in \mathbb{R} \text{ a.e.}) \quad \forall f \in D(B).
\end{cases}
\end{align*}
\tag{1.2.8}
\]

First, we claim that \( A \) is an extension of \( B \) (in formulas, \( B \subset A \)), that is
\[ D(B) \subset D(A) \quad \& \quad Af = Bf \quad \forall f \in D(B). \]

Indeed, by Hölder’s inequality, we have that, for all \( f \in W^{1,p}(\mathbb{R}) \) and all \( t > 0 \),
\[
\int_{-\infty}^{+\infty} \left| \frac{f(x + t) - f(x)}{t} - f'(x) \right|^p dx \\
= \int_{-\infty}^{+\infty} \frac{1}{t} \int_0^t (f'(x + s) - f'(x)) ds \bigg| dx
\leq \frac{1}{t} \int_{-\infty}^{+\infty} dx \int_0^t \left| f'(x + s) - f'(x) \right|^p ds
\leq \frac{1}{t} \int_0^t ds \int_{-\infty}^{+\infty} \left| f'(x + s) - f'(x) \right|^p dx.
\]

Now, owing to the translation continuity of the integral the last integral can be made arbitrarily small by taking \( t > 0 \) small enough. So, \( B \subset A \).

In order to conclude that \( A = B \) it suffices to prove that \( D(A) \subset D(B) \).

For this purpose, for any fixed \( f \in D(A) \) and any \( \varepsilon > 0 \) let \( f_\varepsilon = f \ast \rho_\varepsilon \), where \( \{\rho_\varepsilon\}_{\varepsilon > 0} \) is a \( C^\infty \) approximate unity with support in \( [-\varepsilon, \varepsilon] \). Then \( f_\varepsilon \in D(B) \) and \( f_\varepsilon \to f \) as \( \varepsilon \to 0 \). Since \( B \) is a closed operator, if we show that \( Bf_\varepsilon \to Af \) then \( f \in D(B) \) and \( A = B \). To check that \( Bf_\varepsilon \to Af \) observe that
\[
\int_{-\infty}^{+\infty} |f'_\varepsilon(x) - Af(x)|^p dx \\
\leq \liminf_{t \downarrow 0} \int_{-\infty}^{+\infty} \left| \frac{f_\varepsilon(x + t) - f_\varepsilon(x)}{t} - Af(x) \right|^p dx
\leq \liminf_{t \downarrow 0} \int_{-\infty}^{+\infty} \int_{-\varepsilon}^{\varepsilon} \left| \frac{f(x - y + t) - f(x - y)}{t} - Af(x) \right|^p \rho_\varepsilon(y) dy dx
\leq \liminf_{t \downarrow 0} \int_{-\infty}^{+\infty} \int_{-\varepsilon}^{\varepsilon} \left| \frac{f(x - y + t) - f(x - y)}{t} - Af(x) \right|^p \rho_\varepsilon(y) dy dx
\]
1.3. ASYMPTOTIC PROPERTIES OF $C_0$-SEMIGROUPS

Now, since

\[
\int_{-\infty}^{+\infty} \int_{-\varepsilon}^{\varepsilon} \left| \frac{f(x-y+t) - f(x-y)}{t} - Af(x) \right|^p \rho_\varepsilon(y) \, dy \, dx \\
\leq 2^{p-1} \left\{ \int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon(y) \, dy \int_{-\infty}^{+\infty} \left| \frac{f(x-y+t) - f(x-y)}{t} - Af(x-y) \right|^p \, dx \\
+ \int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon(y) \, dy \int_{-\infty}^{+\infty} \left| Af(x-y) - Af(x) \right|^p \, dx \right\},
\]

the conclusion follows recalling the translation continuity of the integral and the fact that

\[
\int_{-\infty}^{+\infty} \left| \frac{f(x+t) - f(x)}{t} - Af(x) \right|^p \, dx \quad (t \to 0).
\]

Exercise 4 Show that $W^{1,p}(\mathbb{R})$ is a Banach space with the above norm and that operator $B : D(B) \subset X \to X$ defined in (1.2.8) is closed.

1.3. Asymptotic properties of $C_0$-semigroups

Let $S$ be a $C_0$-semigroup of bounded linear operators on $X$.

**Definition 4** The number

\[
\omega_0(S) = \inf_{t \geq 0} \frac{\log \| S(t) \|}{t}
\]

(1.3.1)

is called the type or growth bound of $S$.

**Proposition 3** The growth bound of $S$ satisfies

\[
\omega_0(S) = \lim_{t \to \infty} \frac{\log \| S(t) \|}{t} < +\infty.
\]

(1.3.2)

Moreover, for any $\varepsilon > 0$ there exists $M_\varepsilon \geq 1$ such that

\[
\| S(t) \| \leq M_\varepsilon e^{(\omega_0(S) + \varepsilon)t} \quad \forall t \geq 0.
\]

(1.3.3)

**Proof.** The fact that $\omega_0(S) < +\infty$ is a direct consequence of (1.3.1). In order to prove (1.3.2) it suffices to show that

\[
\limsup_{t \to \infty} \frac{\log \| S(t) \|}{t} \leq \omega_0(S).
\]

(1.3.4)

For any $\varepsilon > 0$ let $t_\varepsilon > 0$ be such that

\[
\frac{\log \| S(t_\varepsilon) \|}{t_\varepsilon} < \omega_0(S) + \varepsilon.
\]

(1.3.5)
Let us write any \( t \geq t_\varepsilon \) as \( t = n(t)\varepsilon + r(t) \) with \( n(t) \in \mathbb{N} \) and \( r(t) \in [0, t_\varepsilon) \).

Then, by (1.2.5) and (1.3.5),

\[
\|S(t)\| \leq \|S(t_\varepsilon)\|^{n(t)}\|S(r(t))\| \leq e^{t_\varepsilon n(t)(\omega_0(S) + \varepsilon)} N_{t_\varepsilon} \leq N_{t_\varepsilon} e^{(\omega_0(S) + \varepsilon)t}
\]

which proves (1.3.3). Moreover, taking the logarithm of both sides of the above inequality we get

\[
\frac{\log \|S(t)\|}{t} \leq \omega_0(S) + \varepsilon + \frac{N_{t_\varepsilon}}{t}
\]

and (1.3.4) follows as \( t \to \infty \).

\[\square\]

**Example 4**

It is immediate to realize that the left-translation semigroup of Example 2 satisfies \( \|S(t)\| = 1 \) for all \( t \geq 0 \). So, \( \omega_0(S) = 0 \).

### 1.4 Spectral properties of generators

**Resolvent set and spectrum**

Let \( A : D(A) \subset X \to X \) be a closed operator on a complex Banach space \( X \).

**Definition 5** The resolvent set of \( A \), \( \rho(A) \), is the set of all \( \lambda \in \mathbb{C} \) such that \( \lambda I - A : D(A) \to X \) is bijective and its complement \( \sigma(A) = \mathbb{C} \setminus \rho(A) \) is called the spectrum of \( A \). For any \( \lambda \in \rho(A) \) the inverse \( R(\lambda, A) = (\lambda I - A)^{-1} \) is called the resolvent of \( A \) at \( \lambda \).

**Remark 2** Observe that, by the closed graph theorem, \( R(\lambda, A) \) is a bounded linear operator on \( X \). Also, the identity

\[
AR(\lambda, A) = \lambda R(\lambda, A) - I \quad \forall \lambda \in \rho(A)
\]

is easy to check. Moreover, the following *resolvent identity* holds:

\[
R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \quad \forall \lambda, \mu \in \rho(A).
\]

Indeed, by (1.4.1) we have that

\[
[\lambda R(\lambda, A) - AR(\lambda, A)]R(\mu, A) = R(\mu, A)
\]

and

\[
R(\lambda, A)[\mu R(\mu, A) - AR(\mu, A)] = R(\lambda, A).
\]

Since \( AR(\lambda, A) = R(\lambda, A)A \) on \( D(A) \), (1.4.2) follows.
Proposition 4 Let $A : D(A) \subset X \to X$ be a closed operator. Then $\rho(A)$ is open in $\mathbb{C}$ and for any $\mu \in \rho(A)$ the resolvent $R(\lambda, A)$ is given by the series

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1} \quad (1.4.3)$$

for all $\lambda \in \mathbb{C}$ satisfying $|\mu - \lambda| < 1/\|R(\mu, A)\|$. Consequently, $\lambda \mapsto R(\lambda, A)$ is analytic on $\rho(A)$ and for all $n \in \mathbb{N}$

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1}. \quad (1.4.4)$$

Proof. For all $\lambda \in \mathbb{C}$ we have that

$$\lambda I - A = \mu I - A + (\lambda - \mu)I = [I - (\mu - \lambda)R(\mu, A)](\mu I - A).$$

This operator is bijective if and only if $[I - (\mu - \lambda)R(\mu, A)]$ is invertible, which is the case for $|\mu - \lambda| < 1/\|R(\mu, A)\|$. In this case

$$R(\lambda, A) = R(\mu, A)[I - (\mu - \lambda)R(\mu, A)]^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1}.$$ 

The analyticity of $R(\lambda, A)$ and (1.4.4) follow from (1.4.3). □

Example 5 On $X = C([0, 1])$ with the uniform norm consider the closed operator $A : D(A) \subset X \to X$ defined by

$$\begin{align*}
D(A) &= C^1([0, 1]) \\
Af &= f', \quad \forall f \in D(A).
\end{align*}$$

Then $\sigma(A) = \mathbb{C}$ because for any $\lambda \in \mathbb{C}$ the function $f_\lambda(x) = e^{\lambda x}$ satisfies

$$\lambda f_\lambda(x) - f_\lambda'(x) = 0 \quad \forall x \in [0, 1].$$

On the other hand, for the closed operator $A_0$ defined by

$$\begin{align*}
D(A_0) &= \{ f \in C^1([0, 1]) : f(0) = 0 \} \\
A_0 f &= f', \quad \forall f \in D(A_0),
\end{align*}$$

we have that $\sigma(A_0) = \emptyset$. Indeed, for any $g \in X$ the problem

$$\begin{align*}
\lambda f(x) - f'(x) &= g(x) \quad x \in [0, 1] \\
f(0) &= 0
\end{align*}$$

admits the unique solution

$$f(x) = -\int_0^x e^{\lambda(x-s)} g(s) \, dx \quad (x \in [0, 1])$$

which belongs to $D(A_0)$.
Spectral properties of the infinitesimal generator

Let $M \geq 0$ and let $\omega \in \mathbb{R}$.

**Definition 6** We denote by $G(M, \omega)$ the class of all $C_0$-semigroups of bounded linear operators on $X$ such that

$$
\|S(t)\| \leq Me^{\omega t} \quad \forall t \geq 0.
$$

(1.4.5)

When $M = 1$ and $\omega = 0$ we say that $S(t)$ is a contraction semigroup.

**Proposition 5 (Integral representation)** Let $A : D(A) \subset X \to X$ be the infinitesimal generator of $S \in G(M, \omega)$. Then $\rho(A)$ contains the half-plane

$$
\Pi_\omega = \{ \lambda \in \mathbb{C} : \Re \lambda > \omega \}
$$

(1.4.6)

and

$$
R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t) x \, dt \quad \forall x \in X, \forall \lambda \in \Pi_\omega.
$$

(1.4.7)

**Proof.** We must prove that, given any $\lambda \in \Pi_\omega$ and $x \in X$, the equation

$$
\lambda u - Au = x
$$

(1.4.8)

has a unique solution given by (1.4.7).

**Existence:** observe that $u := \int_0^\infty e^{-\lambda t} S(t) x \, dt \in X$ because $\Re \lambda > \omega$. Moreover, for all $h > 0$,

$$
\frac{S(h)u - u}{h} = \frac{1}{h} \left\{ \int_0^\infty e^{-\lambda (t+h)} S(t) x \, dt - \int_0^\infty e^{-\lambda t} S(t) x \, dt \right\}
$$

$$
= \frac{1}{h} \left\{ e^{\lambda h} \int_0^\infty e^{-\lambda t} S(t) x \, dt - \int_0^\infty e^{-\lambda t} S(t) x \, dt \right\}
$$

$$
= \frac{e^{\lambda h} - 1}{h} u - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} S(t) x \, dt.
$$

So

$$
\lim_{h \to 0} \frac{S(h)u - u}{h} = \lambda u - x
$$

which in turn yields that $u \in D(A)$ and (1.4.8) holds true.

**Uniqueness:** let $u \in D(A)$ be a solution of (1.4.8). Then

$$
\int_0^\infty e^{-\lambda t} S(t)(\lambda u - Au) \, dt = \lambda \int_0^\infty e^{-\lambda t} S(t) u \, dt - \int_0^\infty e^{-\lambda t} \frac{d}{dt} S(t) u \, dt = u
$$

which implies that $u$ is given by (1.4.7).
1.4. SPECTRAL PROPERTIES OF GENERATORS

Definition 7 For any operator \( A : D(A) \subset X \to X \) we define the spectral bound of \( A \) as
\[
s(A) = \sup \{ \Re \lambda : \lambda \in \sigma(A) \}.
\]

Corollary 1 Let \( S \) be a \( C_0 \)-semigroup on \( X \) with infinitesimal generator \( A \). Then
\[
-\infty \leq s(A) \leq \omega_0(S) < +\infty.
\]

Proposition 6 Let \( A : D(A) \subset X \to X \) and \( B : D(B) \subset X \to X \) be closed linear operators in \( X \) and suppose \( B \subset A \), that is, \( D(B) \subset D(A) \) and \( Ax = Bx \) for all \( x \in D(B) \). If \( \rho(A) \cap \rho(B) \neq \emptyset \), then \( A = B \).

Proof. Let \( \lambda \in \rho(A) \cap \rho(B) \), let \( x \in D(A) \), and set
\[
y = \lambda x - Ax \quad \text{and} \quad z = R(\lambda, B)y.
\]
Then \( z \in D(B) \) and \( \lambda z - Bz = \lambda x - Ax \). Since \( B \subset A \), \( \lambda z - Bz = \lambda z - Az \).
Thus, \( (\lambda - A)(x - z) = 0 \). So, \( x = z \in D(B) \) and \( A = B \). \( \square \)

Example 6 (Right-translation semigroup) On the real Banach space
\[
X = \{ f \in BU C(\mathbb{R}_+) : f(0) = 0 \}
\]
of all bounded uniformly continuous functions \( f : \mathbb{R}_+ \to \mathbb{R} \) with the uniform norm, consider the right-translation semigroup
\[
(S(t)f)(x) = \begin{cases} f(x-t) & \text{if } x > t \\ 0 & \text{if } x \in [0,t] \end{cases} \quad \forall x, t \geq 0.
\]
It is easy to check that \( S \) is a \( C_0 \)-semigroup on \( X \) with \( \|S(t)\| = 1 \) for all \( t \geq 0 \). In order to characterize its infinitesimal generator \( A \), let us consider the operator \( B : D(B) \subset X \to X \) defined by
\[
\begin{align*}
D(B) &= \{ f \in X : f' \in X \} \\
Bf &= -f', \quad \forall f \in D(B).
\end{align*}
\]
We claim that:

(i) \( B \subset A \)

Proof. Let \( f \in D(B) \). Then, for all \( x, t \geq 0 \) we have
\[
\frac{(S(t)f)(x) - f(x)}{t} = \begin{cases} -\frac{f(x)}{t} = -f'(x_t), & 0 \leq x \leq t \\
\frac{f(x-t) - f(x)}{-t} = -f'(x_t) & x \geq t
\end{cases}
\]
with $0 \leq x - x_t \leq t$. Therefore
\[
\sup_{x \geq 0} \left| \frac{(S(t)f)(x) - f(x)}{t} + f'(x) \right| \leq \sup_{|x-y| \leq t} |f'(x) - f'(y)| \to 0 \quad \text{as} \quad t \downarrow 0
\]
because $f'$ is uniformly continuous. \qed

(ii) $1 \in \rho(B)$

Proof. For any $g \in X$ the unique solution $f$ of the problem
\[
\begin{cases}
    f \in D(B) \\
    f(x) + f'(x) = g(x) \quad \forall x \geq 0
\end{cases}
\]
is given by
\[
f(x) = \int_0^x e^{s-x} g(s) \, ds \quad (x \geq 0).
\]

Since $1 \in \rho(A)$ by Proposition 5, Proposition 6 yields that $A = B$.

**Proposition 7 (Laundau-Kolmogorov)** Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a contraction semigroup $S$. Then
\[
|Ax|^2 \leq 4 |x| |A^2 x| \quad \forall x \in D(A^2), \quad (1.4.9)
\]
where
\[
\begin{aligned}
D(A^2) &= \{ x \in D(A) : Ax \in D(A) \} \\
A^2 x &= A(Ax), \quad \forall x \in D(A^2).
\end{aligned}
\]

Proof. For any $x \in D(A^2)$ and all $t \geq 0$ we have that
\[
\int_0^t (t-s)S(s)A^2 x \, ds = \left[ (t-s)S(s)Ax \right]_{s=0}^{s=t} + \int_0^t S(s)Ax \, ds \\
&= -tAx + \left[ S(s)x \right]_{s=0}^{s=t} = -tAx + S(t)x - x.
\]
Therefore, for all $t > 0$,
\[
|Ax| \leq \frac{1}{t} |S(t)x - x| + \frac{1}{t} \int_0^t |t-s)|S(s)A^2 x| \, ds \\
&\leq \frac{2}{t} |x| + \frac{t}{2} |A^2 x|.
\]

If $A^2 x = 0$, then the above inequality yields $Ax = 0$ by letting $t \to \infty$. So, (1.4.9) is true in this case. On the other hand, for $A^2 x \neq 0$ the function of $t$ on the right-hand side of (1.4.11) attains its minimum at
\[
t_0 = \frac{2|x|^{1/2}}{|A^2 x|^{1/2}}.
\]
By taking $t = t_0$ in (1.4.11) we obtain (1.4.9) once again. \qed
Example 7 Let us recall that the infinitesimal generator $A : D(A) \subset X \to X$ of the left-translation semigroup $S$ on $X = L^p(\mathbb{R})$ introduced in Example 2 is given by
\[
\begin{cases}
D(A) = W^{1,p}(\mathbb{R}) \\
Af(x) = f'(x) \quad (x \in \mathbb{R} \text{ a.e.}) \quad \forall f \in D(A).
\end{cases}
\]
Since
\[
D(A^2) = \{ f \in W^{1,p}(\mathbb{R}) : f' \in W^{1,p}(\mathbb{R}) \} =: W^{2,p}(\mathbb{R}),
\]
by Proposition 7 we deduce the interpolation inequality
\[
|f'|_p \leq 2 \left( \frac{1}{|f|_p} |f''|_p \right) \quad \forall f \in W^{2,p}(\mathbb{R}).
\]

1.5 The Hille-Yosida generation theorem

Theorem 2 Let $M \geq 0$ and $\omega \in \mathbb{R}$. For a linear operator $A : D(A) \subset X \to X$ the following conditions are equivalent:

(a) $A$ is closed, $D(A)$ is dense in $X$, and for every $\lambda \in \mathbb{C}$ with $\Re \lambda > \omega$ one has that $\lambda \in \rho(A)$ and
\[
\|R(\lambda, A)^k\| \leq \frac{M}{(\Re \lambda - \omega)^k} \quad \forall k \geq 1. \tag{1.5.1}
\]

(b) $A$ is the infinitesimal generator of a $C_0$-semigroup $S \in \mathcal{G}(M, \omega)$.

Proof. (b) $\Rightarrow$ (a) The fact that $A$ is closed and $D(A)$ is dense in $X$ has already been proved, see propositions 1 and 2. In order to prove (1.5.1) observe that, by using (1.4.7) to compute the $k$-th derivative of the resolvent of $A$, we obtain
\[
\frac{d^k}{d\lambda^k} R(\lambda, A)x = (-1)^k \int_0^\infty t^k e^{-\lambda t} S(t)x \, dt \quad \forall x \in X, \ \forall \lambda \in \Pi_\omega,
\]
where $\Pi_\omega$ is defined in (1.4.6). Therefore,
\[
\left\| \frac{d^k}{d\lambda^k} R(\lambda, A) \right\| \leq M \int_0^\infty t^k e^{-(\Re \lambda - \omega)t} \, dt = \frac{M k!}{(\Re \lambda - \omega)^{k+1}}
\]
where the integral is easily computed by an induction argument. The conclusion follows recalling (1.4.4).

(a) $\Rightarrow$ (b) The reasoning will be split into four steps.

Step 1: the Yosida approximation of $A$.
For all $n > \omega$ we define $A_n \in \mathcal{L}(X)$, called Yosida approximations of $A$, by
\[
A_n = n^2 R(n, A) - n = nAR(n, A), \tag{1.5.2}
\]
where the last identity follows from (1.4.1). We claim that
\[
\lim_{n \to \infty} A_n x = Ax \quad \forall x \in X.
\] (1.5.3)
Indeed, let us first take \( x \in D(A) \) and set \( J_n = nR(n,A) \). Then, we have that
\[
A_n x = AJ_n x = J_n Ax
\] (1.5.4)
and
\[
\lim_{n \to \infty} (J_n x - x) = \lim_{n \to \infty} R(n,A)Ax = 0 \quad \forall x \in D(A)
\]
in view of (1.5.1). In fact, observing that
\[
\|J_n\| \leq M_n n^{\omega - \omega} \quad (1.5.5)
\]
we conclude that
\[
\lim_{n \to \infty} J_n x = x \quad \forall x \in X \quad (1.5.6)
\]
because \( D(A) \) is dense in \( X \). This together with (1.5.4) yields (1.5.3).

**Step 2: construction of an approximate semigroup.**
For all \( n > \omega \) we define
\[
S_n(t) = e^{tA_n} = e^{-nt} \sum_{k=0}^\infty \frac{n^{2k}t^k R(n,A)^k}{k!}, \quad \forall t \geq 0.
\]
Observe that, in view of (1.5.1),
\[
\|S_n(t)\| \leq Me^{-nt} \sum_{k=0}^\infty \frac{n^{2k}t^k}{k!(n-\omega)^k} = Me^{\frac{n\omega t}{n-\omega}} \leq Me^{2\omega t} \quad (1.5.7)
\]
for all \( t \geq 0 \) and \( n > \omega \).

**Step 3: uniform convergence on compact sets.**
For any \( x \in X \), \( u_n(t) := S_n(t)x = e^{tA_n}x \) satisfies
\[
\begin{align*}
(u_n - u_m)'(t) &= A_n(u_n - u_m)(t) + (A_n - A_m)u_m(t), \quad \forall t \geq 0 \\
(u_n - u_m)(0) &= 0.
\end{align*}
\]
Therefore
\[
(u_n - u_m)(t) = \int_0^t e^{(t-s)A_n}e^{sA_m}(A_n - A_m)x \, ds, \quad \forall t \geq 0
\]
which in turn yields, by (1.5.7),
\[
|(u_n - u_m)(t)| \leq M^2 t e^{2\omega t}|(A_n - A_m)x|.
\]
Thanks to (1.5.3), the above estimate implies that \( \{u_n\} \) is a Cauchy sequence on all compact subsets of \( \mathbb{R}_+ \) for \( x \in D(A) \) and (1.5.7) guarantees that the same is true for all \( x \in X \). Consequently, the limit (uniform on all \([0,T] \subset \mathbb{R}_+\) )

\[
S(t)x = \lim_{n \to \infty} S_n(t)x, \quad \forall x \in X, \quad (1.5.8)
\]
defines a \( C_0 \)-semigroup of bounded linear operators on \( X \). Moreover, passing to the limit as \( n \to \infty \) in (1.5.7) we conclude that \( S \in G(M, \omega) \).

**Step 4: identification of the infinitesimal generation.**

First, we show that \( S(t)x \) is differentiable for all \( x \in D(A) \). Indeed,

\[
|S'_n(t)x - S(t)Ax| \leq |S_n(t)A_nx - S_n(t)Ax| + |S_n(t)Ax - S(t)Ax|
\]

where, by (1.5.3) and (1.5.7),

\[
|S_n(t)A_nx - S_n(t)Ax| \leq Me^{2\omega t} |A_nx - Ax| \to 0 \quad (n \to \infty)
\]

and

\[
|S_n(t)Ax - S(t)Ax| \to 0 \quad (n \to \infty)
\]

uniformly on all compact subsets of \( \mathbb{R}_+ \) by (1.5.8). Therefore

\[
S'(t)x = S(t)Ax, \quad \forall x \in D(A), \quad \forall t \geq 0. \quad (1.5.9)
\]

Now, let \( B : D(B) \subset X \to X \) be the infinitesimal generator of \( S \). Then \( A \subset B \) in view of (1.5.9). Moreover, \( \Pi_{\omega} \subset \rho(A) \) by assumption (a) and \( \Pi_{\omega} \subset \rho(B) \) by Proposition 5. So, on account of Proposition 6, \( A = B \). \( \square \)

**Remark 3** The above proof shows that condition (a) in Theorem 2 can be relaxed as follows

(a') \( A \) is closed, \( D(A) \) is dense in \( X \), \( (\omega, \infty) \subset \rho(A) \) and

\[
\|R(n,A)^k\| \leq \frac{M}{(n-\omega)^k} \quad \forall k \geq 1, \quad \forall n > \omega.
\]

**Remark 4** When \( M = 1 \) the countably many bounds in condition (a) follow from (1.5.1) for \( k = 1 \), that is,

\[
\|R(\lambda,A)\| \leq \frac{1}{|\Re \lambda - \omega|} \quad \forall k \geq 1, \quad \forall \lambda \in \Pi_{\omega}.
\]

**Exercise 5** Given any \( S \in G(M,0) \) with \( M \geq 1 \), define

\[
|x|_S = \sup_{t \geq 0} |S(t)x|, \quad \forall x \in X. \quad (1.5.10)
\]

Show that:
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1. $| \cdot |_S$ is a norm on $X$,
2. $|x| \leq |x|_S \leq M|x|$ for all $x \in X$, and
3. $S$ is a contraction semigroup with respect to $| \cdot |_S$.

**Remark 5** Let $A : D(A) \subset X \to X$ be a closed operator satisfying (1.5.1) but suppose $D(A)$ fails to be dense in $X$. In the Banach space $Y := D(A)$ let us define the operator $B$, called the part of $A$ in $Y$, by

\[
\begin{cases}
D(B) = \{ x \in D(A) : Ax \in Y \} \\
Bx = Ax \quad \forall x \in D(B).
\end{cases}
\]

Then $R(\lambda, A)(Y) \subset D(B)$ for all $\lambda \in \mathbb{C}$ such that $\Re \lambda > \omega$. Moreover, owing to (1.4.1) for all $x \in D(A)$ we have that

\[
\lim_{n \to \infty} nR(n, A)x = \lim_{n \to \infty} \{ R(n, A)Ax + x \} = x. \quad (1.5.11)
\]

Since $\|nR(n, A)\|$ is bounded, (1.5.11) holds true for all $x \in Y$. Hence, $D(B)$ is dense in $Y$. Consequently, $B$ satisfies in $Y$ all the assumptions of Theorem 2.

### 1.6 The homogeneous Cauchy problem

**Proposition 8** Let $S$ be a $\mathcal{C}_0$-semigroup of bounded linear operators on $X$ and $A : D(A) \subset X \to X$ be its infinitesimal generator. Then for every $x \in D(A)$ the Cauchy problem

\[
\begin{cases}
y'(t) = Ay(t) \\
y(0) = x
\end{cases}
\]

has a unique solution $y \in \mathcal{C}^1([0, \infty); X) \cap \mathcal{C}([0, \infty); D(A))$ given by

\[
y(t) = S(t)x \quad \forall t \geq 0.
\]

**Proof.** The fact that $y(t) = S(t)x$ satisfies (1.6.1) has already been proved (Theorem 1). Let us show that this is the unique solution of the problem. Let $z \in \mathcal{C}^1([0, \infty); X) \cap \mathcal{C}([0, \infty); D(A))$ be a solution of (1.6.1), fix $t > 0$, and set

\[
u(s) = S(t - s)z(s), \quad \forall s \in [0, t].
\]

Then

\[
u'(s) = -AS(t - s)z(s) + S(t - s)Az(s) = 0, \quad \forall s \in [0, t].
\]

Therefore, $z(t) = u(t) = u(0) = y(t)$. □
Example 8 (Transport equation in $L^p(\mathbb{R})$) Let $p \geq 1$. Recalling the analysis of the left-translation semigroup on $L^p(\mathbb{R})$ developed in examples 2 and 3, by Proposition 8 we conclude that for each $f \in W^{1,p}(\mathbb{R})$ the unique solution of the problem
\[
\begin{aligned}
&\frac{\partial u}{\partial t}(t, x) = \frac{\partial u}{\partial x}(t, x) \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\
u(0, x) = f(x) & \quad x \in \mathbb{R}
\end{aligned}
\]
is given by $u(t, x) = f(x + t)$.

Example 9 (Heat equation in $L^p(0, \pi)$) Let $p \geq 2$. On $X = L^p(0, \pi)$ consider the operator defined by
\[
\begin{aligned}
D(A) &= W^{2,p}(0, \pi) \cap W^{1,p}_0(0, \pi) \\
Af(x) &= f''(x) & x \in (0, \pi) \text{ a.e.}
\end{aligned}
(1.6.2)
\]
where
\[
W^{1,p}_0(0, \pi) = \{ f \in W^{1,p}(0, \pi) : f(0) = 0 = f(\pi) \}.
\]
Since $C_c^\infty(0, \pi) \subset D(A)$, we have that $D(A)$ is dense in $X$. Moreover, $A$ can be shown to be closed (see Exercise 6 below). We now show that $A$ satisfies condition $(a')$ of Remark 3 with $M = 1$ and $\omega = 0$ so that Theorem 2 will imply that $A$ generates a $C_0$-semigroup of contractions on $X$.

Step 1: $\sigma(A) = \{-n^2 : n \in \mathbb{N}\}$.

Fix any $g \in X$. We will show that, for all $\lambda \neq n^2(n \geq 1)$, the Sturm-Liouville system
\[
\begin{aligned}
\lambda f(x) - f''(x) &= g(x), & 0 < x < \pi \\
f(0) &= 0 = f(\pi)
\end{aligned}
(1.6.3)
\]
adopts a unique solution $f \in D(A)$. Denoting by
\[
g(x) = \sum_{n=1}^{\infty} g_n \sin(nx) & \quad (x \in [0, \pi])
\]
the Fourier series of $g$, we seek a candidate solution $f$ of the form
\[
f(x) = \sum_{n=1}^{\infty} f_n \sin(nx) & \quad (x \in [0, \pi]).
\]
In order to satisfy (1.6.3) one must have
\[
(\lambda + n^2)f_n = g_n & \quad \forall n \geq 1.
\]
So, for any $\lambda \neq -n^2$, (1.6.3) has a unique solution given by
\[
f(x) = \sum_{n=1}^{\infty} \frac{g_n}{\lambda + n^2} \sin(nx) & \quad (x \in [0, \pi]).
\]
From the above representation it follows that \( f \in H^2(0, \pi) \cap H^1_0(0, \pi) \). In fact, returning to the equation in (1.6.3) one concludes that \( f \in D(A) \).

**Step 2: resolvent estimate.**

By multiplying both members of the equation in (1.6.3) by \( |f|^{p-2}f \) and integrating over \((0, \pi)\) one obtains, for all \( \lambda > 0 \),

\[
\lambda \int_0^\pi |f(x)|^p \, dx + (p - 1) \int_0^\pi |f(x)|^{p-2}|f'(x)|^2 \, dx = \int_0^\pi g(x)|f(x)|^{p-2}f(x) \, dx
\]

which yields

\[
|f|_p \leq \frac{1}{\lambda} |g|_p \quad \forall \lambda > 0.
\]

**Step 3: conclusion.**

By Proposition 8 we conclude that for each \( f \in W^{2,p}(0, \pi) \cap W^{1,p}_0(0, \pi) \) the unique solution of

\[
\begin{cases}
\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) & (t,x) \in \mathbb{R}_+ \times (0, \pi) \\
u(t,0) = 0 = u(t, \pi) & t \geq 0 \\
u(0,x) = f(x) & x \in (0, \pi)
\end{cases}
\]

is given by \( u(t,x) = (S(t)f)(x) \).

**Exercise 6** Prove that operator \( A \) defined in (1.6.2) is closed.

**Example 10 (Heat equation in \( L^p(\mathbb{R}) \))** Let \( f \in W^{2,p}(\mathbb{R}) \) with \( p \geq 2 \). By following the reasoning of Example 9, let us solve the Cauchy problem

\[
\begin{cases}
\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\
u(0,x) = f(x) & x \in \mathbb{R}.
\end{cases}
\]

The operator defined by

\[
\begin{cases}
D(A) = W^{2,p}(\mathbb{R}) \\
Af(x) = f''(x) & x \in \mathbb{R} \ a.e.
\end{cases}
\]

is densely defined and closed. Let us begin by studying the problem

\[
\begin{cases}
f \in D(A) \\
\lambda f - f'' = g \in X
\end{cases}
\]

in the special case \( p = 2 \). Taking the Fourier transform of both members of the above equation we find

\[(\lambda + \xi^2)\hat{f}(\xi) = \hat{g}(\xi) \quad \forall \xi \in \mathbb{R}.
\]
So, for any \( \lambda > 0 \) we have that the solution to problem (1.6.4) is given by

\[
\phi_\lambda(x) = \frac{e^{-\sqrt{\lambda}|x|}}{2\sqrt{\lambda}} ,
\]

that is,

\[
f(x) = \frac{1}{2\sqrt{\lambda}} \left\{ \int_{-\infty}^{x} g(y)e^{-\sqrt{\lambda}(x-y)}dy + \int_{x}^{\infty} g(y)e^{-\sqrt{\lambda}(y-x)}dy \right\}.
\]

Moreover, the above representation formula holds true for any \( p \geq 2 \). We have thus proved that \((0, \infty) \subset \rho(A)\). Finally, by multiplying both members of the equation in (1.6.3) by \(|f|^{p-2}f\) and integrating over \( \mathbb{R} \) we obtain as in Example 9

\[
\lambda \int_{-\infty}^{\infty} |f|^p dx + (p-1) \int_{-\infty}^{\infty} |f|^{p-2}|f'|^2 dx = \int_{-\infty}^{\infty} g|f|^{p-2}f dx
\]

which yields

\[
|f|_p \leq \frac{1}{\lambda} |g|_p.
\]

Therefore, \( A \) satisfies condition \((a')\) of Remark 3 and generates a \( C_0\)-semigroup of bounded linear operators on \( X \) which gives the solution of our problem.

**Proposition 9** Let \( A : D(A) \subset X \to X \) be a densely defined closed linear operator satisfying, for some \( M, \omega \geq 0 \), the following conditions:

(i) \( \sigma(A) \subset \{ \lambda \in \mathbb{C} : |\Re \lambda| \leq \omega \} \)

(ii) for all \( k \geq 1 \)

\[
|\Re \lambda| > \omega \implies \| R(\lambda, A)^k \| \leq \frac{M}{(|\Re \lambda| - \omega)^k}.
\]

Then (1.6.1) has a unique solution \( y \in C^1(\mathbb{R}; X) \cap C(\mathbb{R}; D(A)) \).

**Proof.** Observe that both \( A \) and \( -A \) generate a \( C_0\)-semigroup of bounded linear operators on \( X \). Denote by \( S_\pm \) the semigroup generated by \( \pm A \) and set, for any \( x \in X \),

\[
y(t) = \begin{cases} 
S_+(t)x & (t \geq 0) \\
S_-(t)x & (t < 0).
\end{cases}
\]

Then \( y \in C(\mathbb{R}; X) \). Moreover, for \( x \in D(A) \), \( y \in C^1(\mathbb{R}; X) \cap C(\mathbb{R}; D(A)) \) and satisfies (1.6.1). \( \square \)
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Remark 6 In fact, the above assumptions are necessary and sufficient for $A$ to be the infinitesimal generator of a $C_0$ group of bounded linear operators on $X$ (see, for instance, [4, Section I.3.11]). Moreover, like in Remark 3, conditions (i) and (ii) can be weakened as follows:

(i) $(-\infty, -\omega) \cup (\omega, \infty) \subset \rho(A)$

(ii) for all $k \geq 1$ and $|n| > \omega$

\[ ||R(n, A)^k|| \leq \frac{M}{(|n| - \omega)^k}. \]

1.7 Problems

1. Let $X$ be a Banach space and let $A : D(A) \subset X \rightarrow X$ be the infinitesimal generator of a $C_0$-semigroup of bounded linear operators on $X$. Prove that, for every $n \geq 1$,

$D(A^n) := \{ x \in D(A^{n-1}) : Ax \in D(A) \}$

is dense in $X$.

Solution. For $n = 1$ the conclusion follows from Proposition 1. Let the conclusion be true for some $n \geq 1$ and fix any $y \in X$. Then, for any $\varepsilon > 0$ there exists $x_\varepsilon \in D(A^n)$ such that $|x_\varepsilon - y| < \varepsilon$. As shown in the proof of Proposition 1, $M_{0,t}A^n x_\varepsilon \in D(A)$ for all $t > 0$, where $M_{0,t}$ is defined in (1.2.3). Since $M_{0,t}A^n x_\varepsilon = A^n M_{0,t} x_\varepsilon$, we conclude that $M_{0,t} x_\varepsilon \in D(A^{n+1})$. Moreover, there exists $t_\varepsilon > 0$ such that

$|M_{0,t_\varepsilon} x_\varepsilon - y| \leq |M_{0,t_\varepsilon} x_\varepsilon - x_\varepsilon| + |x_\varepsilon - y| < 2\varepsilon. \quad \square$

2. For fixed $T > 0$ and $p \geq 1$ let $X = L^p(0, T)$ and

$(S(t)f)(x) = \begin{cases} f(x - t) & x \in [t, T] \\ 0 & x \in [0, t] \end{cases} \quad \forall x \in [0, T], \forall t \geq 0.$

Prove that $S$ is a $C_0$-semigroup of bounded linear operators on $X$ which satisfies $\|S(t)\| \leq 1$ for all $t \geq 0$. Moreover, observe that $S$ is nilpotent, that is, we have $S(t) \equiv 0$, $\forall t \geq T$. Deduce that $\omega_0(S) = -\infty$.

3. On $X = \{ f \in C([0, \pi]) : f(0) = 0 = f(\pi) \}$ with the uniform norm, consider the linear operator $A : D(A) \subset X \rightarrow X$ defined by

\[ \begin{cases} D(A) = \{ f \in C^2([0, 1]) : f(0) = f(\pi) = 0 = f''(0) = f''(\pi) \} \\ Af = f'' \end{cases}, \forall f \in D(A). \]
Apply Theorem 2 to show that $A$ generates a $C_0$-semigroup of contractions on $X$ and derive the initial-boundary value problem which is solved by such semigroup.

**Solution.** We only prove that $\|R(\lambda, A)\| \leq 1/\lambda$ for all $\lambda > 0$. Fix any $g \in X$ and let $f = R(\lambda, A)g$. Let $x_0 \in [0, \pi]$ be such that $|f(x_0)| = |f|_\infty$. If $f(x_0) > 0$, then $x_0 \in (0, \pi)$ is a maximum point of $f$. So, $f''(x_0) < 0$ and we have that

$$\lambda |f|_\infty = \lambda f(x_0) \leq \lambda f(x_0) - f''(x_0) = g(x_0) \leq |g|_\infty.$$ 

On the other hand, if $f(x_0) < 0$, then $x_0 \in (0, \pi)$ once again and $x_0$ is a minimum point of $f$. Thus, $f''(x_0) > 0$ and

$$\lambda |f|_\infty = -\lambda f(x_0) \leq -\lambda f(x_0) + f''(x_0) = -g(x_0) \leq |g|_\infty.$$ 

In any case, we have that $\lambda |f|_\infty \leq |g|_\infty$. □

4. Let $S$ be $C_0$-semigroup of bounded linear operators on $X$ and let $K \subset X$ be compact. Prove that for every $t_0 \geq 0$

$$\lim_{t \to t_0} \sup_{x \in K} |S(t)x - S(t_0)x| = 0.$$ (1.7.1)

**Solution.** We may assume $S \in \mathcal{G}(M, 0)$ for some $M > 0$ without loss of generality. Let $t_0 > 0$ and fix any $\varepsilon > 0$. Since $K$ is totally bounded, there exist $x_1, \ldots, x_{N_\varepsilon} \in X$ such that

$$K \subset \bigcup_{n=1}^{N_\varepsilon} B\left(x_n, \frac{\varepsilon}{M}\right).$$

Moreover, there exists $\tau > 0$ such that

$$|t - t_0| < \tau \implies |S(t)x_n - S(t_0)x_n| < \varepsilon \quad \forall n = 1, \ldots, N_\varepsilon.$$ 

Thus, for all $|t - t_0| < \tau$ we have that, if $x \in K$ is such that $x \in B\left(x_n, \frac{\varepsilon}{M}\right)$, then

$$|S(t)x - S(t_0)x| \leq |S(t)x - S(t)x_n| + |S(t)x_n - S(t_0)x_n| + |S(t_0)x_n - S(t_0)x| \leq 2M|x - x_n| + \varepsilon < 3\varepsilon.$$ 

So, the limit of $|S(t)x - S(t_0)x|$ as $t \to t_0$ is uniform on $K$. □

5. Use the resolvent identity (1.4.2) to prove that $R(\lambda, A)$ commutes with $R(\mu, A)$ for all $\lambda, \mu \in \rho(A)$. 

CHAPTER 1. GENERATION OF $C_0$-SEMIGROUPS

Solution. For all $\lambda, \mu \in \rho(A)$ we have that

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A).$$

So,

$$R(\mu, A) - R(\lambda, A) = (\lambda - \mu)R(\lambda, A)R(\mu, A)$$

but also

$$R(\mu, A) - R(\lambda, A) = (\lambda - \mu)R(\mu, A)R(\lambda, A).$$

Therefore $R(\lambda, A)R(\mu, A) = R(\mu, A)R(\lambda, A).$ \qed

6. Prove that if $A : D(A) \subset X \to X$ is a closed operator and $B \in \mathcal{L}(X)$, then $A + B : D(A) \subset X \to X$ is also closed.

7. Prove that $A : D(A) \subset X \to X$ is closed if and only if for any sequence $\{x_n\} \subset D(A)$

$$\begin{align*}
x_n &\to x \\
x_n &\to y \implies x \in D(A) \text{ and } Ax = y.
\end{align*}$$

8. Let $S \in \mathcal{G}(M, \omega)$ with $\omega \geq 0$. Prove that $\omega_0(S) < 0$ if and only if

$$\lim_{t \to +\infty} \|S(t)\| = 0. \quad (1.7.2)$$

Solution. One only needs to show that (1.7.2) implies that $\omega_0(S) < 0$. Let $t_0 > 0$ be such that $\|S(t_0)\| < 1/e$. For any $t > 0$ let $n(t) \in \mathbb{N}$ be the unique integer such that

$$n(t)t_0 \leq t < (n(t) + 1)t_0. \quad (1.7.3)$$

Then

$$\|S(t)\| = \|S(n(t)t_0)S(t - n(t)t_0)\| \leq \frac{Me^{\omega(t-n(t))}}{e^{n(t)}} \leq \frac{Me^{\omega t_0}}{e^{n(t)}}.$$ 

Therefore, on account of (1.7.2), we conclude that

$$\begin{align*}
\log \frac{\|S(t)\|}{t} &\leq \frac{\log (Me^{\omega t_0})}{t} - \frac{n(t)}{t} \\
&\leq \frac{\log (Me^{\omega t_0})}{t} - \left(1 - \frac{1}{t_0}\right) \quad \forall t > 0.
\end{align*}$$

Taking the limit as $t \to +\infty$ we conclude that $\omega_0(S) < 0.$ \qed
9. Let $S$ be the heat semigroup constructed in Example 9. Prove that, for any $f \in L^p(0, \pi)$,

$$(S(t)f)(x) = \int_0^\pi K(t, x, y) f(y) \, dy, \quad \forall t \geq 0, \ x \in (0, \pi) \text{ a.e.}$$

where

$$K(t, x, y) = \frac{2}{\pi} \sum_{k=1}^{\infty} e^{-k^2t} \sin(kx) \sin(ky).$$

2 Special classes of semigroups

2.1 Contraction semigroups

In this section we assume that $X$ is an Hilbert space with scalar product $\langle \cdot, \cdot \rangle$.

**Definition 8** An operator $A : D(A) \subset X \to X$ is said to be dissipative if

$$\Re \langle Ax, x \rangle \leq 0 \quad \forall x \in D(A). \quad (2.1.1)$$

**Remark 7** Observe that, if $A$ is dissipative, then for every $\lambda > 0$

$$\| (\lambda I - A)x \| \leq \lambda \| x \| - 2 \Re \langle Ax, x \rangle + \| Ax \|^2 \geq \lambda^2 \| x \|^2 \quad \forall x \in D(A).$$

Hence

$$\| (\lambda I - A)x \| \geq \lambda \| x \| - 2 \Re \langle Ax, x \rangle + \| Ax \|^2 \geq \lambda^2 \| x \|^2 \quad \forall x \in D(A) \quad \text{and} \quad \lambda > 0. \quad (2.1.2)$$

Consequently, $\lambda I - A$ is injective for all $\lambda > 0$. So, if $(\lambda_0 I - A)X = X$ for some $\lambda_0 > 0$, then (2.1.2) implies that $\lambda_0 \in \rho(A)$ and $\| R(\lambda_0, A) \| \leq 1/\lambda_0$. Moreover, since $R(\lambda_0, A)$ is closed, $\lambda_0 I - A$ is closed and therefore $A$ is closed as well.

**Proposition 10** For a dissipative operator $A : D(A) \subset X \to X$ the following properties are equivalent:

(a) $(\lambda_0 I - A)X = X$ for some $\lambda_0 > 0$, and

(b) $(\lambda I - A)X = X$ for all $\lambda > 0$.

**Proof.** The only implication that require a proof is (a) $\Rightarrow$ (b). By Remark 7 the set

$$\Lambda = \{ \lambda \in (0, \infty) : (\lambda I - A)X = X \}$$

is contained in $\rho(A)$ which is open in $\mathbb{C}$. This implies that $\Lambda$ is also open. Let us show that $\Lambda$ is closed: let $\Lambda \ni \lambda_n \to \lambda > 0$ and fix any $y \in X$. There exists an $x_n \in D(A)$ such that

$$\lambda_n x_n - Ax_n = y. \quad (2.1.3)$$
From (2.1.2) it follows that \( |x_n| \leq |y|/\lambda_n \leq C \) for some \( C > 0 \). Again by (2.1.2),
\[
\lambda_m |x_n - x_m| \leq |\lambda_m (x_n - x_m) - A(x_n - x_m)| \\
\leq |\lambda_m - \lambda_n| |x_n| + |\lambda_n x_n - Ax_n - (\lambda_m x_m - Ax_m)| \\
\leq C|\lambda_m - \lambda_n|.
\]
Therefore \( \{x_n\} \) is a Cauchy sequence. Let \( x_n \to x \). Then \( Ax_n \to \lambda x - y \) by (2.1.3). Since \( A \) is closed by Remark 7, \( x \in D(A) \) and \( \lambda x - Ax = y \). This show that \( \lambda I - A \) is surjective and implies that \( \lambda \in \Lambda \). Thus \( \Lambda \) is both open and closed in \( (0, \infty) \). Moreover, \( \Lambda \neq \emptyset \) because \( \lambda_0 \in \Lambda \). So, \( \Lambda = (0, \infty) \). 

**Proposition 11** Let \( A : D(A) \subset X \to X \) be dissipative with \( (I - A)X = X \). Then \( D(A) \) is dense in \( X \).

*Proof.* Let \( z \in X \) be such that \( \langle z, x \rangle = 0 \) for all \( x \in D(A) \). We will show that \( z = 0 \) or, equivalently since \( (I - A) \) is surjective, that
\[
0 = \langle z, x - Ax \rangle = \langle z, Ax \rangle \quad \forall x \in D(A).
\]

Let \( x \in D(A) \). Then by Proposition 10 there exists a sequence \( \{x_n\} \subset D(A) \) such that
\[
nx = nx_n - Ax_n \quad \forall n \geq 1. \tag{2.1.4}
\]

Since \( Ax_n = n(x_n - x) \in D(A) \), we have that \( x_n \in D(A^2) \) and
\[
Ax = Ax_n - \frac{1}{n} A^2 x_n \quad \text{or} \quad Ax_n = \left( I - \frac{1}{n} A \right)^{-1} Ax.
\]

Since \( \|(I - \frac{1}{n} A)^{-1}\| \leq 1 \) by (2.1.2), the above identity yields \( |Ax_n| \leq |Ax| \). So, by (2.1.4) we obtain
\[
|x_n - x| \leq \frac{1}{n} |Ax_n| \leq \frac{1}{n} |Ax|.
\]

Therefore, \( x_n \to x \). Moreover, since \( \{Ax_n\} \) is bounded, there is a subsequence \( Ax_{n_k} \) such that \( Ax_{n_k} \to y \). Since \( A \) is closed by Remark 7 we deduce that \( y = Ax \) (see Problems 1.7). Now, recall that \( \langle z, x \rangle = 0 \) for all \( x \in D(A) \) to deduce that
\[
\langle z, Ax_{n_k} \rangle = n_k \langle z, x_{n_k} - x \rangle = 0 \quad \forall k \geq 1.
\]

Letting \( k \to \infty \) in the above identity we conclude that \( \langle z, Ax \rangle = 0 \). \hfill \Box

**Proposition 12** For an operator \( A : D(A) \subset X \to X \) the following properties are equivalent:

(a) \( A \) is the infinitesimal generator of a contraction semigroup on \( X \);
(b) $A$ is dissipative and $(\lambda_0 I - A)X = X$ for some $\lambda_0 > 0$.

(c) $A$ is dissipative and $(\lambda I - A)X = X$ for all $\lambda > 0$.

Proof. In view of Proposition 10, the only implications that require a proof are $(a) \Rightarrow (b)$ and $(c) \Rightarrow (a)$.

$(a) \Rightarrow (b)$ Let $A$ be the infinitesimal generator of a contraction semigroup $S$. Then $(0, \infty) \subset \rho(A)$ by Theorem 2 and $A$ is dissipative because $\Re \langle Ax, x \rangle = \lim_{t \to 0} \Re \langle S(t)x - x, x \rangle \leq 0 \quad \forall x \in D(A)$.

$(c) \Rightarrow (a)$ Assume $(c)$. Then $D(A)$ is dense in $X$ by Proposition 11. Moreover, by Remark 7, $A$ is closed, $(0, \infty) \subset \rho(A)$, and $\|R(\lambda, A)\| \leq 1/\lambda$ for all $\lambda > 0$. The conclusion follows by Theorem 2. □

The above results can be completed by looking at $A^*$, the adjoint of $A$, the definition of which we recall below. Given $A : D(A) \subset X \to X$, with $D(A)$ dense in $X$, let $D(A^*)$ denote the subspace of $X$ consisting of all $y \in X$ for which there exists a constant $C_y \geq 0$ such that

$$|\langle Ax, y \rangle| \leq C_y |x| \quad \forall x \in D(A). \quad (2.1.5)$$

Observe that, since $D(A)$ is dense in $X$, $(2.1.5)$ yields that $x \mapsto \langle Ax, y \rangle$ can be extended to a unique bounded linear functional $\tilde{\phi}_y \in X^*$. Denoting by $j : X^* \to X$ the Riesz isomorphism, we define

$$A^*y = j(\tilde{\phi}_y) \quad \forall y \in D(A^*). \quad (2.1.6)$$

Then the following adjoint identity holds true

$$\langle Ax, y \rangle = \langle x, A^*y \rangle \quad \forall x \in D(A), \forall y \in D(A^*). \quad (2.1.7)$$

**Exercise 7** Check that $D(A^*)$ is a subspace of $X$, that $A^* : D(A^*) \subset X \to X$ is a linear operator, and that $A^*$ is closed.

**Solution.** We only prove that $A^*$ is closed. Let $\{y_n\} \subset D(A^*)$ and $y, z \in X$ be such that

$$\begin{cases} y_n \to y \\ A^*y_n \to z \end{cases} \quad (n \to \infty)$$

Then $\{A^*y_n\}$ is bounded, say $|A^*y_n| \leq C$. So, recalling $(2.1.7)$,

$$|\langle Ax, y_n \rangle| = |\langle x, A^*y_n \rangle| \leq C|x| \quad \forall x \in D(A)$$

yields

$$|\langle Ax, y \rangle| \leq C|x| \quad \forall x \in D(A)$$
implying that \( y \in D(A^*) \). Moreover

\[
\langle Ax, y \rangle = \lim_{n \to \infty} \langle Ax, y_n \rangle = \langle x, z \rangle \quad \forall x \in D(A).
\]

Thus, \( \langle x, A^*y - z \rangle = 0 \) for all \( x \in D(A) \). Since \( D(A) \) is dense, \( A^*y = z \). \( \square \)

**Remark 8** If \( A \in \mathcal{L}(X) \), then \( A^* \) is also bounded and we have that \( A^{**} = A \).

**Theorem 3 (Lumer-Phillips)** Let \( A : D(A) \subset X \to X \) be a densely defined closed linear operator. If \( A \) and \( A^* \) are dissipative, then \( A \) is the infinitesimal generator of a contraction semigroup on \( X \).

**Proof.** In view of Proposition 12 it suffices to show that \( (0, \infty) \subset \rho(A) \).

**Step 1:** \( \lambda I - A \) and \( \lambda I - A^* \) are injective for every \( \lambda > 0 \).

This follows from Remark 7.

**Step 2:** \( (\lambda I - A)(D(A)) \) is dense in \( X \) for every \( \lambda > 0 \).

Let \( y \in X \) be such that

\[
\langle \lambda x - Ax, y \rangle = 0 \quad \forall x \in D(A).
\]

Then \( \langle Ax, y \rangle = \lambda \langle x, y \rangle \) implies that \( y \in D(A^*) \) and

\[
\langle x, \lambda y - A^* y \rangle = 0 \quad \forall x \in X.
\]

So, \( \lambda y - A^* y = 0 \) which, by Step 1, yields \( y = 0 \).

**Step 3:** \( \lambda I - A \) is surjective for every \( \lambda > 0 \).

Fix any \( y \in X \). By Step 1, there exists \( \{x_n\} \subset D(A) \) such that

\[
\lambda x_n - Ax_n =: y_n \to y \quad \text{as} \quad n \to \infty.
\]

By (2.1.2) we deduce that, for all \( n, m \geq 1 \),

\[
|x_n - x_m| \leq \frac{1}{\lambda} |y_n - y_m|
\]

which yields that \( \{x_n\} \) is a Cauchy sequence in \( X \). Therefore, there exists \( x \in X \) such that

\[
\begin{cases}
x_n \to x \\
 Ax_n = \lambda x_n - y_n \to \lambda x - y \quad (n \to \infty)
\end{cases}
\]

Since \( A \) is closed, \( x \in D(A) \) and \( \lambda x - Ax = y \). \( \square \)

**Remark 9** The notion of dissipative operators can be given in Banach spaces and Theorem 3 remains valid in such settings. However, Proposition 11 is true only if \( X \) is reflexive (see, for instance, [6, Section 1.4]).
Example 11 (Wave equation in \( L^2(0, \pi) \)) Let us set \( H^1_0(0, \pi) = W^{1,2}_0(0, \pi) \) and \( H^2(0, \pi) = W^{2,2}(0, \pi) \). For any given \( f \in H^2(0, \pi) \cap H^1_0(0, \pi) \) and \( g \in H^1_0(0, \pi) \) we want to solve the problem

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) & (t, x) \in \mathbb{R} \times (0, \pi) \\
u(t, 0) = 0 = u(t, \pi) & t \in \mathbb{R} \\
u(0, x) = f(x), \quad \frac{\partial u}{\partial x}(0, x) = g(x) & x \in (0, \pi).
\end{cases}
\] (2.1.8)

Let \( X \) be the Hilbert space \( H^1_0(0, \pi) \times L^2(0, \pi) \) with the scalar product

\[
\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \rangle = \int_0^\pi [u'(s)\bar{u}'(s) + v(s)\bar{v}(s)] \, ds.
\]

Denoting by \( A : D(A) \subset L^2(0, \pi) \to L^2(0, \pi) \) the second derivative with homogeneous Dirichlet boundary conditions studied in Example 9, define \( A : D(A) \subset X \to X \) by

\[
\begin{cases}
D(A) = (H^2(0, \pi) \cap H^1_0(0, \pi)) \times H^1_0(0, \pi) \\
A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ Au \end{pmatrix}
\end{cases}
\] (2.1.9)

The fact that \( A \) is closed and \( D(A) \) is dense can be easily checked. We claim that \( \mathbb{R} \setminus \{0\} \subset \rho(A) \) and

\[
R(\lambda, A) = \begin{pmatrix} \lambda & 1 \\ A & \lambda \end{pmatrix} R(\lambda^2, A) \quad \forall \lambda \neq 0.
\] (2.1.10)

Indeed, for any \((f, g) \in X\) the resolvent equation

\[
(\lambda I - A) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}
\]

is equivalent to the system

\[
\begin{cases}
\lambda u - v = f \\
\lambda v - Au = g.
\end{cases}
\]

Hence, \( v = \lambda u - f \) and, by solving the equation

\[
\lambda^2 u - Au = g + \lambda f,
\]

we find

\[
\begin{cases}
u = \lambda R(\lambda^2, A)f + R(\lambda^2, A)g \\
u = [\lambda^2 R(\lambda^2, A) - 1] f + \lambda R(\lambda^2, A)g.
\end{cases}
\]

Since \( \lambda^2 R(\lambda^2, A) - 1 = AR(\lambda^2, A) \) by (1.4.1), the above identities yield (2.1.10).
Now, integrating by parts we obtain
\[
\left( A\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right) = \int_0^\pi \left[ u'(s)v'(s) + u(s)v''(s) \right] ds = 0, \quad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in D(A).
\]

Therefore \( A \) is dissipative and so it generates a contraction semigroup \( e^{tA} \) on \( X \) by Proposition 12. In fact, \( e^{tA} \) is a \( C_0 \) group thanks to Proposition 9 and Remark 6. Consequently, problem (2.1.8) has a unique solution
\[
u \in C^2(\mathbb{R}; L^2(0, \pi)) \cap C^1(\mathbb{R}; H^1_0(0, \pi)) \cap C(\mathbb{R}; H^2(0, \pi) \cap H^1_0(0, \pi))
\]
which is given by the first component of \( e^{tA}(f,g) \).

**Definition 9** A densely defined closed linear operator \( A : D(A) \subset X \to X \) is said to be symmetric if \( A \subset A^* \), that is,
\[
D(A) \subset D(A^*) \quad \text{and} \quad Ax = A^*x \quad \forall x \in D(A).
\]

A is said to be self-adjoint if \( A = A^* \).

Clearly, a symmetric operator \( A \) is self-adjoint if and only if \( D(A) = D(A^*) \). This is always the case when \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup of bounded linear operators on \( X \), as our next result guarantees.

**Proposition 13** Let \( A : D(A) \subset X \to X \) be a densely defined closed linear operator such that \( \rho(A) \cap \mathbb{R} \neq \emptyset \). If \( A \) is symmetric, then \( A \) is self-adjoint.

**Proof.** We will prove that \( D(A^*) \subset D(A) \) in two steps. Let \( \lambda \in \rho(A) \cap \mathbb{R} \).

**Step 1:** \( R(\lambda, A) = R(\lambda, A)^* \).

Since \( R(\lambda, A) \in \mathcal{L}(X) \) it suffices to show that
\[
\langle R(\lambda, A)x, y \rangle = \langle x, R(\lambda, A)y \rangle \quad \forall x, y \in X.
\]

Fix any \( x, y \in X \) and set
\[
u = R(\lambda, A)x \quad \text{and} \quad v = R(\lambda, A)y
\]
so that
\[
\lambda u - Au = x \quad \text{and} \quad \lambda v - Av = y.
\]

Since \( A \) is symmetric, we have that
\[
\langle R(\lambda, A)x, y \rangle = \langle u, y \rangle = \langle u, \lambda v - Av \rangle = \langle \lambda u - Au, v \rangle = \langle x, R(\lambda, A)y \rangle.
\]

**Step 2:** \( D(A^*) \subset D(A) \).

Let \( u \in D(A^*) \) and set \( x = \lambda u - A^*u \). Observe that, for all \( v \in D(A) \),
\[
\langle x, v \rangle = \langle \lambda u - A^*u, v \rangle = \langle u, \lambda v - Av \rangle.
\]
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Now, take any $y \in X$ and let $v = R(\lambda, A)y$. Then the above identity yields
\[ \langle x, R(\lambda, A)y \rangle = \langle u, y \rangle \quad \forall y \in X. \]
This identity and Step 1 imply that $u = R(\lambda, A)^*x = R(\lambda, A)x \in D(A)$. □

The following property of self-adjoint operators is very useful.

**Corollary 2 (Stone)** Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator. If $A$ is self-adjoint, then $B := iA$ is the infinitesimal generator of a $C_0$ unitary group on $X$.

**Proof.** Since $A$ is self-adjoint, we have that
\[ \langle Ax, x \rangle = \langle x, A^*x \rangle = \langle x, Ax \rangle \quad \forall x \in D(A). \]
Thus, $\langle Ax, x \rangle$ is real and
\[ \Re \langle Bx, x \rangle = \Re \langle iAx, x \rangle = 0 \quad \forall x \in D(B), \]
which implies that $B$ and $B^* = -B$ are dissipative. So, $B$ and $-B$ generate contraction semigroups on $X$. Therefore $B$ generates a $C_0$ unitary group. □

**Exercise 8** On $X = L^2(0, \pi; \mathbb{C})$ let $A : D(A) \subset X \to X$ be the operator

\[
\begin{align*}
D(A) &= H^2(0, \pi; \mathbb{C}) \cap H^1_0(0, \pi; \mathbb{C}) \\
Af(x) &= f''(x) \\
x \in (0, \pi) \text{ a.e.}
\end{align*}
\] (2.1.11)

Show that $A$ is self-adjoint and dissipative.

**Solution.** We begin by observing that
\[ \langle Af, f \rangle = \int_0^\pi f''(x)\overline{f(x)}dx = -\int_0^\pi |f'(x)|^2dx \quad \forall f \in D(A). \]
Therefore $A$ is dissipative.

Moreover, $A$ is symmetric. Indeed, that for all $g \in D(A)$ we have
\[ \langle Af, g \rangle = \int_0^\pi f''(x)\overline{g(x)}dx = \int_0^\pi f(x)\overline{g''(x)}dx \quad \forall f \in D(A). \] (2.1.12)

Therefore $|\langle Af, g \rangle| \leq |g''|_2|f|_2$ for all $f \in D(A)$, which yields $g \in D(A^*)$. Then (2.1.7), together with (2.1.12), implies that $A^*g = g''$ for all $g \in D(A)$.

Thus, in order to prove that $A$ is self-adjoint it suffices to check that $1 \in \rho(A)$. In fact, we will show that—as in Example 9—$\sigma(A) = \{-n^2 : n \in \mathbb{N}\}$. Fix any $g \in X$ and consider the Sturm-Liouville system
\[
\begin{align*}
\lambda f(x) - f''(x) &= g(x), \quad 0 < x < \pi \\
f(0) &= 0 = f(\pi).
\end{align*}
\] (2.1.13)
Let us consider the odd extension of $g$ to $[-\pi, \pi]$ and denote by $g(x) = \sum_{n \in \mathbb{Z}^*} \hat{g}(n)e^{inx}$ $(x \in [0, \pi])$ the Fourier series of such a function. We seek a solution $f$ of the form $f(x) = \sum_{n \in \mathbb{Z}^*} \hat{f}(n)e^{inx}$ $(x \in [0, \pi])$.

In order to satisfy (2.1.13) one must have $(\lambda + n^2)\hat{f}(n) = \hat{g}(n)$ $\forall n \in \mathbb{Z}^*$. So, for any $\lambda \in \mathbb{C} \setminus \{-n^2 : n \geq 1\}$, (2.1.13) has a unique solution.

Example 12 (Schrödinger equation) By Corollary 2 and Exercise 8 we have that, for any $f \in H^2(0, \pi) \cap H^1_0(0, \pi)$, there exists a unique solution $u \in C^1(\mathbb{R}; L^2(0, \pi)) \cap C(\mathbb{R}; H^2(0, \pi) \cap H^1_0(0, \pi))$ of the problem

$$\begin{cases}
\frac{\partial u}{\partial t}(t, x) = i \frac{\partial^2 u}{\partial x^2}(t, x) & (t, x) \in \mathbb{R} \times (0, \pi) \\
u(t, 0) = u(t, \pi) & t \in \mathbb{R} \\
u(0, x) = f(x) & x \in (0, \pi).
\end{cases}$$

2.2 Analytic semigroups

We recall that, for any $\omega \in \mathbb{R}$, we have denoted by $\Pi_{\omega}$ the complex half-plane in (1.4.6). Moreover, for any $\theta \in (0, \pi]$ we define

$$\Sigma_{\omega, \theta} = \{ \lambda \in \mathbb{C} \setminus \{\omega\} : |\arg(\lambda - \omega)| < \theta\}. \quad (2.2.1)$$

Let $\omega \in \mathbb{R}$ and $\theta_0 \in (\pi/2, \pi]$.

Definition 10 A densely defined closed linear operator $A : D(A) \subset X \to X$ on a Banach space $X$ is called sectorial of base point $\omega \in \mathbb{R}$ and angle $\theta_0$ if:

(a) $\Sigma_{\omega, \theta_0} \subset \rho(A)$, and

(b) there exists a nondecreasing function $M : (0, \theta_0) \to (0, +\infty)$ such that

$$\|R(\lambda, A)\| \leq \frac{M(\theta)}{|\lambda - \omega|} \quad \forall \theta \in (0, \theta_0), \forall \lambda \in \Sigma_{\omega, \theta}. \quad (2.2.2)$$
2.2. ANALYTIC SEMIGROUPS

Let $A : D(A) \subset X \to X$ be a sectorial operator of base point $\omega$ and angle $\theta_0$ on a Banach space $X$. For any $\varepsilon > 0$ and $\theta \in (\pi/2, \theta_0)$, let

$$\gamma_{\varepsilon, \theta} = \gamma_{\varepsilon, \theta}^+ \cup \gamma_{\varepsilon, \theta}^- \cup \gamma_{\varepsilon, \theta}^0$$ (2.2.3)

where

$$\gamma_{\varepsilon, \theta}^\pm = \{ z \in \mathbb{C} : z = \omega + re^{\pm i\theta}, \; r \geq \varepsilon \}$$

and

$$\gamma_{\varepsilon, \theta}^0 = \{ z \in \mathbb{C} : z = \omega + \varepsilon e^{i\eta}, \; |\eta| \leq \theta \}.$$

**Proposition 14 (Dunford integral)** Let $\varepsilon > 0$ and $\theta \in (\pi/2, \theta_0)$ be fixed. Then for each $t \geq 0$

$$S(t) = \begin{cases} \frac{1}{2\pi i} \int_{\gamma_{\varepsilon, \theta}} e^{\lambda t} R(\lambda, A) \, d\lambda & t > 0 \\ I & t = 0 \end{cases}$$ (2.2.4)

is a bounded linear operator on $X$.

**Proof.** Since $\gamma_{\varepsilon, \theta}^0, \gamma_{\varepsilon, \theta}^\pm \subset \Sigma_{\omega, \theta}$, by (2.2.2) we deduce that for any $t > 0$

$$|e^{\lambda t} R(\lambda, A)| \leq e^{t(\omega + r \cos \theta)} \frac{M(\theta)}{r} \quad \forall \lambda = \omega + re^{\pm i\theta} \in \gamma_{\varepsilon, \theta}^\pm.$$
and
\[ \| e^{\lambda t} R(\lambda, A) \| \leq e^{t(\omega + \varepsilon \cos \eta)} \frac{M(\theta)}{\varepsilon} \quad \forall \lambda = \omega + \varepsilon e^{i\eta} \in \gamma_{\varepsilon, \theta}. \]

Because \( \cos \theta < 0 \), the above inequalities ensure the convergence of the integral in (2.2.4). The completeness of \( \mathcal{L}(X) \) yields \( S(t) \in \mathcal{L}(X) \).

Exercise 9 Use Cauchy’s theorem for holomorphic functions to show that, for any \( 0 < \varepsilon_1 < \varepsilon_2 \) and \( \theta_1, \theta_2 \in (\pi/2, \theta_0) \), we have
\[
\int_{\gamma_{\varepsilon_1, \theta_1}} e^{\lambda t} R(\lambda, A) d\lambda - \int_{\gamma_{\varepsilon_2, \theta_2}} e^{\lambda t} R(\lambda, A) d\lambda = 0.
\]

Solution. It suffices to observe that, owing to Cauchy’s theorem,
\[
\int_{\gamma_{\varepsilon_1, \theta_1}} e^{\lambda t} R(\lambda, A) d\lambda - \int_{\gamma_{\varepsilon_2, \theta_2}} e^{\lambda t} R(\lambda, A) d\lambda = 0.
\]

Similarly,
\[
\int_{\gamma_{\varepsilon_2, \theta_1}} e^{\lambda t} R(\lambda, A) d\lambda - \int_{\gamma_{\varepsilon_2, \theta_2}} e^{\lambda t} R(\lambda, A) d\lambda = 0.
\]
Theorem 4 Let $A : D(A) \subset X \to X$ be a sectorial operator of base point $\omega$ and angle $\theta \in (0, \theta_0)$ on a Banach space $X$. Fix any $0 < \varepsilon < 1$ and $\theta \in (\pi/2, \theta_0)$, and define $S : \mathbb{R}_+ \to \mathcal{L}(X)$ as in (2.2.4). Then the following properties hold true.

(a) $S \in \mathcal{C}^1(\mathbb{R}_+^*; \mathcal{L}(X))$ and $S'(t) = AS(t)$ for all $t > 0$.

(b) There exist constants $M, N \geq 0$ such that

\[
\|S(t)\| \leq Me^{\omega t} \quad \forall t \geq 0
\]  

(2.2.5)

and

\[
\|(A - \omega I)S(t)\| \leq \frac{N}{t} e^{\omega t} \quad \forall t > 0.
\]  

(2.2.6)

(c) $S$ is a $\mathcal{C}_0$-semigroup and $A$ is its infinitesimal generator.

Proof. Without loss of generality we can restrict the analysis to the case of $\omega = 0$. Indeed, the general case can be treated replacing $A$ by $A_\omega := A - \omega I$ which is easily seen to be sectorial of base point 0. Since $R(\mu, A_\omega) = R(\mu + \omega, A)$, one

\footnote{For instance, one can take $\varepsilon = \frac{1}{2}$ and $\theta = \frac{\pi}{4} + \frac{\theta_0}{2}$.}
recovers $S$, the semigroup generated by $A$, from the semigroup $S_\omega$ generated by $A_\omega$ via the formula $S(t) = e^{\omega t}S_\omega(t)$.

**Step 1: proof of (a).**

The fact that $S \in \mathcal{C}^1(\mathbb{R}^*_+; \mathcal{L}(X))$ follows by differentiating under the integral sign: by (2.2.2) we deduce that for any $\lambda \in \gamma_{\epsilon,\theta}^+$

$$\left\| \frac{\partial}{\partial t} e^{\lambda t} R(\lambda, A) \right\| = \left\| \lambda e^{\lambda t} R(\lambda, A) \right\| \leq e^{\epsilon R \lambda} M(\theta) = e^{t|\lambda| \cos \theta} M(\theta)$$

which guarantees the convergence of the integral $\frac{1}{2\pi i} \int_{\gamma_{\epsilon,\theta}} \frac{\partial}{\partial t} e^{\lambda t} R(\lambda, A) \, d\lambda$, in the space $\mathcal{L}(X)$, because $\cos \theta < 0$. Moreover, recalling the identity $\lambda R(\lambda, A) = I + AR(\lambda, A)$, we obtain

$$S'(t) = \frac{1}{2\pi i} \int_{\gamma_{\epsilon,\theta}} \lambda e^{\lambda t} R(\lambda, A) \, d\lambda$$

$$= \frac{I}{2\pi i} \int_{\gamma_{\epsilon,\theta}} e^{\lambda t} \, d\lambda + \frac{1}{2\pi i} \int_{\gamma_{\epsilon,\theta}} Ae^{\lambda t} R(\lambda, A) \, d\lambda = AS(t)$$

for all $t > 0$ because

$$\int_{\gamma_{\epsilon,\theta}} e^{\lambda t} \, d\lambda = \lim_{R \to \infty} \int_{\gamma_{\epsilon,\theta}^R} e^{\lambda t} \, d\lambda = 0$$
Step 2: proof of (b).

The change of variable \( \lambda t = \xi \) transforms the integral in (2.2.4) into

\[
S(t) = \frac{1}{2\pi i} \int_{\gamma_{\epsilon,\theta}} e^{\xi} R\left(\frac{\xi}{t}, A\right) \frac{d\xi}{t} = \frac{1}{2\pi i} \int_{\gamma_{\epsilon,\theta}} e^{\xi} R\left(\frac{\xi}{t}, A\right) \frac{d\xi}{t},
\]

where we have used Exercise 9. Therefore,

\[
S(t) = \frac{1}{2\pi i} \left\{ \int_{\xi}^{\infty} e^{re^{i\eta}} R\left(\frac{re^{i\theta}}{t}, A\right) \frac{re^{i\theta}}{t} \, dr - \int_{\xi}^{\infty} e^{re^{-i\theta}} R\left(\frac{re^{-i\theta}}{t}, A\right) \frac{re^{-i\theta}}{t} \, dr + \int_{-\theta}^{\theta} e^{re^{i\eta}} R\left(\frac{e^{i\eta}}{t}, A\right) i e^{i\eta} \frac{d\eta}{t} \right\}
\]

Now, appealing to (2.2.2) we have that

\[
\| S(t) \| \leq \frac{M(\theta)}{2\pi} \left\{ 2 \int_{\xi}^{\infty} e^{r \cos \theta} \frac{dr}{r} + \int_{-\theta}^{\theta} e^{r \cos \eta} \, d\eta \right\} =: M.
\]
By the same change of variable in (2.2.7), a computation similar to the one above leads to
\[
\|S'(t)\| \leq \frac{M(\theta)}{2\pi t} \left\{ 2 \int_0^\infty e^{r\cos \theta} dr + \varepsilon \int_-^\theta e^{\varepsilon \cos \eta} d\eta \right\}.
\]

Here we can also use the fact that the above inequality holds true for all \(\varepsilon > 0\).
So, passing to the limit as \(\varepsilon \downarrow 0\) we obtain
\[
\|S'(t)\| \leq \frac{M(\theta)}{\pi t |\cos \theta|} =: \frac{N}{t}.
\]

**Step 3: S is strongly continuous.**
In view of (2.2.5) it suffices to show that \(S(t)x \to x\) as \(t \downarrow 0\) for all \(x \in D(A)\).
So, fix \(x \in D(A)\) and let \(y = x - Ax\). Then \(x = R(1, A)y\) and, recalling that \(0 < \varepsilon < 1^2\), for all \(t > 0\) the resolvent identity (1.4.2) yields
\[
S(t)x = S(t)R(1, A)y = \frac{1}{2\pi i} \int_{\gamma_\varepsilon, \theta} e^{\lambda t} R(\lambda, A)R(1, A)y d\lambda
\]
\[
= \frac{1}{2\pi i} \int_{\gamma_\varepsilon, \theta} e^{\lambda t} \frac{R(\lambda, A)y}{1 - \lambda} d\lambda - \frac{1}{2\pi i} \int_{\gamma_\varepsilon, \theta} \frac{e^{\lambda t}}{1 - \lambda} R(1, A)y d\lambda
\]
\[
= \frac{1}{2\pi i} \int_{\gamma_\varepsilon, \theta} e^{\lambda t} \frac{R(\lambda, A)y}{1 - \lambda} d\lambda
\]

because, by Cauchy’s theorem,
\[
\int_{\gamma_\varepsilon, \theta} \frac{e^{\lambda t}}{1 - \lambda} d\lambda = \lim_{R \to \infty} \int_{\gamma_R, \theta} \frac{e^{\lambda t}}{1 - \lambda} d\lambda = 0.
\]

Therefore, by Cauchy’s integral formula
\[
\lim_{t \downarrow 0} S(t)x = \frac{1}{2\pi i} \int_{\gamma_0, \theta} \frac{R(\lambda, A)y}{1 - \lambda} d\lambda
\]
\[
= \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\gamma_R, \theta} \frac{R(\lambda, A)y}{1 - \lambda} d\lambda = R(1, A)y = x.
\]

\(^2\)Without the restriction \(\varepsilon \in (0, 1)\), here one should take \(\varepsilon > \varepsilon\) and define \(y = \varepsilon_0 x - Ax\).
Step 4: $S(t + s) = S(t)S(s)$.

Fix any $\theta' \in (\pi/2, \theta)$. Then for all $t, s > 0$ we have that

$$S(t)S(s) = \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{\epsilon, \theta}} e^{\lambda t} R(\lambda, A) d\lambda \cdot \int_{\gamma_{2\epsilon, \theta'}} e^{\mu s} R(\mu, A) d\mu.$$ 

So, by the resolvent identity (1.4.2) we obtain

$$S(t)S(s) = \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{\epsilon, \theta}} e^{\lambda t} R(\lambda, A) d\lambda \cdot \int_{\gamma_{2\epsilon, \theta'}} e^{\mu s} R(\mu, A) d\mu - \int_{\gamma_{2\epsilon, \theta'}} e^{\mu s} R(\mu, A) d\mu \int_{\gamma_{\epsilon, \theta}} e^{\lambda t} \frac{d\lambda}{\mu - \lambda} = S(t + s)$$

because for each $\lambda \in \gamma_{\epsilon, \theta}$ and $\mu \in \gamma_{2\epsilon, \theta'}$ we have that

$$\frac{1}{2\pi i} \int_{\gamma_{2\epsilon, \theta'}} \frac{e^{\mu s}}{\mu - \lambda} d\mu = e^{\lambda s} \quad \text{and} \quad \frac{1}{2\pi i} \int_{\gamma_{\epsilon, \theta}} \frac{e^{\lambda t}}{\mu - \lambda} d\lambda = 0.$$
Step 5: A is the infinitesimal generator of S.
Let $B : D(B) \subset X \to X$ be the infinitesimal generator of $S$. Then $A \subset B$ in view of (a). Moreover, $\Pi_0 \subset \rho(A)$ by assumption and $\Pi_0 \subset \rho(B)$ by Proposition 5. So, on account of Proposition 6, $A = B$. \qed

Exercise 10 Let $A : D(A) \subset X \to X$ be a sectorial operator which generates a $C_0$-group. Show that $A \in \mathcal{L}(X)$.

Theorem 5 Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $C_0$-semigroup $S \in \mathcal{G}(M, \omega)$. Then the following properties are equivalent.

(a) There exists $\theta_0 \in (\pi/2, \pi]$ such that $A$ is sectorial of base point $\omega$ and angle $\theta_0$.

(b) $S \in C^1(\mathbb{R}_+^* ; \mathcal{L}(X))$ and there exists $N > 0$ such that

$$\|(A - \omega I)S(t)\| \leq \frac{N}{t} e^{\omega t} \quad \forall t > 0. \quad (2.2.8)$$

(c) There exists $\theta \in (0, \pi/2)$ such $S$ has an analytic extension to $\Sigma_{\omega, \theta}$ and $z \mapsto e^{-\omega z}S(z)$ is bounded on $\Sigma_{\omega, \theta'}$ for all $0, \theta' < \theta$. 


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Proof. First, we observe Theorem 4 ensures that \((a) \Rightarrow (b)\). In the proof of the remaining statements we note that one can assume \(\omega = 0\), as we did in the proof of Theorem 4, without loss of generality.

\[(b) \Rightarrow (c)\] We claim that \(S \in C^\infty(\mathbb{R}_+^*; \mathcal{L}(X))\) and

\[
\begin{aligned}
S(t)X &\subset D(A^n) \\
S^{(n)}(t) &= A^n S(t) = \left(AS\left(\frac{t}{n}\right)\right)^n = \left(S'\left(\frac{t}{n}\right)\right)^n \\
\forall n \geq 1, \forall t > 0.
\end{aligned}
\] (2.2.9)

Indeed, (2.2.9) holds true for \(n = 1\). Assuming it holds for some \(n \geq 1\) we have that

\[
S\left(\frac{t}{n+1}\right) A^n S\left(\frac{nt}{n+1}\right) X \subset D(A).
\]

This shows that \(S(t)X \subset D(A^{n+1})\) and

\[
\left(S'\left(\frac{t}{n+1}\right)\right)^{n+1} = \left(AS\left(\frac{t}{n+1}\right)\right)^{n+1} = A^{n+1} S(t) = S^{(n+1)}(t).
\]

Next, by (2.2.8) and (2.2.9) we deduce that

\[
\|S^{(n)}(t)\| \leq \frac{n^n N^n}{t^n} \quad \forall n \geq 1, \forall t > 0.
\]

Therefore, for every \(t > 0\),

\[
\sum_{n=0}^{\infty} \frac{|z - t|^n}{n!} \|S^{(n)}(t)\| \leq \sum_{n=0}^{\infty} \frac{|z - t|^n n^n N^n}{t^n} < \infty
\]

(2.2.10)

for all complex numbers \(z\) in the disc

\[
C\left(t, \frac{t}{Ne}\right) := \left\{z \in \mathbb{C} : |z - t| < \frac{t}{Ne}\right\}.
\]

Consequently, the series \(\sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} S^{(n)}(t)\) defines an analytic function \(F_t\) on \(C\left(t, \frac{t}{Ne}\right)\). Taking \(\theta = \arctan\left(\frac{1}{Ne}\right)\), we conclude that there is a unique analytic function \(F\) on

\[
\Sigma_{0, \theta} \subset \bigcup_{t>0} C\left(t, \frac{t}{Ne}\right)
\]

which coincides with \(F_t\) on any \(C\left(t, \frac{t}{Ne}\right)\) and therefore with \(S\) on \(\mathbb{R}_+^*\).

Finally, in order to show that \(S(z) = F(z)\) is bounded on every subsector of \(\Sigma_{0, \theta}\), fix any \(0 < q < 1\) and set \(\theta' = \arctan\left(\frac{q}{Ne}\right)\). Then, by (2.2.10), for all \(z \in \Sigma_{0, \theta'}\) we have that

\[
\|S(z)\| \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} \|S^{(n)}(z)\| \leq \sum_{n=0}^{\infty} \frac{|z|^n n^n N^n}{(z_n)^n n!} \leq \sum_{n=0}^{\infty} \frac{n^n}{n! e^n q^n} < \infty.
\]
We show that, since $S(z)$ is analytic on a sector of the form $\Sigma_{0,\theta}$, where $0 < \theta < \pi/2$, the integral representation formula (1.4.7) for $R(\lambda, A)$—which holds for $\Re \lambda > 0$—can be extended to the sector $\Sigma_{0, \frac{\pi + \theta}{2}}$. Observe that any $\lambda$ in such a sector can be written in the form

$$\lambda = |\lambda|e^{i\alpha} \quad \text{with} \quad |\alpha| < \frac{\pi + \theta}{2}. \quad (2.2.11)$$

Let us consider the case of $\alpha \geq 0$ first. Define

$$R(\lambda, A) = \int_{\gamma} e^{-\lambda z} S(z) \, dz \quad (2.2.12)$$

where

$$\gamma = \{z \in \mathbb{C} : z = re^{-i\frac{3}{4}\theta}, r \geq 0\}.$$ 

We now show that the integral in (2.2.12) converges to the resolvent of $A$ and (2.2.2) holds. Indeed, denoting by $M$ an upper bound for $\|S(z)\|$ on $\gamma$, since

$$R(\lambda, A) = \int_{0}^{+\infty} e^{-\lambda re^{-i\frac{3}{4}\theta}} S(re^{-i\frac{3}{4}\theta}) e^{-i\frac{3}{4}\theta} \, dr$$

we have that

$$\|R(\lambda, A)\| \leq M \int_{0}^{+\infty} e^{-r\Re[|\lambda|e^{i(\alpha - \frac{3}{4}\theta)}]} \, dr = M \int_{0}^{+\infty} e^{-r|\lambda| \cos(\alpha - \frac{3}{4}\theta)} \, dr,$$
where
\[
\cos \left( \alpha - \frac{3}{4} \theta \right) \geq \min \left\{ \cos \left( \frac{3}{4} \theta \right), \sin \left( \frac{\theta}{4} \right) \right\} =: K_\theta > 0
\]

because
\[
-\frac{3}{4} \theta < \alpha - \frac{3}{4} \theta < \frac{\pi}{2} - \frac{\theta}{4}.
\]

Therefore, the integral in (2.2.12) converges and
\[
\| R(\lambda, A) \| \leq \frac{M}{K_\theta |\lambda|}
\]

for all \( \lambda \) of the form (2.2.11) with \( \alpha \geq 0 \). On the other hand, for \( \alpha < 0 \) one can repeat the above argument replacing \( \gamma_+ \) by
\[
\gamma_- := \{ z \in \mathbb{C} : z = re^{i\frac{3}{4} \theta}, r \geq 0 \}.
\]

Finally, to prove that the integral in (2.2.12) gives the resolvent of \( A \) it suffices to observe that, for all \( \lambda \) of the form (2.2.11) with \( \alpha \geq 0 \), we have
\[
(\lambda I - A)R(\lambda, A) = \int_{\gamma_+} e^{-\lambda z} (\lambda S(z) - AS(z)) \, dz = \int_{\gamma_+} e^{-\lambda z} (\lambda S(z) - S'(z)) \, dz = I.
\]

This shows that \( \Sigma_{0, \frac{\pi + \theta}{2}} \subset \rho(A) \) and completes the proof. \( \Box \)

**Definition 11** A \( C_0 \)-semigroup is called analytic if verifies any of the conditions of Theorem 5.

The following proposition provides a useful sufficient condition for an operator to be sectorial.

**Proposition 15** Let \( A : D(A) \subset X \to X \) be a densely defined closed linear operator such that, for some \( \omega \in \mathbb{R} \) and \( M > 0 \), \( \Pi_\omega \subset \rho(A) \) and
\[
\| R(\lambda, A) \| \leq \frac{M}{|\lambda - \omega|} \quad \forall \lambda \in \Pi_\omega.
\]

Then \( A \) is the infinitesimal generator of an analytic semigroup.

**Proof.** As is by now well known, we can develop the reasoning assuming that \( \omega = -1 \), which in turn implies that \( \rho(A) \) contains the imaginary axis. Since \( i\beta \in \rho(A) \) for all \( \beta \in \mathbb{R} \), by Proposition 4 we conclude that
\[
C \left( \left[ i\beta, \frac{1}{\| R(i\beta, A) \|} \right] \right) \subset \rho(A) \quad \forall \beta \neq 0.
\]
So, owing to (2.2.13), $C(i\beta, |\beta|/M) \subset \rho(A)$ for all $\beta \in \mathbb{R}\setminus\{0\}$. Therefore
\[
\Sigma_{0,\theta} = \bigcup_{\beta \neq 0} C(i\beta, |\beta|/M) \cup \Pi_{-1} \subset \rho(A)
\]

with
\[
\theta = \frac{\pi}{2} + \arctan\left(\frac{1}{M}\right).
\]

Now, fix any $0 < q < 1$ and let $\theta' = \frac{\pi}{2} + \arctan\left(\frac{q}{M}\right)$. Then (1.4.3) yields
\[
R(\lambda, A) = \sum_{n=0}^{\infty} (-1)^n (\Re \lambda)^n R(i3\lambda, A)^{n+1} \quad \forall \lambda \in \Sigma_{0,\theta} \setminus \Pi_0.
\]

Hence
\[
\|R(\lambda, A)\| \leq \sum_{n=0}^{\infty} |\Re \lambda|^n \left(\frac{M}{|3\lambda|}\right)^{n+1} \leq \frac{M}{|3\lambda|} \sum_{n=0}^{\infty} q^n = \frac{M}{1-q} \frac{1}{|3\lambda|}. \quad (2.2.14)
\]

Moreover, for all $\lambda \in \Sigma_{0,\theta} \setminus \Pi_0$ we have
\[
|\lambda|^2 = (\Re \lambda)^2 + (3\lambda)^2 \leq \left(\frac{q}{M}\right)^2 + \frac{1}{3\lambda} (3\lambda)^2,
\]

which, combined with (2.2.14), yields
\[
\|R(\lambda, A)\| \leq \frac{\sqrt{q^2 + M^2}}{(1-q)|\lambda|} \quad \forall \lambda \in \Sigma_{0,\theta} \setminus \Pi_0. \quad \square
\]
2.2. ANALYTIC SEMIGROUPS

Exercise 11 Let $A : D(A) \subset X \to X$ be a self-adjoint dissipative operator on an Hilbert space $(X, \langle \cdot, \cdot \rangle)$. Then $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions on $X$, by Theorem 3. Prove that $S$ is analytic.

**Solution.** Fix any $\lambda \in \Pi_0$ and $y \in X$. Then $x = R(\lambda, A)y$ satisfies $\lambda x - Ax = y$ and, taking the scalar product with $x$ we obtain

$$
\Re \lambda |x|^2 \leq \Re \lambda |x|^2 - \langle Ax, x \rangle = \Re \langle x, y \rangle
$$

$$
\Im \lambda |x|^2 = \Im \langle x, y \rangle
$$

because $\langle Ax, x \rangle \leq 0$. Thus, since $\Re \lambda > 0$ for all $\lambda \in \Pi_0$ we have that

$$
[(\Re \lambda)^2 + (\Im \lambda)^2] |x|^4 \leq (\Re \langle x, y \rangle)^2 + (\Im \langle x, y \rangle)^2 = |\langle x, y \rangle|^2 \leq |y|^2 |x|^2
$$

which yields $\|R(\lambda, A)\| \leq 1/|\lambda|$. The conclusion follows by Proposition 15. □

As a first consequence of analyticity, we now give a result due to Triggiani [8] on the asymptotic behavior of $S(t)$.

**Proposition 16 (Triggiani)** Let $A : D(A) \subset X \to X$ be the infinitesimal generator of an analytic semigroup $S$. Then $s(A) = \omega_0(S)$.

**Proof.** Proceeding by contradiction, let us suppose that

$$
s(A) \leq -2\varepsilon < 0 = \omega_0(S). \tag{2.2.15}
$$

Since $A$ is sectorial and $\omega_0 = 0$,

$$
\Sigma_{-\varepsilon, \theta} \subset \rho(A) \quad \text{for some} \quad \theta \in \left(\frac{\pi}{2}, \pi\right].
$$

Fix any $\eta \in \left(\frac{\pi}{2}, \theta\right)$ and let

$$
\gamma_{\eta}^\pm = \{ z \in \mathbb{C} : z = -\varepsilon + re^{\pm i\eta}, \ r \geq 0 \}.
$$
Then for all \( t > 0 \)

\[
S(t) = \frac{1}{2\pi i} \left\{ \int_{\gamma} e^{\lambda t} R(\lambda, A) \, d\lambda + \int_{\gamma'} e^{\lambda t} R(\lambda, A) \, d\lambda \right\}.
\]

Therefore

\[
\|S(t)\| \leq \frac{M(\eta)}{2\pi} \int_{0}^{+\infty} e^{(r \cos \eta - \varepsilon)t} \left\{ \frac{1}{|re^{i\eta} - \varepsilon|} + \frac{1}{|re^{-i\eta} - \varepsilon|} \right\} \, dr.
\]

Since

\[
|re^{\pm i\eta} - \varepsilon|^2 = r^2 + \varepsilon^2 - 2r\varepsilon \cos \eta \geq 2r\varepsilon(1 - \cos \eta),
\]

we conclude that

\[
\|S(t)\| \leq \frac{M(\eta)}{\pi} e^{-\varepsilon t} \int_{0}^{+\infty} e^{r t \cos \eta} \frac{1}{\sqrt{2r\varepsilon(1 - \cos \eta)}} \, dr = \frac{M(\eta)}{\pi \sqrt{2\varepsilon(1 - \cos \eta)}} e^{-\varepsilon t} \int_{0}^{+\infty} \frac{e^{s \cos \eta} \, ds}{\sqrt{s}} \quad \forall t > 0,
\]

which contradicts \( \omega_0(S) = 0 \).

\[\square\]

**Example 13** We return to the heat equation studied in Example 9 and Exercise 8 to show that the associated semigroup is analytic.

Let us consider the case \( p = 2 \) first. Then operator \( A \) in (2.1.11) is self-adjoint and dissipative. Therefore the corresponding semigroup is analytic thanks to Exercise 11.

Next, for \( p > 2 \) let \( X = L^p(0, \pi; \mathbb{C}) \) and consider the operator defined by

\[
\begin{cases}
D(A) = W^{2,p}(0, \pi; \mathbb{C}) \cap W_0^{1,p}(0, \pi; \mathbb{C}) \\
Af(x) = f''(x) \quad x \in (0, \pi) \text{ a.e.}
\end{cases}
\] (2.2.16)

Proceeding as in Exercise 8 one can show that \( \sigma(A) = \{-n^2 : n \geq 1\} \). Let \( \Re \lambda > 0 \). For any fixed \( g \in X \) consider the Sturm-Liouville system

\[
\begin{cases}
\lambda f(x) - f''(x) = g(x), \quad 0 < x < \pi \\
u(0) = 0 = u(\pi).
\end{cases}
\] (2.2.17)

By multiplying both members of the equation in (2.2.17) by \( f |f|^{p-2} \) and integrating by parts over \( (0, \pi) \) one obtains

\[
\lambda \int_{0}^{\pi} |f(x)|^p \, dx + \frac{p}{2} \int_{0}^{\pi} |f'(x)|^2 |f(x)|^{p-2} \, dx + \frac{p-2}{2} \int_{0}^{\pi} |f(x)|^{p-4} (f'(x))^2 \, dx = \int_{0}^{\pi} g(x)\overline{f(x)} |f(x)|^{p-2} \, dx
\]
which in turn yields
\[
\Re \lambda \int_0^\pi |f(x)|^p \, dx + \frac{p}{2} \int_0^\pi |f'(x)| f(x)|^{p-2} \, dx + \frac{p-2}{2} \Re \int_0^\pi |f(x)|^{p-4}(f'(x))^2 \, dx \leq |g|_p |f|_{p}^{p-1} \tag{2.2.18}
\]
and
\[
\Im \lambda \int_0^\pi |f(x)|^p \, dx + \frac{p-2}{2} \Im \int_0^\pi |f(x)|^{p-4}(f'(x))^2 \, dx = \Im \int_0^\pi g(x)f(x)|f(x)|^{p-2} \, dx. \tag{2.2.19}
\]
Since
\[
\Re \int_0^\pi |f(x)|^{p-4}(f'(x))^2 \, dx \geq - \int_0^\pi |f(x)|^{p-2}|f'(x)|^2 \, dx,
\]
from (2.2.18) it follows that
\[
\Re \lambda \int_0^\pi |f(x)|^p \, dx + \int_0^\pi |f'(x)|^2 |f(x)|^{p-2} \, dx \leq |g|_p |f|_{p}^{p-1}.
\]
Hence, recalling that \(\Re \lambda > 0\),
\[
\Re \lambda |f|_p \leq |g|_p \tag{2.2.20}
\]
and
\[
\int_0^\pi |f'(x)|^2 |f(x)|^{p-2} \, dx \leq |g|_p |f|_{p}^{p-1}. \tag{2.2.21}
\]
Similarly, since
\[
|\Im \int_0^\pi |f(x)|^{p-4}(f'(x))^2 \, dx| \leq \int_0^\pi |f'(x)|^2 |f(x)|^{p-2} \, dx,
\]
by (2.2.19) and (2.2.21) we deduce that
\[
|\Im \lambda| \int_0^\pi |f(x)|^p \, dx \leq \frac{p}{2} |g|_p |f|_{p}^{p-1}.
\]
or
\[
|\Im \lambda| |f|_p \leq \frac{p}{2} |g|_p. \tag{2.2.22}
\]
Finally, by combining (2.2.20) and (2.2.22) we obtain
\[
|f|_p \leq \sqrt{\frac{4 + p^2}{2|\lambda|}} |g|_p \quad \forall \Re \lambda > 0 \tag{2.2.23}
\]
which ensures that the corresponding semigroup is analytic even for \(p > 2\).
2.3 Compact semigroups

We recall that an operator $\Lambda \in \mathcal{L}(X)$ is called compact if it maps bounded sets into relatively compact sets. Equivalently, $\Lambda$ is compact if, denoting by $B_1$ the unit ball of $X$, one has that $\Lambda(B_1)$ is compact in $X$.

Typical examples of compact operators are operators of finite rank, that is, such that $\dim \Lambda(X) < \infty$. Observe that the identity map $I : X \to X$ is compact if and only if $\dim X < \infty$.

The family of all compact operators on $X$ is a closed subspace of $\mathcal{L}(X)$ (see for instance [3]), here denoted by $\mathcal{K}(X)$.

Exercise 12 Let $A : D(A) \subset X \to X$ be a closed operator. Prove that the following properties are equivalent:

(a) $R(\lambda, A) \in \mathcal{K}(X)$ for all $\lambda \in \rho(A)$;

(b) $R(\lambda_0, A) \in \mathcal{K}(X)$ for some $\lambda_0 \in \rho(A)$.

Solution. Observe that, by the resolvent identity (1.4.2) on has that

$$R(\lambda, A) = [(\lambda_0 - \lambda)R(\lambda, A) + I] R(\lambda_0, A).$$

\[ \square \]

Let now $S$ be a $C_0$-semigroup of bounded linear operators on $X$.

Definition 12 $S$ is called compact if $S(t) \in \mathcal{K}(X)$ for all $t > 0$ and eventually compact if there exists $t_0 > 0$ such that $S(t_0) \in \mathcal{K}(X)$.

Lemma 2 If $S(t_0) \in \mathcal{K}(X)$ for some $t_0 > 0$ then

(a) $S(t) \in \mathcal{K}(X)$ for all $t \geq t_0$;

(b) $S \in \mathcal{C}([t_0, \infty); \mathcal{L}(X))$.

Proof. Property (a) is an easy consequence of the semigroup property

$$S(t) = S(t - t_0)S(t_0)$$

since the product of a bounded operator with a compact one is compact.

As for (b), since $S(t_0)(B_1)$ is compact in $X$, recalling (1.7.1) we have that

$$\|S(t + h) - S(t)\| = \sup_{x \in B_1} |S(t + h)x - S(t)x|$$

$$= \sup_{x \in B_1} \left| (S(t + h - t_0)x - S(t - t_0))S(t_0)x \right| \to 0 \ (t \to t_0)$$

for all $t \geq t_0$.

\[ \square \]

Theorem 6 Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $C_0$-semigroup $S \in \mathcal{G}(M, \omega)$. Then the following properties are equivalent:
2.3. COMPACT SEMIGROUPS

(a) $S$ is compact;

(b) $S \in \mathcal{C}(\mathbb{R}_+^*; \mathcal{L}(X))$ and $R(\lambda, A) \in \mathcal{K}(X)$ for some (hence for all) $\lambda \in \rho(A)$.

Proof. ([a] $\Rightarrow$ [b]) Since $S \in \mathcal{C}(\mathbb{R}_+^*; \mathcal{L}(X))$ by Lemma 2, recalling the integral representation formula (1.4.6) we have that

$$R(\lambda, A) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} e^{-\lambda t} S(t) \, dt \quad \forall \lambda \in \Pi_\omega, \hspace{1cm} (2.3.1)$$

where the integral $\int_{\varepsilon}^{\infty} e^{-\lambda t} S(t) \, dt$ converges in $\mathcal{L}(X)$ and the limit exists because $S$ is bounded near zero. Moreover, for every $\varepsilon > 0$ the operator

$$R_\varepsilon(\lambda, A) := \int_{\varepsilon}^{\infty} e^{-\lambda t} S(t) \, dt \quad (\lambda \in \Pi_\omega)$$

is compact because $\mathcal{K}(X)$ is closed. Since

$$\|R(\lambda, A) - R_\varepsilon(\lambda, A)\| \leq \left\| \int_{\varepsilon}^{0} e^{-\lambda t} S(t) \, dt \right\| \leq M\varepsilon \to 0 \text{ as } \varepsilon \downarrow 0,$$

we conclude that $R(\lambda, A) \in \mathcal{K}(X)$, again by the fact that $\mathcal{K}(X)$ is closed.

([b] $\Rightarrow$ [a]) Since $S \in \mathcal{C}(\mathbb{R}_+^*; \mathcal{L}(X))$, we have that (2.3.1) holds in the uniform operator topology and, for any fixed $s > 0$,

$$\lambda R(\lambda, A)S(s) - S(s) = \int_{0}^{\infty} \lambda e^{-\lambda t}(S(t + s) - S(s)) \, dt \quad \forall \lambda \in \Pi_\omega.$$

Therefore, taking $\lambda > \omega$, for all $\delta > 0$ we have that

$$\|\lambda R(\lambda, A)S(s) - S(s)\| \leq \int_{0}^{\delta} \lambda e^{-\lambda t}\|S(t + s) - S(s)\| \, dt + \int_{\delta}^{\infty} \lambda e^{-\lambda t}\|S(t + s) - S(s)\| \, dt$$

Now,

$$\int_{0}^{\delta} \lambda e^{-\lambda t}\|S(t + s) - S(s)\| \, dt \leq \sup_{0 \leq t \leq \delta} \|S(t + s) - S(s)\|.$$

On the other hand,

$$\int_{\delta}^{\infty} \lambda e^{-\lambda t}\|S(t + s) - S(s)\| \, dt \leq M \int_{\delta}^{\infty} \lambda e^{-\lambda t}(e^{\omega(t+s)} + e^{\omega s}) \, dt$$

$$\leq Me^{\omega s}\left(\frac{e^{(\omega-\lambda)\delta}}{\lambda-\omega} + e^{-\lambda\delta}\right) \to 0 \text{ as } \lambda \uparrow \infty.$$
Thus,
\[
\limsup_{\lambda \to \infty} \| \lambda R(\lambda, A) S(s) - S(s) \| \leq \sup_{0 \leq t \leq \delta} \| S(t + s) - S(s) \|
\]
which in turn implies that \( \lambda R(\lambda, A) S(s) \to S(s) \) as \( \lambda \to \infty \) because \( \delta \) is arbitrary. Since \( \lambda R(\lambda, A) S(s) \) is compact for all \( \lambda > \omega \), so is \( S(s) \) for all \( s > 0 \) because \( \mathcal{K}(X) \) is closed.

\[\square\]

**Corollary 3** Let \( A : D(A) \subset X \to X \) be the infinitesimal generator of a \( C_0 \)-semigroup of bounded linear operators on \( X \) denoted by \( S \). If

(a) \( S \in \mathcal{C}([t_0, \infty); X) \) for some \( t_0 > 0 \), and

(b) \( R(\lambda, A) \in \mathcal{K}(X) \) for some (hence for all) \( \lambda \in \rho(A) \),

then \( S(t) \in \mathcal{K}(X) \) for all \( t \geq t_0 \).

**Example 14** Let us consider the heat semigroup of Example 13 in the case of \( p = 2 \). In view of Theorem 6, in order to prove that such a semigroup is compact, it suffices to show that \( R(\lambda, A) \in \mathcal{K}(X) \) for some \( \lambda \in \rho(A) \). Now, taking \( \lambda = 1 \) from (2.2.19) and (2.2.20) it follows that \( |f'|_2 \leq |g|_2 \). In other words, \( R(1, A) \) maps the unit ball of \( L^2(0, \pi; \mathbb{C}) \) into the unit ball of \( H^1_0(0, \pi; \mathbb{C}) \). Since the immersion \( H^1_0(0, \pi; \mathbb{C}) \subset C^{0,1/2}(0, \pi; \mathbb{C}) \) is continuous, by Ascoli’s theorem we conclude that \( R(1, A) \) is compact. Therefore the heat semigroup on \( (0, \pi) \) is compact.

### 2.4 Problems

1. Consider the heat equation in \( L^p(\mathbb{R}) \) with \( p \geq 2 \) that we studied in Example 10. Prove that the associated semigroup is analytic but not compact.

### 3 Perturbation of semigroups

#### 3.1 Perturbation by bounded operators

In this chapter we shall stress the connection between a \( C_0 \)-semigroup, \( S(t) \), and its infinitesimal generator \( A \) by adopting the equivalent notation

\[ S(t) = e^{tA} \quad \forall t \geq 0. \]
Theorem 7. Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on $X$ such that $\| e^{tA} \| \leq M e^{\omega t}$ and let $B \in \mathcal{L}(X)$. Then $A + B : D(A) \subseteq X \to X$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on $X$ satisfying
\[ \| e^{t(A+B)} \| \leq M e^{\omega t + M\|B\|} t \quad \forall t \geq 0. \] (3.1.1)

Proof. Step 1: the special case $\omega = 0$ and $M = 1$.

In view of Proposition 5 we have that $\rho(A) \supseteq \mathbb{R}^+_\ast$ and
\[ \lambda I - (A + B) = [I - BR(\lambda, A)](\lambda I - A) \quad \forall \lambda > 0. \] (3.1.2)

Therefore
\[ \lambda \in \rho(A + B) \iff [I - BR(\lambda, A)]^{-1} \in \mathcal{L}(X). \]

Now, for all $\lambda \in \Pi_{\|B\|}$ we have that $\|BR(\lambda, A)\| < 1$. So $\lambda \in \rho(A + B)$ and
\[ R(\lambda, A + B) = R(\lambda, A) \sum_{n=0}^{\infty} (BR(\lambda, A))^n. \] (3.1.3)

Moreover
\[ \|R(\lambda, A + B)\| \leq \frac{1}{\Re \lambda} \sum_{n=0}^{\infty} \left( \frac{\|B\|}{\Re \lambda} \right)^n = \frac{1}{\Re \lambda - \|B\|}. \] (3.1.4)

Then, since $A + B : D(A) \subseteq X \to X$ is closed, by Theorem 2 we conclude that $A + B$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on $X$ satisfying
\[ \| e^{t(A+B)} \| \leq e^{\|B\|t} \quad \forall t \geq 0. \]

Step 2: the general case.

Let us consider $A_{\omega} = A - \omega I$. The corresponding semigroup $e^{tA_{\omega}} = e^{-\omega t} e^{tA}$ belongs to $\mathcal{G}(M, 0)$. Now, denote by $||| \cdot |||$ the equivalent norm defined in (1.5.10) for which $e^{tA_{\omega}}$ turns out to be a contraction semigroup and observe that
\[ |||Bx||| \leq M \|B\| |x| \leq M \|B\| |||x||| \quad \forall x \in X. \]

By Step 1, $A_{\omega} + B$ generates a $\mathcal{C}_0$-semigroup of bounded linear operators on $X$ satisfying
\[ |||e^{t(A_{\omega}+B)}||| \leq e^{|||B|||t} \leq e^{M\|B\|t} \quad \forall t \geq 0. \]

Therefore, for all $x \in X$,
\[ |e^{t(A_{\omega}+B)}x| \leq |||e^{t(A_{\omega}+B)}x||| \leq e^{M\|B\|t} |||x||| \leq Me^{M\|B\|t}|x| \quad \forall t \geq 0. \]

Since $e^{t(A_{\omega}+B)} = e^{-\omega t} e^{t(A+B)}$, the conclusion follows. \qed
Lemma 3 Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $C_0$-semigroup on $X$ and let $B \in \mathcal{L}(X)$. Then $e^{t(A+B)}$ is a solution of the Volterra integral equation

\[ V(t)x = e^{tA}x + \int_0^t e^{(t-s)A}BV(s)x \, ds \quad \forall x \in X. \quad (3.1.5) \]

Proof. For any $x \in D(A)$ and $t > 0$ the function $H : [0, t] \to X$ defined by

\[ H(t) = e^{tA}x + \int_0^t e^{(t-s)A}BV(s)x \, ds \quad \forall x \in D(A). \]

is continuously differentiable and satisfies for all $0 < s < t$

\[ H'(s) = -Ae^{(t-s)A}e^{s(A+B)}x + e^{(t-s)A}(A + B)e^{s(A+B)}x. \]

By integrating the above relation over $[0, t]$ we obtain

\[ e^{t(A+B)}x = e^{tA}x + \int_0^t e^{(t-s)A}Be^{s(A+B)}x \, ds \quad \forall x \in D(A). \]

Since all the operators in the above equation are continuous, the identity holds for all $x \in X$ and the conclusion follows. \qed

For any $T > 0$ we denote by $B(0, T; \mathcal{L}(X))$ the Banach space of all maps $\Lambda : [0, T] \to \mathcal{L}(X)$ such that

\[ \sup_{t \in [0, T]} \|\Lambda(t)\| < \infty. \]

Proposition 17 Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $C_0$-semigroup on $X$ such that $\|e^{tA}\| \leq Me^{\omega t}$ and let $B \in \mathcal{L}(X)$. Then there exists a unique family $\{V(t)\}_{t \geq 0}$ such that

(a) $V(t) \in \mathcal{L}(X)$ for all $t \geq 0$,

(b) $t \mapsto V(t)x$ is continuous on $\mathbb{R}_+$ for all $x \in X$, and

(c) $(3.1.5)$ is satisfied for all $t \geq 0$.

Moreover

\[ V(t) = \sum_{n=0}^{\infty} V_n(t) \quad \forall t \geq 0 \quad (3.1.6) \]

where $\{V_n(t)\}_{t \geq 0}$ is defined by

\[ V_0(t) = e^{tA} \quad \text{and} \quad V_{n+1}(t)x = \int_0^t e^{(t-s)A}BV_n(s)x \, ds \quad \forall x \in X. \quad (3.1.7) \]

Furthermore, the series in $(3.1.6)$ converges in $B(0, T; \mathcal{L}(X))$ for all $T \geq 0$. 
3.1. PERTURBATION BY BOUNDED OPERATORS

Proof. Define \( \{V_n(t)\}_{t \geq 0} \) by (3.1.7). Then \( t \mapsto V_n(t)x \) is continuous on \( \mathbb{R}_+ \) for all \( x \in X \). Moreover, proceeding by induction one can easily prove that

\[
\|V_n(t)\| \leq Me^{\omega t} \frac{M^n t^n}{n!} \quad \forall t \geq 0, \, \forall n \geq 0
\]

which in turn shows that the series in (3.1.6) converges in \( L^\infty(0,T;\mathcal{L}(X)) \) for all \( T \geq 0 \). So, \( t \mapsto V(t)x \) is continuous on \( \mathbb{R}_+ \) and satisfies (3.1.5). This shows the existence part of the conclusion. As for uniqueness, let \( \{U(t)\}_{t \geq 0} \) be another family of operators satisfying \((a),(b),(c)\). Then for all \( x \in X \)

\[
\|(V(t) - U(t))x\| \leq M\|B\| \int_0^t e^{\omega(t-s)} \|(V(s) - U(s))x\| \, ds \quad \forall t \geq 0.
\]

Now, Gronwall’s lemma ensures that \( U \equiv V \). □

Corollary 4 Let \( A : D(A) \subset X \to X \) be the infinitesimal generator of a \( C_0 \)-semigroup on \( X \) such that \( \|e^{tA}\| \leq Me^{\omega t} \) and let \( B \in \mathcal{L}(X) \). Then

\[
\|e^{t(A+B)} - e^{tA}\| \leq Me^{\omega t}(e^{M\|B\|t} - 1).
\]

Proof. By Proposition 17 and Theorem 7 we obtain

\[
\left| e^{t(A+B)}x - e^{tA}x \right| \leq \int_0^t \|e^{(t-s)A}\| \|B\| \|e^{s(A+B)}\| \|x\| \, ds \\
\leq \int_0^t Me^{\omega(t-s)} \|B\| Me^{(\omega + M\|B\|)s} \|x\| \, ds \\
= Me^{\omega t}(e^{M\|B\|t} - 1) \|x\|
\]

The conclusion follows. □

Theorem 8 Let \( A : D(A) \subset X \to X \) be the infinitesimal generator of a compact \( C_0 \)-semigroup of bounded linear operators on \( X \) and let \( B \in \mathcal{L}(X) \). Then \( A + B : D(A) \subset X \to X \) is the infinitesimal generator of a compact \( C_0 \)-semigroup of bounded linear operators on \( X \).

Proof. By Theorem 6 we have that \( e^{tA} \) is continuous in the uniform operator topology for \( t > 0 \) and \( R(\lambda, A) \) is compact for all \( \lambda \in \rho(A) \). Moreover, for all \( \lambda > \omega \) we have that \( \|R(\lambda, A)\| \leq M/(\lambda - \omega) \) and so, for \( \lambda > \omega + M\|B\| + 1 \), the series

\[
R(\lambda, A + B) = R(\lambda, A) \sum_{n=0}^{\infty} \left( BR(\lambda, A) \right)^n
\]

converges in \( \mathcal{L}(X) \). Since each term on the right-hand side is compact, so is \( R(\lambda, A + B) \) for all \( \lambda \in \rho(A + B) \). Thus, appealing to Theorem 6 once again, it suffices to show that \( e^{t(A+B)} \) is continuous in the uniform operator topology.
for \( t > 0 \). Since \( e^{t(A+B)} \) is given by the series in (3.1.6), this continuity property follows from the fact that, being \( e^{tA} \) continuous in the uniform operator topology for \( t > 0 \), so is each \( V_n \) in (3.1.7) and also their sum because the series converges in \( B(0;T;L(X)) \) for all \( T \geq 0 \). □

**Example 15** Let us apply Theorem 7 to solve the wave equation with lower order terms

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + a(x) \frac{\partial u}{\partial t} + b(x) \frac{\partial u}{\partial x} + c(x)u \\
u(t, 0) &= 0 = u(t, \pi) \\
u(0, x) &= u_0(x), \quad \frac{\partial u}{\partial t}(0, x) = u_1(x) \quad x \in (0, \pi).
\end{aligned}
\]

In Example 11, we have shown that the operator \( A \) defined in (2.1.9) is the infinitesimal generator of a unitary group on \( X = H^1_0(0, \pi) \times L^2(0, \pi) \). Let

\[
B\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ B_1u + B_2v \end{pmatrix} \quad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in X,
\]

where

\[
\begin{aligned}
B_1u &= b(x)u' + c(x)u \quad \forall u \in H^1_0(0, \pi) \\
B_2v &= a(x)v \quad \forall v \in L^2(0, \pi).
\end{aligned}
\]

Assuming \( a, b, c \in L^\infty(0, \pi) \), one obtains that

\[
\|B\begin{pmatrix} u \\ v \end{pmatrix}\| \leq M \|\begin{pmatrix} u \\ v \end{pmatrix}\| \quad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in X
\]

with

\[
M = \sqrt{|a|^2 + |b|^2 + |c|^2}.
\]

Therefore \( A + B \) is the infinitesimal generator of a \( C_0 \)-group on \( X \) which provides the solution of (3.1.9). Moreover, by (3.1.1) we conclude that

\[
\|e^{t(A+B)}\| \leq e^{M|t|} \quad \forall t \in \mathbb{R}.
\]

### 3.2 Perturbation of sectorial operators

**Theorem 9** Let \( A : D(A) \subset X \to X \) be the infinitesimal generator of an analytic semigroup and let \( B : D(B) \subset X \to X \) be a closed linear operator satisfying

(a) \( D(B) \supset D(A) \), and

(b) \( \exists a, b \geq 0 \) such that \( |Bx| \leq a|Ax| + b|x| \) for all \( x \in D(A) \).
There exists \( \alpha > 0 \) such that if \( 0 \leq a \leq \alpha \) then \( A + B : D(A) \subset X \to X \) is the infinitesimal generator of an analytic semigroup.

**Proof. Step 1: the case \( \omega = 0 \).**

Owing to Theorem 5 we have that there is an angle \( \theta_0 \in (\frac{\pi}{2}, \pi] \) such that \( \Sigma_{0, \theta_0} \subset \rho(A) \) and

\[
\|R(\lambda, A)\| \leq \frac{M_\theta}{|\lambda|} \quad \forall \theta \in (0, \theta_0), \forall \lambda \in \Sigma_{0, \theta}.
\]

Fix any \( \theta \in (\pi/2, \theta_0) \) and let \( \lambda \in \Sigma_{0, \theta} \). Then for every \( x \in X \)

\[
|BR(\lambda, A)x| \leq a|AR(\lambda, A)x| + b|R(\lambda, A)x| \\
\leq a(M_\theta + 1)|x| + \frac{bM_\theta}{|\lambda|}|x|.
\]

Hence, fixing any \( \varepsilon > 0 \) and choosing \( \alpha = \frac{1}{2(M_\theta + 1)} \) and \( |\lambda| \geq 2(bM_\theta + \varepsilon) \),

we have that

\[
\|BR(\lambda, A)\| \leq \frac{1}{2} + \frac{bM_\theta}{2(bM_\theta + \varepsilon)} = \frac{2bM_\theta + \varepsilon}{2(bM_\theta + \varepsilon)} < 1. \tag{3.2.1}
\]

Therefore \( I - BR(\lambda, A) \) is invertible and, recalling (3.1.2), by (3.2.1) we obtain

\[
\|R(\lambda, A + B)\| \leq \|[I - BR(\lambda, A)]^{-1}\| \|R(\lambda, A)\| \leq \frac{2(bM_\theta + \varepsilon)M_\theta}{\varepsilon|\lambda|}
\]

for all \( \lambda \in \Pi_{2(bM_\theta + \varepsilon)} \). By Proposition 15 we conclude that \( A + B \) is the infinitesimal generator of an analytic semigroup.

**Step 2: the general case.**

Consider \( A_\omega = A - \omega I \) with the associated semigroup \( e^{tA_\omega} = e^{-\omega t}e^{tA} \) which belongs to \( G(M, 0) \). Assumption (b) implies that

\[
|Bx| \leq a|A_\omega x| + (a\omega + b)|x| \quad \forall x \in D(A).
\]

By Step 1, \( A_\omega + B = A + B - \omega I \) is the infinitesimal generator of an analytic semigroup and the same is true for \( A + B \).

**Corollary 5** Let \( A : D(A) \subset X \to X \) be the infinitesimal generator of an analytic semigroup and let \( B \in \mathcal{L}(X) \). Then \( A + B : D(A) \subset X \to X \) is the infinitesimal generator of an analytic semigroup.
Example 16 Consider the problem
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + b(x) \frac{\partial u}{\partial x} + c(x) u \\
u(t, 0) &= u(t, \pi) = 0 = u(t, \pi) \\
u(0, x) &= u_0(x) \\
\end{aligned}
\tag{3.2.2}
\]
with \(u_0 \in X = L^2(0, \pi; \mathbb{C})\). Denote by \(A\) the operator in (2.2.16) (with \(p = 2\)) and define \(B : D(B) \subset X \to X\) by
\[
\begin{aligned}
D(B) &= H^1_0(0, \pi; \mathbb{C}) \\
Bf(x) &= b(x)f'(x) + c(x)f(x) \\
\end{aligned}
\]
As shown in Example 13, \(A\) is the infinitesimal generator of an analytic semigroup on \(X\). Assume now \(b \in L^\infty(0, \pi; \mathbb{C})\) and \(c \in L^2(0, \pi; \mathbb{C})\).

Then, in view of (3.5.2) and (3.5.3), we have that for all \(f \in D(A)\)
\[
|Bf| \leq |b|_{\infty}|f'|_2 + |c|_2|f|_\infty \leq \left(|b|_{\infty} + \sqrt{\frac{\pi}{2}}|c|_2\right)|f'|_2
\]
\[
\leq \left(|b|_{\infty} + \frac{\sqrt{\pi}}{2}|c|_2\right)\sqrt{|f''|_2|f|_2}.
\]
So, by the elementary inequality
\[
xy \leq \frac{\varepsilon}{2} x^2 + \frac{1}{2\varepsilon} y^2, \tag{3.2.3}
\]
which holds for all \(x, y \in \mathbb{R}\) and all \(\varepsilon > 0\), we conclude that
\[
|Bf| \leq \varepsilon|Af| + b_\varepsilon|f| \quad \forall \in D(A)
\]
for some constant \(b_\varepsilon > 0\).

Therefore, by Theorem 9, \(A + B\) generates an analytic semigroup which gives the unique solution of (3.2.2).

### 3.3 Perturbation of dissipative operators

Let \(X\) be an Hilbert space with scalar product \(\langle \cdot, \cdot \rangle\). We recall that a dissipative operator \(A : D(A) \subset X \to X\) is called \(m\)-dissipative if \(I - A\) is surjective.

**Theorem 10** Let \(A : D(A) \subset X \to X\) and \(B : D(B) \subset X \to X\) be linear operators satisfying
\(a)\) \(D(B) \supset D(A)\), and
(b) \( \exists a \in [0, 1), \exists b \geq 0 \) such that \( |Bx| \leq a|x| + b|x| \) for all \( x \in D(A) \).

If, in addition,

(c) \( A + tB \) is dissipative for all \( 0 \leq t \leq 1 \), and

(d) \( \exists t_0 \in [0, 1] \) such that \( A + t_0B \) is m-dissipative,

then \( A + tB \) is m-dissipative for all \( 0 \leq t \leq 1 \).

Proof. It suffices to show that there exists \( \delta > 0 \) such that, if \( A + t_0B \) is m-dissipative for some \( t_0 \in [0, 1] \), then \( A + tB \) is m-dissipative for all \( t \in [0, 1] \) satisfying \( |t_0 - t| \leq \delta \).

Assume that \( A + t_0B \) is m-dissipative for some \( t_0 \in [0, 1] \). Then

\[
R(t_0) := [I - (A + t_0B)]^{-1} \text{ satisfies } \|R(t_0)\| \leq 1.
\]

We now proceed to show that \( BR(t_0) \) is bounded. Owing to assumption (b), for all \( x \in D(A) \) we have that

\[
|Bx| \leq a|((A + t_0B)x + t_0|Bx|) + b|x| \\
\leq a|(A + t_0B)x| + a|Bx| + b|x|
\]

and so

\[
|Bx| \leq \frac{a}{1-a} |(A + t_0B)x| + \frac{b}{1-a} |x| \quad \forall x \in D(A). \tag{3.3.1}
\]

Since \( R(t_0)(X) \subset D(A) \) and \( (A + t_0B)R(t_0) = R(t_0) - I \), by (3.3.1) we get

\[
|BR(t_0)x| \leq \frac{a}{1-a} |R(t_0)x - x| + \frac{b}{1-a} |R(t_0)x| \leq \frac{2a + b}{1-a} |x| \tag{3.3.2}
\]

for all \( x \in X \), which shows that \( BR(t_0) \) is bounded.

Now, since

\[
I - (A + tB) = I - (A + t_0B) + (t_0 - t)B \\
= [I + (t_0 - t)BR(t_0)] [I - (A + t_0B)]
\]

and \( I - (A + t_0B) \), we deduce that \( I - (A + tB) \) is invertible if and only if \( I + (t_0 - t)BR(t_0) \) is invertible. In view of (3.3.2), this is definitely the case if

\[
||(t_0 - t)BR(t_0)|| \leq |t_0 - t| \frac{2a + b}{1-a} < 1.
\]

So, the proof is completed choosing \( \delta = \frac{1-a}{2a+b+1} \). \( \square \)

**Corollary 6** Let \( A : D(A) \subset X \rightarrow X \) be the infinitesimal generator of a \( C_0 \)-semigroup of contractions on \( X \) and let \( B : D(B) \subset X \rightarrow X \) be a dissipative operator satisfying
(a) \( D(B) \supset D(A) \), and

(b) \( \exists a \in [0, 1), \exists b \geq 0 \) such that \( |Bx| \leq a |Ax| + b |x| \) for all \( x \in D(A) \).

Then \( A + B : D(A) \subset X \to X \) is the infinitesimal generator of a \( C_0 \)-semigroup of contractions on \( X \).

**Example 17** The Schrödinger equation with potential \( V : (0, \pi) \to \mathbb{C} \)

\[
\begin{aligned}
\frac{\partial u}{\partial t} (t, x) &= i \frac{\partial^2 u}{\partial x^2} + V(x)u \quad (t, x) \in \mathbb{R} \times (0, \pi) \\
u(t, 0) &= 0 = u(t, \pi) \quad t \in \mathbb{R} \\
u(0, x) &= u_0(x) \quad x \in (0, \pi)
\end{aligned}
\]

can be studied using Theorem 7. We know that \( A \) defined in (2.1.11) is self-adjoint and dissipative, so that \( iA \) generates a unitary group on \( L^2(0, \pi; \mathbb{C}) \) by Stone’s theorem. Therefore, if \( V \in L^\infty(0, \pi; \mathbb{C}) \), then setting

\[Bf(x) = V(x)f(x) \quad \forall f \in X,
\]

from Theorem 7 if follows that \( iA + B \) is the infinitesimal generator of a \( C_0 \)-group on \( X \) satisfying

\[\|e^{t(iA+B)}\| \leq e^{\|V\|_{\infty} |t|} \quad \forall t \in \mathbb{R}.
\]

We can say more about this problem by using Corollary 6. Indeed, since

\[\Re\langle Bf, f \rangle = \int_0^\pi V(x)|f(x)|^2 \, dx \quad \forall f \in X,
\]

we conclude that if \( \Re V(x) \leq 0 \) for a.e. \( x \in (0, \pi) \), then

\[\|e^{t(iA+B)}\| \leq 1 \quad \forall t \geq 0.
\]

### 3.4 Stability under compact perturbations

A useful stability result due to Gibson [5] ensures that, perturbing the generator of an exponentially stable semigroup by a compact operator, one obtains an exponentially stable semigroup again, provided the perturbed semigroup is strongly stable. The original proof given in [5] used approximation by finite dimensional subspaces and a contradiction argument. The topic was then investigated by several authors including Triggiani [7]. Following [1], we now give a completely different proof of Gibson’s theorem based on a simple direct argument, extending the analysis to Banach spaces, and relaxing the original compactness assumptions.
3.4. STABILITY UNDER COMPACT PERTURBATIONS

**Theorem 11** Let $X$ be a reflexive Banach space and let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $C_0$-semigroup, $e^{tA}$, satisfying
\[
\|e^{tA}\| \leq M_0 e^{-\omega_0 t} \quad \forall t \geq 0 \tag{3.4.1}
\]
for some constants $M_0, \omega_0 > 0$. Let $B \in \mathcal{L}(X)$ be such that
\[
\lim_{t \to +\infty} e^{t(A+B)}x = 0 \quad \forall x \in X, \text{ and } \quad B e^{tA} \text{ is compact } \forall t > 0. \tag{3.4.2}
\]
Then, for some constants $M_B, \omega_B > 0$,
\[
\|e^{t(A+B)}\| \leq M_B e^{-\omega_B t} \quad \forall t \geq 0.
\]

**Proof.** To begin with, observe that, as in Remark 1, by the Banach-Steinhaus Theorem we deduce from (3.4.2) that, for some constant $M_1 > 0$,
\[
\|e^{t(A+B)}\| \leq M_1 \quad \forall t \geq 0. \tag{3.4.4}
\]
Now, appealing to a well-known characterization of exponential stability for strongly continuous semigroups (see Problems 1.7), we conclude that
\[
\lim_{t \to +\infty} \|e^{t(A+B)}\| = 0 \tag{3.4.5}
\]
suffices to obtain the desired conclusion. In order to prove (3.4.5), define
\[
\Lambda_t x = \int_0^t e^{(t-s)(A+B)} Be^{sA} x ds, \quad \forall x \in X, \forall t \geq 0.
\]
By Lemma 3 applied to $\tilde{A} := A + B$ and $\tilde{B} := -B$ we have that
\[
e^{t(A+B)}x = e^{tA}x + \Lambda_t x, \quad \forall x \in X, \forall t \geq 0. \tag{3.4.6}
\]
In view of (3.4.6) and (3.4.1) we have that, for every $t \geq 0$,
\[
\|e^{t(A+B)}\| = \sup_{|x| \leq 1} |e^{t(A+B)}x| \\
\leq \sup_{|x| \leq 1} |e^{tA}x| + \sup_{|x| \leq 1} |\Lambda_t x| \leq M_0 e^{-\omega_0 t} + \sup_{|x| \leq 1} |\Lambda_t x| \tag{3.4.7}
\]
Next, let $t_n$ be any sequence of positive numbers such that $t_n \to +\infty$ as $n \to \infty$ and choose vectors $x_n$, with $|x_n| \leq 1$, such that
\[
\sup_{|x| \leq 1} |\Lambda_{t_n} x| < |\Lambda_{t_n} x_n| + \frac{1}{n}. \tag{3.4.8}
\]
Now, extract a weakly convergent subsequence—still labeled $x_n$—to some limit $\bar{x} \in X$. We claim that
\[
\Lambda_{t_n}(x_n - \bar{x}) \to 0 \quad \text{strongly as } \ n \to \infty. \tag{3.4.9}
\]
Indeed, consider the sequence of vector-valued functions
\[ \phi_n(s) := Be^{sA}(x_n - \bar{x}) \quad s \geq 0. \]

Owing to assumption (3.4.1), for all \( n \in \mathbb{N} \) we have
\[ |\phi_n(s)| \leq 2M_0\|B\|e^{-\omega_0 s} \quad \forall s \geq 0. \]

Moreover, on account of (3.4.3), \( \phi_n(s) \) strongly converges to 0, as \( n \to \infty \), for all \( s > 0 \). Therefore, invoking Lebesgue’s dominated convergence theorem for vector-valued functions, we conclude that \( \phi_n \to 0 \) in \( L^1(0, \infty; X) \) as \( n \to \infty \).

Consequently, thanks to (3.4.4),
\[ |\Lambda t_n(x_n - \bar{x})| \leq M_1 \int_0^\infty |\phi_n(s)| \, ds \to 0 \quad \text{as} \quad n \to \infty, \]
which proves (3.4.9).

Finally, combining (3.4.7), (3.4.8) and (3.4.9) we obtain
\[ \|e^{t_n(A+B)}\| \leq M_0e^{-\omega_0 t_n} + \frac{1}{n} + |\Lambda t_n(x_n - \bar{x})| + |\Lambda t_n \bar{x}| \to 0 \quad \text{as} \quad n \to \infty, \quad (3.4.10) \]

where, according to (3.4.6), the fact that
\[ \Lambda t_n \bar{x} = e^{t_n(A+B)} \bar{x} - e^{t_n A} \bar{x} \to 0 \quad \text{as} \quad n \to \infty \]
follows from assumptions (3.4.1) and (3.4.2). Since \( \{t_n\} \) is an arbitrary sequence going to \( \infty \), (3.4.10) yields (3.4.5) and completes the proof. \( \square \)

**Example 18** Consider the heat equation with potential \( V \) on \( (0, \pi) \)

\[ \begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + V(x)u & (t, x) \in \mathbb{R}_+ \times (0, \pi) \\ u(t, 0) = u(t, \pi) = 0 & t \geq 0 \\ u(0, x) = u_0(x) & x \in (0, \pi) \end{cases} \quad (3.4.11) \]

with \( u_0 \in X = L^2(0, \pi) \) and \( V \in \mathcal{C}([0, \pi]) \).

Denote by \( A \) the operator in (2.2.16) \( (p = 2) \) and define
\[ Bf(x) = V(x)f(x) \quad \text{a.e.} \quad x \in (0, \pi), \quad \forall f \in X. \]

Then \( B \in \mathcal{L}(X) \) and
\[ \|B\| = |V|_\infty. \quad (3.4.12) \]

Moreover, as shown in Example 13, \( A \) is the infinitesimal generator of an analytic semigroup on \( X \) and \( \sigma(A) = \{ -n^2 : n \geq 1 \} \). Let us prove that
\[ \|e^{tA}\| \leq e^{-t} \quad \forall t \geq 0. \quad (3.4.13) \]
Indeed, for any \( u_0 \in X \setminus \{0\} \) the function \( v(t, x) := e^{tA}u_0(x) \) satisfies
\[
\frac{1}{2} \frac{d}{dt} \int_0^\pi v^2(t, x)dx = -\int_0^\pi \left( \frac{\partial v}{\partial x}(t, x) \right)^2 dx \leq -\int_0^\pi v^2(t, x)dx \quad (t > 0)
\]
thanks to Poincaré’s inequality (3.5.1). Thus, \( \frac{d}{dt} \log |e^{tA}u_0| \leq -2 \) and
\[
|e^{tA}u_0| \leq e^{-t|u_0|} \quad \forall t \geq 0.
\]

Now, appealing to Theorem 7, by (3.4.12) we conclude that
\[
\|e^{t(A+B)}\| \leq e^{(|V|_\infty - 1)t} \quad \forall t \geq 0
\]
(3.4.14)
so that \( e^{t(A+B)} \) remains exponentially stable if \( |V|_\infty < 1 \).

Let us prove that the same holds true under the weaker assumption
\[
M := \max_{[0, \pi]} V < 1.
\]
Fix any \( u_0 \in X \) and let \( u(t, x) = e^{t(A+B)}u_0(x) \). Proceeding as above we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_0^\pi u^2(t, x)dx = \int_0^\pi \left\{ V(x)u^2(t, x) - \left( \frac{\partial u}{\partial x}(t, x) \right)^2 \right\} dx
\leq (M - 1) \int_0^\pi u^2(t, x)dx
\]
thanks to Poincaré’s inequality. So,
\[
|e^{tA}u_0| \leq e^{(M-1)t|u_0|} \quad \forall t \geq 0
\]
(3.4.16)
which yields exponential stability since \( M < 1 \).

Finally, observe that, for \( V \equiv 1 \), (3.4.11) fails to be even strongly stable because it admits the stationary solution \( u(t, x) = \sin x \).

**Example 19** Continuing the analysis of the previous example we now want to prove the exponential decay at \( \infty \) of the solution of (3.4.11) when \( V \) satisfies the conditions
\[
\begin{align*}
(a) \quad & V(x) \leq 1 \quad \forall x \in [0, \pi] \\
(b) \quad & \exists (\alpha, \beta) \subset [0, \pi] : V(x) < 1 \quad \forall x \in (\alpha, \beta).
\end{align*}
\]
(3.4.17)
which are weaker than (3.4.15).

Recalling that \( e^{tA} \) is compact for \( t > 0 \) (see Example 14) and appealing to Theorem 11, we conclude that \( e^{t(A+B)} \) is exponentially stable if it is strongly stable. Let us show the last assertion by La Salle’s invariance argument. Fix any \( u_0 \in X \) and let \( u(t, x) = e^{t(A+B)}u_0(x) \). Then, for all \( t > 0 \),
\[
\frac{1}{2} \frac{d}{dt} \int_0^\pi u^2(t, x)dx = \int_0^\pi \left\{ V(x)u^2(t, x) - \left( \frac{\partial u}{\partial x}(t, x) \right)^2 \right\} dx \leq 0
\]
(3.4.18)
thanks to Poincaré’s inequality and assumption (3.4.17)-(a). The above inequality proves that 
\[ E(t) := |e^{tA}u_0|^2 \]
is non-decreasing and so
\[ E(t) \downarrow E_\infty \text{ as } t \to +\infty. \tag{3.4.19} \]
Take any sequence \( t_n \geq 0 \) such that \( t_n \uparrow +\infty \) (for instance, \( t_n = n \)). Since 
\[ E(t) \leq E(0), \]
there exists a subsequence, still labeled \( t_n \), such that
\[ u_n := e^{t_n(A+B)}u_0 \to u_\infty \quad (n \to \infty) \]
and, since \( e^{t(A+B)} \) is compact for \( t > 0 \),
\[ \lim_{n \to \infty} e^{t(A+B)}u_n = e^{t(A+B)}u_\infty \quad \forall t > 0. \]
So, in view of (3.4.19),
\[ |e^{t(A+B)}u_\infty| = \lim_{n \to \infty} |e^{t(A+B)}u_n| = \lim_{n \to \infty} |e^{(t_n+t)(A+B)}u_0| \]
\[ = \lim_{t \to +\infty} |e^{t(A+B)}u_0| = \sqrt{E_\infty}, \]
which implies that \( |e^{t(A+B)}u_\infty|^2 = E_\infty \) for all \( t \geq 0 \). By differentiating such an identity we have that 
\[ U(t, x) = e^{t(A+B)}u_\infty(x) \]
satisfies
\[ 0 = \frac{1}{2} \frac{d}{dt} \int_0^\pi U^2(t, x)dx = \int_0^\pi \left\{ V(x)U^2(t, x) - \left( \frac{\partial U}{\partial x} (t, x) \right)^2 \right\} dx \]
\[ \leq \int_0^\pi \left\{ U^2(t, x) - \left( \frac{\partial U}{\partial x} (t, x) \right)^2 \right\} dx \leq 0. \]
In other terms, the above are all equalities and \( U(t, \cdot) \) is a function realizing the identity in Poincaré’s inequality. Therefore,
\[ U(t, x) = c(t) \sin x \]
and, since \( U \) solves the equation in (3.4.11),
\[ c'(t) \sin x = -c(t) \sin x + c(t)V(x) \sin x. \]
Consequently, either \( V \) is constant, say \( V(x) = M \) for all \( x \in [0, \pi] \), or \( c \equiv 0 \). In the former case, we must have \( M < 1 \) by (3.4.17)-(b). Then (3.4.16) implies that \( e^{t(A+B)} \) is exponentially stable. In the latter, we get that \( e^{t(A+B)} \) is strongly stable, hence exponentially stable by Theorem 11.

3.5 Problems

1. In the situation considered in Theorem 9, suppose that \( \|e^{tA}\| \leq Me^{\omega t} \).
   Show that
   \[ \|e^{t(A+B)}\| \leq M(b)e^{(\omega+\Lambda(b))t} \]
where $\Lambda(b) > 0$ satisfies
\[
\lim_{b \to 0} \Lambda(b) = 0.
\]

**Solution.** By Theorem 9 we have that $S_B(t) := e^{(A+B)t}$ is analytic provided that $0 \leq a \leq \alpha$. Then $\omega_0(S_B) = s(A + B)$ by Proposition 16. Moreover, the proof of Theorem 9 shows that
\[
s(A + B) \leq \omega + 2bM_\theta.
\]
The conclusion follows from (1.3.3).

2. Prove that for every $f \in H^1_0(0, \pi)$ the following inequalities hold:

- **Poincaré inequality**
  \[
  |f|_2 \leq |f'|_2. \quad (3.5.1)
  \]

- **Sobolev inequality**
  \[
  |f|_\infty \leq \frac{\sqrt{\pi}}{2} |f'|_2. \quad (3.5.2)
  \]
  Moreover, show that both inequalities are sharp (i.e., for each inequality find a function $f \in H^1_0(0, \pi)$ for which equality holds).

3. Prove that for every $f \in H^2(0, \pi) \cap H^1_0(0, \pi)$ the following Gagliardo-Nirenberg inequality holds:
\[
|f'|_2 \leq \sqrt{|f''|_2} |f|_2. \quad (3.5.3)
\]

4. Let $X = L^p(\mathbb{R}_+)$ with $p \geq 1$. Prove that the left-translation semigroup
\[
(S(t)f)(x) = f(x + t) \quad x \in \mathbb{R}_+ \text{ a.e.}
\]
is strongly stable on $X$ but not exponentially stable.

4 The inhomogeneous Cauchy problem

4.1 The Bochner integral

Let $X$ be a separable Banach space and let $f : J \to X$ be a Borel function on some interval $J = (\alpha, \beta) \subset \mathbb{R}$. Observe that, since the norm is continuous, $t \mapsto |f(t)|$ is also a Borel function.

**Definition 13** A vector-valued Borel function $f : J \to X$ is called Bochner integrable if
\[
\int_J |f(t)| \, dt < \infty.
\]
We summarize here the main properties of the Bochner integral.

- The function \( f : J \to X \) is called simple if it can be represented as
  \[
  f(t) = \sum_{k=1}^{m} x_k \cdot \chi_{J_k}(t)
  \]
  for some choice of elements \( x_k \in X \) and disjoint (Lebesgue) measurable subsets \( J_k \subset J \) such that \( J = \bigcup_{k=1}^{m} J_k \), where we have denoted by \( \chi_{J_k} \) the characteristic functions of the set \( J_k \).

- The Bochner integral of a simple function \( f : J \to X \) is defined as
  \[
  \int_{J} f(t) \, dt = \sum_{k=1}^{m} x_k \cdot |J_k|
  \]
  where \( |J_k| \) denotes the Lebesgue measure of \( J_k \). One can show that the above definition is independent of the representation of \( f \) in (4.1.1). Moreover, for any simple function \( f : J \to X \) we have that
  \[
  \left| \int_{J} f(t) \, dt \right| \leq \int_{J} |f(t)| \, dt.
  \] (4.1.2)

- If \( f : J \to X \) is Bochner integrable, then there exists a sequence \( \{f_n\} \) of simple functions such that
  \[
  \forall t \in J \quad |f_n(t) - f(t)| \downarrow 0 \quad \text{as} \quad n \to \infty. \quad (4.1.3)
  \]

  \textbf{Proof.} Let \( \{e_j\}_{j \in \mathbb{N}} \) a dense countable subset of \( X \). Define
  \[
  \gamma_n(t) = \min\{|f(t) - e_j| : 1 \leq j \leq n\}
  \]
  \[
  j_n(t) = \min\{j \leq n : \gamma_n(t) = |f(t) - e_j|\}.
  \]
  Then
  \[
  f_n(t) := e_{j_n(t)} \quad (n \geq 1, \ t \in J)
  \]
  is a Borel simple function and \( \{f_n\} \) satisfies (4.1.3). \hfill \Box

- Observe that, in view of (4.1.3),
  \[
  \lim_{n \to \infty} \int_{J} |f_n(t) - f(t)| \, dt = 0.
  \]
  This together with (4.1.2) implies that \( \{\int_{J} f_n(t) \, dt\} \) is a Cauchy sequence in \( X \). Therefore we can define
  \[
  \int_{J} f(t) \, dt = \lim_{n \to \infty} \int_{J} f_n(t) \, dt
  \]
  where \( \{f_n\} \) is any sequence of simple functions satisfying (4.1.3).
4.1. THE BOCHNER INTEGRAL

- Estimate (4.1.2) holds true for any Bochner integrable function \( f \).

- For any sequence \( g_n : J \to X \) of Bochner integrable functions
  
  \[
  \lim_{n \to \infty} \int_J |g_n(t) - f(t)| \, dt = 0 \implies \lim_{n \to \infty} \int_J g_n(t) \, dt = \int_J f(t) \, dt.
  \]

- Lebesgue’s dominated convergence theorem holds true:
  for any sequence \( f_n : J \to X \) of Bochner integrable functions, if
  
  \[
  \begin{cases}
  (a) & f_n(t) \to f(t) \text{ a.e. as } n \to \infty \\
  (b) & |f_n(t)| \leq \phi(t) \text{ a.e. with } \phi \in L^1(J),
  \end{cases}
  \]
  then
  
  \[
  \begin{cases}
  (a) & f \text{ is Bochner integrable} \\
  (b) & \lim_{n \to \infty} \int_J |f_n(t) - f(t)| \, dt = 0.
  \end{cases}
  \]

- Let \( A : D(A) \subset X \to X \) be a closed operator. If \( F : J \to X \) is a Bochner integrable function such that
  
  \[
  \begin{cases}
  (a) & f(t) \in D(A) \text{ (} t \in J \text{ a.e.)} \\
  (b) & t \mapsto Af(t) \text{ is Bochner integrable},
  \end{cases}
  \]
  then
  
  \[
  \int_J f(t) \, dt \in D(A) \quad \text{and} \quad A\left(\int_J f(t) \, dt\right) = \int_J Af(t) \, dt. \quad (4.1.4)
  \]

Let now \( p \geq 1 \) and \( -\infty \leq \alpha < \beta \leq +\infty \).

**Definition 14** We denote by \( L^p(\alpha, \beta; X) \) the space of all (equivalence classes of) functions \( f : (\alpha, \beta) \to X \) which are Bochner integrable on each \( J \subset \subset (\alpha, \beta) \) and such that

\[
\|f\|_p := \left( \int_{\alpha}^{\beta} |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty.
\]

Here are some useful properties of \( L^p(\alpha, \beta; X) \).

- \((L^p(\alpha, \beta; X), \|\cdot\|_p)\) is a Banach space.

- If \((X, \langle \cdot, \cdot \rangle)\) is an Hilbert space, then \( L^2(\alpha, \beta; X) \) is an Hilbert space as well with the scalar product

  \[
  \langle f, g \rangle_2 = \int_{\alpha}^{\beta} \langle f(t), g(t) \rangle \, dt \quad \forall f, g \in L^2(\alpha, \beta; X).
  \]
Definition 15 We denote by $W^{1,p}(\alpha, \beta; X)$ the (Sobolev) space of all $f \in L^p(\alpha, \beta; X)$ which possess a continuous representative satisfying

$$f(t) = f(t_0) + \int_{t_0}^{t} g(s) \, ds \quad \forall t \in (\alpha, \beta)$$

for some $t_0 \in (\alpha, \beta)$ and $g \in L^p(\alpha, \beta; X)$.

Some useful properties of $W^{1,p}(\alpha, \beta; X)$ are listed below.

- Every $f \in W^{1,p}(\alpha, \beta; X)$ is differentiable a.e. in $[\alpha, \beta]$ and $f'(t) = g(t)$ for a.e. $t \in [\alpha, \beta]$.

- $W^{1,p}(\alpha, \beta; X)$ is a Banach space with the norm

$$\|f\|_{1,p} = \|f\|_p + \|f'\|_p \quad \forall f \in W^{1,p}(\alpha, \beta; X).$$

- For any $p > 1$, we have that $W^{1,p}(\alpha, \beta; X) \subset C^{0,1-\frac{1}{p}}([\alpha, \beta]; X)$ with continuous embedding. Indeed,

$$|f(t) - f(s)| \leq |t - s|^{1-\frac{1}{p}} \|f'\|_p \quad \forall f \in W^{1,p}(\alpha, \beta; X)$$

by Hölder’s inequality. Consequently, $W^{1,p}(\alpha, \beta; X) \subset C([\alpha, \beta]; X)$ is compact by Ascoli’s theorem. Observe that we also have that

$$W^{1,1}(\alpha, \beta; X) \subset C([\alpha, \beta]; X) \quad (4.1.5)$$

but the embedding fails to be compact.

4.2 Solution of the Cauchy problem

Let $X$ be a separable Banach space and let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $C_0$-semigroup of bounded linear operators on $X$, denoted indifferently by $S(t)$ or $e^{tA}$, which satisfies

$$\|S(t)\| \leq Me^{\omega t} \quad \forall t \geq 0 \quad (4.2.1)$$

for some $M > 0$ and $\omega \in \mathbb{R}$.

We observe that $D(A)$ is a Banach space with the graph norm

$$\|x\|_{D(A)} = |x| + |Ax| \quad \forall x \in D(A).$$

For any fixed $T > 0$, consider the initial value problem

$$\begin{cases}
  u'(t) = Au(t) + f(t), & t \in (0, T) \\
  u(0) = x
\end{cases} \quad (4.2.2)$$

where $x \in X$ and $f \in L^p(0, T; X)$ for a given $p \geq 1$. For the above problem we will give two notions of solutions following [2]. Then we will study the existence, uniqueness, regularity, and asymptotic behavior of solutions.
4.2. SOLUTION OF THE CAUCHY PROBLEM

Notions of solution

Definition 16 Let \( p \geq 1 \) and let \( f \in L^p(0, T; X) \).

- We say that \( u \in W^{1,p}(0, T; X) \cap L^p(0, T; D(A)) \) is a strict solution of problem (4.2.2) if \( u(0) = x \) and
  \[
  u'(t) = Au(t) + f(t) \quad \text{for a.e. } t \in (0, T).
  \]

- We say that \( u \in L^p(0, T; X) \) is a strong solution of problem (4.2.2) if there exists a sequence \( v_n \in W^{1,p}(0, T; X) \cap L^p(0, T; D(A)) \) such that
  \[
  \begin{aligned}
  &v_n \to u \quad \text{and} \quad v_n' - Av_n \to f \quad \text{in } L^p(0, T; X) \\
  &v_n(0) \to x \quad \text{in } X
  \end{aligned} \quad (n \to \infty)
  \]

Definition 17 Let \( f \in C([0, T]; X) \).

- We say that \( u \in C^1([0, T]; X) \cap C([0, T]; D(A)) \) is a strict solution of problem (4.2.2) if \( u(0) = x \) and
  \[
  u'(t) = Au(t) + f(t) \quad \forall t \in (0, T).
  \]

- We say that \( u \in C([0, T]; X) \) is a strong solution of problem (4.2.2) if there exists a sequence \( v_n \in C^1([0, T]; X) \cap C([0, T]; D(A)) \) such that
  \[
  \begin{aligned}
  &v_n \to u \quad \text{and} \quad v_n' - Av_n \to f \quad \text{in } C([0, T]; X) \\
  &v_n(0) \to x \quad \text{in } X
  \end{aligned} \quad (n \to \infty)
  \]

Existence and uniqueness of solutions

Theorem 12 Let \( x \in X \) and let \( f \in L^p(0, T; X) \) (resp. \( f \in C([0, T]; X) \)). Then problem (4.2.2) has a unique strong solution given by
\[
  u(t) = S(t)x + \int_0^t S(t-s)f(s) \, ds \quad (t \in [0, T]).
  \]

Proof. Step 1: existence.
Let \( S \in \mathcal{G}(M, \omega) \) and observe that, since
\[
  \left| \int_0^t S(t-s)f(s) \, ds \right|^p \leq M^p \left( \int_0^t e^{\gamma \omega(t-s)} \, ds \right)^{p-1} \int_0^t |f(s)|^p \, ds \quad (t \in [0, T]),
  \]
the function \( u \) given by (4.2.5) is bounded and therefore belongs to \( L^p(0, T; X) \).
Define
\[
  \begin{cases}
  v_n(t) = nR(n, A)u(t) \\
  f_n(t) = nR(n, A)f(t) \quad \forall n \in \mathbb{N}, \ n > \omega.
  \end{cases}
  \]
By applying \( nR(n, A) \) to all the terms in (4.2.5) we obtain
\[
vn(t) = S(t)x_n + \int_0^t S(t-s)fn(s) \, ds \quad (t \in [0, T]).
\]
(4.2.6)

Since \( x_n \in D(A) \) and \( f_n \in L^p(0,T;D(A)) \) (resp. \( f_n \in C([0,T];X) \)), \( vn \) is differentiable for a.e. \( t \) and we have that \( vn' - Avn = fn \). Moreover, Lebesgue’s dominated convergence theorem and the properties of the Yosida approximation used in Step 1 of the proof of Theorem 2 yield
\[
\begin{align*}
\begin{cases}
v_n \to u & \text{and } v'_n - Av_n \to f \quad \text{in } L^p(0,T;X) \ (\text{resp. } C([0,T];X)) \\
v_n(0) \to x & \text{in } X.
\end{cases}
\end{align*}
\]

So, \( u \) is a strong solution of (4.2.2).

**Step 1: uniqueness.**

Let \( u \) be a strong solution of (4.2.2) and let \( \{v_n\} \) be a sequence satisfying (4.2.3) (resp. (4.2.4)). We set \( fn = v'_n - Av_n \) and \( x_n = v_n(0) \). Then
\[
\frac{d}{ds} (S(t-s)v_n(s)) = S(t-s)fn(s) \quad (s \in [0,t]).
\]

By integrating over \([0,t]\) we deduce that \( vn \) satisfies (4.2.6). Passing to the limit as \( n \to \infty \) we conclude that \( u \) is given by (4.2.5). \( \square \)

The following result provides a useful approximation of strong solutions.

**Proposition 18** Let \( \{x_n\} \subset X \) and \( \{fn\} \subset L^p(0,T;X) \) (\( p \geq 1 \)) be such that
\[
x_n \xrightarrow{X} x \quad \text{and} \quad fn \xrightarrow{L^p(0,T;X)} f \quad (n \to \infty).
\]

Let
\[
\begin{align*}
\begin{cases}
u'_n(t) = Anu_n(t) + fn(t), & t \in (0,T) \\
u_n(0) = x_n
\end{cases}
\end{align*}
\]
where \( An = n^2R(n,A) - n \), \( n > \omega \), is the Yosida approximation of \( A \). Then
\[
u_n \xrightarrow{L^p(0,T;X)} u \quad (n \to \infty)
\]

where \( u \) is the strong solution of (4.2.2).

**Proof.** Since \( An \in L(X) \) we have that
\[
u_n(t) = e^{tAn}x_n + \int_0^t e^{(t-s)An}fn(s) \, ds \quad (t \in [0,T]).
\]
Thus, recalling (1.5.7) and (1.5.8) from the proof of the Hille-Yosida theorem, we obtain
\[
|e^{tAn}x_n - e^{tA}x| \leq Me^{2\omega t}|x_n - x| + |e^{tAn}x - e^{tA}x| \xrightarrow{n \to \infty} 0
\]
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uniformly on \([0, T]\). Moreover,

\[
\left| \int_0^t \left( e^{(t-s)A_n} f_n(s) - e^{(t-s)A} f(s) \right) ds \right|^p \\
\leq 2^{p-1} M^p \int_0^t e^{2\omega p(t-s)} |f_n(s) - f(s)|^p ds \xrightarrow{C([0,T];X)} 0,
\]

\[
+ 2^{p-1} \int_0^t |e^{(t-s)A_n} f(s) - e^{(t-s)A} f(s)|^p ds
\]

where, by Lebesgue’s dominated convergence theorem,

\[
\lim_{n \to \infty} \int_0^T dt \int_0^t |e^{(t-s)A_n} f(s) - e^{(t-s)A} f(s)|^p ds = 0.
\]

The conclusion follows. \(\square\)

Regularity of solutions

Our first result guarantees that the strong solution of (4.2.2) is strict when \(f\) has better “space regularity”.

**Theorem 13** Let \(x \in D(A)\) and let \(f \in L^p(0,T;D(A))\) for some \(p \geq 1\). Then the strong solution \(u\) of problem (4.2.2) is strict in \(L^p(0,T;X)\).

**Proof.** Let \(u\) be the strong solution of problem (4.2.2) and let \(u_n\) be the solution of

\[
\begin{align*}
\begin{cases}
  u_n'(t) = A_n u_n(t) + f(t), & t \in (0,T) \\
  u_n(0) = x
\end{cases}
\end{align*}
\]

where \(A_n = n^2 R(n,A) - n, \ n > \omega,\) is the Yosida approximation of \(A\). Then

\[v_n(t) := A_n u_n(t) \quad (t \in [0,T])\]

satisfies

\[
\begin{align*}
\begin{cases}
  v_n'(t) = A_n v_n(t) + A_n f(t), & t \in (0,T) \\
  v_n(0) = A_n x
\end{cases}
\end{align*}
\]

where

\[A_n x \xrightarrow{X} Ax \quad \text{and} \quad A_n f \xrightarrow{L^p(0,T;X)} A f \quad (n \to \infty).\]

So, Proposition 18 ensures that \(v_n\) converges in \(L^p(0,T;X)\) to the strong solution of

\[
\begin{align*}
\begin{cases}
  v'(t) = Av(t) + Af(t), & t \in (0,T) \\
  v(0) = Ax
\end{cases}
\end{align*}
\]

which, by Theorem 12 is given by

\[v(t) = e^{tA} Ax + \int_0^t e^{(t-s)A} f(s) ds = Au(t) \quad (t \in [0,T] \ \text{a.e.})\]
This shows that \( u \in L^p(0,T; D(A)) \). Moreover

\[
u'_n = A_n u_n + f = v_n + f \xrightarrow{L^p(0,T;X)} v + f \quad (n \to \infty).
\]

Therefore, \( u \in W^{1,p}(0,T; X) \) and \( u' = v + f = Au + f \). \(\square\)

**Corollary 7** Let \( x \in X \) and let \( f \in L^p(0,T; X) \). Then the strong solution \( u \) of problem (4.2.2) belongs to \( C([0,T]; X) \). Moreover, we have that

\[
u_n \xrightarrow{C([0,T];X)} u \quad (n \to \infty),
\]

where \( u_n \) is the strict solution of the problem

\[
\begin{cases}
  u'_n(t) = Au_n(t) + f_n(t), & t \in (0,T) \\
  u_n(0) = x_n
\end{cases}
\]

(4.2.8)

with \( f_n(t) = nR(n,A)f(t) \) and \( x_n = nR(n,A)x \) for all \( n > \omega \).

**Proof.** By (1.5.6) we have that \( D(A) \ni x_n X \xrightarrow{u} u \) as \( n \to \infty \). Moreover, \( f_n \in L^p(0,T; D(A)) \) and

\[
\lim_{n \to \infty} f_n(t) = f(t), \quad \text{and} \quad |f_n(t)| \leq \frac{Mn}{n - \omega} |f(t)| \quad \text{a.e. in } [0,T].
\]

So, \( f_n \xrightarrow{L^p(0,T;X)} f \) by Lebesgue’s theorem. Then Theorem 13 ensures that (4.2.8) has a unique strict solution \( u_n \) which, in particular, belongs to \( C([0,T ]; X) \). Now, the representation formula (4.2.5) implies that, for all \( t \in [0,T] \),

\[
|u_n(t) - u(t)| = \left| e^{tA}(x_n - x) + \int_0^t e^{(t-s)A}[f_n(s) - f(s)] \, ds \right|
\]

\[
\leq Me^{\omega t}|x_n - x| + M \int_0^t e^{\omega(t-s)}|f_n(s) - f(s)| \, ds
\]

\[
\leq C_T (|x_n - x| + \|f_n - f\|_p)
\]

for some constant \( C_T > 0 \). The conclusion follows. \(\square\)

We will now show a similar result for \( f \) with better “time regularity”. We begin by studying the case of \( x = 0 \).

**Lemma 4** Let \( f \in W^{1,p}(0,T; X) \) for some \( p \geq 1 \). Then

\[
u_f(t) := \int_0^t e^{(t-s)A}f(s) \, ds \quad (t \in [0,T])
\]

belongs to \( W^{1,p}(0,T; X) \cap L^p(0,T; D(A)) \) and

\[
u'_f(t) = Au_f(t) + f(t) = e^{tA}f(0) + \int_0^t e^{(t-s)A}f'(s) \, ds \quad (t \in [0,T] \ a.e.)
\]
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Proof. Since
\[ u_f(t) = \int_0^t e^{sA} f(t - s) ds \quad (t \in [0, T]) \]
by differentiating under the integral sign we have that \( u_f \in W^{1,p}(0, T; X) \) and
\[ u_f'(t) = e^{tA} f(0) + \int_0^t e^{(t-s)A} f'(s) ds \quad (t \in [0, T] \text{ a.e.}). \] (4.2.9)

Therefore, we also have
\[ u_f'(t) = \lim_{h \to 0} \frac{1}{h} \left\{ \int_0^{t+h} e^{(t+h-s)A} f(s) ds - \int_0^t e^{(t-s)A} f(s) ds \right\} \]
\[ = \lim_{h \to 0} \frac{e^{hA} - I}{h} \int_0^t e^{(t-s)A} f(s) ds + \frac{1}{h} \int_t^{t+h} e^{(t+h-s)A} f(s) ds \]
\[ = \lim_{h \to 0} \frac{e^{hA} - I}{h} \int_0^t e^{(t-s)A} f(s) ds + f(t). \]
This shows that \( u_f(t) \in D(A) \) and \( Au_f(t) = u_f'(t) - f(x) \). Consequently, \( u_f \in L^p(0, T; D(A)) \) and the conclusion follows recalling (4.2.9). \( \square \)

Theorem 14 Let \( x \in D(A) \) and let \( f \in W^{1,p}(0, T; X) \) for some \( p \geq 1 \). Then the strong solution \( u \) of problem (4.2.2) is strict in \( L^p(0, T; X) \).

Proof. Let \( u \) be the strong solution of problem (4.2.2) and let \( u_n \) be the solution of (4.2.7). Then \( u_n \in C^1([0, T]; X) \) and \( v_n := u_n' \) satisfies
\[
\begin{cases} 
  v_n \in W^{1,p}(0, T; X) \\
  v_n'(t) = A v_n(t) + f'(t), \quad t \in (0, T) \text{ a.e.} \\
  v_n(0) = A x + f(0). 
\end{cases} \] (4.2.10)

So, Proposition 18 ensures that \( v_n \) converges in \( L^p(0, T; X) \) to the strong solution of
\[
\begin{cases} 
  v'(t) = A v(t) + f'(t), \quad t \in (0, T) \\
  v(0) = A x + f(0), 
\end{cases} \]

Therefore, \( u \in W^{1,p}(0, T; X) \) and Lemma 4 yields
\[ u(t) = v(t) = e^{tA}(A x + f(0)) + \int_0^t e^{(t-s)A} f'(s) ds \]
\[ = A u(t) + f(t) \quad t \in (0, T) \text{ a.e.} \]
The conclusion follows. \( \square \)

Remark 10 In general, the strong solution of (4.2.2) fails to be strict for \( f \in C([0, T]; D(A)) \). To see an example, let \( y \notin D(A) \) and take \( f(t) = e^{tA} y \) and \( x = 0 \). Then
\[ u(t) = te^{tA} y \quad \forall t \geq 0 \]
which fails to be differentiable.
4.3 Maximal regularity results

For special classes of generators the strong solution of (4.2.2) in $L^p(0,T;X)$ enjoys additional regularity properties. We investigate below the case of $p = 2$ when $(X,\langle \cdot, \cdot \rangle)$ is a separable Hilbert space, beginning by analyzing the problem for $x = 0$. As in in Lemma 4, we denote by $u_f$ the strong solution of

\[
\begin{cases}
  u'(t) = Au(t) + f(t), & t \in (0,T) \\
  u(0) = 0
\end{cases}
\]  

(4.3.1)

which is given by the function

\[
u_f(t) := \int_0^t e^{(t-s)A} f(s) ds \quad (t \in [0,T]).
\]

(4.3.2)

Theorem 15 Let $A : D(A) \subset X \to X$ be a self-adjoint dissipative operator on a real Hilbert space $X$ and let $f \in L^2(0,T;X)$. Then $u_f$ is the strict solution of (4.3.1) and

\[
\|Au_f\|_2 \leq \|f\|_2.
\]

(4.3.3)

Proof. Consider, as in the proof of Corollary 7, $f_n(t) := nR(n,A)f(t)$ for all $n > \omega$. We have that $f_n \in L^p(0,T;D(A))$ and

\[
\int_0^T |f_n|^2 dt \to f \quad (n \to \infty).
\]

(4.3.4)

Then $u_n := u_{f_n} \in W^{1,2}(0,T;X) \cap L^2(0,T;D(A))$ satisfies

\[
u'_n = Au_n + f_n \quad \text{a.e. in} \quad [0,T].
\]

(4.3.5)

Therefore

\[
\frac{1}{2} \frac{d}{dt} \langle Au_n, u_n \rangle = \langle u'_n, Au_n \rangle = |Au_n|^2 + \langle f_n, Au_n \rangle
\]

and

\[
|Au_n|^2 - \frac{1}{2} \frac{d}{dt} \langle Au_n, u_n \rangle = -\langle f_n, Au_n \rangle \leq \frac{1}{2} \left( |f_n|^2 + |Au_n|^2 \right).
\]

Hence, integrating on $[0,T]$, since $A$ is dissipative we get

\[
\int_0^T |Au_n|^2 dt \leq \int_0^T |Au_n|^2 dt - \langle Au_n(T), u_n(T) \rangle \leq \int_0^T |f_n|^2 dt.
\]

(4.3.6)

Now, applying the above inequality to the difference $u_n - u_m$ we obtain

\[
\|A(u_n - u_m)\|_2 \leq \|f_n - f_m\|_2 \quad \forall m, n > \omega,
\]

which implies that $\{u_n\}$ is a Cauchy sequence in $W^{1,2}(0,T;X) \cap L^2(0,T;D(A))$ in view of (4.3.4). By Corollary 7, $u_n$ converges to $u$ in $C([0,T];X)$ and so,
4.3. MAXIMAL REGULARITY RESULTS

recalling (4.3.5), we conclude that \( u \) is the strict solution of (4.3.1). Finally, (4.3.3) follows from (4.3.6) passing to the limit as \( n \to \infty \). □

Consequently, when \( A \) is self-adjoint and dissipative, \( u'f \) and \( Au f \) have the same regularity in the space \( L^2(0,T;X) \) as the right-hand side \( f \)—a property which is called maximal regularity. Since we know that \( e^{tA} \) is analytic in this case (see Exercise 11), it is natural to ask whether such maximal regularity holds true, more generally, when \( A \) is the infinitesimal generator of an analytic semigroup. In order to show that this is indeed the case we need to recall some properties of the Fourier transform on \( L^2(\mathbb{R};X) \).

Fix an orthonormal basis \( \{e_k\}_{k \geq 1} \) of \( H \) and represent \( g \in L^2(\mathbb{R};X) \) as

\[
g(t) = \sum_{k=1}^{\infty} g_k(t)e_k \quad (t \in \mathbb{R} \text{ a.e.}),
\]

where \( g_k(t) = \langle g(t), e_k \rangle \). Then we have that \( \|g\|^2 = \sum_{k=1}^{\infty} |g_k|^2 \). Denoting by

\[
\hat{g}_k(\tau) = \int_{-\infty}^{+\infty} g_k(t)e^{-i\tau t} dt \quad (t \in \mathbb{R} \text{ a.e.})
\]

the Fourier transform of \( g_k \), we have that

\[
\int_{-\infty}^{+\infty} |g_k(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{g}_k(\tau)|^2 d\tau \quad (k \geq 1).
\]

Then we can define \( \mathcal{F} : L^2(\mathbb{R};X) \to L^2(\mathbb{R};X) \) by

\[
\mathcal{F}[g](\tau) = \hat{g}(\tau) = \sum_{k=1}^{\infty} \hat{g}_k(\tau)e_k \quad (\tau \in \mathbb{R} \text{ a.e.})
\]

We will use the following properties of the Fourier transform on \( L^2(\mathbb{R};X) \).

- **Plancherel identity**: for every \( g \in L^2(\mathbb{R};X) \) we have that

\[
\int_{-\infty}^{+\infty} |g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{g}(\tau)|^2 d\tau \quad (k \geq 1). \tag{4.3.7}
\]

- **Derivation formula**: for every \( g \in W^{1,2}(0,T;X) \) we have that

\[
\mathcal{F}[g'](\tau) = i\tau \hat{g}(\tau) \quad (\tau \in \mathbb{R} \text{ a.e.}) \tag{4.3.8}
\]

- **Action of a closed operator**: for any closed operator \( A : D(A) \subset X \to X \) and any \( g \in L^2(\mathbb{R};D(A)) \) we have that \( F[g] \in L^2(\mathbb{R};D(A)) \) and

\[
AF[g](\tau) = \mathcal{F}[Ag](\tau) \quad (\tau \in \mathbb{R} \text{ a.e.}) \tag{4.3.9}
\]
Theorem 16 Let \( A : D(A) \subset X \to X \) be the infinitesimal generator of an analytic semigroup with negative growth bound and let \( f \in L^2(0,T;X) \). Then \( u_f \) is the strict solution of (4.3.1) and

\[
\|Au_f\|_2 \leq (M + 1)\|f\|_2
\]

(4.3.10)

where \( M > 0 \) is the constant in (4.2.1).

Proof. Let us assume first that \( f \in L^2(0,T;D(A)) \). Then \( u_f \) is the strict solution of (4.3.1). Define

\[
F(t) = \begin{cases} 
  f(t), & t \in [0,T] \\
  0, & t \in \mathbb{R} \setminus [0,T]
\end{cases}
\]

and

\[
U(t) = \begin{cases} 
  0, & t > 0 \\
  u_f(t), & t \in [0,T] \\
  e^{(t-T)A}u_f(T), & t > T.
\end{cases}
\]

Then \( f \in L^2(\mathbb{R};D(A)) \), \( U \in W^{1,2}(\mathbb{R};X) \cap L^2(\mathbb{R};D(A)) \) because \( e^{tA} \) has a negative growth bound, and

\[
U'(t) = AU(t) + F(t) \quad (t \in \mathbb{R} \text{ a.e.})
\]

So, we can take the Fourier transform of both terms of the above identity to obtain, in view of (4.3.8) and (4.3.9),

\[
i\tau \hat{U}(\tau) = A\hat{U}(\tau) + \hat{F}(\tau) \quad (\tau \in \mathbb{R} \text{ a.e.})
\]

So, \( \hat{U}(\tau) = R(i\tau,A)\hat{F}(\tau) \) and, since \( AR(i\tau,A) = i\tau R(i\tau,A) - I \), the resolvent estimate yields

\[
|A\hat{U}(\tau)| = |i\tau R(i\tau,A)\hat{F}(\tau) - \hat{F}(\tau)| \leq (M + 1)|\hat{F}(\tau)| \quad (t \in \mathbb{R} \text{ a.e.})
\]

Therefore

\[
\int_0^T |Au_f|^2 dt \leq \int_{-\infty}^{+\infty} |AU|^2 d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |A\hat{U}|^2 d\tau
\]

\[
\leq \frac{(M + 1)^2}{2\pi} \int_{-\infty}^{+\infty} |\hat{F}|^2 d\tau = (M + 1)^2 \int_0^T |f|^2 dt.
\]

Finally, in order to remove the extra assumption \( f \in L^2(0,T;D(A)) \), let \( f_n(t) := nR(n,A)f(t) \) for all \( n > 0 \). Then

\[
u_n := u_{f_n} \in W^{1,2}(0,T;X) \cap L^2(0,T;D(A))
\]
and the above inequality yields
\[ \|A(u_n - u_m)\|_2 \leq (M + 1)\|f_n - f_m\|_2 \]
and
\[ \|u'_n - u'_m\|_2 \leq (M + 2)\|f_n - f_m\|_2 \]
for all \( n,m \geq 1 \). Thus
\[ \{u_n\} \text{ is a Cauchy sequence in } W^{1,2}(0,T;X) \cap L^2(0,T;D(A)). \]
Since \( \{u_n\} \) converges to \( u \) we conclude that \( u \) is the strict solution of (4.3.1).

Estimate (4.3.10) follows from the analogous inequality for \( u_n \).

\[ \square \]

In order to obtain similar regularity results for (4.2.2), let us set
\[ [D(A),X]_{1/2} = \left\{ x \in X : \int_0^\infty |Ae^{tA}x|^2 dt < \infty \right\}. \quad (4.3.11) \]
It is easy to see that \([D(A),X]_{1/2}\) is a subspace of \( X \) containing \( D(A) \). The following result is a direct consequence of Theorem 16 and definition (4.3.11).

**Corollary 8** Let \( A : D(A) \subset X \to X \) be the infinitesimal generator of an analytic semigroup with negative growth bound. If
\[ x \in [D(A),X]_{1/2} \quad \text{and} \quad f \in L^2(0,T;X) \]
then the strong solution \( u \) of (4.2.2) is strict.

**Example 20** On \( X = L^2(0,\pi) \) let \( A : D(A) \subset X \to X \) be the operator (studied in Exercise 8)
\[
\begin{cases}
D(A) = H^2(0,\pi) \cap H^1_0(0,\pi) \\
Af(x) = f''(x) & x \in (0,\pi) \text{ a.e.}
\end{cases}
\]
We know that \( A \) is self-adjoint and dissipative. Moreover, \( A \) is the infinitesimal generator of an analytic semigroup of negative type (Example 13). We now show that
\[ [D(A),X]_{1/2} = H^1_0(0,\pi). \quad (4.3.12) \]
Let us fix \( f \in H^1_0(0,\pi) \) and consider its the Fourier series
\[ f(x) = \sum_{n=1}^{\infty} f_n \sin(nx) \quad (x \in [0,\pi]). \]
By Parseval’s identity we have
\[ \sum_{n=1}^{\infty} n^2 |f_n|^2 = \frac{2}{\pi} \int_0^\pi |f'(x)|^2 dx. \]
Moreover
\[ Ae^{tA} f(x) = - \sum_{n=1}^{\infty} n^2 e^{-n^2 t} f_n \sin(nx) \quad (x \in [0, \pi]). \]

Therefore
\[
\int_0^\infty |Ae^{tA} f|^2 \, dt = \frac{\pi}{2} \sum_{n=1}^{\infty} \int_0^\infty n^4 |f_n|^2 e^{-2n^2 t} \, dt = \frac{\pi}{4} \sum_{n=1}^{\infty} n^2 |f_n|^2 = \frac{1}{2} \int_0^\pi |f'(x)|^2 \, dx < \infty.
\]

This identity implies \( H_0^1(0, \pi) \subset [D(A), X]_{1/2} \) as well as the converse inclusion.

We can use (4.3.12) to the problem
\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t, x) & (t, x) \in (0, T) \times (0, \pi) \text{ a.e.} \\
u(t, 0) = 0 = u(t, \pi) & t \in (0, T) \\
u(0, x) = u_0(x) & x \in (0, \pi).
\end{cases}
\]

Since
\[ L^2((0, T); L^2(0, \pi)) = L^2((0, T) \times (0, \pi)), \]
by Corollary 8 we conclude that for all
\[ f \in L^2((0, T) \times (0, \pi)) \quad \text{and} \quad u_0 \in H_0^1(0, \pi) \]
problem (4.3.13) has a unique strict solution \( u \) such that
\[ \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} \in L^2((0, T) \times (0, \pi)). \]

4.4 Problems

1. Let \( A : D(A) \subset X \to X \) be the infinitesimal generator of a \( C_0 \)-semigroup of bounded linear operators on \( X \) of negative type. Prove that
\[ |x|_{D(A)} = |Ax| \quad \forall x \in D(A) \]
is a norm on \( D(A) \), equivalent to the graph norm.

2. Give an example to show that (4.1.5) is not a compact embedding.

3. Generalize Corollary 8 removing the assumption \( \omega_0(e^{tA}) < 0 \).

5 Notation

- \( \mathbb{R} = (-\infty, \infty) \) stands for the real line, \( \mathbb{R}_+ \) for \([0, \infty)\), and \( \mathbb{R}_+^* \) for \((0, \infty)\).
• $\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, \ldots\}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\} = \{\pm 1, \pm 2, \ldots\}$

• For any $\lambda \in \mathbb{C}$, $\Re \lambda$ and $\Im \lambda$ denote the real and imaginary parts of $\lambda$, respectively

• $|\cdot|$ stands for the norm of a Banach space $X$, as well as for the absolute value of a real number or the modulus of a complex number

• $\mathcal{L}(X)$ is the Banach space of all bounded linear operators $\Lambda : X \to X$ equipped with the norm $\|\Lambda\| = \sup_{|x|\leq 1} |\Lambda x|$

• $\mathcal{K}(X)$ is the closed subspace of $\mathcal{L}(X)$ of all compact operators $\Lambda : X \to X$

• $\omega_0(S)$ denotes the growth bound of a $C_0$-semigroup of bounded linear operators on $X$ (Definition 4)

• $s(A)$ denotes the spectral bound of a closed operator $A : D(A) \subset X \to X$ (Definition 7)

• $\Pi_\omega = \{ \lambda \in \mathbb{C} : \Re \lambda > \omega \}$ for any $\omega \in \mathbb{R}$

• $\Sigma_{\omega, \theta} = \{ \lambda \in \mathbb{C} \setminus \{\omega\} : |\arg(\lambda - \omega)| < \theta \}$ for any $\omega \in \mathbb{R}$ and $\theta \in (0, \pi]$

• $C(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$ for any $z_0 \in \mathbb{C}$ and $r > 0$

Bibliography


