EAM2 Lecture Notes

Piermarco CANNARSA

aa 2015/2016

Contents

1 Generation of $\mathcal{C}_0$-semigroups

1.1 $\mathcal{C}_0$-semigroups

Exponential of a bounded operator

Let $(X, |\cdot|)$ be a real or complex Banach space and denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators $\Lambda : X \to X$ equipped with the norm

$$\|\Lambda\| = \sup_{|x| \leqslant 1} |\Lambda x|.$$

Let $T > 0$. For any $A \in \mathcal{L}(X)$ the Cauchy problem

$$\begin{cases} y'(t) = Ay(t) \quad (t \in [0, T]) \\ y(0) = x \in X \end{cases} \tag{1.1.1}$$

can easily be solved by a well-known iteration method. Let us set

$$y_0(t) = x\,, \quad y_{n+1}(t) = x + \int_0^t Ay_n(s)ds \quad (t \in [0, T]),$$

where the above integral is understood in the Riemann sense. Then the solution of (1.1.1) is given by

$$y(t) = \lim_{n\to\infty} y_n(t) = e^{tA}x = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x,$$

where the series converges uniformly in $\mathcal{L}(X)$.

Motivated by applications to partial differential equations and other kinds of functional equations, we will extend the theory to problems associated with an unbounded linear operator $A : D(A) \subset X \to X$.

$\mathcal{C}_0$-semigroups

Definition 1 *A $\mathcal{C}_0$-semigroup of bounded linear operators on X is map $S : [0, \infty) \to \mathcal{L}(X)$ with the following properties:*

(a) $S(0) = I$ and $S(t + s) = S(t)S(s)$ for all $t, s \geqslant 0$, and

(b) for every $x \in X$ the map $t \mapsto S(t)x$ is continuous from $[0, \infty)$ to X.

Equivalent notations for the semigroup S are $S(\cdot)$, $\{S(t)\}_{t\geqslant 0}$, and even the simpler form $S(t)$.

Example 1 For any $A \in \mathcal{L}(X)$ the exponential $S(t) = e^{tA}$ is a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Moreover,

(b') the map $S : [0, \infty) \to \mathcal{L}(X)$ is continuous.

Notice that (b') is stronger than (b). Moreover, it is known (see, for instance, [4, Theorem I.3.7]) that if $S(\cdot)$ satisfies (a) and (b'), then there exists $A \in \mathcal{L}(X)$ such that $S(t) = e^{tA}$.

Example 2 For a fixed $p \geqslant 1$ let $X = L^p(\mathbb{R})$ and define, $\forall f \in X$,

$$\big(S(t)f\big)(x) = f(x+t) \quad \forall x \in \mathbb{R}\,, \forall t \geqslant 0. \tag{1.1.2}$$

Then S is $\mathcal{C}_0$-semigroup which fails to satisfy (b'). Indeed, suppose S has property (b') and let $\tau > 0$ be such that $\|S(t) - I\| < 1/2$ for all $t \in [0, \tau]$. Then by taking $f_n(x) = n^{1/p}\chi_{[0,1/n]}(x)$ for $p < \infty$ and $n > 1/\tau$ we have that $|f_n| = 1$ and

$$|S(\tau)f_n - f_n| = \Big(\int_{\mathbb{R}} n|\chi_{[0,1/n]}(x+\tau) - \chi_{[0,1/n]}(x)|^p dx\Big)^{\frac{1}{p}} = 2^{1/p}.$$

Observe that (1.1.2) makes sense for $t < 0$ as well. In this case we say that S is a $\mathcal{C}_0$ group of bounded linear operators on X. On the other hand, if one takes $X = L^p(\mathbb{R}_+)$, then (1.1.2) makes sense only for $t \geqslant 0$.

1.2 The infinitesimal generator of a $\mathcal{C}_0$-semigroup

Let S be a $\mathcal{C}_0$-semigroup of bounded linear operators on X. We are interested in studying the limit

$$\lim_{h \downarrow 0} \frac{S(h)x - x}{h} \tag{1.2.1}$$

as a function of $x \in X$.

Exercise 1 Show that if $A \in \mathcal{L}(X)$ then

$$\lim_{h \downarrow 0} \frac{e^{hA}x - x}{h} = Ax \quad \forall x \in X.$$

Definition 2 *The linear operator $A : D(A) \subset X \to X$ defined by*

$$\begin{cases} D(A) = \big\{x \in X \ : \ \exists \lim_{h\downarrow 0} \frac{S(h)x-x}{h}\big\} \\ Ax = \lim_{h\downarrow 0} \frac{S(h)x-x}{h} & \forall x \in D(A) \end{cases} \tag{1.2.2}$$

is called the infinitesimal generator *of S.*

Exercise 2 Check that (1.2.2) defines a linear operator.

Proposition 1 *$D(A)$ is dense in X.*

Proof. For any $x \in X$ define

$$M_{t,h}x = \frac{1}{h}\int_t^{t+h} S(s)x\,ds \qquad (t \geqslant 0\,,\ h > 0). \tag{1.2.3}$$

Then, by continuity, $\lim_{h\downarrow 0} M_{t,h}x = S(t)x$. Moreover, for any $t, h > 0$,

$$\begin{aligned}
\frac{S(h)-I}{h}\,M_{0,t}x &= \frac{1}{ht}\int_0^t (S(h+s)-S(s))x\,ds \\
&= \frac{1}{ht}\Big\{\int_h^{t+h} S(s)x\,ds - \int_0^t S(s)x\,ds\Big\} \\
&= \frac{1}{ht}\Big\{\int_t^{t+h} S(s)x\,ds - \int_0^h S(s)x\,ds\Big\} = \frac{1}{t}\big\{M_{t,h}x - M_{0,h}x\big\}.
\end{aligned}$$

Therefore

$$\lim_{h\downarrow 0}\frac{S(h)-I}{h}\,M_{0,t}x = \frac{S(t)x-x}{t} \qquad \forall x \in X.$$

This yields $M_{0,t}x \in D(A)$. Since $\lim_{t\to 0} M_{0,t}x = x$, $D(A)$ is dense in X. □

Lemma 1 *For all $x \in D(A)$ we have that $S(t)x \in D(A)$ for every $t \geqslant 0$ and*

$$AS(t)x = S(t)Ax. \tag{1.2.4}$$

Proof. For all $h > 0$ we have that

$$\frac{S(h)-I}{h}\,S(t)x = S(t)\,\frac{S(h)-I}{h}\,x \to S(t)Ax \quad \text{as} \quad h\downarrow 0.$$

Therefore $S(t)x \in D(A)$ and (1.2.4) holds true. □

Remark 1 Fix any $T \geqslant 0$ and observe that for all $x \in X$ there exists a constant $N_{T,x} > 0$ such that $|S(t)x| \leqslant N_{T,x}$ for all $t \in [0,T]$. Then the Uniform Boundedness Principle ensures that, for some constant $N_T > 0$,

$$\|S(t)\| \leqslant N_T \qquad \forall t \in [0,T]. \tag{1.2.5}$$

The following theorem provides a solution to problem (1.1.1) for $x \in D(A)$.

Theorem 1 *For all $x \in D(A)$ we have that $t \mapsto S(t)x$ is differentiable for every $t \geqslant 0$ and*

$$\frac{d}{dt}\,S(t)x = AS(t)x = S(t)Ax.$$

Proof. Fix any $t \geqslant 0$. Then Lemma 1 ensures that

$$\frac{S(t+h)x - S(t)x}{h} = \frac{S(h)-I}{h}\,S(t)x \to AS(t)x \quad \text{as} \quad h \downarrow 0.$$

Hence, $S(t)x$ has the right derivative at t.

In order to complete the proof, let $t > 0$. Then for all $0 < h < t$ we have that

$$\frac{S(t-h)x - S(t)x}{-h} = S(t-h)\,\frac{S(h)-I}{h}\,x.$$

On the other hand, by (1.2.5),

$$\begin{aligned}\Big|S(t-h)\,\frac{S(h)-I}{h}\,x - S(t)Ax\Big|& \\ \leqslant \Big|S(t-h)\Big| \cdot \Big|\frac{S(h)-I}{h}\,x - S(h)Ax\Big|& \\ \leqslant N_t\Big|\frac{S(h)-I}{h}\,x - S(h)Ax\Big| &\longrightarrow 0 \text{ as } h \downarrow 0.\end{aligned}$$

Therefore

$$\frac{S(t-h)x - S(t)x}{-h} \longrightarrow S(t)Ax = AS(t)x \quad \text{as} \quad h \downarrow 0,$$

showing that the left and right derivatives coincide. □

Definition 3 *An operator $A : D(A) \subset X \to X$ is said to be closed if its graph*

$$Graph(A) = \big\{(x, y) \ : \ x \in D(A)\,,\ y = Ax\big\}$$

is a closed subset of the product space $X \times X$.

Exercise 3 Prove that $A : D(A) \subset X \to X$ is closed if and only if for any sequence $\{x_n\} \subset D(A)$

$$\begin{cases} x_n \to x \\ Ax_n \to y \end{cases} \quad \Longrightarrow \quad x \in D(A) \quad \text{and} \quad Ax = y. \tag{1.2.6}$$

Proposition 2 *The infinitesimal generator of a $\mathcal{C}_0$-semigroup is a closed operator.*

Proof. Let A be the infinitesimal generator of S and let $\{x_n\} \subset D(A)$ be as in (1.2.6). By Theorem 1 we have that, for all $t \geqslant 0$

$$S(t)x_n - x_n = \int_0^t S(s)Ax_n dx.$$

Hence, taking the limit as $n \to \infty$ and dividing by t, we obtain

$$\frac{S(t)x - x}{t} = \frac{1}{t}\int_0^t S(s)y dx.$$

Passing to the limit as $t \downarrow 0$, we conclude that $Ax = y$. □

Example 3 Let us identify the generator $A : D(A) \subset X \to X$ of the left-translation semigroup S on $X = L^p(\mathbb{R})$ introduced in Example 2. We denote by $|f|_p$ the norm of f in $L^p(\mathbb{R})$ and by $W^{1,p}(\mathbb{R})$ the Banach space of all locally absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$|f|_{1,p} := |f|_p + |f'|_p < \infty. \tag{1.2.7}$$

We will show that A equals the unbounded operator $B : D(B) \subset X \to X$ defined by

$$\begin{cases} D(B) = W^{1,p}(\mathbb{R}) \\ Bf(x) = f'(x) \quad (x \in \mathbb{R} \text{ a.e.}) \quad \forall f \in D(B). \end{cases} \tag{1.2.8}$$

First, we claim that A is an extension of B (in formulas, $B \subset A$), that is

$$D(B) \subset D(A) \qquad \& \qquad Af = Bf \quad \forall f \in D(B).$$

Indeed, by Hölder's inequality, we have that, for all $f \in W^{1,p}(\mathbb{R})$ and all $t > 0$,

$$\begin{aligned} &\int_{-\infty}^{+\infty} \Big| \frac{f(x+t) - f(x)}{t} - f'(x) \Big|^p dx \\ &= \int_{-\infty}^{+\infty} \frac{1}{t} \Big| \int_0^t \big(f'(x+s) - f'(x)\big) ds \Big|^p dx \\ &\leqslant \frac{1}{t} \int_{-\infty}^{+\infty} dx \int_0^t \big| f'(x+s) - f'(x) \big|^p ds \\ &= \frac{1}{t} \int_0^t ds \int_{-\infty}^{+\infty} \big| f'(x+s) - f'(x) \big|^p dx. \end{aligned}$$

Now, owing to the translation continuity of the integral the last integral can be made arbitrarily small by taking $t > 0$ small enough. So, $B \subset A$.

In order to conclude that $A = B$ it suffices to prove that $D(A) \subset D(B)$. For this purpose, for any fixed $f \in D(A)$ and any $\varepsilon > 0$ let $f_\varepsilon = f * \rho_\varepsilon$, where $\{\rho_\varepsilon\}_{\varepsilon > 0}$ is a $\mathcal{C}^\infty$ approximate unity with support in $[-\varepsilon, \varepsilon]$. Then $f_\varepsilon \in D(B)$ and $f_\varepsilon \to f$ as $\varepsilon \to 0$. Since B is a closed operator, if we show that $Bf_\varepsilon \to Af$ then $f \in D(B)$ and $A = B$. To check that $Bf_\varepsilon \to Af$ observe that

$$\begin{aligned} &\int_{-\infty}^{+\infty} \big| f_\varepsilon'(x) - Af(x) \big|^p dx \\ &\leqslant \liminf_{t \downarrow 0} \int_{-\infty}^{+\infty} \Big| \frac{f_\varepsilon(x+t) - f_\varepsilon(x)}{t} - Af(x) \Big|^p dx \\ &= \liminf_{t \downarrow 0} \int_{-\infty}^{+\infty} \Big| \int_{-\varepsilon}^{\varepsilon} \Big[\frac{f(x-y+t) - f(x-y)}{t} - Af(x) \Big] \rho_\varepsilon(y)\, dy \Big|^p dx \\ &\leqslant \liminf_{t \downarrow 0} \int_{-\infty}^{+\infty} \int_{-\varepsilon}^{\varepsilon} \Big| \frac{f(x-y+t) - f(x-y)}{t} - Af(x) \Big|^p \rho_\varepsilon(y)\, dy\, dx \end{aligned}$$

Now, since

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\varepsilon}^{\varepsilon}\Big|\frac{f(x-y+t)-f(x-y)}{t}-Af(x)\Big|^p\rho_\varepsilon(y)\,dy\,dx \\
&\leqslant 2^{p-1}\Big\{\int_{-\varepsilon}^{\varepsilon}\rho_\varepsilon(y)\,dy\int_{-\infty}^{+\infty}\Big|\frac{f(x-y+t)-f(x-y)}{t}-Af(x-y)\Big|^p dx \\
&\qquad\qquad +\int_{-\varepsilon}^{\varepsilon}\rho_\varepsilon(y)\,dy\int_{-\infty}^{+\infty}\big|Af(x-y)-Af(x)\big|^p dx\Big\},
\end{aligned}$$

the conclusion follows recalling the translation continuity of the integral and the fact that

$$\int_{-\infty}^{+\infty}\Big|\frac{f(x+t)-f(x)}{t}-Af(x)\Big|^p dx \quad (t\to 0).$$

Exercise 4 Show that $W^{1,p}(\mathbb{R})$ is a Banach space with the above norm and that operator $B : D(B)\subset X\to X$ defined in (1.2.8) is closed.

1.3 Asymptotic properties of $\mathcal{C}_0$-semigroups

Let S be a $\mathcal{C}_0$-semigroup of bounded linear operators on X.

Definition 4 *The number*

$$\omega_0(S)=\inf_{t\geqslant 0}\frac{\log\|S(t)\|}{t} \tag{1.3.1}$$

is called the type *or* growth bound *of S.*

Proposition 3 *The growth bound of S satisfies*

$$\omega_0(S)=\lim_{t\to\infty}\frac{\log\|S(t)\|}{t}<+\infty. \tag{1.3.2}$$

Moreover, for any $\varepsilon>0$ there exists $M_\varepsilon\geqslant 1$ such that

$$\|S(t)\|\leqslant M_\varepsilon e^{(\omega_0(S)+\varepsilon)t}\qquad \forall t\geqslant 0. \tag{1.3.3}$$

Proof. The fact that $\omega_0(S)<+\infty$ is a direct consequence of (1.3.1). In order to prove (1.3.2) it suffices to show that

$$\limsup_{t\to\infty}\frac{\log\|S(t)\|}{t}\leqslant\omega_0(S). \tag{1.3.4}$$

For any $\varepsilon>0$ let $t_\varepsilon>0$ be such that

$$\frac{\log\|S(t_\varepsilon)\|}{t_\varepsilon}<\omega_0(S)+\varepsilon. \tag{1.3.5}$$

Let us write any $t \geqslant t_\varepsilon$ as $t = n(t)t_\varepsilon + r(t)$ with $n(t) \in \mathbb{N}$ and $r(t) \in [0, t_\varepsilon)$. Then, by (1.2.5) and (1.3.5),

$$\|S(t)\| \leqslant \|S(t_\varepsilon)\|^{n(t)} \|S(r(t))\| \leqslant e^{t_\varepsilon n(t)(\omega_0(S)+\varepsilon)} N_{t_\varepsilon} \leqslant N_{t_\varepsilon} e^{(\omega_0(S)+\varepsilon)t}$$

which proves (1.3.3). Moreover, taking the logarithm of both sides of the above inequality we get

$$\frac{\log \|S(t)\|}{t} \leqslant \omega_0(S) + \varepsilon + \frac{N_{t_\varepsilon}}{t}$$

and (1.3.4) follows as $t \to \infty$. □

Example 4 It it immediate to realize that the left-translation semigroup of Example 2 satisfies $\|S(t)\| = 1$ for all $t \geqslant 0$. So, $\omega_0(S) = 0$.

1.4 Spectral properties of generators

Resolvent set and spectrum

Let $A : D(A) \subset X \to X$ be a closed operator on a complex Banach space X.

Definition 5 *The* resolvent set *of A, $\rho(A)$, is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A : D(A) \to X$ is bijective and its complement $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called the* spectrum *of A. For any $\lambda \in \rho(A)$ the inverse $R(\lambda, A) = (\lambda I - A)^{-1}$ is called the resolvent of A at λ.*

Remark 2 Observe that, by the closed graph theorem, $R(\lambda, A)$ is a bounded linear operator on X. Also, the identity

$$AR(\lambda, A) = \lambda R(\lambda, A) - I \qquad \forall \lambda \in \rho(A) \tag{1.4.1}$$

is easy to check. Moreover, the following *resolvent identity* holds:

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A) \qquad \forall \lambda, \mu \in \rho(A). \tag{1.4.2}$$

Indeed, by (1.4.1) we have that

$$[\lambda R(\lambda, A) - AR(\lambda, A)] R(\mu, A) = R(\mu, A)$$

and

$$R(\lambda, A)[\mu R(\mu, A) - AR(\mu, A)] = R(\lambda, A).$$

Since $AR(\lambda, A) = R(\lambda, A)A$ on $D(A)$, (1.4.2) follows.

Proposition 4 *Let $A : D(A) \subset X \to X$ be a closed operator. Then $\rho(A)$ is open in $\mathbb{C}$ and for any $\mu \in \rho(A)$ the resolvent $R(\lambda, A)$ is given by the series*

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1} \tag{1.4.3}$$

for all $\lambda \in \mathbb{C}$ satisfying $|\mu - \lambda| < 1/\|R(\mu, A)\|$. Consequently, $\lambda \mapsto R(\lambda, A)$ is analytic on $\rho(A)$ and for all $n \in \mathbb{N}$

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n \, n! \, R(\lambda, A)^{n+1}. \tag{1.4.4}$$

Proof. For all $\lambda \in \mathbb{C}$ we have that

$$\lambda I - A = \mu I - A + (\lambda - \mu)I = [I - (\mu - \lambda)R(\mu, A)](\mu I - A).$$

This operator is bijective if and only if $[I - (\mu - \lambda)R(\mu, A)]$ is invertible, which is the case for $|\mu - \lambda| < 1/\|R(\mu, A)\|$. In this case

$$R(\lambda, A) = R(\mu, A)[I - (\mu - \lambda)R(\mu, A)]^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1}.$$

The analyticity of $R(\lambda, A)$ and (1.4.4) follow from (1.4.3). $\square$

Example 5 On $X = \mathcal{C}([0, 1])$ with the uniform norm consider the closed operator $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) = \mathcal{C}^1([0, 1]) \\ Af = f', \quad \forall f \in D(A). \end{cases}$$

Then $\sigma(A) = \mathbb{C}$ because for any $\lambda \in \mathbb{C}$ the function $f_\lambda(x) = e^{\lambda x}$ satisfies

$$\lambda f_\lambda(x) - f_\lambda'(x) = 0 \qquad \forall x \in [0, 1].$$

On the other hand, for the closed operator A_0 defined by

$$\begin{cases} D(A_0) = \{f \in \mathcal{C}^1([0, 1]) \ : \ f(0) = 0\} \\ A_0 f = f', \quad \forall f \in D(A_0), \end{cases}$$

we have that $\sigma(A_0) = \varnothing$. Indeed, for any $g \in X$ the problem

$$\begin{cases} \lambda f(x) - f'(x) = g(x) \quad x \in [0, 1] \\ f(0) = 0 \end{cases}$$

admits the unique solution

$$f(x) = -\int_0^x e^{\lambda(x-s)} g(s)\, dx \quad (x \in [0, 1])$$

which belongs to $D(A_0)$.

Spectral properties of the infinitesimal generator

Let $M \geqslant 0$ and let $\omega \in \mathbb{R}$.

Definition 6 *We denote by $\mathcal{G}(M,\omega)$ the class of all $\mathcal{C}_0$-semigroups of bounded linear operators on X such that*

$$\|S(t)\| \leqslant Me^{\omega t} \qquad \forall t \geqslant 0. \tag{1.4.5}$$

When $M = 1$ and $\omega = 0$ we say that $S(t)$ is a contraction semigroup.

Proposition 5 (Integral representation) *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of $S \in \mathcal{G}(M,\omega)$. Then $\rho(A)$ contains the half-plane*

$$\Pi_\omega = \big\{\lambda \in \mathbb{C} \ : \ \Re\lambda > \omega\big\} \tag{1.4.6}$$

and

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t)x\,dt \qquad \forall x \in X\,,\ \forall \lambda \in \Pi_\omega. \tag{1.4.7}$$

Proof. We must prove that, given any $\lambda \in \Pi_\omega$ and $x \in X$, the equation

$$\lambda u - Au = x \tag{1.4.8}$$

has a unique solution given by (1.4.7).
Existence: observe that $u := \int_0^\infty e^{-\lambda t} S(t)x\,dt \in X$ because $\Re\lambda > \omega$. Moreover, for all $h > 0$,

$$\begin{aligned}
\frac{S(h)u - u}{h} &= \frac{1}{h}\Big\{\int_0^\infty e^{-\lambda t} S(t+h)x\,dt - \int_0^\infty e^{-\lambda t} S(t)x\,dt\Big\} \\
&= \frac{1}{h}\Big\{e^{\lambda h}\int_h^\infty e^{-\lambda t} S(t)x\,dt - \int_0^\infty e^{-\lambda t} S(t)x\,dt\Big\} \\
&= \frac{e^{\lambda h}-1}{h}\,u - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} S(t)x\,dt.
\end{aligned}$$

So

$$\lim_{h \downarrow 0} \frac{S(h)u - u}{h} = \lambda u - x$$

which in turn yields that $u \in D(A)$ and (1.4.8) holds true.
Uniqueness: let $u \in D(A)$ be a solution of (1.4.8). Then

$$\int_0^\infty e^{-\lambda t} S(t)(\lambda u - Au)\,dt = \lambda \int_0^\infty e^{-\lambda t} S(t)u\,dt - \int_0^\infty e^{-\lambda t}\,\frac{d}{dt}\,S(t)u\,dt = u$$

which implies that u is given by (1.4.7). $\square$

Definition 7 *For any operator $A : D(A) \subset X \to X$ we define the* spectral bound *of A as*

$$s(A) = \sup\{\, \Re\lambda \ : \ \lambda \in \sigma(A)\,\}.$$

Corollary 1 *Let S be a $\mathcal{C}_0$-semigroup on X with infinitesimal generator A. Then*

$$-\infty \leqslant s(A) \leqslant \omega_0(S) < +\infty.$$

Proposition 6 *Let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be closed linear operators in X and suppose $B \subset A$, that is, $D(B) \subset D(A)$ and $Ax = Bx$ for all $x \in D(B)$. If $\rho(A) \cap \rho(B) \neq \varnothing$, then $A = B$.*

Proof. Let $\lambda \in \rho(A) \cap \rho(B)$, let $x \in D(A)$, and set

$$y = \lambda x - Ax \quad \text{and} \quad z = R(\lambda, B)y.$$

Then $z \in D(B)$ and $\lambda z - Bz = \lambda x - Ax$. Since $B \subset A$, $\lambda z - Bz = \lambda z - Az$. Thus, $(\lambda I - A)(x - z) = 0$. So, $x = z \in D(B)$ and $A = B$. □

Example 6 (Right-translation semigroup) On the real Banach space

$$X = \{f \in \mathcal{BUC}(\mathbb{R}_+) \ : \ f(0) = 0\}$$

of all bounded uniformly continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ with the uniform norm, consider the right-translation semigroup

$$\big(S(t)f\big)(x) = \begin{cases} f(x-t) & x > t \\ 0 & x \in [0,t] \end{cases} \qquad \forall x, t \geqslant 0.$$

It is easy to check that S is a $\mathcal{C}_0$-semigroup on X with $\|S(t)\| = 1$ for all $t \geqslant 0$. In order to characterize its infinitesimal generator A, let us consider the operator $B : D(B) \subset X \to X$ defined by

$$\begin{cases} D(B) = \big\{f \in X \ : \ f' \in X\big\} \\ Bf = -f', \quad \forall f \in D(B). \end{cases}$$

We claim that:

(*i*) $B \subset A$

Proof. Let $f \in D(B)$. Then, for all $x, t \geqslant 0$ we have

$$\frac{\big(S(t)f\big)(x) - f(x)}{t} = \begin{cases} -\frac{f(x)}{t} = -f'(x_t), & 0 \leqslant x \leqslant t \\ \frac{f(x-t)-f(x)}{-t} = -f'(x_t) & x \geqslant t \end{cases}$$

with $0 \leqslant x - x_t \leqslant t$. Therefore

$$\sup_{x \geqslant 0} \Big| \frac{\big(S(t)f\big)(x) - f(x)}{t} + f'(x) \Big| \leqslant \sup_{|x-y| \leqslant t} |f'(x) - f'(y)| \to 0 \quad \text{as} \quad t \downarrow 0$$

because f' is uniformly continuous. □

(ii) $1 \in \rho(B)$

Proof. For any $g \in X$ the unique solution f of the problem

$$\begin{cases} f \in D(B) \\ f(x) + f'(x) = g(x) \quad \forall x \geqslant 0 \end{cases}$$

is given by

$$f(x) = \int_0^x e^{s-x} g(s)\, ds \qquad (x \geqslant 0).$$ □

Since $1 \in \rho(A)$ by Proposition 5, Proposition 6 yields that $A = B$.

Proposition 7 (Laundau-Kolmogorov) *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a contraction semigroup S. Then*

$$|Ax|^2 \leqslant 4\,|x|\,|A^2 x| \qquad \forall x \in D(A^2), \tag{1.4.9}$$

where

$$\begin{cases} D(A^2) = \{x \in D(A) \ : \ Ax \in D(A)\} \\ A^2 x = A(Ax)\,, \quad \forall x \in D(A^2). \end{cases} \tag{1.4.10}$$

Proof. For any $x \in D(A^2)$ and all $t \geqslant 0$ we have that

$$\begin{aligned} \int_0^t (t-s) S(s) A^2 x\, ds &= \big[(t-s)S(s)Ax\big]_{s=0}^{s=t} + \int_0^t S(s) Ax\, ds \\ &= -tAx + \big[S(s)x\big]_{s=0}^{s=t} = -tAx + S(t)x - x. \end{aligned}$$

Therefore, for all $t > 0$,

$$\begin{aligned} |Ax| &\leqslant \frac{1}{t}\,|S(t)x - x| + \frac{1}{t}\int_0^t (t-s)|S(s)A^2x|ds \\ &\leqslant \frac{2}{t}\,|x| + \frac{t}{2}\,|A^2 x|. \end{aligned} \tag{1.4.11}$$

If $A^2 x = 0$, then the above inequality yields $Ax = 0$ by letting $t \to \infty$. So, (1.4.9) is true in this case. On the other hand, for $A^2 x \neq 0$ the function of t on the right-hand side of (1.4.11) attains its minimum at

$$t_0 = \frac{2|x|^{1/2}}{|A^2 x|^{1/2}}.$$

By taking $t = t_0$ in (1.4.11) we obtain (1.4.9) once again. □

Example 7 Let us recall that the infinitesimal generator $A : D(A) \subset X \to X$ of the left-translation semigroup S on $X = L^p(\mathbb{R})$ introduced in Example 2 is given by

$$\begin{cases} D(A) = W^{1,p}(\mathbb{R}) \\ Af(x) = f'(x) \quad (x \in \mathbb{R} \text{ a.e.}) \quad \forall f \in D(A). \end{cases}$$

Since

$$D(A^2) = \{f \in W^{1,p}(\mathbb{R}) \ : \ f' \in W^{1,p}(\mathbb{R})\} =: W^{2,p}(\mathbb{R}),$$

by Proposition 7 we deduce the interpolation inequality

$$|f'|_p \leqslant 2\sqrt{|f|_p\,|f''|_p} \qquad \forall f \in W^{2,p}(\mathbb{R}).$$

1.5 The Hille-Yosida generation theorem

Theorem 2 *Let $M \geqslant 0$ and $\omega \in \mathbb{R}$. For a linear operator $A : D(A) \subset X \to X$ the following conditions are equivalent:*

(a) A is closed, $D(A)$ is dense in X, and for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ one has that $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)^k\| \leqslant \frac{M}{(\Re\lambda - \omega)^k} \qquad \forall k \geqslant 1. \tag{1.5.1}$$

(b) A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup $S \in \mathcal{G}(M, \omega)$.

Proof. $\boxed{(b) \Rightarrow (a)}$ The fact that A is closed and $D(A)$ is dense in X has already been proved, see propositions 1 and 2. In order to prove (1.5.1) observe that, by using (1.4.7) to compute the k-th derivative of the resolvent of A, we obtain

$$\frac{d^k}{d\lambda^k} R(\lambda, A)x = (-1)^k \int_0^\infty t^k e^{-\lambda t} S(t)x\,dt \qquad \forall x \in X\,,\ \forall \lambda \in \Pi_\omega,$$

where Π_ω is defined in (1.4.6). Therefore,

$$\left\|\frac{d^k}{d\lambda^k} R(\lambda, A)\right\| \leqslant M \int_0^\infty t^k e^{-(\Re\lambda - \omega)t}\,dt = \frac{M\,k!}{(\Re\lambda - \omega)^{k+1}}$$

where the integral is easily computed by an induction argument. The conclusion follows recalling (1.4.4).

$\boxed{(a) \Rightarrow (b)}$ The reasoning will be split into four steps.

Step 1: *the Yosida approximation of A.*
For all $n > \omega$ we define $A_n \in \mathcal{L}(X)$, called Yosida approximations of A, by

$$A_n = n^2 R(n, A) - n = nAR(n, A), \tag{1.5.2}$$

where the last identity follows from (1.4.1). We claim that

$$\lim_{n\to\infty} A_n x = Ax \qquad \forall x \in X. \tag{1.5.3}$$

Indeed, let us first take $x \in D(A)$ and set $J_n = nR(n, A)$. Then, we have that

$$A_n x = AJ_n x = J_n Ax \tag{1.5.4}$$

and

$$\lim_{n\to\infty} (J_n x - x) = \lim_{n\to\infty} R(n, A)Ax = 0 \quad \forall x \in D(A)$$

in view of (1.5.1). In fact, observing that

$$\|J_n\| \leqslant \frac{Mn}{n-\omega} \tag{1.5.5}$$

we conclude that

$$\lim_{n\to\infty} J_n x = x \qquad \forall x \in X \tag{1.5.6}$$

because $D(A)$ is dense in X. This together with (1.5.4) yields (1.5.3).

Step 2: construction of an approximate semigroup.
For all $n > \omega$ we define

$$S_n(t) = e^{tA_n} = e^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k} t^k R(n, A)^k}{k!}, \quad \forall t \geqslant 0.$$

Observe that, in view of (1.5.1),

$$\|S_n(t)\| \leqslant Me^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k} t^k}{k!(n-\omega)^k} = Me^{\frac{n\omega t}{n-\omega}} \leqslant Me^{2\omega t} \tag{1.5.7}$$

for all $t \geqslant 0$ and $n > \omega$.

Step 3: uniform convergence on compact sets.
For any $x \in X$, $u_n(t) := S_n(t)x = e^{tA_n}x$ satisfies

$$\begin{cases} (u_n - u_m)'(t) = A_n(u_n - u_m)(t) + (A_n - A_m)u_m(t), & \forall t \geqslant 0 \\ (u_n - u_m)(0) = 0. \end{cases}$$

Therefore

$$(u_n - u_m)(t) = \int_0^t e^{(t-s)A_n} e^{sA_m} (A_n - A_m)x \, ds, \quad \forall t \geqslant 0$$

which in turn yields, by (1.5.7),

$$|(u_n - u_m)(t)| \leqslant M^2 \, t \, e^{2\omega t} \, |(A_n - A_m)x|.$$

Thanks to (1.5.3), the above estimate implies that $\{u_n\}$ is a Cauchy sequence on all compact subsets of $\mathbb{R}_+$ for $x \in D(A)$ and (1.5.7) guarantees that the same is true for all $x \in X$. Consequently, the limit (uniform on all $[0,T] \subset \mathbb{R}_+$)

$$S(t)x = \lim_{n\to\infty} S_n(t)x, \quad \forall x \in X, \tag{1.5.8}$$

defines a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Moreover, passing to the limit as $n \to \infty$ in (1.5.7) we conclude that $S \in \mathcal{G}(M,\omega)$.

Step 4: identification of the infinitesimal generation.
First, we show that $S(t)x$ is differentiable for all $x \in D(A)$. Indeed,

$$|S_n'(t)x - S(t)Ax| \leqslant |S_n(t)A_nx - S_n(t)Ax| + |S_n(t)Ax - S(t)Ax|$$

where, by (1.5.3) and (1.5.7),

$$|S_n(t)A_nx - S_n(t)Ax| \leqslant Me^{2\omega t}|A_nx - Ax| \to 0 \quad (n \to \infty)$$

and

$$|S_n(t)Ax - S(t)Ax| \to 0 \quad (n \to \infty)$$

uniformly on all compact subsets of $\mathbb{R}_+$ by (1.5.8). Therefore

$$S'(t)x = S(t)Ax, \quad \forall x \in D(A)\,, \ \forall t \geqslant 0. \tag{1.5.9}$$

Now, let $B : D(B) \subset X \to X$ be the infinitesimal generator of S. Then $A \subset B$ in view of (1.5.9). Moreover, $\Pi_\omega \subset \rho(A)$ by assumption (a) and $\Pi_\omega \subset \rho(B)$ by Proposition 5. So, on account of Proposition 6, $A = B$. □

Remark 3 The above proof shows that condition (a) in Theorem 2 can be relaxed as follows

(a') A is closed, $D(A)$ is dense in X, $(\omega,\infty) \subset \rho(A)$ and

$$\|R(n,A)^k\| \leqslant \frac{M}{(n-\omega)^k} \qquad \forall k \geqslant 1\,, \ \forall n > \omega.$$

Remark 4 When $M = 1$ the countably many bounds in condition (a) follow from (1.5.1) for $k = 1$, that is,

$$\|R(\lambda,A)\| \leqslant \frac{1}{\Re\lambda - \omega} \qquad \forall k \geqslant 1\,, \ \forall \lambda \in \Pi_\omega.$$

Exercise 5 Given any $S \in \mathcal{G}(M,0)$ with $M \geqslant 1$, define

$$|x|_S = \sup_{t\geqslant 0} |S(t)x|\,, \quad \forall x \in X. \tag{1.5.10}$$

Show that:

1. $|\cdot|_S$ is a norm on X,

2. $|x| \leqslant |x|_S \leqslant M|x|$ for all $x \in X$, and

3. S is a contraction semigroup with respect to $|\cdot|_S$.

Remark 5 Let $A : D(A) \subset X \to X$ be a closed operator satisfying (1.5.1) but suppose $D(A)$ fails to be dense in X. In the Banach space $Y := \overline{D(A)}$ let us define the operator B, called the *part of A in Y*, by

$$\begin{cases} D(B) = \big\{x \in D(A) \ : \ Ax \in Y\big\} \\ Bx = Ax \qquad \forall x \in D(B). \end{cases}$$

Then $R(\lambda, A)(Y) \subset D(B)$ for all $\lambda \in \mathbb{C}$ such that $\Re\lambda > \omega$. Moreover, owing to (1.4.1) for all $x \in D(A)$ we have that

$$\lim_{n\to\infty} nR(n, A)x = \lim_{n\to\infty} \big\{R(n, A)Ax + x\big\} = x. \tag{1.5.11}$$

Since $\|nR(n, A)\|$ is bounded, (1.5.11) holds true for all $x \in Y$. Hence, $D(B)$ is dense in Y. Consequently, B satisfies in Y all the assumptions of Theorem 2.

1.6 The homogeneous Cauchy problem

Proposition 8 *Let S be a $\mathcal{C}_0$-semigroup of bounded linear operators on X and $A : D(A) \subset X \to X$ be its infinitesimal generator. Then for every $x \in D(A)$ the Cauchy problem*

$$\begin{cases} y'(t) = Ay(t) \\ y(0) = x \end{cases} \tag{1.6.1}$$

has a unique solution $y \in \mathcal{C}^1([0,\infty); X) \cap \mathcal{C}([0,\infty); D(A))$ given by

$$y(t) = S(t)x \qquad \forall t \geqslant 0.$$

Proof. The fact that $y(t) = S(t)x$ satisfies (1.6.1) has already been proved (Theorem 1). Let us show that this is the unique solution of the problem. Let $z \in \mathcal{C}^1([0,\infty); X) \cap \mathcal{C}([0,\infty); D(A))$ be a solution of (1.6.1), fix $t > 0$, and set

$$u(s) = S(t-s)z(s)\,, \quad \forall s \in [0, t].$$

Then

$$u'(s) = -AS(t-s)z(s) + S(t-s)Az(s) = 0\,, \quad \forall s \in [0, t].$$

Therefore, $z(t) = u(t) = u(0) = y(t)$. □

Example 8 (Transport equation in $L^p(\mathbb{R})$) Let $p \geqslant 1$. Racalling the analysis of the left-translation semigroup on $L^p(\mathbb{R})$ developed in examples 2 and 3, by Proposition 8 we conclude that for each $f \in W^{1,p}(\mathbb{R})$ the unique solution of the problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial u}{\partial x}(t,x) & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ u(0,x) = f(x) & x \in \mathbb{R} \end{cases}$$

is given by $u(t,x) = f(x+t)$.

Example 9 (Heat equation in $L^p(0,\pi)$) Let $p \geqslant 2$. On $X = L^p(0,\pi)$ consider the operator defined by

$$\begin{cases} D(A) = W^{2,p}(0,\pi) \cap W_0^{1,p}(0,\pi) \\ Af(x) = f''(x) & x \in (0,\pi) \text{ a.e.} \end{cases} \tag{1.6.2}$$

where

$$W_0^{1,p}(0,\pi) = \big\{ f \in W^{1,p}(0,\pi) \ : \ f(0) = 0 = f(\pi) \big\}.$$

Since $\mathcal{C}_c^\infty(0,\pi) \subset D(A)$, we have that $D(A)$ is dense in X. Moreover, A can be shown to be closed (see Exercise 6 below). We now show that A satisfies condition (a') of Remark 3 with $M = 1$ and $\omega = 0$ so that Theorem 2 will imply that A generates a $\mathcal{C}_0$-semigroup of contractions on X.

Step 1: $\sigma(A) = \{-n^2 \ : \ n \in \mathbb{N}\}$.

Fix any $g \in X$. We will show that, for all $\lambda \neq n^2 (n \geqslant 1)$, the Sturm-Liouville system

$$\begin{cases} \lambda f(x) - f''(x) = g(x), & 0 < x < \pi \\ f(0) = 0 = f(\pi) \end{cases} \tag{1.6.3}$$

admits a unique solution $f \in D(A)$. Denoting by

$$g(x) = \sum_{n=1}^\infty g_n \sin(nx) \qquad (x \in [0,\pi])$$

the Fourier series of g, we seek a candidate solution f of the form

$$f(x) = \sum_{n=1}^\infty f_n \sin(nx) \qquad (x \in [0,\pi]).$$

In order to satisfy (1.6.3) one must have

$$(\lambda + n^2) f_n = g_n \qquad \forall n \geqslant 1.$$

So, for any $\lambda \neq -n^2$, (1.6.3) has a unique solution given by

$$f(x) = \sum_{n=1}^\infty \frac{g_n}{\lambda + n^2} \sin(nx) \qquad (x \in [0,\pi]).$$

From the above representation it follows that $f \in H^2(0,\pi) \cap H_0^1(0,\pi)$. In fact, returning to the equation in (1.6.3) one concludes that $f \in D(A)$.

Step 2: resolvent estimate.

By multiplying both members of the equation in (1.6.3) by $|f|^{p-2}f$ and integrating over $(0,\pi)$ one obtains, for all $\lambda > 0$,

$$\lambda \int_0^\pi |f(x)|^p dx + (p-1)\int_0^p |f(x)|^{p-2}|f'(x)|^2 dx = \int_0^\pi g(x)|f(x)|^{p-2}f(x)\,dx$$

which yields

$$|f|_p \leqslant \frac{1}{\lambda}\,|g|_p \qquad \forall \lambda > 0.$$

Step 3: conclusion.

By Proposition 8 we conclude that for each $f \in W^{2,p}(0,\pi) \cap W_0^{1,p}(0,\pi)$ the unique solution of

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) & (t,x) \in \mathbb{R}_+ \times (0,\pi) \\ u(t,0) = 0 = u(t,\pi) & t \geqslant 0 \\ u(0,x) = f(x) & x \in (0,\pi) \end{cases}$$

is given by $u(t,x) = (S(t)f)(x)$.

Exercise 6 Prove that operator A defined in (1.6.2) is closed.

Example 10 (Heat equation in $L^p(\mathbb{R})$) Let $f \in W^{2,p}(\mathbb{R})$ with $p \geqslant 2$. By following the reasoning of Example 9, let us solve the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ u(0,x) = f(x) & x \in \mathbb{R}. \end{cases}$$

The operator defined by

$$\begin{cases} D(A) = W^{2,p}(\mathbb{R}) \\ Af(x) = f''(x) & x \in \mathbb{R} \text{ a.e.} \end{cases}$$

is densely defined and closed. Let us begin by studying the problem

$$\begin{cases} f \in D(A) \\ \lambda f - f'' = g \in X \end{cases} \tag{1.6.4}$$

in the special case $p = 2$. Taking the Fourier transform of both members of the above equation we find

$$(\lambda + \xi^2)\widehat{f}(\xi) = \widehat{g}(\xi) \quad \forall \xi \in \mathbb{R}.$$

So, for any $\lambda > 0$ we have that the solution to problem (1.6.4) is given by

$$f(x) = (g * \phi_\lambda)(x) \quad \text{with} \quad \phi_\lambda(x) = \frac{e^{-\sqrt{\lambda}\,|x|}}{2\sqrt{\lambda}}\,,$$

that is,

$$f(x) = \frac{1}{2\sqrt{\lambda}}\Big\{\int_{-\infty}^{x} g(y)e^{-\sqrt{\lambda}\,(x-y)}dy + \int_{x}^{\infty} g(y)e^{-\sqrt{\lambda}\,(y-x)}dy\Big\}.$$

Moreover, the above representation formula holds true for any $p \geqslant 2$. We have thus proved that $(0,\infty) \subset \rho(A)$. Finally, by multiplying both members of the equation in (1.6.3) by $|f|^{p-2}f$ and integrating over $\mathbb{R}$ we obtain as in Example 9

$$\lambda\int_{-\infty}^{\infty}|f|^p dx + (p-1)\int_{-\infty}^{\infty}|f|^{p-2}|f'|^2 dx = \int_{-\infty}^{\infty} g|f|^{p-2}f\,dx$$

which yields

$$|f|_p \leqslant \frac{1}{\lambda}\,|g|_p.$$

Therefore, A satisfies condition (a') of Remark 3 and generates a $\mathcal{C}_0$-semigroup of bounded linear operators on X which gives the solution of our problem.

Proposition 9 *Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator satisfying, for some $M, \omega \geqslant 0$, the following conditions:*

(*i*) $\sigma(A) \subset \big\{\lambda \in \mathbb{C} \ : \ |\Re\,\lambda| \leqslant \omega\big\}$

(*ii*) *for all $k \geqslant 1$*

$$|\Re\,\lambda| > \omega \quad \Longrightarrow \quad \|R(\lambda, A)^k\| \leqslant \frac{M}{(|\Re\,\lambda| - \omega)^k}.$$

Then (1.6.1) *has a unique solution $y \in \mathcal{C}^1(\mathbb{R}; X) \cap \mathcal{C}(\mathbb{R}; D(A))$.*

Proof. Observe that both A and $-A$ generate a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Denote by $S_\pm$ the semigroup generated by $\pm A$ and set, for any $x \in X$,

$$y(t) = \begin{cases} S_+(t)x & (t \geqslant 0) \\ S_-(-t)x & (t < 0). \end{cases}$$

Then $y \in \mathcal{C}(\mathbb{R}; X)$. Moreover, for $x \in D(A)$, $y \in \mathcal{C}^1(\mathbb{R}; X) \cap \mathcal{C}(\mathbb{R}; D(A))$ and satisfies (1.6.1). $\square$

Remark 6 In fact, the above assumptions are necessary and sufficient for A to be the infinitesimal generator of a $\mathcal{C}_0$ group of bounded linear operators on X (see, for instance, [4, Section I.3.11]). Moreover, like in Remark 3, conditions (i) and (ii) can be weakened as follows:

(i) $(-\infty, -\omega) \cup (\omega, \infty) \subset \rho(A)$

(ii) for all $k \geqslant 1$ and $|n > \omega$

$$\|R(n, A)^k\| \leqslant \frac{M}{(|n| - \omega)^k}.$$

1.7 Problems

1. Let X be a Banach space and let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Prove that, for every $n \geqslant 1$,

$$D(A^n) := \{x \in D(A^{n-1}) \ : \ Ax \in D(A)\}$$

is dense in X.

Solution. For $n = 1$ the conclusion follows from Proposition 1. Let the conclusion be true for some $n \geqslant 1$ and fix any $y \in X$. Then, for any $\varepsilon > 0$ there exists $x_\varepsilon \in D(A^n)$ such that $|x_\varepsilon - y| < \varepsilon$. As shown in the proof of Proposition 1, $M_{0,t} A^n x_\varepsilon \in D(A)$ for all $t > 0$, where $M_{0,t}$ is defined in (1.2.3). Since $M_{0,t} A^n x_\varepsilon = A^n M_{0,t} x_\varepsilon$, we conclude that $M_{0,t} x_\varepsilon \in D(A^{n+1})$. Moreover, there exists $t_\varepsilon > 0$ such that

$$|M_{0,t} x_\varepsilon - y| \leqslant |M_{0,t_\varepsilon} x_\varepsilon - x_\varepsilon| + |x_\varepsilon - y| < 2\varepsilon. \qquad \square$$

2. For fixed $T > 0$ and $p \geqslant 1$ let $X = L^p(0, T)$ and

$$\big(S(t)f\big)(x) = \begin{cases} f(x - t) & x \in [t, T] \\ 0 & x \in [0, t) \end{cases} \qquad \forall x \in [0, T]\,,\ \forall t \geqslant 0.$$

Prove that S is a $\mathcal{C}_0$-semigroup of bounded linear operators on X which satisfies $\|S(t)\| \leqslant 1$ for all $t \geqslant 0$. Moreover, observe that S is *nilpotent*, that is, we have $S(t) \equiv 0$, $\forall t \geqslant T$. Deduce that $\omega_0(S) = -\infty$.

3. On $X = \{f \in \mathcal{C}([0, \pi]) \ : \ f(0) = 0 = f(\pi)\}$ with the uniform norm, consider the linear operator $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) = \{f \in \mathcal{C}^2([0, 1]) \ : \ f(0) = f(\pi) = 0 = f''(0) = f''(\pi)\} \\ Af = f'', \quad \forall f \in D(A). \end{cases}$$

Apply Theorem 2 to show that A generates a $\mathcal{C}_0$-semigroup of contractions on X and derive the initial-boundary value problem which is solved by such semigroup.

Solution. We only prove that $\|R(\lambda, A)\| \leqslant 1/\lambda$ for all $\lambda > 0$. Fix any $g \in X$ and let $f = R(\lambda, A)g$. Let $x_0 \in [0, \pi]$ be such that $|f(x_0)| = |f|_\infty$. If $f(x_0) > 0$, then $x_0 \in (0, \pi)$ is a maximum point of f. So, $f''(x_0) \leqslant 0$ and we have that

$$\lambda|f|_\infty = \lambda f(x_0) \leqslant \lambda f(x_0) - f''(x_0) = g(x_0) \leqslant |g|_\infty.$$

On the other hand, if $f(x_0) < 0$, then $x_0 \in (0, \pi)$ once again and x_0 is a minimum point of f. Thus, $f''(x_0) \geqslant 0$ and

$$\lambda|f|_\infty = -\lambda f(x_0) \leqslant -\lambda f(x_0) + f''(x_0) = -g(x_0) \leqslant |g|_\infty.$$

In any case, we have that $\lambda|f|_\infty \leqslant |g|_\infty$. □

4. Let S be $\mathcal{C}_0$-semigroup of bounded linear operators on X and let $K \subset X$ be compact. Prove that for every $t_0 \geqslant 0$

$$\lim_{t \to t_0} \sup_{x \in K} \big|S(t)x - S(t_0)x\big| = 0\,. \tag{1.7.1}$$

Solution. We may assume $S \in \mathcal{G}(M, 0)$ for some $M >$) without loss of generality. Let $t_0 > 0$ and fix any $\varepsilon > 0$. Since K is totally bounded, there exist $x_1, \ldots, x_{N_\varepsilon} \in X$ such that

$$K \subset \bigcup_{n=1}^{N_\varepsilon} B\Big(x_n, \frac{\varepsilon}{M}\Big).$$

Moreover, there exists $\tau > 0$ such that

$$|t - t_0| < \tau \quad \Longrightarrow \quad \big|S(t)x_n - S(t_0)x_n\big| < \varepsilon \qquad \forall n = 1, \ldots, N_\varepsilon.$$

Thus, for all $|t - t_0| < \tau$ we have that, if $x \in K$ is such that $x \in B\big(x_n, \frac{\varepsilon}{M}\big)$, then

$$\begin{aligned}
&\big|S(t)x - S(t_0)x\big| \\
&\quad\leqslant \big|S(t)x - S(t)x_n\big| + \big|S(t)x_n - S(t_0)x_n\big| + \big|S(t_0)x_n - S(t_0)x\big| \\
&\quad\leqslant 2M|x - x_n| + \varepsilon < 3\varepsilon.
\end{aligned}$$

So, the limit of $\big|S(t)x - S(t_0)x\big|$ as $t \to t_0$ is uniform on K. □

5. Use the resolvent identity (1.4.2) to prove that $R(\lambda, A)$ commutes with $R(\mu, A)$ for all $\lambda, \mu \in \rho(A)$.

Solution. For all $\lambda, \mu \in \rho(A)$ we have that

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A).$$

So,

$$R(\mu, A) - R(\lambda, A) = (\lambda - \mu) R(\lambda, A) R(\mu, A)$$

but also

$$R(\mu, A) - R(\lambda, A) = (\lambda - \mu) R(\mu, A) R(\lambda, A).$$

Therefore $R(\lambda, A) R(\mu, A) = R(\mu, A) R(\lambda, A)$. □

6. Prove that if $A : D(A) \subset X \to X$ is a closed operator and $B \in \mathcal{L}(X)$, then $A + B : D(A) \subset X \to X$ is also closed.

7. Prove that $A : D(A) \subset X \to X$ is closed if and only if for any sequence $\{x_n\} \subset D(A)$

$$\begin{cases} x_n \to x \\ Ax_n \rightharpoonup y \end{cases} \quad \Longrightarrow \quad x \in D(A) \quad \text{and} \quad Ax = y.$$

8. Let $S \in \mathcal{G}(M, \omega)$ with $\omega \geqslant 0$. Prove that $\omega_0(S) < 0$ if and only if

$$\lim_{t \to +\infty} \|S(t)\| = 0. \tag{1.7.2}$$

Solution. One only needs to show that (1.7.2) implies that $\omega_0(S) < 0$. Let $t_0 > 0$ be such that $\|S(t_0)\| < 1/e$. For any $t > 0$ let $n(t) \in \mathbb{N}$ be the unique integer such that

$$n(t) t_0 \leqslant t < \big(n(t) + 1\big) t_0. \tag{1.7.3}$$

Then

$$\|S(t)\| = \big\|S\big(n(t)t_0\big) S\big(t - n(t)t_0\big)\big\| \leqslant \frac{Me^{\omega(t - n(t))}}{e^{n(t)}} \leqslant \frac{Me^{\omega t_0}}{e^{n(t)}}.$$

Therefore, on account of (1.7.2), we conclude that

$$\begin{aligned} \frac{\log \|S(t)\|}{t} &\leqslant \frac{\log\big(Me^{\omega t_0}\big)}{t} - \frac{n(t)}{t} \\ &\leqslant \frac{\log\big(Me^{\omega t_0}\big)}{t} - \Big(\frac{1}{t_0} - \frac{1}{t}\Big) \qquad \forall t > 0. \end{aligned}$$

Taking the limit as $t \to +\infty$ we conclude that $\omega_0(S) < 0$. □

9. Let S be the heat semigroup constructed in Example 9. Prove that, for any $f \in L^p(0,\pi)$,

$$(S(t)f)(x) = \int_0^\pi K(t,x,y)f(y)\,dy\,, \quad \forall t \geqslant 0,\ x \in (0,\pi) \text{ a.e.}$$

where

$$K(t,x,y) = \frac{2}{\pi}\sum_{k=1}^{\infty} e^{-k^2 t}\sin(kx)\sin(ky).$$

2 Special classes of semigroups

2.1 Contraction semigroups

In this section we assume that X is an Hilbert space with scalar product $\langle \cdot, \cdot \rangle$.

Definition 8 *An operator $A : D(A) \subset X \to X$ is said to be* dissipative *if*

$$\Re \langle Ax, x\rangle \leqslant 0 \qquad \forall x \in D(A). \tag{2.1.1}$$

Remark 7 Observe that, if A is dissipative, then for every $\lambda > 0$

$$|(\lambda I - A)x|^2 = \lambda^2|x|^2 - 2\Re\langle Ax, x\rangle + |Ax|^2 \geqslant \lambda^2|x|^2 \qquad \forall x \in D(A).$$

Hence

$$|(\lambda I - A)x| \geqslant \lambda|x| \qquad \forall x \in D(A) \quad \text{and} \quad \lambda > 0. \tag{2.1.2}$$

Consequently, $\lambda I - A$ is injective for all $\lambda > 0$. So, if $(\lambda_0 I - A)X = X$ for some $\lambda_0 > 0$, then (2.1.2) implies that $\lambda_0 \in \rho(A)$ and $\|R(\lambda_0, A)\| \leqslant 1/\lambda_0$. Moreover, since $R(\lambda_0, A)$ is closed, $\lambda_0 I - A$ is closed and therefore A is closed as well.

Proposition 10 *For a dissipative operator $A : D(A) \subset X \to X$ the following properties are equivalent:*

(a) *$(\lambda_0 I - A)X = X$ for some $\lambda_0 > 0$, and*

(b) *$(\lambda I - A)X = X$ for all $\lambda > 0$.*

Proof. The only implication that require a proof is $(a) \Rightarrow (b)$. By Remark 7 the set

$$\Lambda = \big\{\lambda \in (0,\infty)\ :\ (\lambda I - A)X = X\big\}$$

is contained in $\rho(A)$ which is open in $\mathbb{C}$. This implies that Λ is also open. Let us show that Λ is closed: let $\Lambda \ni \lambda_n \to \lambda > 0$ and fix any $y \in X$. There exists an $x_n \in D(A)$ such that

$$\lambda_n x_n - Ax_n = y. \tag{2.1.3}$$

From (2.1.2) it follows that $|x_n| \leqslant |y|/\lambda_n \leqslant C$ for some $C > 0$. Again by (2.1.2),

$$\begin{aligned}\lambda_m|x_n - x_m| &\leqslant |\lambda_m(x_n - x_m) - A(x_n - x_m)| \\ &\leqslant |\lambda_m - \lambda_n|\,|x_n| + |\lambda_n x_n - Ax_n - (\lambda_m x_m - Ax_m)| \\ &\leqslant C|\lambda_m - \lambda_n|.\end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence. Let $x_n \to x$. Then $Ax_n \to \lambda x - y$ by (2.1.3). Since A is closed by Remark 7, $x \in D(A)$ and $\lambda x - Ax = y$. This show that $\lambda I - A$ is surjective and implies that $\lambda \in \Lambda$. Thus Λ is both open and closed in $(0, \infty)$. Moreover, $\Lambda \neq \varnothing$ because $\lambda_0 \in \Lambda$. So, $\Lambda = (0, \infty)$. □

Proposition 11 *Let $A : D(A) \subset X \to X$ be dissipative with $(I - A)X = X$. Then $D(A)$ is dense in X.*

Proof. Let $z \in X$ be such that $\langle z, x\rangle = 0$ for all $x \in D(A)$. We will show that $z = 0$ or, equivalently since $(I - A)$ is surjective, that

$$0 = \langle z, x - Ax\rangle = \langle z, Ax\rangle \qquad \forall x \in D(A).$$

Let $x \in D(A)$. Then by Proposition 10 there exists a sequence $\{x_n\} \subset D(A)$ such that

$$nx = nx_n - Ax_n \qquad \forall n \geqslant 1. \tag{2.1.4}$$

Since $Ax_n = n(x_n - x) \in D(A)$, we have that $x_n \in D(A^2)$ and

$$Ax = Ax_n - \frac{1}{n}\,A^2 x_n \quad \text{or} \quad Ax_n = \Big(I - \frac{1}{n}\,A\Big)^{-1} Ax.$$

Since $\|(I - \frac{1}{n}\,A)^{-1}\| \leqslant 1$ by (2.1.2), the above identity yields $|Ax_n| \leqslant |Ax|$. So, by (2.1.4) we obtain

$$|x_n - x| \leqslant \frac{1}{n}\,|Ax_n| \leqslant \frac{1}{n}\,|Ax|.$$

Therefore, $x_n \to x$. Moreover, since $\{Ax_n\}$ is bounded, there is a subsequence Ax_{n_k} such that $Ax_{n_k} \rightharpoonup y$. Since A is closed by Remark 7 we deduce that $y = Ax$ (see Problems 1.7). Now, recall that $\langle z, x\rangle = 0$ for all $x \in D(A)$ to deduce that

$$\langle z, Ax_{n_k}\rangle = n_k\langle z, x_{n_k} - x\rangle = 0 \qquad \forall k \geqslant 1.$$

Letting $k \to \infty$ in the above identity we conclude that $\langle z, Ax\rangle = 0$. □

Proposition 12 *For an operator $A : D(A) \subset X \to X$ the following properties are equivalent:*

(*a*) *A is the infinitesimal generator of a contraction semigroup on X;*

(b) *A is dissipative and $(\lambda_0 I - A)X = X$ for some $\lambda_0 > 0$.*

(c) *A is dissipative and $(\lambda I - A)X = X$ for all $\lambda > 0$.*

Proof. In view of Proposition 10, the only implications that require a proof are $(a) \Rightarrow (b)$ and $(c) \Rightarrow (a)$.

$\boxed{(a) \Rightarrow (b)}$ Let A be the infinitesimal generator of a contraction semigroup S. Then $(0, \infty) \subset \rho(A)$ by Theorem 2 and A is dissipative because

$$\Re \langle Ax, x \rangle = \lim_{t \downarrow 0} \Re \Big\langle \frac{S(t)x - x}{t}, x \Big\rangle \leqslant 0 \qquad \forall x \in D(A).$$

$\boxed{(c) \Rightarrow (a)}$ Assume (c). Then $D(A)$ is dense in X by Proposition 11. Moreover, by Remark 7, A is closed, $(0, \infty) \subset \rho(A)$, and $\|R(\lambda, A)\| \leqslant 1/\lambda$ for all $\lambda > 0$. The conclusion follows by Theorem 2. □

The above results can be completed by looking at A^*, the *adjoint* of A, the definition of which we recall below. Given $A : D(A) \subset X \to X$, with $D(A)$ dense in X, let $D(A^*)$ denote the subspace of X consisting of all $y \in X$ for which there exists a constant $C_y \geqslant 0$ such that

$$|\langle Ax, y \rangle| \leqslant C_y |x| \qquad \forall x \in D(A). \tag{2.1.5}$$

Observe that, since $D(A)$ is dense in X, (2.1.5) yields that $x \mapsto \langle Ax, y \rangle$ can be extended to a unique bounded linear functional $\phi_y \in X^*$. Denoting by $j : X^* \to X$ the Riesz isomorphism, we define

$$A^* y = j(\phi_y) \qquad \forall y \in D(A^*). \tag{2.1.6}$$

Then the following *adjoint identity* holds true

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \qquad \forall x \in D(A), \forall y \in D(A^*). \tag{2.1.7}$$

Exercise 7 Check that $D(A^*)$ is a subspace of X, that $A^* : D(A^*) \subset X \to X$ is a linear operator, and that A^* is closed.

Solution. We only prove that A^* is closed. Let $\{y_n\} \subset D(A^*)$ and $y, z \in X$ be such that

$$\begin{cases} y_n \to y \\ A^* y_n \to z \end{cases} \qquad (n \to \infty)$$

Then $\{A^* y_n\}$ is bounded, say $|A^* y_n| \leqslant C$. So, recalling (2.1.7),

$$|\langle Ax, y_n \rangle| = |\langle x, A^* y_n \rangle| \leqslant C|x| \qquad \forall x \in D(A)$$

yields

$$|\langle Ax, y \rangle| \leqslant C|x| \qquad \forall x \in D(A)$$

implying that $y \in D(A^*)$. Moreover

$$\langle Ax, y\rangle = \lim_{n\to\infty} \langle Ax, y_n\rangle = \langle x, z\rangle \qquad \forall x \in D(A).$$

Thus, $\langle x, A^*y - z\rangle = 0$ for all $x \in D(A)$. Since $D(A)$ is dense, $A^*y = z$. $\square$

Remark 8 If $A \in \mathcal{L}(X)$, then A^* is also bounded and we have that $A^{**} = A$.

Theorem 3 (Lumer-Phillips) *Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator. If A and A^* are dissipative, then A is the infinitesimal generator of a contraction semigroup on X.*

Proof. In view of Proposition 12 it suffices to show that $(0, \infty) \subset \rho(A)$.

Step 1: *$\lambda I - A$ and $\lambda I - A^*$ are injective for every $\lambda > 0$.*
This follows from Remark 7.

Step 2: *$(\lambda I - A)(D(A))$ is dense in X for every $\lambda > 0$.*
Let $y \in X$ be such that

$$\langle \lambda x - Ax, y\rangle = 0 \qquad \forall x \in D(A).$$

Then $\langle Ax, y\rangle = \lambda\langle x, y\rangle$ implies that $y \in D(A^*)$ and

$$\langle x, \lambda y - A^*y\rangle = 0 \qquad \forall x \in X.$$

So, $\lambda y - A^*y = 0$ which, by Step 1, yields $y = 0$.

Step 3: *$\lambda I - A$ is surjective for every $\lambda > 0$.*
Fix any $y \in X$. By Step 1, there exists $\{x_n\} \subset D(A)$ such that

$$\lambda x_n - Ax_n =: y_n \to y \quad \text{as} \quad n \to \infty.$$

By (2.1.2) we deduce that, for all $n, m \geqslant 1$,

$$|x_n - x_m| \leqslant \frac{1}{\lambda}\,|y_n - y_m|$$

which yields that $\{x_n\}$ is a Cauchy sequence in X. Therefore, there exists $x \in X$ such that

$$\begin{cases} x_n \to x \\ Ax_n = \lambda x_n - y_n \to \lambda x - y \end{cases} \qquad (n \to \infty)$$

Since A is closed, $x \in D(A)$ and $\lambda x - Ax = y$. $\square$

Remark 9 The notion of dissipative operators can be given in Banach spaces and Theorem 3 remains valid in such settings. However, Proposition 11 is true only if X is reflexive (see, for instance, [6, Section 1.4]).

Example 11 (Wave equation in $L^2(0,\pi)$) Let us set $H_0^1(0,\pi) = W_0^{1,2}(0,\pi)$ and $H^2(0,\pi) = W^{2,2}(0,\pi)$. For any given $f \in H^2(0,\pi) \cap H_0^1(0,\pi)$ and $g \in H_0^1(0,\pi)$ we want to solve the problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) & (t,x) \in \mathbb{R} \times (0,\pi) \\ u(t,0) = 0 = u(t,\pi) & t \in \mathbb{R} \\ u(0,x) = f(x)\,, \ \frac{\partial u}{\partial t}(0,x) = g(x) & x \in (0,\pi). \end{cases} \tag{2.1.8}$$

Let $\mathcal{X}$ be the Hilbert space $H_0^1(0,\pi) \times L^2(0,\pi)$ with the scalar product

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \bar u \\ \bar v \end{pmatrix} \right\rangle = \int_0^\pi \big[u'(s)\bar u'(s) + v(s)\bar v(s)\big]ds.$$

Denoting by $A : D(A) \subset L^2(0,\pi) \to L^2(0,\pi)$ the second derivative with homogeneous Dirichlet boundary conditions studied in Example 9, define $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ by

$$\begin{cases} D(\mathcal{A}) = \big(H^2(0,\pi) \cap H_0^1(0,\pi)\big) \times H_0^1(0,\pi) \\ \mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ Au \end{pmatrix} \end{cases} \tag{2.1.9}$$

The fact that $\mathcal{A}$ is closed and $D(\mathcal{A})$ is dense can be easily checked. We claim that $\mathbb{R} \setminus \{0\} \subset \rho(\mathcal{A})$ and

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} \lambda & 1 \\ A & \lambda \end{pmatrix} R(\lambda^2, A) \qquad \forall \lambda \neq 0. \tag{2.1.10}$$

Indeed, for any $(f,g) \in \mathcal{X}$ the resolvent equation

$$(\lambda I - \mathcal{A})\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

is equivalent to the system

$$\begin{cases} \lambda u - v = f \\ \lambda v - Au = g. \end{cases}$$

Hence, $v = \lambda u - f$ and, by solving the equation

$$\lambda^2 u - Au = g + \lambda f,$$

we find

$$\begin{cases} u = \lambda R(\lambda^2, A)f + R(\lambda^2, A)g \\ v = \big[\lambda^2 R(\lambda^2, A) - 1\big]f + \lambda R(\lambda^2, A)g. \end{cases}$$

Since $\lambda^2 R(\lambda^2, A) - 1 = AR(\lambda^2, A)$ by (1.4.1), the above identities yield (2.1.10).

Now, integrating by parts we obtain

$$\Big\langle \mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \Big\rangle = \int_0^\pi \big[u'(s)v'(s) + u(s)v''(s)\big]ds = 0, \ \forall \begin{pmatrix} u \\ v \end{pmatrix} \in D(\mathcal{A}).$$

Therefore $\mathcal{A}$ is dissipative and so it generates a contraction semigroup $e^{t\mathcal{A}}$ on $\mathcal{X}$ by Proposition 12. In fact, $e^{t\mathcal{A}}$ is a $\mathcal{C}_0$ group thanks to Proposition 9 and Remark 6. Consequently, problem (2.1.8) has a unique solution

$$u \in \mathcal{C}^2(\mathbb{R}; L^2(0,\pi)) \cap \mathcal{C}^1(\mathbb{R}; H_0^1(0,\pi)) \cap \mathcal{C}(\mathbb{R}; H^2(0,\pi) \cap H_0^1(0,\pi))$$

which is given by the first component of $e^{t\mathcal{A}}(f,g)$.

Definition 9 *A densely defined closed linear operator $A : D(A) \subset X \to X$ is said to be* symmetric *if $A \subset A^*$, that is,*

$$D(A) \subset D(A^*) \quad \textit{and} \quad Ax = A^*x \qquad \forall x \in D(A).$$

A is said to be self-adjoint *if $A = A^*$.*

Clearly, a symmetric operator A is self-adjoint if and only if $D(A) = D(A^*)$. This is always the case when A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X, as our next result guarantees.

Proposition 13 *Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator such that $\rho(A) \cap \mathbb{R} \neq \varnothing$. If A is symmetric, then A is self-adjoint.*

Proof. We will prove that $D(A^*) \subset D(A)$ in two steps. Let $\lambda \in \rho(A) \cap \mathbb{R}$.

Step 1: $R(\lambda, A) = R(\lambda, A)^*$.
Since $R(\lambda, A) \in \mathcal{L}(X)$ it suffices to show that

$$\langle R(\lambda, A)x, y\rangle = \langle x, R(\lambda, A)y\rangle \qquad \forall x, y \in X.$$

Fix any $x, y \in X$ and set

$$u = R(\lambda, A)x \quad \text{and} \quad v = R(\lambda, A)y$$

so that

$$\lambda u - Au = x \quad \text{and} \quad \lambda v - Av = y.$$

Since A is symmetric, we have that

$$\langle R(\lambda, A)x, y\rangle = \langle u, y\rangle = \langle u, \lambda v - Av\rangle = \langle \lambda u - Au, v\rangle = \langle x, R(\lambda, A)y\rangle.$$

Step 2: $D(A^*) \subset D(A)$.
Let $u \in D(A^*)$ and set $x = \lambda u - A^*u$. Observe that, for all $v \in D(A)$,

$$\langle x, v\rangle = \langle \lambda u - A^*u, v\rangle = \langle u, \lambda v - Av\rangle.$$

Now, take any $y \in X$ and let $v = R(\lambda, A)y$. Then the above identity yields

$$\langle x, R(\lambda, A)y\rangle = \langle u, y\rangle \qquad \forall y \in X.$$

This identity and Step 1 imply that $u = R(\lambda, A)^* x = R(\lambda, A)x \in D(A)$. $\square$

The following property of self-adjoint operators is very useful.

Corollary 2 (Stone) *Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator. If A is self-adjoint, then $B := iA$ is the infinitesimal generator of a $\mathcal{C}_0$ unitary group on X.*

Proof. Since A is self-adjoint, we have that

$$\langle Ax, x\rangle = \langle x, A^* x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle} \qquad \forall x \in D(A).$$

Thus, $\langle Ax, x\rangle$ is real and

$$\Re\langle Bx, x\rangle = \Re\langle iAx, x\rangle = 0 \qquad \forall x \in D(B),$$

which implies that B and $B^* = -B$ are dissipative. So, B and $-B$ generate contraction semigroups on X. Therefore B generates a $\mathcal{C}_0$ unitary group. $\square$

Exercise 8 On $X = L^2(0, \pi; \mathbb{C})$ let $A : D(A) \subset X \to X$ be the operator

$$\begin{cases} D(A) = H^2(0, \pi; \mathbb{C}) \cap H_0^1(0, \pi; \mathbb{C}) \\ Af(x) = f''(x) \qquad\qquad\qquad\qquad x \in (0, \pi) \text{ a.e.} \end{cases} \tag{2.1.11}$$

Show that A is self-adjoint and dissipative.
Solution. We begin by observing that

$$\langle Af, f\rangle = \int_0^\pi f''(x)\overline{f(x)}dx = -\int_0^\pi |f'(x)|^2 dx \qquad \forall f \in D(A).$$

Therefore A is dissipative.

Moreover, A is symmetric. Indeed, that for all $g \in D(A)$ we have

$$\langle Af, g\rangle = \int_0^\pi f''(x)\overline{g(x)}dx = \int_0^\pi f(x)\overline{g''(x)}dx \quad \forall f \in D(A). \tag{2.1.12}$$

Therefore $|\langle Af, g\rangle| \leqslant |g''|_2 |f|_2$ for all $f \in D(A)$, which yields $g \in D(A^*)$. Then (2.1.7), together with (2.1.12), implies that $A^* g = g''$ for all $g \in D(A)$.

Thus, in order to prove that A is self-adjoint it suffices to check that $1 \in \rho(A)$. In fact, we will show that—as in Example 9— $\sigma(A) = \{-n^2 \ : \ n \in \mathbb{N}\}$. Fix any $g \in X$ and consider the Sturm-Liouville system

$$\begin{cases} \lambda f(x) - f''(x) = g(x), \quad 0 < x < \pi \\ f(0) = 0 = f(\pi). \end{cases} \tag{2.1.13}$$

Let us consider the odd extension of g to $[-\pi,\pi]$ and denote by

$$g(x)=\sum_{n\in\mathbb{Z}^*}\widehat{g}(n)e^{inx}\qquad (x\in[0,\pi])$$

the Fourier series of such a function. We seek a solution f of the form

$$f(x)=\sum_{n\in\mathbb{Z}^*}\widehat{f}(n)e^{inx}\qquad (x\in[0,\pi]).$$

In order to satisfy (2.1.13) one must have

$$(\lambda+n^2)\widehat{f}(n)=\widehat{g}(n)\qquad \forall n\in\mathbb{Z}^*.$$

So, for any $\lambda\in\mathbb{C}\setminus\{-n^2\ :\ n\geqslant 1\}$, (2.1.13) has a unique solution. □

Example 12 (Schrödinger equation) By Corollary 2 and Exercise 8 we have that, for any $f\in H^2(0,\pi)\cap H^1_0(0,\pi)$, there exists a unique solution

$$u\in\mathcal{C}^1(\mathbb{R};L^2(0,\pi))\cap\mathcal{C}(\mathbb{R};H^2(0,\pi)\cap H^1_0(0,\pi))$$

of the problem

$$\begin{cases}\frac{\partial u}{\partial t}\,(t,x)=i\,\frac{\partial^2 u}{\partial x^2}\,(t,x) & (t,x)\in\mathbb{R}\times(0,\pi)\\ u(t,0)=0=u(t,\pi) & t\in\mathbb{R}\\ u(0,x)=f(x) & x\in(0,\pi).\end{cases}$$

2.2 Analytic semigroups

We recall that, for any $\omega\in\mathbb{R}$, we have denoted by Π_ω the complex half-plane in (1.4.6). Moreover, for any $\theta\in(0,\pi]$ we define

$$\Sigma_{\omega,\theta}=\big\{\lambda\in\mathbb{C}\setminus\{\omega\}\ :\ \big|\arg(\lambda-\omega)\big|<\theta\big\}.\tag{2.2.1}$$

Let $\omega\in\mathbb{R}$ and $\theta_0\in(\pi/2,\pi]$.

Definition 10 *A densely defined closed linear operator $A:D(A)\subset X\to X$ on a Banach space X is called* sectorial *of base point $\omega\in\mathbb{R}$ and angle θ_0 if:*

(a) $\Sigma_{\omega,\theta_0}\subset\rho(A)$, and

(b) there exists a nondecreasing function $M:(0,\theta_0)\to(0,+\infty)$ such that

$$\|R(\lambda,A)\|\leqslant\frac{M(\theta)}{|\lambda-\omega|}\qquad\forall\theta\in(0,\theta_0),\ \forall\lambda\in\Sigma_{\omega,\theta}.\tag{2.2.2}$$

Let $A : D(A) \subset X \to X$ be a sectorial operator of base point ω and angle θ_0 on a Banach space X. For any $\varepsilon > 0$ and $\theta \in (\pi/2, \theta_0)$, let

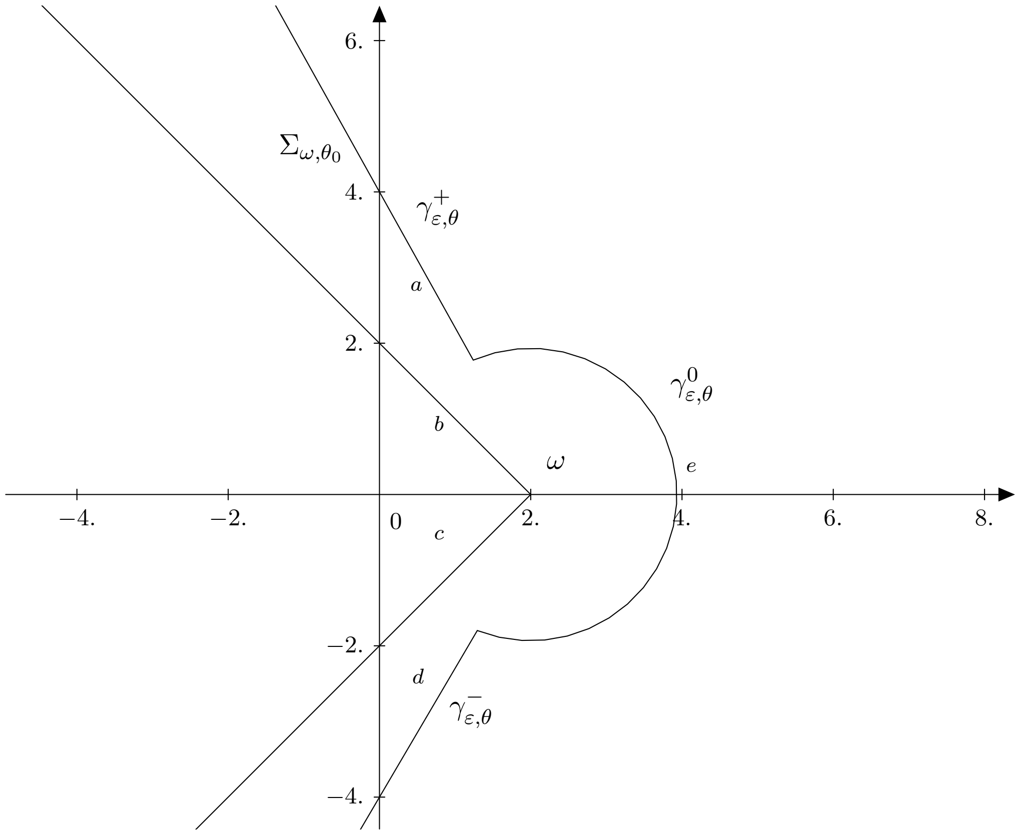
$$\gamma_{\varepsilon,\theta} = \gamma^+_{\varepsilon,\theta} \cup \gamma^-_{\varepsilon,\theta} \cup \gamma^0_{\varepsilon,\theta} \tag{2.2.3}$$

where

$$\gamma^\pm_{\varepsilon,\theta} = \{z \in \mathbb{C} \ : \ z = \omega + re^{\pm i\theta}, \ r \geqslant \varepsilon\}$$

and

$$\gamma^0_{\varepsilon,\theta} = \{z \in \mathbb{C} \ : \ z = \omega + \varepsilon e^{i\eta}, \ |\eta| \leqslant \theta\}.$$



Proposition 14 (Dunford integral) *Let $\varepsilon > 0$ and $\theta \in (\pi/2, \theta_0)$ be fixed. Then for each $t \geqslant 0$*

$$S(t) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_{\varepsilon,\theta}} e^{\lambda t} R(\lambda, A)\, d\lambda & t > 0 \\ I & t = 0 \end{cases} \tag{2.2.4}$$

is a bounded linear operator on X.

Proof. Since $\gamma^0_{\varepsilon,\theta}, \gamma^\pm_{\varepsilon,\theta} \subset \Sigma_{\omega,\theta}$, by (2.2.2) we deduce that for any $t > 0$

$$\|e^{\lambda t} R(\lambda, A)\| \leqslant e^{t(\omega + r\cos\theta)} \frac{M(\theta)}{r} \qquad \forall \lambda = \omega + re^{\pm i\theta} \in \gamma^\pm_{\varepsilon,\theta}$$

and

$$\|e^{\lambda t} R(\lambda, A)\| \leqslant e^{t(\omega+\varepsilon\cos\eta)} \frac{M(\theta)}{\varepsilon} \qquad \forall \lambda = \omega + \varepsilon e^{i\eta} \in \gamma^0_{\varepsilon,\theta}.$$

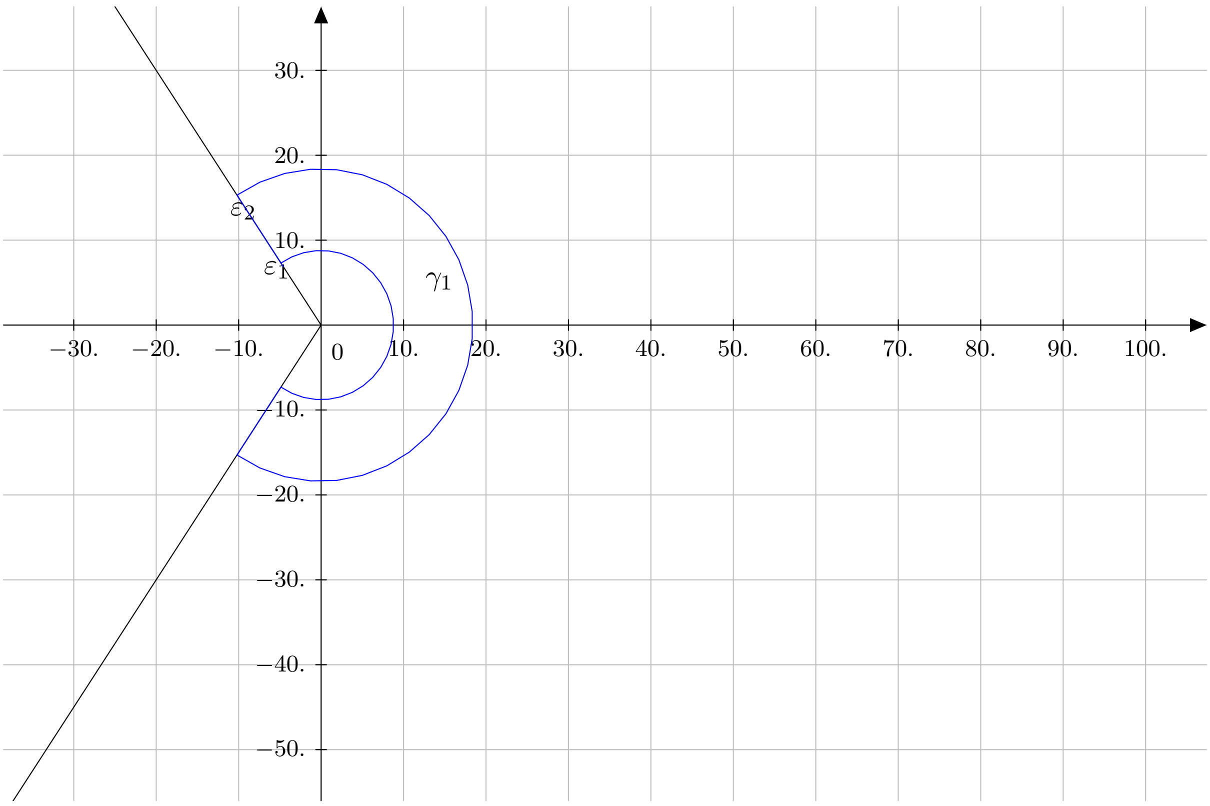
Because $\cos\theta < 0$, the above inequalities ensure the convergence of the integral in (2.2.4). The the completeness of $\mathcal{L}(X)$ yields $S(t) \in \mathcal{L}(X)$. □

Exercise 9 Use Cauchy's theorem for holomorphic functions to show that, for any $0 < \varepsilon_1 < \varepsilon_2$ and $\theta_1, \theta_2 \in (\pi/2, \theta_0)$, we have

$$\int_{\gamma_{\varepsilon_1,\theta_1}} e^{\lambda t} R(\lambda, A)\, d\lambda = \int_{\gamma_{\varepsilon_2,\theta_2}} e^{\lambda t} R(\lambda, A)\, d\lambda \qquad \forall t > 0.$$

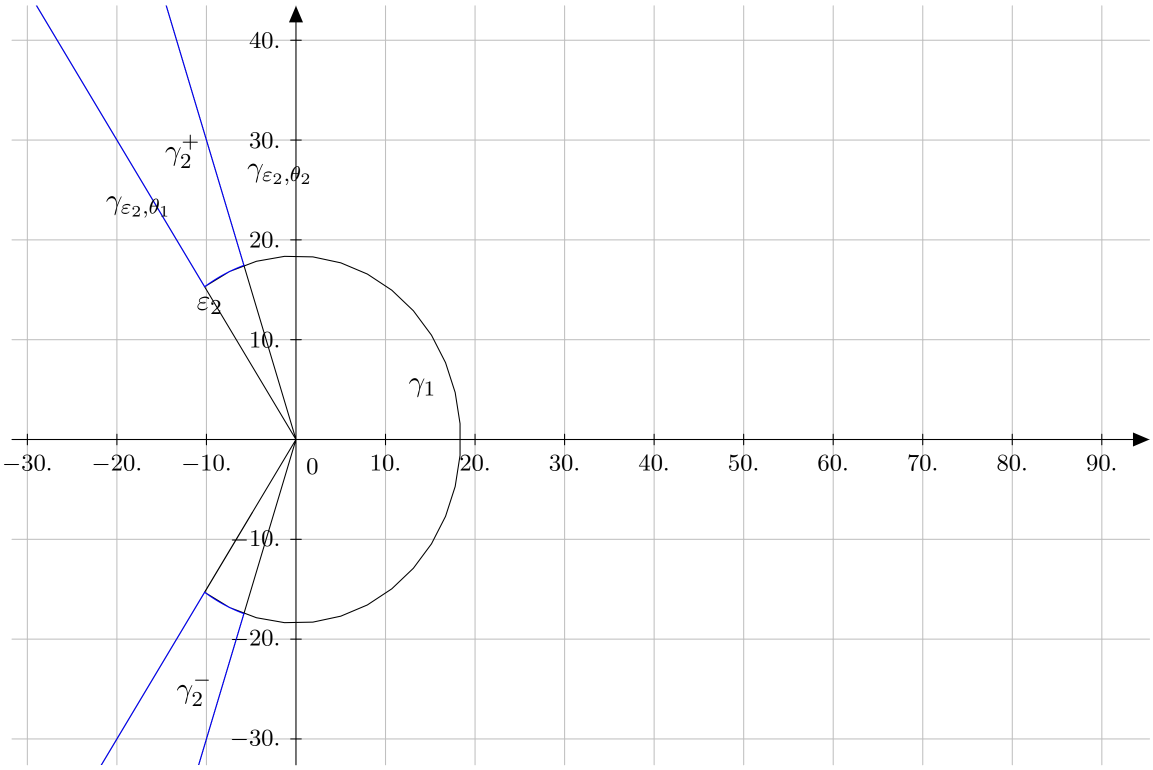
Solution. It suffices to observe that, owing to Cauchy's theorem,

$$\int_{\gamma_{\varepsilon_1,\theta_1}} e^{\lambda t} R(\lambda, A)\, d\lambda - \int_{\gamma_{\varepsilon_2,\theta_1}} e^{\lambda t} R(\lambda, A)\, d\lambda = \int_{\gamma_1} e^{\lambda t} R(\lambda, A)\, d\lambda = 0.$$



Similarly,

$$\int_{\gamma_{\varepsilon_2,\theta_1}} e^{\lambda t} R(\lambda, A)\, d\lambda - \int_{\gamma_{\varepsilon_2,\theta_2}} e^{\lambda t} R(\lambda, A)\, d\lambda = \int_{\gamma_2^+ \cup \gamma_2^-} e^{\lambda t} R(\lambda, A)\, d\lambda = 0$$



again by Cauchy's theorem and a simple asymptotic argument. □

Theorem 4 *Let $A : D(A) \subset X \to X$ be a sectorial operator of base point ω and angle θ_0 on a Banach space X. Fix any $0 < \varepsilon < 1$ and $\theta \in (\pi/2, \theta_0)$*[1]*, and define $S : \mathbb{R}_+ \to \mathcal{L}(X)$ as in (2.2.4). Then the following properties hold true.*

(*a*) *$S \in \mathcal{C}^1(\mathbb{R}_+^*; \mathcal{L}(X))$ and $S'(t) = AS(t)$ for all $t > 0$.*

(*b*) *There exist constants $M, N \geqslant 0$ such that*

$$\|S(t)\| \leqslant Me^{\omega t} \qquad \forall t \geqslant 0 \tag{2.2.5}$$

and

$$\|(A - \omega I)S(t)\| \leqslant \frac{N}{t}\, e^{\omega t} \qquad \forall t > 0. \tag{2.2.6}$$

(*c*) *S is a $\mathcal{C}_0$-semigroup and A is its infinitesimal generator.*

Proof. Without loss of generality we can restrict the analysis to the case of $\omega = 0$. Indeed, the general case can be treated replacing A by $A_\omega := A - \omega I$ which is easily seen to be sectorial of base point 0. Since $R(\mu, A_\omega) = R(\mu + \omega, A)$, one

[1] For instance, one can take $\varepsilon = \frac{1}{2}$ and $\theta = \frac{\pi}{4} + \frac{\theta_0}{2}$.

recovers S, the semigroup generated by A, from the semigroup S_ω generated by A_ω via the formula $S(t) = e^{\omega t} S_\omega(t)$.

Step 1: proof of (a).
The fact that $S \in \mathcal{C}^1(\mathbb{R}_+^*; \mathcal{L}(X))$ follows by differentiating under the integral sign: by (2.2.2) we deduce that for any $\lambda \in \gamma_{\varepsilon,\theta}^{\pm}$

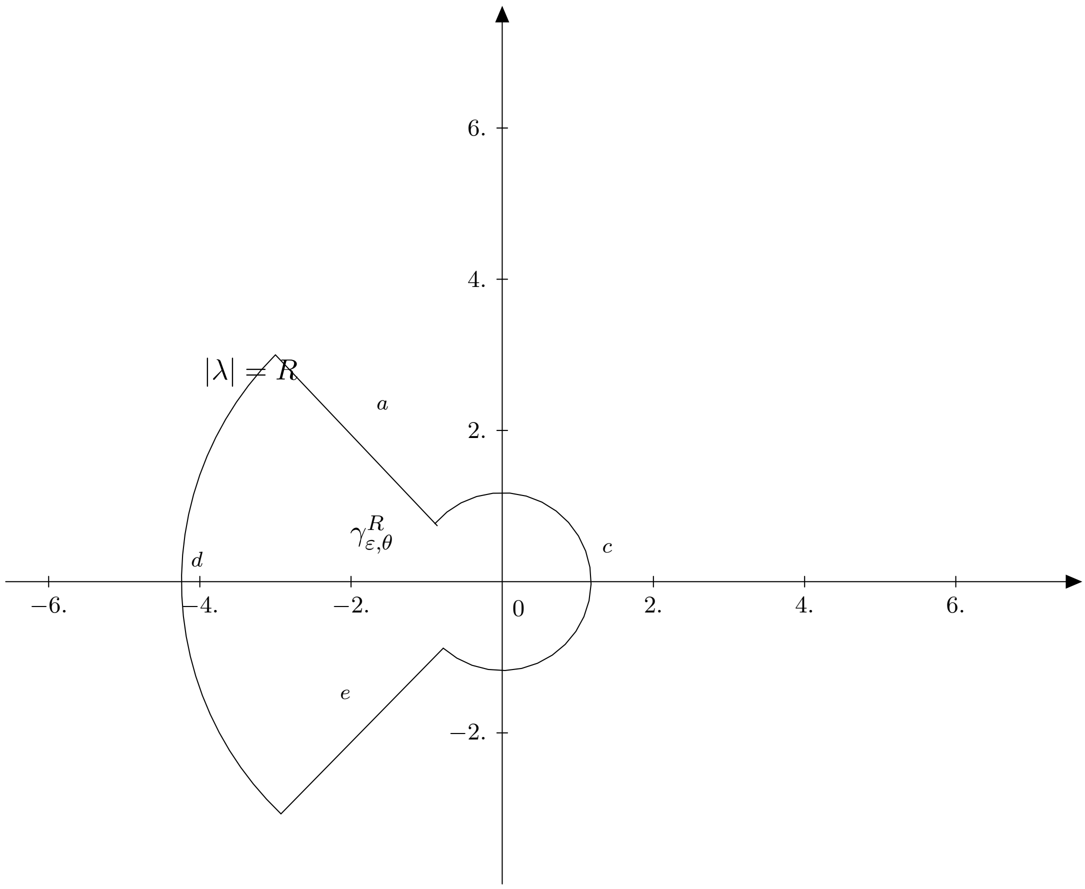
$$\left\| \frac{\partial}{\partial t} e^{\lambda t} R(\lambda, A) \right\| = \| \lambda e^{\lambda t} R(\lambda, A) \| \leqslant e^{t \Re \lambda} M(\theta) = e^{t|\lambda| \cos \theta} M(\theta)$$

which guarantees the convergence of the integral $\frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} \frac{\partial}{\partial t} e^{\lambda t} R(\lambda, A)\, d\lambda$, in the space $\mathcal{L}(X)$, because $\cos \theta < 0$. Moreover, recalling the identity $\lambda R(\lambda, A) = I + AR(\lambda, A)$, we obtain

$$\begin{aligned} S'(t) &= \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} \lambda e^{\lambda t} R(\lambda, A)\, d\lambda \\ &= \frac{I}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} e^{\lambda t}\, d\lambda + \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} A e^{\lambda t} R(\lambda, A)\, d\lambda = AS(t) \end{aligned} \tag{2.2.7}$$

for all $t > 0$ because

$$\int_{\gamma_{\varepsilon,\theta}} e^{\lambda t}\, d\lambda = \lim_{R \to \infty} \int_{\gamma_{\varepsilon,\theta}^R} e^{\lambda t}\, d\lambda = 0$$



Step 2: *proof of* (*b*).
The change of variable $\lambda t = \xi$ transforms the integral in (2.2.4) into

$$S(t) = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon t,\theta}} e^{\xi} R\Big(\frac{\xi}{t}, A\Big) \frac{d\xi}{t} = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} e^{\xi} R\Big(\frac{\xi}{t}, A\Big) \frac{d\xi}{t}\,,$$

where we have used Exercise 9. Therefore,

$$\begin{aligned} S(t) = \frac{1}{2\pi i}\Big\{ \int_{\varepsilon}^{\infty} e^{re^{i\theta}} R\Big(\frac{re^{i\theta}}{t}, A\Big) \frac{e^{i\theta}}{t}\, dr \\ - \int_{\varepsilon}^{\infty} e^{re^{-i\theta}} R\Big(\frac{re^{-i\theta}}{t}, A\Big) \frac{e^{-i\theta}}{t}\, dr \\ + \int_{-\theta}^{\theta} e^{\varepsilon e^{i\eta}} R\Big(\frac{\varepsilon e^{i\eta}}{t}, A\Big)\, i\varepsilon\, e^{i\eta} \frac{d\eta}{t} \Big\} \end{aligned}$$

Now, appealing to (2.2.2) we have that

$$\|S(t)\| \leqslant \frac{M(\theta)}{2\pi} \Big\{ 2 \int_{\varepsilon}^{\infty} \frac{e^{r\cos\theta}}{r}\, dr + \int_{-\theta}^{\theta} e^{\varepsilon\cos\eta}\, d\eta \Big\} =: M.$$

By the same change of variable in (2.2.7), a computation similar to the one above leads to

$$\|S'(t)\| \leqslant \frac{M(\theta)}{2\pi t}\Big\{2\int_\varepsilon^\infty e^{r\cos\theta}\,dr + \varepsilon\int_{-\theta}^{\theta} e^{\varepsilon\cos\eta}\,d\eta\Big\}.$$

Here we can also use the fact that the above inequality holds true for all $\varepsilon > 0$. So, passing to the limit as $\varepsilon \downarrow 0$ we obtain

$$\|S'(t)\| \leqslant \frac{M(\theta)}{\pi t|\cos\theta|} =: \frac{N}{t}.$$

Step 3: *S is strongly continuous.*
In view of (2.2.5) it suffices to show that $S(t)x \to x$ as $t \downarrow 0$ for all $x \in D(A)$. So, fix $x \in D(A)$ and let $y = x - Ax$. Then $x = R(1, A)y$ and, recalling that $0 < \varepsilon < 1$[2], for all $t > 0$ the resolvent identity (1.4.2) yields

$$\begin{aligned} S(t)x = S(t)R(1,A)y &= \frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\theta}} e^{\lambda t}R(\lambda,A)R(1,A)y\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\theta}} \frac{e^{\lambda t}}{1-\lambda}R(\lambda,A)y\,d\lambda - \frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\theta}} \frac{e^{\lambda t}}{1-\lambda}R(1,A)y\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\theta}} \frac{e^{\lambda t}}{1-\lambda}R(\lambda,A)y\,d\lambda \end{aligned}$$

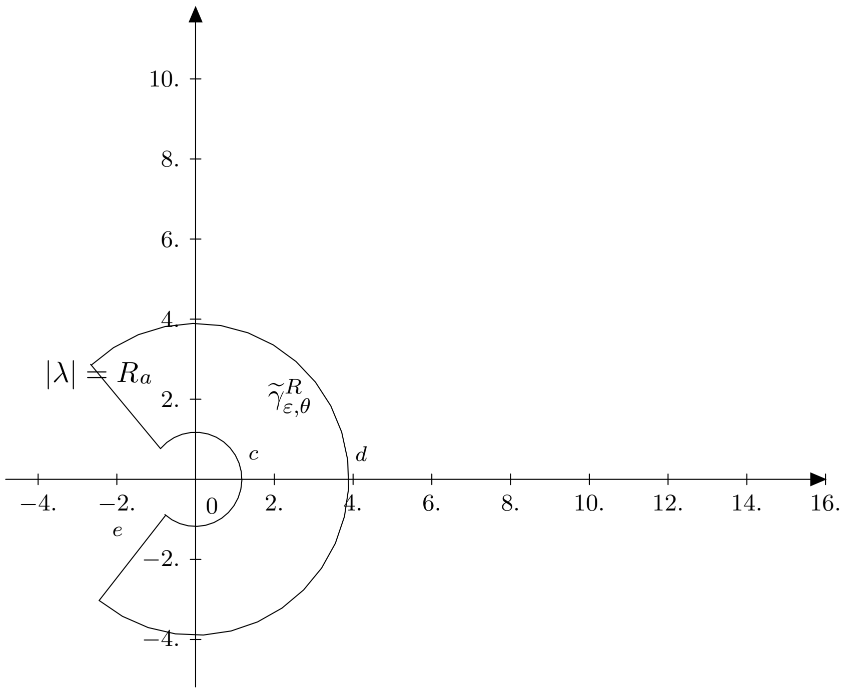
because, by Cauchy's theorem,

$$\int_{\gamma_{\varepsilon,\theta}} \frac{e^{\lambda t}}{1-\lambda}\,d\lambda = \lim_{R\to\infty}\int_{\gamma^R_{\varepsilon,\theta}} \frac{e^{\lambda t}}{1-\lambda}\,d\lambda = 0.$$

Therefore, by Cauchy's integral formula

$$\begin{aligned} \lim_{t\downarrow 0} S(t)x &= \frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\theta}} \frac{R(\lambda,A)y}{1-\lambda}\,d\lambda \\ &= \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\tilde{\gamma}^R_{\varepsilon,\theta}} \frac{R(\lambda,A)y}{1-\lambda}\,d\lambda = R(1,A)y = x. \end{aligned}$$

[2] Without the restriction $\varepsilon \in (0,1)$, here one should take $\varepsilon_0 > \varepsilon$ and define $y = \varepsilon_0 x - Ax$.



Step 4: $S(t+s) = S(t)S(s)$.
Fix any $\theta' \in (\pi/2, \theta)$. Then for all $t, s > 0$ we have that

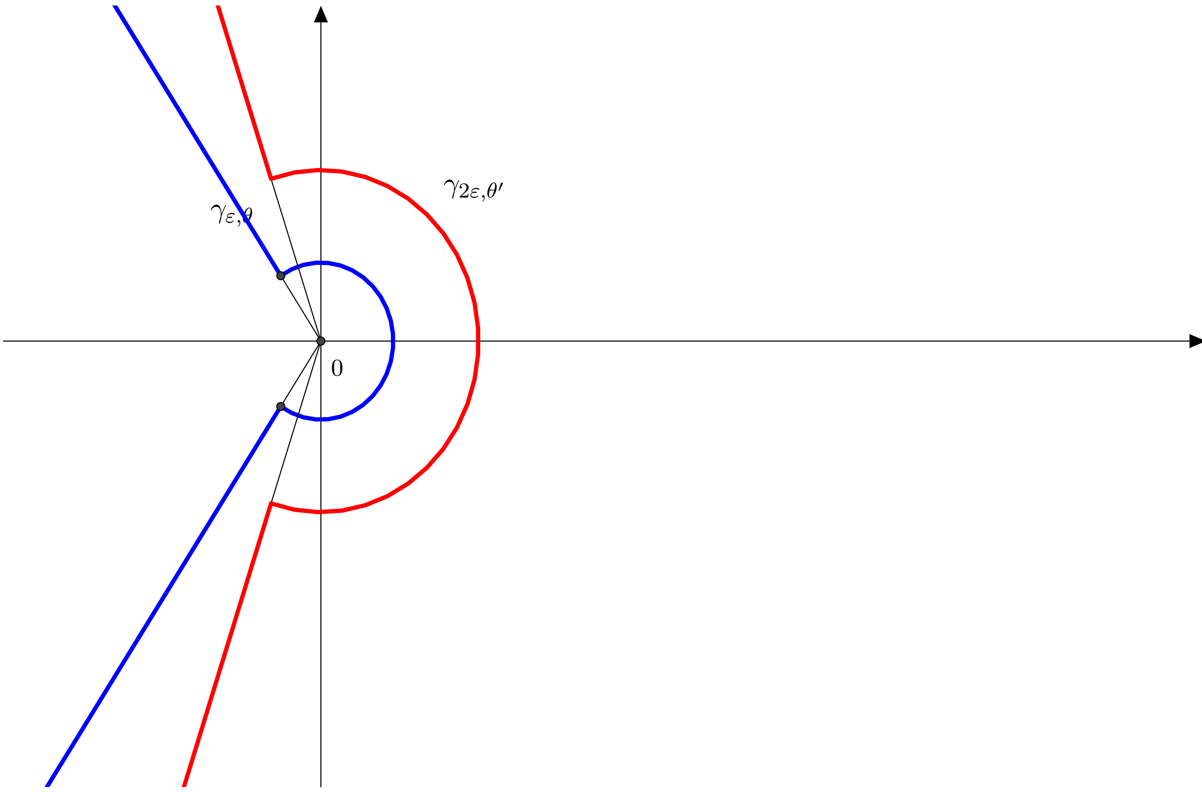
$$S(t)S(s) = \Big(\frac{1}{2\pi i}\Big)^2 \int_{\gamma_{\varepsilon,\theta}} e^{\lambda t} R(\lambda, A)\, d\lambda \ \cdot \int_{\gamma_{2\varepsilon,\theta'}} e^{\mu s} R(\mu, A)\, d\mu.$$

So, by the resolvent identity (1.4.2) we obtain

$$\begin{aligned} S(t)S(s) &= \Big(\frac{1}{2\pi i}\Big)^2 \int_{\gamma_{\varepsilon,\theta}} \int_{\gamma_{2\varepsilon,\theta'}} e^{\lambda t + \mu s}\, \frac{R(\lambda, A) - R(\mu, A)}{\mu - \lambda}\, d\lambda d\mu \\ &= \Big(\frac{1}{2\pi i}\Big)^2 \int_{\gamma_{\varepsilon,\theta}} e^{\lambda t} R(\lambda, A)\, d\lambda \int_{\gamma_{2\varepsilon,\theta'}} \frac{e^{\mu s}}{\mu - \lambda}\, d\mu \\ &\quad - \Big(\frac{1}{2\pi i}\Big)^2 \int_{\gamma_{2\varepsilon,\theta'}} e^{\mu s} R(\mu, A)\, d\mu \int_{\gamma_{\varepsilon,\theta}} \frac{e^{\lambda t}}{\mu - \lambda}\, d\lambda = S(t+s) \end{aligned}$$

because for each $\lambda \in \gamma_{\varepsilon,\theta}$ and $\mu \in \gamma_{2\varepsilon,\theta'}$ we have that

$$\frac{1}{2\pi i} \int_{\gamma_{2\varepsilon,\theta'}} \frac{e^{\mu s}}{\mu - \lambda}\, d\mu = e^{\lambda s} \quad \text{and} \quad \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} \frac{e^{\lambda t}}{\mu - \lambda}\, d\lambda = 0.$$



Step 5: *A is the infinitesimal generator of S.*
Let $B : D(B) \subset X \to X$ be the infinitesimal generator of S. Then $A \subset B$ in view of (a). Moreover, $\Pi_0 \subset \rho(A)$ by assumption and $\Pi_0 \subset \rho(B)$ by Proposition 5. So, on account of Proposition 6, $A = B$. $\square$

Exercise 10 Let $A : D(A) \subset X \to X$ be a sectorial operator which generates a $\mathcal{C}_0$-group. Show that $A \in \mathcal{L}(X)$.

Theorem 5 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup $S \in \mathcal{G}(M, \omega)$. Then the following properties are equivalent.*

(a) *There exists $\theta_0 \in (\pi/2, \pi]$ such that A is sectorial of base point ω and angle θ_0.*

(b) *$S \in \mathcal{C}^1(\mathbb{R}_+^*; \mathcal{L}(X))$ and there exists $N > 0$ such that*

$$\|(A - \omega I)S(t)\| \leqslant \frac{N}{t} e^{\omega t} \qquad \forall t > 0. \tag{2.2.8}$$

(c) *There exists $\theta \in (0, \pi/2)$ such S has an analytic extension to $\Sigma_{\omega,\theta}$ and $z \mapsto e^{-\omega z} S(z)$ is bounded on $\Sigma_{\omega,\theta'}$ for all $0, \theta' < \theta$.*

Proof. First, we observe Theorem 4 ensures that $(a) \Rightarrow (b)$. In the proof of the remaining statements we note that one can assume $\omega = 0$, as we did in the proof of Theorem 4, without loss of generality.

$\boxed{(b) \Rightarrow (c)}$ We claim that $S \in \mathcal{C}^\infty(\mathbb{R}_+^*; \mathcal{L}(X))$ and

$$\begin{cases} S(t)X \subset D(A^n) \\ S^{(n)}(t) = A^n S(t) = \Big(AS(\frac{t}{n})\Big)^n = \Big(S'(\frac{t}{n})\Big)^n \end{cases} \qquad \forall n \geqslant 1\,,\ \forall t > 0. \tag{2.2.9}$$

Indeed, (2.2.9) holds true for $n = 1$. Assuming it holds for some $n \geqslant 1$ we have that

$$S\Big(\frac{t}{n+1}\Big)A^n S\Big(\frac{nt}{n+1}\Big)X \subset D(A).$$

This shows that $S(t)X \subset D(A^{n+1})$ and

$$\Big(S'(\frac{t}{n+1})\Big)^{n+1} = \Big(AS(\frac{t}{n+1})\Big)^{n+1} = A^{n+1}S(t) = S^{(n+1)}(t).$$

Next, by (2.2.8) and (2.2.9) we deduce that

$$\|S^{(n)}(t)\| \leqslant \frac{n^n N^n}{t^n} \qquad \forall n \geqslant 1\,,\ \forall t > 0.$$

Therefore, for every $t > 0$,

$$\sum_{n=0}^{\infty} \frac{|z-t|^n}{n!}\|S^{(n)}(t)\| \leqslant \sum_{n=0}^{\infty} \frac{|z-t|^n}{n!}\frac{n^n N^n}{t^n} < \infty \tag{2.2.10}$$

for all complex numbers z in the disc

$$C\Big(t, \frac{t}{Ne}\Big) := \Big\{z \in \mathbb{C} \ :\ |z-t| < \frac{t}{Ne}\Big\}.$$

Consequently, the series $\sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} S^{(n)}(t)$ defines an analytic function F_t on $C(t, \frac{t}{Ne})$. Taking $\theta = \arctan(\frac{1}{Ne})$, we conclude that there is a unique analytic function F on

$$\Sigma_{0,\theta} \subset \bigcup_{t>0} C\Big(t, \frac{t}{Ne}\Big)$$

which coincides with F_t on any $C(t, \frac{t}{Ne})$ and therefore with S on $\mathbb{R}_+^*$.

Finally, in order to show that $S(z) = F(z)$ is bounded on every subsector of $\Sigma_{0,\theta}$, fix any $0 < q < 1$ and set $\theta' = \arctan(\frac{q}{Ne})$. Then, by (2.2.10), for all $z \in \Sigma_{0,\theta'}$ we have that

$$\|S(z)\| \leqslant \sum_{n=0}^{\infty} \frac{|\Im z|^n}{n!}\|S^{(n)}(\Re z)\| \leqslant \sum_{n=0}^{\infty} \frac{|\Im z|^n}{(\Re z)^n}\frac{n^n N^n}{n!} \leqslant \sum_{n=0}^{\infty} \frac{n^n}{n! e^n}\, q^n < \infty.$$

$\boxed{(c) \Rightarrow (a)}$ We show that, since $S(z)$ is analytic on a sector of the form $\Sigma_{0,\theta}$, where $0 < \theta < \pi/2$, the integral representation formula (1.4.7) for $R(\lambda, A)$—which holds for $\Re\lambda > 0$—can be extended to the sector $\Sigma_{0,\frac{\pi+\theta}{2}}$. Observe that any λ in such a sector can be written in the form

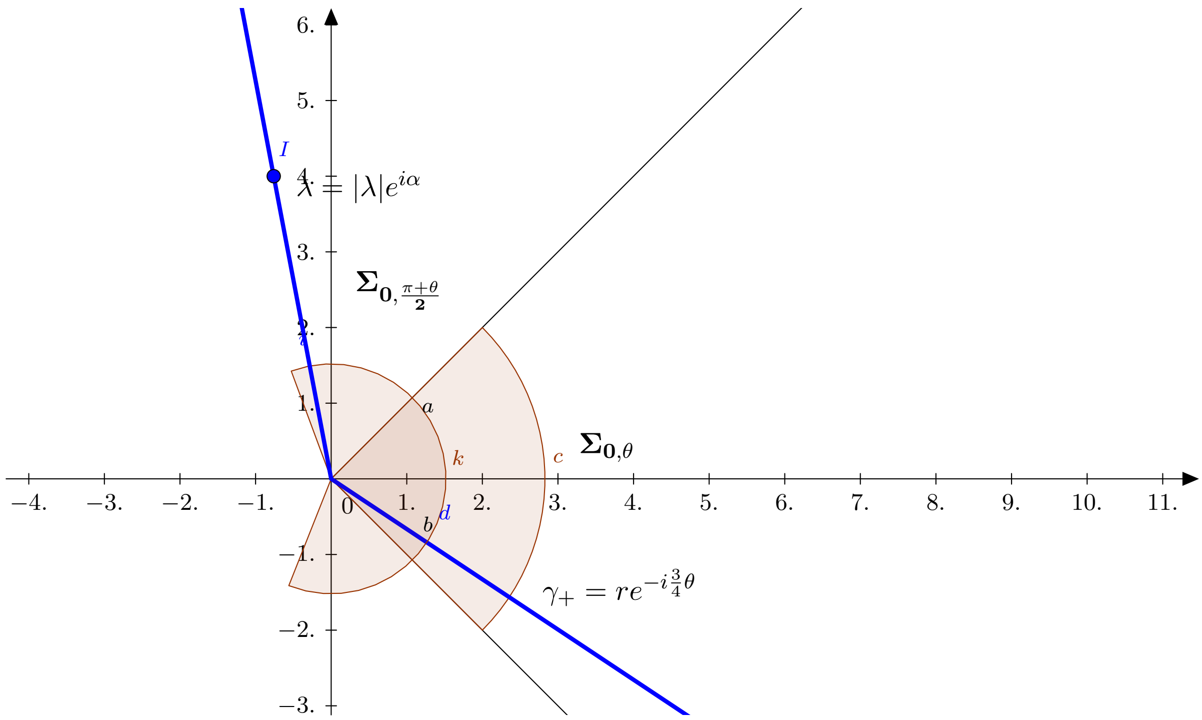
$$\lambda = |\lambda| e^{i\alpha} \quad \text{with} \quad |\alpha| < \frac{\pi+\theta}{2}. \tag{2.2.11}$$

Let us consider the case of $\alpha \geqslant 0$ first. Define

$$R(\lambda, A) = \int_{\gamma_+} e^{-\lambda z} S(z)\, dz \tag{2.2.12}$$

where

$$\gamma_+ = \big\{ z \in \mathbb{C} \ : \ z = re^{-i\frac{3}{4}\theta}\,,\ r \geqslant 0 \big\}.$$



We now show that the integral in (2.2.12) converges to the resolvent of A and (2.2.2) holds. Indeed, denoting by M an upper bound for $\|S(z)\|$ on γ_+, since

$$R(\lambda, A) = \int_0^{+\infty} e^{-\lambda r e^{-i\frac{3}{4}\theta}} S\big(re^{-i\frac{3}{4}\theta}\big) e^{-i\frac{3}{4}\theta}\, dr$$

we have that

$$\|R(\lambda, A)\| \leqslant M \int_0^{+\infty} e^{-r\Re(|\lambda| e^{i(\alpha - \frac{3}{4}\theta)})}\, dr = M \int_0^{+\infty} e^{-r|\lambda|\cos(\alpha - \frac{3}{4}\theta)}\, dr\,,$$

where

$$\cos\Big(\alpha - \frac{3}{4}\theta\Big) \geqslant \min\Big\{\cos\Big(\frac{3}{4}\theta\Big), \sin\Big(\frac{\theta}{4}\Big)\Big\} =: K_\theta > 0$$

because

$$-\frac{3}{4}\theta < \alpha - \frac{3}{4}\theta < \frac{\pi}{2} - \frac{\theta}{4}\,.$$

Therefore, the integral in (2.2.12) converges and

$$\|R(\lambda, A)\| \leqslant \frac{M}{K_\theta |\lambda|}$$

for all λ of the form (2.2.11) with $\alpha \geqslant 0$. On the other hand, for $\alpha < 0$ one can repeat the above argument replacing γ_+ by

$$\gamma_- := \big\{z \in \mathbb{C} \ : \ z = re^{i\frac{3}{4}\theta}\,,\ r \geqslant 0\big\}.$$

Finally, to prove that the integral in (2.2.12) gives the resolvent of A it suffices to observe that, for all λ of the form (2.2.11) with $\alpha \geqslant 0$, we have

$$\begin{aligned}(\lambda I - A)R(\lambda, A) &= \int_{\gamma_+} e^{-\lambda z}\big(\lambda S(z) - AS(z)\big)\,dz \\ &= \int_{\gamma_+} e^{-\lambda z}\big(\lambda S(z) - S'(z)\big)\,dz = I\,.\end{aligned}$$

This shows that $\Sigma_{0,\frac{\pi+\theta}{2}} \subset \rho(A)$ and completes the proof. □

Definition 11 *A $\mathcal{C}_0$-semigroup is called* analytic *if verifies any of the conditions of Theorem 5.*

The following proposition provides a useful sufficient condition for an operator to be sectorial.

Proposition 15 *Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator such that, for some $\omega \in \mathbb{R}$ and $M > 0$, $\Pi_\omega \subset \rho(A)$ and*

$$\|R(\lambda, A)\| \leqslant \frac{M}{|\lambda - \omega|} \qquad \forall \lambda \in \Pi_\omega\,. \tag{2.2.13}$$

Then A is the infinitesimal generator of an analytic semigroup.

Proof. As is by now well known, we can develop the reasoning assuming that $\omega = -1$, which in turn implies that $\rho(A)$ contains the imaginary axis. Since $i\beta \in \rho(A)$ for all $\beta \in \mathbb{R}$, by Proposition 4 we conclude that

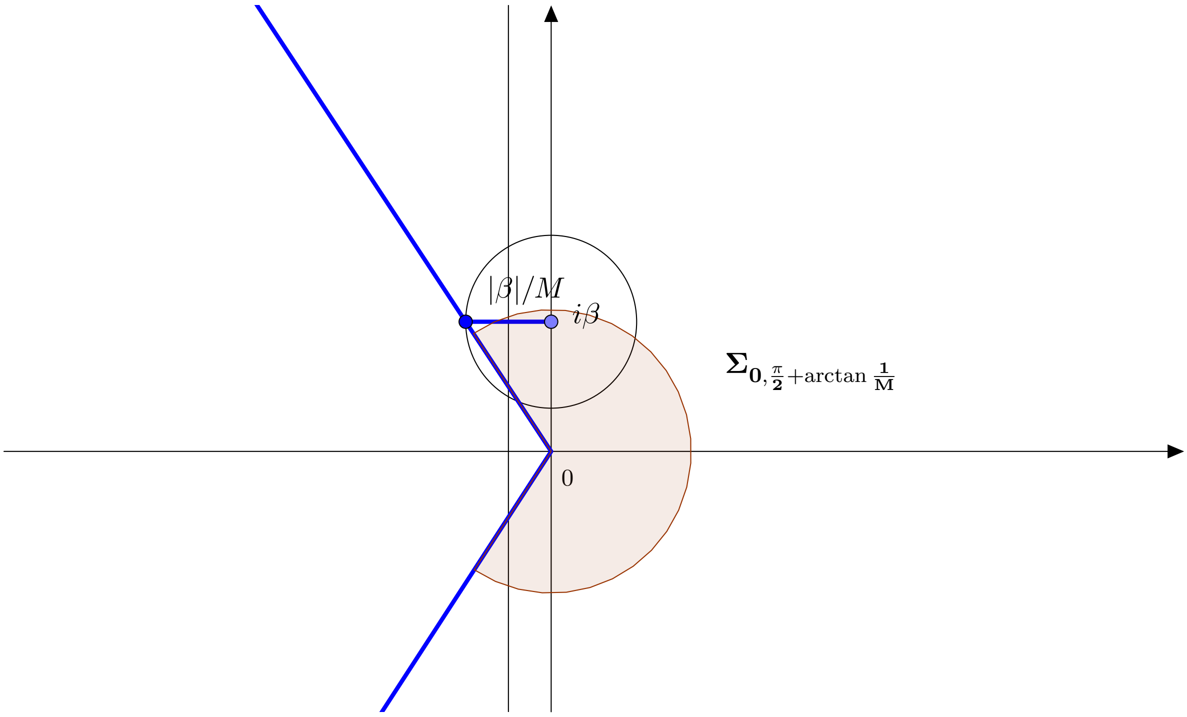
$$C\Big(i\beta, \frac{1}{\|R(i\beta, A)\|}\Big) \subset \rho(A) \qquad \forall \beta \neq 0.$$

So, owing to (2.2.13), $C(i\beta, |\beta|/M) \subset \rho(A)$ for all $\beta \in \mathbb{R} \setminus \{0\}$. Therefore

$$\Sigma_{0,\theta} \subset \bigcup_{\beta \neq 0} C(i\beta, |\beta|/M) \cup \Pi_{-1} \subset \rho(A)$$

with

$$\theta = \frac{\pi}{2} + \arctan\left(\frac{1}{M}\right).$$



Now, fix any $0 < q < 1$ and let $\theta' = \frac{\pi}{2} + \arctan(\frac{q}{M})$. Then(1.4.3) yields

$$R(\lambda, A) = \sum_{n=0}^{\infty} (-1)^n (\Re\lambda)^n R(i\Im\lambda, A)^{n+1} \qquad \forall \lambda \in \Sigma_{0,\theta} \setminus \Pi_0.$$

Hence

$$\|R(\lambda, A)\| \leqslant \sum_{n=0}^{\infty} |\Re\lambda|^n \left(\frac{M}{|\Im\lambda|}\right)^{n+1} \leqslant \frac{M}{|\Im\lambda|} \sum_{n=0}^{\infty} q^n = \frac{M}{1-q} \frac{1}{|\Im\lambda|}. \tag{2.2.14}$$

Moreover, for all $\lambda \in \Sigma_{0,\theta} \setminus \Pi_0$ we have

$$|\lambda|^2 = (\Re\lambda)^2 + (\Im\lambda)^2 \leqslant \left[\left(\frac{q}{M}\right)^2 + 1\right] (\Im\lambda)^2,$$

which, combined with (2.2.14), yields

$$\|R(\lambda, A)\| \leqslant \frac{\sqrt{q^2 + M^2}}{(1-q)|\lambda|} \qquad \forall \lambda \in \Sigma_{0,\theta} \setminus \Pi_0. \qquad \square$$

Exercise 11 Let $A : D(A) \subset X \to X$ be a self-adjoint dissipative operator on an Hilbert space $(X, \langle\cdot,\cdot\rangle)$. Then A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions on X, S, by Theorem 3. Prove that S is analytic.

Solution. Fix any $\lambda \in \Pi_0$ and $y \in X$. Then $x = R(\lambda, A)y$ satisfies $\lambda x - Ax = y$ and, taking the scalar product with x we obtain

$$\begin{aligned} \Re\lambda |x|^2 &\leqslant \Re\lambda |x|^2 - \langle Ax, x\rangle = \Re\langle x, y\rangle \\ \Im\lambda |x|^2 &= \Im\langle x, y\rangle \end{aligned}$$

because $\langle Ax, x\rangle \leqslant 0$. Thus, since $\Re\lambda > 0$ for all $\lambda \in \Pi_0$ we have that

$$\left[(\Re\lambda)^2 + (\Im\lambda)^2\right] |x|^4 \leqslant (\Re\langle x, y\rangle)^2 + (\Im\langle x, y\rangle)^2 = |\langle x, y\rangle|^2 \leqslant |y|^2|x|^2$$

which yields $\|R(\lambda, A)\| \leqslant 1/|\lambda|$. The conclusion follows by Proposition 15. □

As a first consequence of analyticity, we now give a result due to Triggiani [8] on the asymptotic behavior of $S(t)$.

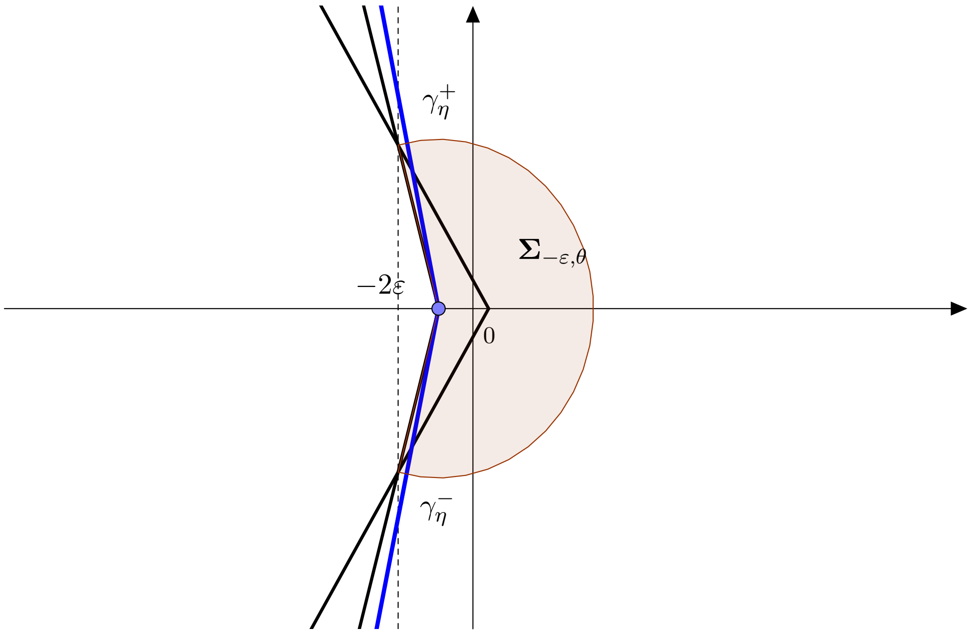
Proposition 16 (Triggiani) *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of an analytic semigroup S. Then $s(A) = \omega_0(S)$.*

Proof. Proceeding by contradiction, let us suppose that

$$s(A) \leqslant -2\varepsilon < 0 = \omega_0(S). \tag{2.2.15}$$

Since A is sectorial and $\omega_0 = 0$,

$$\Sigma_{-\varepsilon,\theta} \subset \rho(A) \quad \text{for some} \quad \theta \in \left(\frac{\pi}{2}, \pi\right].$$



Fix any $\eta \in (\frac{\pi}{2}, \theta)$ and let

$$\gamma_\eta^\pm = \left\{z \in \mathbb{C} \ : \ z = -\varepsilon + re^{\pm i\eta}\,,\ r \geqslant 0\right\}.$$

Then for all $t > 0$

$$S(t) = \frac{1}{2\pi i}\Big\{\int_{\gamma_\eta^-} e^{\lambda t} R(\lambda, A)\, d\lambda + \int_{\gamma_\eta^+} e^{\lambda t} R(\lambda, A)\, d\lambda\Big\}.$$

Therefore

$$\|S(t)\| \leqslant \frac{M(\eta)}{2\pi}\int_0^{+\infty} e^{(r\cos\eta-\varepsilon)t}\Big\{\frac{1}{|re^{i\eta}-\varepsilon|} + \frac{1}{|re^{-i\eta}-\varepsilon|}\Big\}dr.$$

Since

$$|re^{\pm i\eta}-\varepsilon|^2 = r^2+\varepsilon^2-2r\varepsilon\cos\eta \geqslant 2r\varepsilon(1-\cos\eta),$$

we conclude that

$$\begin{aligned}\|S(t)\| &\leqslant \frac{M(\eta)}{\pi}\, e^{-\varepsilon t}\int_0^{+\infty}\frac{e^{rt\cos\eta}}{\sqrt{2r\varepsilon(1-\cos\eta)}}\,dr\\ &= \frac{M(\eta)}{\pi\sqrt{2\varepsilon(1-\cos\eta)}}\,\frac{e^{-\varepsilon t}}{\sqrt{t}}\int_0^{+\infty}\frac{e^{s\cos\eta}}{\sqrt{s}}\,ds \qquad \forall t>0,\end{aligned}$$

which contradicts $\omega_0(S) = 0$. □

Example 13 We return to the heat equation studied in Example 9 and Exercise 8 to show that the associated semigroup is analytic.

Let us consider the case $p = 2$ first. Then operator A in (2.1.11) is self-adjoint and dissipative. Therefore the corresponding semigroup is analytic thanks to Exercise 11.

Next, for $p > 2$ let $X = L^p(0,\pi;\mathbb{C})$ and consider the operator defined by

$$\begin{cases} D(A) = W^{2,p}(0,\pi;\mathbb{C}) \cap W_0^{1,p}(0,\pi;\mathbb{C}) \\ Af(x) = f''(x) \qquad\qquad x \in (0,\pi) \text{ a.e.}\end{cases} \tag{2.2.16}$$

Proceeding as in Exercise 8 one can show that $\sigma(A) = \{-n^2 \ :\ n \geqslant 1\}$. Let $\Re\lambda > 0$. For any fixed $g \in X$ consider the Sturm-Liouville system

$$\begin{cases} \lambda f(x) - f''(x) = g(x), \quad 0 < x < \pi \\ u(0) = 0 = u(\pi).\end{cases} \tag{2.2.17}$$

By multiplying both members of the equation in (2.2.17) by $\overline{f}|f|^{p-2}$ and integrating by parts over $(0,\pi)$ one obtains

$$\begin{aligned}&\lambda\int_0^\pi |f(x)|^p\,dx + \frac{p}{2}\int_0^\pi |f'(x)|^2|f(x)|^{p-2}\,dx\\ &+\ \frac{p-2}{2}\int_0^\pi |f(x)|^{p-4}(\overline{f})^2(f'(x))^2\,dx = \int_0^\pi g(x)\overline{f(x)}|f(x)|^{p-2}\,dx\end{aligned}$$

which in turn yields

$$\Re\lambda \int_0^\pi |f(x)|^p\,dx + \frac{p}{2}\int_0^\pi |f'(x)|^2|f(x)|^{p-2}\,dx \\ + \frac{p-2}{2}\,\Re\int_0^\pi |f(x)|^{p-4}(\overline{f})^2(f'(x))^2\,dx \leqslant |g|_p\,|f|_p^{p-1} \quad (2.2.18)$$

and

$$\Im\lambda \int_0^\pi |f(x)|^p\,dx + \frac{p-2}{2}\,\Im\int_0^\pi |f(x)|^{p-4}(\overline{f})^2(f'(x))^2\,dx \\ = \Im\int_0^\pi g(x)\overline{f(x)}|f(x)|^{p-2}\,dx. \quad (2.2.19)$$

Since

$$\Re\int_0^\pi |f(x)|^{p-4}(\overline{f})^2(f'(x))^2\,dx \geqslant -\int_0^\pi |f(x)|^{p-2}|f'(x)|^2\,dx\,,$$

from (2.2.18) if follows that

$$\Re\lambda\int_0^\pi |f(x)|^p\,dx + \int_0^\pi |f'(x)|^2|f(x)|^{p-2}\,dx \leqslant |g|_p\,|f|_p^{p-1}.$$

Hence, recalling that $\Re\lambda > 0$,

$$\Re\lambda|f|_p \leqslant |g|_p \quad (2.2.20)$$

and

$$\int_0^\pi |f'(x)|^2|f(x)|^{p-2}\,dx \leqslant |g|_p\,|f|_p^{p-1}. \quad (2.2.21)$$

Similarly, since

$$\left|\Im\int_0^\pi |f(x)|^{p-4}(\overline{f})^2(f'(x))^2\,dx\right| \leqslant \int_0^\pi |f'(x)|^2|f(x)|^{p-2}\,dx\,,$$

by (2.2.19) and (2.2.21) we deduce that

$$|\Im\lambda|\int_0^\pi |f(x)|^p\,dx \leqslant \frac{p}{2}\,|g|_p\,|f|_p^{p-1}\,.$$

or

$$|\Im\lambda|\,|f|_p \leqslant \frac{p}{2}\,|g|_p\,. \quad (2.2.22)$$

Finally, by combining (2.2.20) and (2.2.22) we obtain

$$|f|_p \leqslant \frac{\sqrt{4+p^2}}{2|\lambda|}\,|g|_p \qquad \forall\Re\lambda > 0 \quad (2.2.23)$$

which ensures that the corresponding semigroup is analytic even for $p > 2$.

2.3 Compact semigroups

We recall that an operator $\Lambda \in \mathcal{L}(X)$ is called *compact* if it maps bounded sets into relatively compact sets. Equivalently, Λ is compact if, denoting by B_1 the unit ball of X, one has that $\overline{\Lambda(B_1)}$ is compact in X.

Typical examples of compact operators are operators of finite rank, that is, such that $\dim \Lambda(X) < \infty$. Observe that the identity map $I : X \to X$ is compact if and only if $\dim X < \infty$.

The family of all compact operators on X is a closed subspace of $\mathcal{L}(X)$ (see for instance [3]), here denoted by $\mathcal{K}(X)$.

Exercise 12 Let $A : D(A) \subset X \to X$ be a closed operator. Prove that the following properties are equivalent:

(a) $R(\lambda, A) \in \mathcal{K}(X)$ for all $\lambda \in \rho(A)$;

(b) $R(\lambda_0, A) \in \mathcal{K}(X)$ for some $\lambda_0 \in \rho(A)$.

Solution. Observe that, by the resolvent identity (1.4.2) on has that

$$R(\lambda, A) = \big[(\lambda_0 - \lambda)R(\lambda, A) + I\big]R(\lambda_0, A)\,. \qquad \square$$

Let now S be a $\mathcal{C}_0$-semigroup of bounded linear operators on X.

Definition 12 *S is called* compact *if $S(t) \in \mathcal{K}(X)$ for all $t > 0$ and* eventually compact *if there exists $t_0 > 0$ such that $S(t_0) \in \mathcal{K}(X)$.*

Lemma 2 *If $S(t_0) \in \mathcal{K}(X)$ for some $t_0 > 0$ then*

(a) $S(t) \in \mathcal{K}(X)$ for all $t \geqslant t_0$;

(b) $S \in \mathcal{C}\big([t_0, \infty); \mathcal{L}(X)\big)$.

Proof. Property (a) is an easy consequence of the semigroup property

$$S(t) = S(t - t_0)S(t_0)$$

since the product of a bounded operator with a compact one is compact.

As for (b), since $\overline{S(t_0)(B_1)}$ is compact in X, recalling (1.7.1) we have that

$$\begin{aligned}\|S(t+h) - S(t)\| &= \sup_{x \in B_1} |S(t+h)x - S(t)x| \\ &= \sup_{x \in B_1} \big|\big(S(t+h-t_0)x - S(t-t_0)\big)S(t_0)x\big| \to 0 \ (t \to t_0)\end{aligned}$$

for all $t \geqslant t_0$. $\square$

Theorem 6 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup $S \in \mathcal{G}(M, \omega)$. Then the following properties are equivalent:*

(a) *S is compact;*

(b) $S \in \mathcal{C}\big(\mathbb{R}_+^*; \mathcal{L}(X)\big)$ *and* $R(\lambda, A) \in \mathcal{K}(X)$ *for some (hence for all)* $\lambda \in \rho(A)$.

Proof. $\boxed{(a) \Rightarrow (b)}$ Since $S \in \mathcal{C}\big(\mathbb{R}_+^*; \mathcal{L}(X)\big)$ by Lemma 2, recalling the integral representation formula (1.4.6) we have that

$$R(\lambda, A) = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty e^{-\lambda t} S(t)\, dt \qquad \forall \lambda \in \Pi_\omega, \tag{2.3.1}$$

where the integral $\int_\varepsilon^\infty e^{-\lambda t} S(t)\, dt$ converges in $\mathcal{L}(X)$ and the limit exists because S is bounded near zero. Moreover, for every $\varepsilon > 0$ the operator

$$R_\varepsilon(\lambda, A) := \int_\varepsilon^\infty e^{-\lambda t} S(t)\, dt \qquad (\lambda \in \Pi_\omega)$$

is compact because $\mathcal{K}(X)$ is closed. Since

$$\big\|R(\lambda, A) - R_\varepsilon \lambda, A)\big\| \leqslant \Big\| \int_0^\varepsilon e^{-\lambda t} S(t)\, dt \Big\| \leqslant M\varepsilon \to 0 \text{ as } \varepsilon \downarrow 0,$$

we conclude that $R(\lambda, A) \in \mathcal{K}(X)$, again by the fact that $\mathcal{K}(X)$ is closed.

$\boxed{(b) \Rightarrow (a)}$ Since $S \in \mathcal{C}\big(\mathbb{R}_+^*; \mathcal{L}(X)\big)$, we have that (2.3.1) holds in the uniform operator topology and, for any fixed $s > 0$,

$$\lambda R(\lambda, A) S(s) - S(s) = \int_0^\infty \lambda e^{-\lambda t} \big(S(t+s) - S(s)\big) dt \qquad \forall \lambda \in \Pi_\omega.$$

Therefore, taking $\lambda > \omega$, for all $\delta > 0$ we have that

$$\begin{aligned} &\big\|\lambda R(\lambda, A) S(s) - S(s)\big\| \\ &\quad \leqslant \int_0^\delta \lambda e^{-\lambda t} \big\|S(t+s) - S(s)\big\| dt + \int_\delta^\infty \lambda e^{-\lambda t} \big\|S(t+s) - S(s)\big\| dt \end{aligned}$$

Now,

$$\int_0^\delta \lambda e^{-\lambda t} \big\|S(t+s) - S(s)\big\| dt \leqslant \sup_{0 \leqslant t \leqslant \delta} \big\|S(t+s) - S(s)\big\|.$$

On the other hand,

$$\begin{aligned} \int_\delta^\infty \lambda e^{-\lambda t} \big\|S(t+s) - S(s)\big\| dt &\leqslant M \int_\delta^\infty \lambda e^{-\lambda t} \big(e^{\omega(t+s)} + e^{\omega s}\big) dt \\ &\leqslant M e^{\omega s} \Big(\frac{\lambda e^{(\omega - \lambda)\delta}}{\lambda - \omega} + e^{-\lambda \delta} \Big) \to 0 \text{ as } \lambda \uparrow \infty. \end{aligned}$$

Thus,

$$\limsup_{\lambda\uparrow\infty} \big\|\lambda R(\lambda,A)S(s) - S(s)\big\| \leqslant \sup_{0\leqslant t\leqslant\delta} \big\|S(t+s) - S(s)\big\|$$

which in turn implies that $\lambda R(\lambda,A)S(s) \to S(s)$ as $\lambda \uparrow \infty$ because δ is arbitrary. Since $\lambda R(\lambda,A)S(s)$ is compact for all $\lambda > \omega$, so is $S(s)$ for all $s > 0$ because $\mathcal{K}(X)$ is closed. $\square$

Corollary 3 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X denoted by S. If*

(a) *$S \in \mathcal{C}\big([t_0,\infty);X\big)$ for some $t_0 > 0$, and*

(b) *$R(\lambda,A) \in \mathcal{K}(X)$ for some (hence for all) $\lambda \in \rho(A)$,*

then $S(t) \in \mathcal{K}(X)$ for all $t \geqslant t_0$.

Example 14 Let us consider the heat semigroup of Example 13 in the case of $p = 2$. In view of Theorem 6, in order to prove that such a semigroup is compact, it suffices to show that $R(\lambda,A) \in \mathcal{K}(X)$ for some $\lambda \in \rho(A)$. Now, taking $\lambda = 1$ from (2.2.19) and (2.2.20) it follows that $|f'|_2 \leqslant |g|_2$. In other words, $R(1,A)$ maps the unit ball of $L^2(0,\pi;\mathbb{C})$ into the unit ball of $H^1_0(0,\pi;\mathbb{C})$. Since the immersion $H^1_0(0,\pi;\mathbb{C}) \subset \mathcal{C}^{0,1/2}([0,\pi];\mathbb{C})$ is continuous, by Ascoli's theorem we conclude that $R(1,A)$ is compact. Therefore the heat semigroup on $(0,\pi)$ is compact.

2.4 Problems

1. Consider the heat equation in $L^p(\mathbb{R})$ with $p \geqslant 2$ that we studied in Example 10. Prove that the associated semigroup is analytic but not compact.

3 Perturbation of semigroups

3.1 Perturbation by bounded operators

In this chapter we shall stress the connection between a $\mathcal{C}_0$-semigroup, $S(t)$, and its infinitesimal generator A by adopting the equivalent notation

$$S(t) = e^{tA} \qquad \forall t \geqslant 0.$$

Theorem 7 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X such that $\|e^{tA}\| \leqslant Me^{\omega t}$ and let $B \in \mathcal{L}(X)$. Then $A + B : D(A) \subset X \to X$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X satisfying*

$$\|e^{t(A+B)}\| \leqslant Me^{(\omega+M\|B\|)t} \qquad \forall t \geqslant 0. \tag{3.1.1}$$

Proof. Step 1: the special case $\omega = 0$ and $M = 1$.
In view of Proposition 5 we have that $\rho(A) \supset \mathbb{R}_+^*$ and

$$\lambda I - (A + B) = \big[I - BR(\lambda, A)\big](\lambda I - A) \qquad \forall \lambda > 0. \tag{3.1.2}$$

Therefore

$$\lambda \in \rho(A + B) \quad \Longleftrightarrow \quad \big[I - BR(\lambda, A)\big]^{-1} \in \mathcal{L}(X)\,.$$

Now, for all $\lambda \in \Pi_{\|B\|}$ we have that $\|BR(\lambda, A)\| < 1$. So $\lambda \in \rho(A + B)$ and

$$R(\lambda, A + B) = R(\lambda, A) \sum_{n=0}^{\infty} \big(BR(\lambda, A)\big)^n. \tag{3.1.3}$$

Moreover

$$\|R(\lambda, A + B)\| \leqslant \frac{1}{\Re\lambda} \sum_{n=0}^{\infty} \Big(\frac{\|B\|}{\Re\lambda}\Big)^n = \frac{1}{\Re\lambda - \|B\|}\,. \tag{3.1.4}$$

Then, since $A + B : D(A) \subset X \to X$ is closed, by Theorem 2 we conclude that $A + B$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X satisfying

$$\|e^{t(A+B)}\| \leqslant e^{\|B\|t} \qquad \forall t \geqslant 0.$$

Step 2: the general case.
Let us consider $A_\omega = A - \omega I$. The corresponding semigroup $e^{tA_\omega} = e^{-\omega t}e^{tA}$ belongs to $\mathcal{G}(M, 0)$. Now, denote by $|||\cdot|||$ the equivalent norm defined in (1.5.10) for which e^{tA_ω} turns out to be a contraction semigroup and observe that

$$|||Bx||| \leqslant M\,\|B\|\,|x| \leqslant M\,\|B\|\,|||x||| \qquad \forall x \in X.$$

By Step 1, $A_\omega + B$ generates a $\mathcal{C}_0$-semigroup of bounded linear operators on X satisfying

$$|||e^{t(A_\omega+B)}||| \leqslant e^{|||B|||t} \leqslant e^{M\|B\|t} \qquad \forall t \geqslant 0.$$

Therefore, for all $x \in X$,

$$|e^{t(A_\omega+B)}x| \leqslant |||e^{t(A_\omega+B)}x||| \leqslant e^{M\|B\|t}|||x||| \leqslant Me^{M\|B\|t}|x| \qquad \forall t \geqslant 0.$$

Since $e^{t(A_\omega+B)} = e^{-\omega t}e^{t(A+B)}$, the conclusion follows. □

Lemma 3 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup on X and let $B \in \mathcal{L}(X)$. Then $e^{t(A+B)}$ is a solution of the Volterra integral equation*

$$V(t)x = e^{tA}x + \int_0^t e^{(t-s)A}BV(s)x\,ds \qquad \forall x \in X. \tag{3.1.5}$$

Proof. For any $x \in D(A)$ and $t > 0$ the function $H : [0,t] \to X$ defined by

$$H(t) = e^{(t-s)A}e^{s(A+B)}x$$

is continuously differentiable and satisfies for all $0 < s < t$

$$\begin{aligned} H'(s) &= -Ae^{(t-s)A}e^{s(A+B)}x + e^{(t-s)A}(A+B)e^{s(A+B)}x \\ &= e^{(t-s)A}Be^{s(A+B)}x. \end{aligned}$$

By integrating the above relation over $[0,t]$ we obtain

$$e^{t(A+B)}x = e^{tA}x + \int_0^t e^{(t-s)A}Be^{s(A+B)}x\,ds \qquad \forall x \in D(A).$$

Since all the operators in the above equation are continuos, the identity holds for all $x \in X$ and the conclusion follows. $\square$

For any $T > 0$ we denote by $\mathcal{B}\big(0,T;\mathcal{L}(X)\big)$ the Banach space of all maps $\Lambda : [0,T] \to \mathcal{L}(X)$ such that

$$\sup_{t\in[0,T]} \|\Lambda(t)\| < \infty.$$

Proposition 17 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup on X such that $\|e^{tA}\| \leqslant Me^{\omega t}$ and let $B \in \mathcal{L}(X)$. Then there exists a unique family $\{V(t)\}_{t\geqslant 0}$ such that*

(a) $V(t) \in \mathcal{L}(X)$ *for all* $t \geqslant 0$,

(b) $t \mapsto V(t)x$ *is continuous on* $\mathbb{R}_+$ *for all* $x \in X$, *and*

(c) (3.1.5) *is satisfied for all* $t \geqslant 0$.

Moreover

$$V(t) = \sum_{n=0}^{\infty} V_n(t) \qquad \forall t \geqslant 0 \tag{3.1.6}$$

where $\{V_n(t)\}_{t\geqslant 0}$ *is defined by*

$$V_0(t) = e^{tA} \quad \text{and} \quad V_{n+1}(t)x = \int_0^t e^{(t-s)A}BV_n(s)x\,ds \qquad \forall x \in X. \tag{3.1.7}$$

Furthermore, the series in (3.1.6) *converges in* $\mathcal{B}\big(0,T;\mathcal{L}(X)\big)$ *for all* $T \geqslant 0$.

Proof. Define $\{V_n(t)\}_{t\geqslant 0}$ by (3.1.7). Then $t \mapsto V_n(t)x$ is continuous on $\mathbb{R}_+$ for all $x \in X$. Moreover, proceeding by induction one can easily prove that

$$\|V_n(t)\| \leqslant Me^{\omega t}\,\frac{M^n\|B\|^n t^n}{n!} \qquad \forall t \geqslant 0\,,\ \forall n \geqslant 0$$

which in turn shows that the series in (3.1.6) converges in $L^\infty\big(0,T;\mathcal{L}(X)\big)$ for all $T \geqslant 0$. So, $t \mapsto V(t)x$ is continuous on $\mathbb{R}_+$ and satisfies (3.1.5). This shows the existence part of the conclusion. As for uniqueness, let $\{U(t)\}_{t\geqslant 0}$ be another family of operators satisfying (a), (b), and (c). Then for all $x \in X$

$$\big\|\big(V(t) - U(t)\big)x\big\| \leqslant M\|B\| \int_0^t e^{\omega(t-s)}\big\|\big(V(s) - U(s)\big)x\big\|\,ds \qquad \forall t \geqslant 0.$$

Now, Gronwall's lemma ensures that $U \equiv V$. □

Corollary 4 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup on X such that $\|e^{tA}\| \leqslant Me^{\omega t}$ and let $B \in \mathcal{L}(X)$. Then*

$$\|e^{t(A+B)} - e^{tA}\| \leqslant Me^{\omega t}(e^{M\|B\|t} - 1).$$

Proof. By Proposition 17 and Theorem 7 we obtain

$$\begin{aligned}
\big|e^{t(A+B)}x - e^{tA}x\big| &\leqslant \int_0^t \|e^{(t-s)A}\|\,\|B\|\,\|e^{s(A+B)}\|\,|x|\,ds \\
&\leqslant \int_0^t Me^{\omega(t-s)}\,\|B\|\,Me^{(\omega+M\|B\|)s}\,|x|\,ds \\
&= Me^{\omega t}\big(e^{M\|B\|t} - 1\big)|x|
\end{aligned}$$

The conclusion follows. □

Theorem 8 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a compact $\mathcal{C}_0$-semigroup of bounded linear operators on X and let $B \in \mathcal{L}(X)$. Then $A + B : D(A) \subset X \to X$ is the infinitesimal generator of a compact $\mathcal{C}_0$-semigroup of bounded linear operators on X.*

Proof. By Theorem 6 we have that e^{tA} is continuous in the uniform operator topology for $t > 0$ and $R(\lambda, A)$ is compact for all $\lambda \in \rho(A)$. Moreover, for all $\lambda > \omega$ we have that $\|R(\lambda, A)\| \leqslant M/(\lambda - \omega)$ and so, for $\lambda > \omega + M\|B\| + 1$, the series

$$R(\lambda, A + B) = R(\lambda, A)\sum_{n=0}^{\infty}\big(BR(\lambda, A)\big)^n \tag{3.1.8}$$

converges in $\mathcal{L}(X)$. Since each term on the right-hand side is compact, so is $R(\lambda, A + B)$ for all $\lambda \in \rho(A + B)$. Thus, appealing to Theorem 6 once again, it suffices to show that $e^{t(A+B)}$ is continuous in the uniform operator topology for $t > 0$. Since $e^{t(A+B)}$ is given by the series in (3.1.6), this continuity property follows from the fact that, being e^{tA} continuous in the uniform operator topology for $t > 0$, so is each V_n in (3.1.7) and also their sum because the series converges in $\mathcal{B}\big(0,T;\mathcal{L}(X)\big)$ for all $T \geqslant 0$. $\square$

Example 15 Let us apply Theorem 7 to solve the wave equation with lower order terms

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + a(x)\,\frac{\partial u}{\partial t} + b(x)\,\frac{\partial u}{\partial x} + c(x)u & (t,x) \in \mathbb{R}\times(0,\pi) \\ u(t,0) = 0 = u(t,\pi) & t \in \mathbb{R} \\ u(0,x) = u_0(x)\,,\ \frac{\partial u}{\partial t}\,(0,x) = u_1(x) & x \in (0,\pi). \end{cases} \tag{3.1.9}$$

In Example 11, we have shown that the operator $\mathcal{A}$ defined in (2.1.9) is the infinitesimal generator of a unitary group on $\mathcal{X} = H_0^1(0,\pi)\times L^2(0,\pi)$. Let

$$\mathcal{B}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ B_1 u + B_2 v \end{pmatrix} \qquad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{X},$$

where

$$\begin{cases} B_1 u = b(x)u' + c(x)u & \forall u \in H_0^1(0,\pi) \\ B_2 v = a(x)v & \forall v \in L^2(0,\pi). \end{cases}$$

Assuming

$$a, b, c \in L^\infty(0,\pi),$$

one obtains that

$$\left\|\mathcal{B}\begin{pmatrix} u \\ v \end{pmatrix}\right\| \leqslant M\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\| \qquad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{X}$$

with

$$M = \sqrt{|a|_\infty^2 + |b|_\infty^2 + |c|_\infty^2}.$$

Therefore $\mathcal{A}+\mathcal{B}$ is the infinitesimal generator of a $\mathcal{C}_0$-group on $\mathcal{X}$ which provides the solution of (3.1.9). Moreover, by (3.1.1) we conclude that

$$\|e^{t(\mathcal{A}+\mathcal{B})}\| \leqslant e^{M|t|} \qquad \forall t \in \mathbb{R}.$$

3.2 Perturbation of sectorial operators

Theorem 9 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of an analytic semigroup and let $B : D(B) \subset X \to X$ be a closed linear operator satisfying*

(*a*) *$D(B) \supset D(A)$, and*

(*b*) *$\exists\, a, b \geqslant 0$ such that $|Bx| \leqslant a|Ax| + b|x|$ for all $x \in D(A)$.*

There exists $\alpha > 0$ such that if $0 \leqslant a \leqslant \alpha$ then $A+B : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup.

Proof. Step 1: the case $\omega = 0$.
Owing to Theorem 5 we have that there is an angle $\theta_0 \in (\frac{\pi}{2}, \pi]$ such that $\Sigma_{0,\theta_0} \subset \rho(A)$ and

$$\|R(\lambda, A)\| \leqslant \frac{M_\theta}{|\lambda|} \qquad \forall \theta \in (0, \theta_0),\ \forall \lambda \in \Sigma_{0,\theta}.$$

Fix any $\theta \in (\pi/2, \theta_0)$ and let $\lambda \in \Sigma_{0,\theta}$. Then for every $x \in X$

$$\begin{aligned} |BR(\lambda, A)x| &\leqslant a|AR(\lambda, A)x| + b|R(\lambda, A)x| \\ &\leqslant a(M_\theta + 1)|x| + \frac{bM_\theta}{|\lambda|}\,|x|. \end{aligned}$$

Hence, fixing any $\varepsilon > 0$ and choosing

$$\alpha = \frac{1}{2(M_\theta + 1)} \quad \text{and} \quad |\lambda| \geqslant 2(bM_\theta + \varepsilon)\,,$$

we have that

$$\|BR(\lambda, A)\| \leqslant \frac{1}{2} + \frac{bM_\theta}{2(bM_\theta + \varepsilon)} = \frac{2bM_\theta + \varepsilon}{2(bM_\theta + \varepsilon)} < 1\,. \tag{3.2.1}$$

Therefore $I - BR(\lambda, A)$ is invertible and, recalling (3.1.2), by (3.2.1) we obtain

$$\|R(\lambda, A+B)\| \leqslant \big\|\big[I - BR(\lambda, A)\big]^{-1}\big\|\,\|R(\lambda, A)\| \leqslant \frac{2(bM_\theta + \varepsilon)M_\theta}{\varepsilon|\lambda|}$$

for all $\lambda \in \Pi_{2(bM_\theta + \varepsilon)}$. By Proposition 15 we conclude that $A + B$ is the infinitesimal generator of an analytic semigroup.

Step 2: the general case.
Consider $A_\omega = A - \omega I$ with the associated semigroup $e^{tA_\omega} = e^{-\omega t} e^{tA}$ which belongs to $\mathcal{G}(M, 0)$. Assumption (b) implies that

$$|Bx| \leqslant a|A_\omega x| + (a\omega + b)|x| \qquad \forall x \in D(A).$$

By Step 1, $A_\omega + B = A + B - \omega I$ is the infinitesimal generator of an analytic semigroup and the same is true for $A + B$. □

Corollary 5 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of an analytic semigroup and let $B \in \mathcal{L}(X)$. Then $A + B : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup.*

Example 16 Consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x} + c(x)u & (t,x) \in \mathbb{R}_+ \times (0,\pi) \\ u(t,0) = 0 = u(t,\pi) & t \geqslant 0 \\ u(0,x) = u_0(x) & x \in (0,\pi) \end{cases} \tag{3.2.2}$$

with $u_0 \in X = L^2(0,\pi;\mathbb{C})$. Denote by A the operator in (2.2.16) (with $p = 2$) and define $B : D(B) \subset X \to X$ by

$$\begin{cases} D(B) = H_0^1(0,\pi;\mathbb{C}) \\ Bf(x) = b(x)f'(x) + c(x)f(x) \quad x \in (0,\pi) \text{ a.e.} \end{cases}$$

As shown in Example 13, A is the infinitesimal generator of an analytic semigrooup on X. Assume now

$$b \in L^\infty(0,\pi;\mathbb{C}) \quad \text{and} \quad c \in L^2(0,\pi;\mathbb{C}).$$

Then, in view of (3.5.2) and (3.5.3), we have that for all $f \in D(A)$

$$\begin{aligned} |Bf| &\leqslant |b|_\infty |f'|_2 + |c|_2 |f|_\infty \leqslant \Big(|b|_\infty + \frac{\sqrt{\pi}}{2}\,|c|_2\Big)|f'|_2 \\ &\leqslant \Big(|b|_\infty + \frac{\sqrt{\pi}}{2}\,|c|_2\Big)\sqrt{|f''|_2|f|_2}. \end{aligned}$$

So, by the elementary inequality

$$xy \leqslant \frac{\varepsilon}{2}\,x^2 + \frac{1}{2\varepsilon}\,y^2, \tag{3.2.3}$$

which holds for all $x, y \in \mathbb{R}$ and all $\varepsilon > 0$, we conclude that

$$|Bf| \leqslant \varepsilon|Af| + b_\varepsilon|f| \qquad \forall \in D(A)$$

for some constant $b_\varepsilon > 0$.

Therefore, by Theorem 9, $A + B$ generates an analytic semigroup which gives the unique solution of (3.2.2).

3.3 Perturbation of dissipative operators

Let X be an Hilbert space with scalar product $\langle\cdot,\cdot\rangle$. We recall that a dissipative operator $A : D(A) \subset X \to X$ is called *m-dissipative* if $I - A$ is surjective.

Theorem 10 *Let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be linear operators satisfying*

(a) $D(B) \supset D(A)$, and

(*b*) $\exists\, a \in [0,1)$, $\exists\, b \geqslant 0$ *such that* $|Bx| \leqslant a|Ax| + b|x|$ *for all* $x \in D(A)$.

If, in addition,

(*c*) $A + tB$ *is dissipative for all* $0 \leqslant t \leqslant 1$, *and*

(*d*) $\exists\, t_0 \in [0,1]$ *such that* $A + t_0 B$ *is* m*-dissipative,*

then $A + tB$ *is* m*-dissipative for all* $0 \leqslant t \leqslant 1$.

Proof. It suffices to show that there exists $\delta > 0$ such that, if $A + t_0 B$ is m-dissipative for some $t_0 \in [0,1]$, then $A + tB$ is m-dissipative for all $t \in [0,1]$ satisfying $|t_0 - t| \leqslant \delta$.

Assume that $A + t_0 B$ is m-dissipative for some $t_0 \in [0,1]$. Then

$$R(t_0) := \big[I - (A + t_0 B)\big]^{-1} \quad \text{satisfies} \quad \|R(t_0)\| \leqslant 1.$$

We now proceed to show that $BR(t_0)$ is bounded. Owing to assumption (*b*), for all $x \in D(A)$ we have that

$$\begin{array}{rcl} |Bx| & \leqslant & a\big(|(A + t_0 B)x| + t_0|Bx|\big) + b|x| \\ & \leqslant & a|(A + t_0 B)x| + a|Bx| + b|x| \end{array}$$

and so

$$|Bx| \leqslant \frac{a}{1-a}\,|(A + t_0 B)x| + \frac{b}{1-a}\,|x| \qquad \forall x \in D(A). \tag{3.3.1}$$

Since $R(t_0)(X) \subset D(A)$ and $(A + t_0 B)R(t_0) = R(t_0) - I$, by (3.3.1) we get

$$|BR(t_0)x| \leqslant \frac{a}{1-a}\,|R(t_0)x - x| + \frac{b}{1-a}\,|R(t_0)x| \leqslant \frac{2a+b}{1-a}\,|x| \tag{3.3.2}$$

for all $x \in X$, which shows that $BR(t_0)$ is bounded.

Now, since

$$\begin{array}{rcl} I - (A + tB) & = & I - (A + t_0 B) + (t_0 - t)B \\ & = & \big[I + (t_0 - t)BR(t_0)\big]\,\big[I - (A + t_0 B)\big] \end{array}$$

and $I - (A + t_0 B)$, we deduce that $I - (A + tB)$ is invertible if and only if $I + (t_0 - t)BR(t_0)$ is invertible. In view of (3.3.2), this is definitely the case if

$$\|(t_0 - t)BR(t_0)\| \leqslant |t_0 - t|\,\frac{2a+b}{1-a} < 1.$$

So, the proof is completed choosing $\delta = \frac{1-a}{2a+b+1}$. □

Corollary 6 *Let* $A : D(A) \subset X \to X$ *be the infinitesimal generator of a* $\mathcal{C}_0$*-semigroup of contractions on* X *and let* $B : D(B) \subset X \to X$ *be a dissipative operator satisfying* *(a)* $D(B) \supset D(A)$*, and*

(b) $\exists\, a \in [0,1)$*,* $\exists\, b \geqslant 0$ *such that* $|Bx| \leqslant a|Ax| + b|x|$ *for all* $x \in D(A)$*.*

Then $A+B : D(A) \subset X \to X$ *is the infinitesimal generator of a* $\mathcal{C}_0$*-semigroup of contractions on* X*.*

Example 17 The Schrödinger equation with potential $V : (0,\pi) \to \mathbb{C}$

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = i\,\frac{\partial^2 u}{\partial x^2} + V(x)u & (t,x) \in \mathbb{R} \times (0,\pi) \\ u(t,0) = 0 = u(t,\pi) & t \in \mathbb{R} \\ u(0,x) = u_0(x) & x \in (0,\pi) \end{cases}$$

can be studied using Theorem 7. We know that A defined in (2.1.11) is self-adjoint and dissipative, so that iA generates a unitary group on $L^2(0,\pi;\mathbb{C})$ by Stone's theorem. Therefore, if $V \in L^\infty(0,\pi;\mathbb{C})$, then setting

$$Bf(x) = V(x)f(x) \qquad \forall f \in X,$$

from Theorem 7 if follows that $iA + B$ is the infinitesimal generator of a $\mathcal{C}_0$-group on X satisfying

$$\|e^{t(iA+B)}\| \leqslant e^{|V|_\infty |t|} \qquad \forall t \in \mathbb{R}.$$

We can say more about this problem by using Corollary 6. Indeed, since

$$\Re\langle Bf, f\rangle = \int_0^\pi V(x)|f(x)|^2 dx \qquad \forall f \in X,$$

we conclude that if $\Re V(x) \leqslant 0$ for a.e. $x \in (0,\pi)$, then

$$\|e^{t(iA+B)}\| \leqslant 1 \qquad \forall t \geqslant 0.$$

3.4 Stability under compact perturbations

A useful stability result due to Gibson [5] ensures that, perturbing the generator of an exponentially stable semigroup by a compact operator, one obtains an exponentially stable semigroup again, provided the perturbed semigroup is strongly stable. The original proof given in [5] used approximation by finite dimensional subspaces and a contradiction argument. The topic was then investigated by several authors including Triggiani [7]. Following [1], we now give a completely different proof of Gibson's theorem based on a simple direct argument, extending the analysis to Banach spaces, and relaxing the original compactness assumptions.

Theorem 11 *Let X be a reflexive Banach space and let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0-$semigroup, e^{tA}, satisfying*

$$\|e^{tA}\| \leqslant M_0 e^{-\omega_0 t} \qquad \forall t \geq 0 \tag{3.4.1}$$

for some constants $M_0, \omega_0 > 0$. Let $B \in \mathcal{L}(X)$ be such that

$$\lim_{t \to +\infty} e^{t(A+B)} x = 0 \qquad \forall x \in X\,, \text{ and} \tag{3.4.2}$$

$$Be^{tA} \quad \text{is compact} \quad \forall t > 0\,. \tag{3.4.3}$$

Then, for some constants $M_B, \omega_B > 0$,

$$\|e^{t(A+B)}\| \leqslant M_B e^{-\omega_B t} \qquad \forall t \geq 0\,.$$

Proof. To begin with, observe that, as in Remark 1, by the Banach-Steinhaus Theorem we deduce from (3.4.2) that, for some constant $M_1 > 0$,

$$\|e^{t(A+B)}\| \leqslant M_1 \qquad \forall t \geq 0. \tag{3.4.4}$$

Now, appealing to a well-known characterization of exponential stability for strongly continuous semigroups (see Problems 1.7), we conclude that

$$\lim_{t \to +\infty} \|e^{t(A+B)}\| = 0 \tag{3.4.5}$$

suffices to obtain the desired conclusion. In order to prove (3.4.5), define

$$\Lambda_t x = \int_0^t e^{(t-s)(A+B)} B e^{sA} x \, ds, \qquad \forall x \in X\,,\ \forall t \geq 0\,.$$

By Lemma 3 applied to $\widetilde{A} := A + B$ and $\widetilde{B} := -B$ we have that

$$e^{t(A+B)} x = e^{tA} x + \Lambda_t x\,, \qquad \forall x \in X\,,\ \forall t \geq 0\,. \tag{3.4.6}$$

In view of (3.4.6) and (3.4.1) we have that, for every $t \geq 0$,

$$\begin{aligned} \|e^{t(A+B)}\| &= \sup_{|x| \leqslant 1} |e^{t(A+B)} x| \\ &\leqslant \sup_{|x| \leqslant 1} |e^{tA} x| + \sup_{|x| \leqslant 1} |\Lambda_t x| \leqslant M_0 e^{-\omega_0 t} + \sup_{|x| \leqslant 1} |\Lambda_t x| \end{aligned} \tag{3.4.7}$$

Next, let t_n be any sequence of positive numbers such that $t_n \to +\infty$ as $n \to \infty$ and choose vectors x_n, with $|x_n| \leqslant 1$, such that

$$\sup_{|x| \leqslant 1} |\Lambda_{t_n} x| < |\Lambda_{t_n} x_n| + \frac{1}{n}\,. \tag{3.4.8}$$

Now, extract a weakly convergent subsequence—still labeled x_n—to some limit $\bar{x} \in X$. We claim that

$$\Lambda_{t_n}(x_n - \bar{x}) \to 0 \quad \text{strongly as} \quad n \to \infty\,. \tag{3.4.9}$$

Indeed, consider the sequence of vector-valued functions

$$\phi_n(s) := Be^{sA}(x_n - \bar{x}) \qquad s \geq 0\,.$$

Owing to assumption (3.4.1), for all $n \in \mathbb{N}$ we have

$$|\phi_n(s)| \leqslant 2M_0\|B\|e^{-\omega_0 s} \qquad \forall s \geq 0\,.$$

Moreover, on account of (3.4.3), $\phi_n(s)$ strongly converges to 0, as $n \to \infty$, for all $s > 0$. Therefore, invoking Lebesgue's dominated convergence theorem for vector-valued functions, we conclude that $\phi_n \to 0$ in $L^1(0,\infty;X)$ as $n \to \infty$. Consequently, thanks to (3.4.4),

$$|\Lambda_{t_n}(x_n - \bar{x})| \leqslant M_1 \int_0^\infty |\phi_n(s)|\, ds \;\to\; 0 \quad \text{as} \quad n \to \infty\,,$$

which proves (3.4.9).

Finally, combining (3.4.7), (3.4.8) and (3.4.9) we obtain

$$\begin{aligned}&\|e^{t_n(A+B)}\| \\ &\qquad \leqslant M_0 e^{-\omega_0 t_n} + \frac{1}{n} + |\Lambda_{t_n}(x_n - \bar{x})| + |\Lambda_{t_n}\bar{x}| \;\to\; 0 \quad \text{as} \quad n \to \infty\,,\end{aligned} \tag{3.4.10}$$

where, according to (3.4.6), the fact that

$$\Lambda_{t_n}\bar{x} = e^{t_n(A+B)}\bar{x} - e^{t_n A}\bar{x} \;\to\; 0 \quad \text{as} \quad n \to \infty$$

follows from assumptions (3.4.1) and (3.4.2). Since $\{t_n\}$ is an arbitrary sequence going to ∞, (3.4.10) yields (3.4.5) and completes the proof. $\square$

Example 18 Consider the heat equation with potential V on $(0,\pi)$

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + V(x)u & (t,x) \in \mathbb{R}_+ \times (0,\pi) \\ u(t,0) = 0 = u(t,\pi) & t \geqslant 0 \\ u(0,x) = u_0(x) & x \in (0,\pi) \end{cases} \tag{3.4.11}$$

with $u_0 \in X = L^2(0,\pi)$ and $V \in \mathcal{C}([0,\pi])$.

Denote by A the operator in (2.2.16) ($p = 2$) and define

$$Bf(x) = V(x)f(x) \quad \text{a.e. } x \in (0,\pi),\ \forall f \in X.$$

Then $B \in \mathcal{L}(X)$ and

$$\|B\| = |V|_\infty. \tag{3.4.12}$$

Moreover, as shown in Example 13, A is the infinitesimal generator of an analytic semigroup on X and $\sigma(A) = \{-n^2 \,:\, n \geqslant 1\}$. Let us prove that

$$\|e^{tA}\| \leqslant e^{-t} \qquad \forall t \geqslant 0. \tag{3.4.13}$$

Indeed, for any $u_0 \in X \setminus \{0\}$ the function $v(t,x) := e^{tA}u_0(x)$ satisfies

$$\frac{1}{2}\frac{d}{dt}\int_0^\pi v^2(t,x)dx = -\int_0^\pi \Big(\frac{\partial v}{\partial x}(t,x)\Big)^2 dx \leqslant -\int_0^\pi v^2(t,x)dx \quad (t>0)$$

thanks to Poincaré's inequality (3.5.1). Thus, $\frac{d}{dt}\log|e^{tA}u_0|^2 \leqslant -2$ and

$$|e^{tA}u_0| \leqslant e^{-t}|u_0| \qquad \forall t \geqslant 0.$$

Now, appealing to Theorem 7, by (3.4.12) we conclude that

$$\|e^{t(A+B)}\| \leqslant e^{(|V|_\infty - 1)t} \qquad \forall t \geqslant 0 \tag{3.4.14}$$

so that $e^{t(A+B)}$ remains exponentially stable if $|V|_\infty < 1$.

Let us prove that the same holds true under the weaker assumption

$$M := \max_{[0,\pi]} V < 1. \tag{3.4.15}$$

Fix any $u_0 \in X$ and let $u(t,x) = e^{t(A+B)}u_0(x)$. Proceeding as above we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^\pi u^2(t,x)dx &= \int_0^\pi \Big\{V(x)u^2(t,x) - \Big(\frac{\partial v}{\partial x}(t,x)\Big)^2\Big\}dx \\ &\leqslant (M-1)\int_0^\pi u^2(t,x)\,dx\end{aligned}$$

thanks to Poincaré's inequality. So,

$$|e^{tA}u_0| \leqslant e^{(M-1)t}|u_0| \qquad \forall t \geqslant 0 \tag{3.4.16}$$

which yields exponential stability since $M < 1$.

Finally, observe that, for $V \equiv 1$, (3.4.11) fails to be even strongly stable because it admits the stationary solution $u(t,x) = \sin x$.

Example 19 Continuing the analysis of the previous example we now want to prove the exponential decay at ∞ of the solution of (3.4.11) when V satisfies the conditions

$$\begin{cases} (a) \quad V(x) \leqslant 1 \qquad \forall x \in [0,\pi] \\ (b) \quad \exists (\alpha,\beta) \subset [0,\pi] \ : \ V(x) < 1 \qquad \forall x \in (\alpha,\beta). \end{cases} \tag{3.4.17}$$

which are weaker than (3.4.15).

Recalling that e^{tA} is compact for $t > 0$ (see Example 14) and appealing to Theorem 11, we conclude that $e^{t(A+B)}$ is exponentially stable if it is strongly stable. Let us show the last assertion by La Salle's invariance argument. Fix any $u_0 \in X$ and let $u(t,x) = e^{t(A+B)}u_0(x)$. Then, for all $t > 0$,

$$\frac{1}{2}\frac{d}{dt}\int_0^\pi u^2(t,x)dx = \int_0^\pi \Big\{V(x)u^2(t,x) - \Big(\frac{\partial u}{\partial x}(t,x)\Big)^2\Big\}dx \leqslant 0 \tag{3.4.18}$$

thanks to Poincaré's inequality and assumption (3.4.17)-(a). The above inequality proves that $E(t) := |e^{tA}u_0|^2$ is nondcreasing and so

$$E(t) \downarrow E_\infty \quad \text{as} \quad t \to +\infty. \tag{3.4.19}$$

Take any sequence $t_n \geqslant 0$ such that $t_n \uparrow +\infty$ (for instance, $t_n = n$). Since $E(t) \leqslant E(0)$, there exists a subsequence, still labeled t_n, such that

$$u_n := e^{t_n(A+B)}u_0 \rightharpoonup u_\infty \quad (n \to \infty)$$

and, since $e^{t(A+B)}$ is compact for $t > 0$,

$$\lim_{n\to\infty} e^{t(A+B)}u_n = e^{t(A+B)}u_\infty \qquad \forall t > 0.$$

So, in view of (3.4.19),

$$\begin{aligned} |e^{t(A+B)}u_\infty| &= \lim_{n\to\infty} |e^{t(A+B)}u_n| = \lim_{n\to\infty} |e^{(t_n+t)(A+B)}u_0| \\ &= \lim_{t\to+\infty} |e^{t(A+B)}u_0| = \sqrt{E_\infty}\,, \end{aligned}$$

which implies that $|e^{t(A+B)}u_\infty|^2 = E_\infty$ fr all $t \geqslant 0$. By differentiating such an identity we have that $U(t,x) = e^{t(A+B)}u_\infty(x)$ satisfies

$$\begin{aligned} 0 &= \frac{1}{2}\frac{d}{dt}\int_0^\pi U^2(t,x)dx = \int_0^\pi \Big\{V(x)U^2(t,x) - \Big(\frac{\partial U}{\partial x}(t,x)\Big)^2\Big\}dx \\ &\leqslant \int_0^\pi \Big\{U^2(t,x) - \Big(\frac{\partial U}{\partial x}(t,x)\Big)^2\Big\}dx \leqslant 0\,. \end{aligned}$$

In other terms, the above are all equalities and $U(t,\cdot)$ is a function realizing the identity in Poincaré's inequality. Therefore,

$$U(t,x) = c(t)\sin x$$

and, since U solves the equation in (3.4.11),

$$c'(t)\sin x = -c(t)\sin x + c(t)V(x)\sin x.$$

Consequently, either V is constant, say $V(x) = M$ for all $x \in [0,\pi]$, or $c \equiv 0$. In the former case, we must have $M < 1$ by (3.4.17)-(b). Then (3.4.16) implies that $e^{t(A+B)}$ is exponentially stable. In the latter, we get that $e^{t(A+B)}$ is strongly stable, hence exponentially stable by Theorem 11.

3.5 Problems

1. In the situation considered in Theorem 9, suppose that $\|e^{tA}\| \leqslant Me^{\omega t}$. Show that
$$\|e^{t(A+B)}\| \leqslant M(b)e^{(\omega+\Lambda(b))t}$$

where $\Lambda(b) > 0$ satisfies

$$\lim_{b \downarrow 0} \Lambda(b) = 0.$$

Solution. By Theorem 9 we have that $S_B(t) := e^{t(A+B)}$ is analytic provided that $0 \leqslant a \leqslant \alpha$. Then $\omega_0(S_B) = s(A+B)$ by Proposition 16. Moreover, the proof of Theorem 9 shows that

$$s(A+B) \leqslant \omega + 2bM_\theta.$$

The conclusion follows from (1.3.3). □

2. Prove that for every $f \in H_0^1(0,\pi)$ the following inequalities hold:

   - *Poincaré inequality*
   $$|f|_2 \leqslant |f'|_2. \tag{3.5.1}$$
   - *Sobolev inequality*
   $$|f|_\infty \leqslant \frac{\sqrt{\pi}}{2} |f'|_2. \tag{3.5.2}$$

   Moreover, show that both inequalities are sharp (i.e., for each inequality find a function $f \in H_0^1(0,\pi)$ for which equality holds).

3. Prove that for every $f \in H^2(0,\pi) \cap H_0^1(0,\pi)$ the following *Gagliardo-Nirenberg inequality* holds:
$$|f'|_2 \leqslant \sqrt{|f''|_2 |f|_2}. \tag{3.5.3}$$

4. Let $X = L^p(\mathbb{R}_+)$ with $p \geqslant 1$. Prove that the left-translation semigroup
$$\big(S(t)f\big)(x) = f(x+t) \qquad x \in \mathbb{R}_+ \text{ a.e.}$$
is strogly stable on X but not exponentially stable.

4 The inhomogeneous Cauchy problem

4.1 The Bochner integral

Let X be a separable Banach space and let $f : J \to X$ be a Borel function on some interval $J = (\alpha, \beta) \subset \mathbb{R}$. Observe that, since the norm is continuous, $t \mapsto |f(t)|$ is also a Borel function.

Definition 13 *A vector-valued Borel function $f : J \to X$ is called* Bochner integrable *if*

$$\int_J |f(t)|\, dt < \infty.$$

We summarize here the main properties of the Bochner integral.

- The function $f : J \to X$ is called *simple* if it can be represented as

$$f(t) = \sum_{k=1}^{m} x_k \cdot \chi_{J_k}(t) \tag{4.1.1}$$

for some choice of elements $x_k \in X$ and disjoint (Lebesgue) measurable subsets $J_k \subset J$ such that $J = \cup_{k=1}^{m} J_k$, where we have denoted by χ_{J_k} the characteristic functions of the set J_k.

- The Bochner integral of a simple function $f : J \to X$ is defined as

$$\int_J f(t)\, dt = \sum_{k=1}^{m} x_k \cdot |J_k|$$

where $|J_k|$ denotes the Lebesgue measure of J_k. One can show that the above definition is independent of the representation of f in (4.1.1). Moreover, for any simple function $f : J \to X$ we have that

$$\Big| \int_J f(t)\, dt \Big| \leqslant \int_J |f(t)|\, dt. \tag{4.1.2}$$

- If $f : J \to X$ is Bochner integrable, then there exists a sequence $\{f_n\}$ of simple functions such that

$$\forall t \in J \quad |f_n(t) - f(t)| \downarrow 0 \quad \text{as} \quad n \to \infty. \tag{4.1.3}$$

Proof. Let $\{e_j\}_{j \in \mathbb{N}}$ a dense countable subset of X. Define

$$\begin{aligned} \gamma_n(t) &= \min\{|f(t) - e_j| \ : \ 1 \leqslant j \leqslant n\} \\ j_n(t) &= \min\{j \leqslant n \ : \ \gamma_n(t) = |f(t) - e_j|\}. \end{aligned}$$

Then

$$f_n(t) := e_{j_n(t)} \quad (n \geqslant 1\,,\ t \in J)$$

is a Borel simple function and $\{f_n\}$ satisfies (4.1.3). □

- Observe that, in view of (4.1.3),

$$\lim_{n \to \infty} \int_J |f_n(t) - f(t)|\, dt = 0.$$

This together with (4.1.2) implies that $\{\int_J f_n(t)\, dt\}$ is a Cauchy sequence in X. Therefore we can define

$$\int_J f(t)\, dt = \lim_{n \to \infty} \int_J f_n(t)\, dt$$

where $\{f_n\}$ is any sequence of simple functions satisfying (4.1.3).

- Estimate (4.1.2) holds true for any Bochner integrable function f.
- For any sequence $g_n : J \to X$ of Bochner integrable functions

$$\lim_{n\to\infty} \int_J |g_n(t) - f(t)|\, dt = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \int_J g_n(t)\, dt = \int_J f(t)\, dt.$$

- *Lebesgue's dominated convergence theorem* holds true:

  for any sequence $f_n : J \to X$ of Bochner integrable functions, if

$$\begin{cases} (a) & f_n(t) \to f(t) \ \text{ a.e. as } \ n \to \infty \\ (b) & |f_n(t)| \leqslant \phi(t) \ \text{ a.e. with } \ \phi \in L^1(J), \end{cases}$$

  then

$$\begin{cases} (a) & f \ \text{ is Bochner integrable} \\ (b) & \lim_{n\to\infty} \int_J |f_n(t) - f(t)|\, dt = 0. \end{cases}$$

- Let $A : D(A) \subset X \to X$ be a closed operator. If $F : J \to X$ is a Bochner integrable function such that

$$\begin{cases} (a) & f(t) \in D(A) \ \ (t \in J \text{ a.e.}) \\ (b) & t \mapsto Af(t) \ \text{ is Bochner integrable,} \end{cases}$$

  then

$$\int_J f(t)\, dt \in D(A) \quad \text{and} \quad A\Big(\int_J f(t)\, dt\Big) = \int_J Af(t)\, dt. \qquad (4.1.4)$$

Let now $p \geqslant 1$ and $-\infty \leqslant \alpha < \beta \leqslant +\infty$.

Definition 14 *We denote by $L^p(\alpha, \beta; X)$ the space of all (equivalence classes of) functions $f : (\alpha, \beta) \to X$ which are Bochner integrable on each $J \subset\subset (\alpha, \beta)$ and such that*

$$\|f\|_p := \Big(\int_\alpha^\beta |f(t)|^p dt\Big)^{\frac{1}{p}} < \infty.$$

Here are some useful properties of $L^p(\alpha, \beta; X)$.

- $\big(L^p(\alpha, \beta; X), \|\cdot\|_p\big)$ is a Banach space.
- If $\big(X, \langle\cdot,\cdot\rangle\big)$ is an Hilbert space, then $L^2(\alpha, \beta; X)$ is an Hilbert space as well with the scalar product

$$\langle f, g\rangle_2 = \int_\alpha^\beta \langle f(t), g(t)\rangle dt \qquad \forall f, g \in L^2(\alpha, \beta; X).$$

Definition 15 *We denote by $W^{1,p}(\alpha, \beta; X)$ the (Sobolev) space of all $f \in L^p(\alpha, \beta; X)$ which possess a continuous representative satisfying*

$$f(t) = f(t_0) + \int_{t_0}^{t} g(s)\, ds \qquad \forall t \in (\alpha, \beta)$$

for some $t_0 \in (\alpha, \beta)$ and $g \in L^p(\alpha, \beta; X)$.

Some useful properties of $W^{1,p}(\alpha, \beta; X)$ are listed below.

- Every $f \in W^{1,p}(\alpha, \beta; X)$ is differentiable a.e. in $[\alpha, \beta]$ and $f'(t) = g(t)$ for a.e. $t \in [\alpha, \beta]$.
- $W^{1,p}(\alpha, \beta; X)$ is a Banach space with the norm
$$\|f\|_{1,p} = \|f\|_p + \|f'\|_p \qquad \forall f \in W^{1,p}(\alpha, \beta; X).$$
- For any $p > 1$, we have that
$$W^{1,p}(\alpha, \beta; X) \subset \mathcal{C}^{0,1-\frac{1}{p}}([\alpha, \beta]; X)$$
with continuous embedding. Indeed,
$$|f(t) - f(s)| \leqslant |t - s|^{1-\frac{1}{p}} \|f'\|_p \qquad \forall f \in W^{1,p}(\alpha, \beta; X)$$
by Hölder's inequality. Consequently, $W^{1,p}(\alpha, \beta; X) \subset \mathcal{C}([\alpha, \beta]; X)$ is compact by Ascoli's theorem. Observe that we also have that
$$W^{1,1}(\alpha, \beta; X) \subset \mathcal{C}([\alpha, \beta]; X) \tag{4.1.5}$$
but the embedding fails to be compact.

4.2 Solution of the Cauchy problem

Let X be a separable Banach space and let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X, denoted indifferently by $S(t)$ or e^{tA}, which satisfies

$$\|S(t)\| \leqslant M e^{\omega t} \qquad \forall t \geqslant 0 \tag{4.2.1}$$

for some $M > 0$ and $\omega \in \mathbb{R}$.

We observe that $D(A)$ is a Banach space with the graph norm

$$\|x\|_{D(A)} = |x| + |Ax| \qquad \forall x \in D(A).$$

For any fixed $T > 0$, consider the initial value problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in (0, T) \\ u(0) = x \end{cases} \tag{4.2.2}$$

where $x \in X$ and $f \in L^p(0, T; X)$ for a given $p \geqslant 1$. For the above problem we will give two notions of solutions following [2]. Then we will study the existence, uniqueness, regularity, and asymptotic behavior of solutions.

Notions of solution

Definition 16 *Let $p \geqslant 1$ and let $f \in L^p(0,T;X)$.*

- *We say that $u \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ is a* strict solution *of problem* (4.2.2) *if $u(0) = x$ and*

$$u'(t) = Au(t) + f(t) \quad \text{for a.e. } t \in (0,T).$$

- *We say that $u \in L^p(0,T;X)$ is a* strong solution *of problem* (4.2.2) *if there exists a sequence $v_n \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ such that*

$$\begin{cases} v_n \to u \quad \text{and} \quad v_n' - Av_n \to f & \text{in } L^p(0,T;X) \\ v_n(0) \to x & \text{in } X \end{cases} \qquad (n \to \infty) \tag{4.2.3}$$

Definition 17 *Let $f \in \mathcal{C}([0,T];X)$.*

- *We say that $u \in \mathcal{C}^1([0,T];X) \cap \mathcal{C}([0,T];D(A))$ is a* strict solution *of problem* (4.2.2) *if $u(0) = x$ and*

$$u'(t) = Au(t) + f(t) \qquad \forall t \in (0,T).$$

- *We say that $u \in C([0,T];X)$ is a* strong solution *of problem* (4.2.2) *if there exists a sequence $v_n \in \mathcal{C}^1([0,T];X) \cap \mathcal{C}([0,T];D(A))$ such that*

$$\begin{cases} v_n \to u \quad \text{and} \quad v_n' - Av_n \to f & \text{in } C([0,T];X) \\ v_n(0) \to x & \text{in } X \end{cases} \qquad (n \to \infty) \tag{4.2.4}$$

Existence and uniqueness of solutions

Theorem 12 *Let $x \in X$ and let $f \in L^p(0,T;X)$ (resp. $f \in \mathcal{C}([0,T];X)$). Then problem* (4.2.2) *has a unique strong solution given by*

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds \quad (t \in [0,T]). \tag{4.2.5}$$

Proof. Step 1: existence.
Let $S \in \mathcal{G}(M,\omega)$ and observe that, since

$$\Big|\int_0^t S(t-s)f(s)\,ds\Big|^p \leqslant M^p\Big(\int_0^t e^{p'\omega(t-s)}\,ds\Big)^{p-1}\int_0^t |f(s)|^p\,ds \quad (t \in [0,T]),$$

the function u given by (4.2.5) is bounded and therefore belongs to $L^p(0,T;X)$. Define

$$\begin{cases} v_n(t) = nR(n,A)u(t) \\ f_n(t) = nR(n,A)f(t) \\ x_n = nR(n,A)x \end{cases} \qquad \forall n \in \mathbb{N},\ n > \omega.$$

By applying $nR(n,A)$ to all the terms in (4.2.5) we obtain

$$v_n(t) = S(t)x_n + \int_0^t S(t-s)f_n(s)\,ds \quad (t \in [0,T]). \tag{4.2.6}$$

Since $x_n \in D(A)$ and $f_n \in L^p(0,T;D(A))$ (resp. $f_n \in \mathcal{C}([0,T];X)$), v_n is differentiable for a.e. t and we have that $v_n' - Av_n = f_n$. Moreover, Lebesgue's dominated convergence theorem and the properties of the Yosida approximation used in Step 1 of the proof of Theorem 2 yield

$$\begin{cases} v_n \to u \quad \text{and} \quad v_n' - Av_n \to f & \text{in } L^p(0,T;X) \text{ (resp. } \mathcal{C}([0,T];X)) \\ v_n(0) \to x & \text{in } X. \end{cases}$$

So, u is a strong solution of (4.2.2).

Step 1: uniqueness.
Let u be a strong solution of (4.2.2) and let $\{v_n\}$ be a sequence satisfying (4.2.3) (resp. (4.2.4)). We set $f_n = v_n' - Av_n$ and $x_n = v_n(0)$. Then

$$\frac{d}{ds}\big(S(t-s)v_n(s)\big) = S(t-s)f_n(s) \qquad (s \in [0,t]).$$

By integrating over $[0,t]$ we deduce that v_n satisfies (4.2.6). Passing to the limit as $n \to \infty$ we conclude that u is given by (4.2.5). $\square$

The following result provides a useful approximation of strong solutions.

Proposition 18 *Let $\{x_n\} \subset X$ and $\{f_n\} \subset L^p(0,T;X)$ $(p \geqslant 1)$ be such that*

$$x_n \xrightarrow{X} x \quad \text{and} \quad f_n \overset{L^p(0,T;X)}{\longrightarrow} f \quad (n \to \infty).$$

Let

$$\begin{cases} u_n'(t) = A_n u_n(t) + f_n(t), & t \in (0,T) \\ u_n(0) = x_n \end{cases}$$

where $A_n = n^2R(n,A) - n$, $n > \omega$, is the Yosida approximation of A. Then

$$u_n \overset{L^p(0,T;X)}{\longrightarrow} u \quad (n \to \infty)$$

where u is the strong solution of (4.2.2).

Proof. Since $A_n \in \mathcal{L}(X)$ we have that

$$u_n(t) = e^{tA_n}x_n + \int_0^t e^{(t-s)A_n} f_n(s)\,ds \quad (t \in [0,T]).$$

Thus, recalling (1.5.7) and (1.5.8) from the proof of the Hille-Yosida theorem, we obtain

$$|e^{tA_n}x_n - e^{tA}x| \leqslant Me^{2\omega t}|x_n - x| + |e^{tA_n}x - e^{tA}x| \overset{n\to\infty}{\longrightarrow} 0$$

uniformly on $[0,T]$. Moreover,

$$\Big|\int_0^t \Big(e^{(t-s)A_n}f_n(s) - e^{(t-s)A}f(s)\Big)ds\Big|^p$$
$$\leqslant 2^{p-1}M^p \int_0^t e^{2\omega p(t-s)}|f_n(s)-f(s)|^p ds \overset{\mathcal{C}([0,T];X)}{\longrightarrow} 0,$$
$$+ 2^{p-1}\int_0^t |e^{(t-s)A_n}f(s) - e^{(t-s)A}f(s)|^p ds$$

where, by Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty}\int_0^T dt \int_0^t |e^{(t-s)A_n}f(s) - e^{(t-s)A}f(s)|^p ds = 0.$$

The conclusion follows. □

Regularity of solutions

Our first result guarantees that the strong solution of (4.2.2) is strict when f has better "space regularity".

Theorem 13 *Let $x \in D(A)$ and let $f \in L^p(0,T;D(A))$ for some $p \geqslant 1$. Then the strong solution u of problem* (4.2.2) *is strict in $L^p(0,T;X)$.*

Proof. Let u be the strong solution of problem (4.2.2) and let u_n be the solution of

$$\begin{cases} u_n'(t) = A_n u_n(t) + f(t), & t \in (0,T) \\ u_n(0) = x \end{cases} \tag{4.2.7}$$

where $A_n = n^2R(n,A) - n$, $n > \omega$, is the Yosida approximation of A. Then

$$v_n(t) := A_n u_n(t) \quad (t \in [0,T])$$

satisfies

$$\begin{cases} v_n'(t) = A_n v_n(t) + A_n f(t), & t \in (0,T) \\ v_n(0) = A_n x \end{cases}$$

where

$$A_n x \xrightarrow{X} Ax \quad \text{and} \quad A_n f \overset{L^p(0,T;X)}{\longrightarrow} Af \quad (n\to\infty).$$

So, Proposition 18 ensures that v_n converges in $L^p(0,T;X)$ to the strong solution of

$$\begin{cases} v'(t) = Av(t) + Af(t), & t \in (0,T) \\ v(0) = Ax \end{cases}$$

which, by Theorem 12 is given by

$$v(t) = e^{tA}Ax + \int_0^t e^{(t-s)A}Af(s)\,ds = Au(t) \quad (t \in [0,T] \text{ a.e.})$$

This shows that $u \in L^p(0,T;D(A))$. Moreover

$$u_n' = A_n u_n + f = v_n + f \overset{L^p(0,T;X)}{\longrightarrow} v + f \quad (n \to \infty).$$

Therefore, $u \in W^{1,p}(0,T;X)$ and $u' = v + f = Au + f$. □

Corollary 7 *Let $x \in X$ and let $f \in L^p(0,T;X)$. Then the strong solution u of problem (4.2.2) belongs to $\mathcal{C}([0,T];X)$. Moreover, we have that*

$$u_n \overset{\mathcal{C}([0,T];X)}{\longrightarrow} u \quad (n \to \infty),$$

where u_n is the strict solution of the problem

$$\begin{cases} u_n'(t) = Au_n(t) + f_n(t), & t \in (0,T) \\ u_n(0) = x_n \end{cases} \tag{4.2.8}$$

with $f_n(t) = nR(n,A)f(t)$ and $x_n = nR(n,A)x$ for all $n > \omega$.

Proof. By (1.5.6) we have that $D(A) \ni x_n \overset{X}{\longrightarrow} u$ as $n \to \infty$. Moreover, $f_n \in L^p(0,T;D(A))$ and

$$\lim_{n\to\infty} f_n(t) = f(t), \quad \text{and} \quad |f_n(t)| \leqslant \frac{Mn}{n-\omega}\,|f(t)| \quad \text{a.e. in } [0,T].$$

So, $f_n \overset{L^p(0,T;X)}{\longrightarrow} f$ by Lebesgue's theorem. Then Theorem 13 ensures that (4.2.8) has a unique strict solution u_n which, in particular, belongs to $\mathcal{C}([0,T];X)$. Now, the representation formula (4.2.5) implies that, for all $t \in [0,T]$,

$$\begin{aligned} |u_n(t) - u(t)| &= \Big| e^{tA}(x_n - x) + \int_0^t e^{(t-s)A}[f_n(s) - f(s)]\,ds \Big| \\ &\leqslant Me^{\omega t}|x_n - x| + M\int_0^t e^{\omega(t-s)}|f_n(s) - f(s)|\,ds \\ &\leqslant C_T\big(|x_n - x| + \|f_n - f\|_p\big) \end{aligned}$$

for some constant $C_T > 0$. The conclusion follows. □

We will now show a similar result for f with better "time regularity". We begin by studying the case of $x = 0$.

Lemma 4 *Let $f \in W^{1,p}(0,T;X)$ for some $p \geqslant 1$. Then*

$$u_f(t) := \int_0^t e^{(t-s)A} f(s)ds \quad (t \in [0,T])$$

belongs to $W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ and

$$u_f'(t) = Au_f(t) + f(t) = e^{tA}f(0) + \int_0^t e^{(t-s)A}f'(s)ds \quad (t \in [0,T] \text{ a.e.})$$

Proof. Since

$$u_f(t) = \int_0^t e^{sA} f(t-s)ds \quad (t \in [0,T])$$

by differentiating under the integral sign we have that $u_f \in W^{1,p}(0,T;X)$ and

$$u_f'(t) = e^{tA} f(0) + \int_0^t e^{(t-s)A} f'(s)ds \quad (t \in [0,T] \text{ a.e.}). \tag{4.2.9}$$

Therefore, we also have

$$\begin{aligned} u_f'(t) &= \lim_{h\downarrow 0} \frac{1}{h} \Big\{ \int_0^{t+h} e^{(t+h-s)A} f(s)\,ds - \int_0^t e^{(t-s)A} f(s)\,ds \Big\} \\ &= \lim_{h\downarrow 0} \Big\{ \frac{e^{hA}-I}{h} \int_0^t e^{(t-s)A} f(s)\,ds + \frac{1}{h} \int_t^{t+h} e^{(t+h-s)A} f(s)\,ds \Big\} \\ &= \lim_{h\downarrow 0} \frac{e^{hA}-I}{h} \int_0^t e^{(t-s)A} f(s)\,ds + f(t). \end{aligned}$$

This shows that $u_f(t) \in D(A)$ and $Au_f(t) = u_f'(t) - f(x)$. Consequently, $u_f \in L^p(0,T;D(A))$ and the conclusion follows recalling (4.2.9). □

Theorem 14 *Let $x \in D(A)$ and let $f \in W^{1,p}(0,T;X)$ for some $p \geqslant 1$. Then the strong solution u of problem* (4.2.2) *is strict in $L^p(0,T;X)$.*

Proof. Let u be the strong solution of problem (4.2.2) and let u_n be the solution of (4.2.7). Then $u_n \in \mathcal{C}^1([0,T];X)$ and $v_n := u_n'$ satisfies

$$\begin{cases} v_n \in W^{1,p}(0,T;X) \\ v_n'(t) = A_n v_n(t) + f'(t), \quad t \in (0,T) \text{ a.e.} \\ v_n(0) = A_n x + f(0). \end{cases} \tag{4.2.10}$$

So, Proposition 18 ensures that v_n converges in $L^p(0,T;X)$ to the strong solution of

$$\begin{cases} v'(t) = Av(t) + f'(t), \quad t \in (0,T) \\ v(0) = Ax + f(0), \end{cases}$$

Therefore, $u \in W^{1,p}(0,T;X)$ and Lemma 4 yields

$$\begin{aligned} u'(t) &= v(t) = e^{tA}\big(Ax + f(0)\big) + \int_0^t e^{(t-s)A} f'(s)\,ds \\ &= Au(t) + f(t) \qquad t \in (0,T) \text{ a.e.} \end{aligned}$$

The conclusion follows. □

Remark 10 In general, the strong solution of (4.2.2) fails to be strict for $f \in C([0,T];D(A))$. To see an example, let $y \notin D(A)$ and take $f(t) = e^{tA}y$ and $x = 0$. Then

$$u(t) = te^{tA}y \qquad \forall t \geqslant 0$$

which fails to be differentiable.

4.3 Maximal regularity results

For special classes of generators the strong solution of (4.2.2) in $L^p(0,T;X)$ enjoys additional regularity properties. We investigate below the case of $p = 2$ when $(X, \langle\cdot,\cdot\rangle)$ is a separable Hilbert space, beginning by analyzing the problem for $x = 0$. As in in Lemma 4, we denote by u_f the strong solution of

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in (0,T) \\ u(0) = 0 \end{cases} \tag{4.3.1}$$

which is given by the function

$$u_f(t) := \int_0^t e^{(t-s)A} f(s)ds \quad (t \in [0,T]). \tag{4.3.2}$$

Theorem 15 *Let $A : D(A) \subset X \to X$ be a self-adjoint dissipative operator on a real Hilbert space X and let $f \in L^2(0,T;X)$. Then u_f is the strict solution of* (4.3.1) *and*

$$\|Au_f\|_2 \leqslant \|f\|_2. \tag{4.3.3}$$

Proof. Consider, as in the proof of Corollary 7, $f_n(t) := nR(n,A)f(t)$ for all $n > \omega$. We have that $f_n \in L^p(0,T;D(A))$ and

$$f_n \overset{L^2(0,T;X)}{\longrightarrow} f \quad (n \to \infty). \tag{4.3.4}$$

Then $u_n := u_{f_n} \in W^{1,2}(0,T;X) \cap L^2(0,T;D(A))$ satisfies

$$u_n' = Au_n + f_n \quad \text{a.e. in} \quad [0,T]. \tag{4.3.5}$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\langle Au_n, u_n\rangle = \langle u_n', Au_n\rangle = |Au_n|^2 + \langle f_n, Au_n\rangle$$

and

$$|Au_n|^2 - \frac{1}{2}\frac{d}{dt}\langle Au_n, u_n\rangle = -\langle f_n, Au_n\rangle \leqslant \frac{1}{2}\Big(|f_n|^2 + |Au_n|^2\Big).$$

Hence, integrating on $[0,T]$, since A is dissipative we get

$$\int_0^T |Au_n|^2\,dt \leqslant \int_0^T |Au_n|^2\,dt - \langle Au_n(T), u_n(T)\rangle \leqslant \int_0^T |f_n|^2\,dt. \tag{4.3.6}$$

Now, applying the above inequality to the difference $u_n - u_m$ we obtain

$$\|A(u_n - u_m)\|_2 \leqslant \|f_n - f_m\|_2 \qquad \forall m, n > \omega,$$

which implies that $\{u_n\}$ is a Cauchy sequence in $W^{1,2}(0,T;X) \cap L^2(0,T;D(A))$ in view of (4.3.4). By Corollary 7, u_n converges to u in $\mathcal{C}([0,T];X)$ and so, recalling (4.3.5), we conclude that u is the strict solution of (4.3.1). Finally, (4.3.3) follows from (4.3.6) passing to the limit as $n \to \infty$. $\square$

Consequently, when A is self-adjoint and dissipative, u_f' and Au_f have the same regularity in the space $L^2(0,T;X)$ as the right-hand side f—a property which is called *maximal regularity*. Since we know that e^{tA} is analytic in this case (see Exercise 11), it is natural to ask whether such maximal regularity holds true, more generally, when A is the infinitesimal generator of an analytic semigroup. In order to show that this is indeed the case ne need to recall some properties of the Fourier transform on $L^2(\mathbb{R};X)$.

Fix an orthonormal basis $\{e_k\}_{k\geqslant 1}$ of H and represent $g \in L^2(\mathbb{R};X)$ as

$$g(t) = \sum_{k=1}^{\infty} g_k(t)e_k \quad (t \in \mathbb{R} \text{ a.e.}),$$

where $g_k(t) = \langle g(t), e_k\rangle$. Then we have that $\|g\|_2^2 = \sum_{k=1}^{\infty} |g_k|^2$. Denoting by

$$\widehat{g}_k(\tau) = \int_{-\infty}^{+\infty} g_k(t)e^{-i\tau t}dt \quad (t \in \mathbb{R} \text{ a.e.})$$

the Fourier transform of g_k, we have that

$$\int_{-\infty}^{+\infty} |g_k(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\widehat{g}_k(\tau)|^2 d\tau \quad (k \geqslant 1).$$

Then we can define $\mathcal{F} : L^2(\mathbb{R};X) \to L^2(\mathbb{R};X)$ by

$$\mathcal{F}[g](\tau) = \widehat{g}(\tau) = \sum_{k=1}^{\infty} \widehat{g}_k(\tau)e_k \quad (\tau \in \mathbb{R} \text{ a.e.})$$

We will use the following properties of the Fourier transform on $L^2(\mathbb{R};X)$.

- *Plancherel identity:* for every $g \in L^2(\mathbb{R};X)$ we have that

$$\int_{-\infty}^{+\infty} |g(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\widehat{g}(\tau)|^2 d\tau \quad (k \geqslant 1). \tag{4.3.7}$$

- *Derivation formula:* for every $g \in W^{1,2}(0,T;X)$ we have that

$$\mathcal{F}[g'](\tau) = i\tau\widehat{g}(\tau) \quad (\tau \in \mathbb{R} \text{ a.e.}) \tag{4.3.8}$$

- *Action of a closed operator:* for any closed operator $A : D(A) \subset X \to X$ and any $g \in L^2(\mathbb{R};D(A))$ we have that $F[g] \in L^2(\mathbb{R};D(A))$ and

$$A\mathcal{F}[g](\tau) = \mathcal{F}[Ag](\tau) \quad (\tau \in \mathbb{R} \text{ a.e.}) \tag{4.3.9}$$

Theorem 16 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of an analytic semigroup with negative growth bound and let $f \in L^2(0,T;X)$. Then u_f is the strict solution of* (4.3.1) *and*

$$\|Au_f\|_2 \leqslant (M+1)\|f\|_2 \tag{4.3.10}$$

where $M > 0$ is the constant in (4.2.1).

Proof. Let us assume first that $f \in L^2(0,T;D(A))$. Then u_f is the strict solution of (4.3.1). Define

$$F(t) = \begin{cases} f(t), & t \in [0,T] \\ 0, & t \in \mathbb{R} \setminus [0,T] \end{cases}$$

and

$$U(t) = \begin{cases} 0, & t > 0 \\ u_f(t) & t \in [0,T] \\ e^{(t-T)A}u_f(T) & t > T. \end{cases}$$

Then $f \in L^2(\mathbb{R};D(A))$, $U \in W^{1,2}(\mathbb{R};X) \cap L^2(\mathbb{R};D(A))$ because e^{tA} has a negative growth bound, and

$$U'(t) = AU(t) + F(t) \quad (t \in \mathbb{R} \text{ a.e.})$$

So, we can take the Fourier transform of both terms of the above identity to obtain, in view of (4.3.8) and (4.3.9),

$$i\tau\widehat{U}(\tau) = A\widehat{U}(\tau) + \widehat{F}(\tau) \quad (\tau \in \mathbb{R} \text{ a.e.})$$

So, $\widehat{U}(\tau) = R(i\tau, A)\widehat{F}(\tau)$ and, since $AR(i\tau, A) = i\tau R(i\tau, A) - I$, the resolvent estimate yields

$$|A\widehat{U}(\tau)| = |i\tau R(i\tau, A)\widehat{F}(\tau) - \widehat{F}(\tau)| \leqslant (M+1)|\widehat{F}(\tau)| \quad (t \in \mathbb{R} \text{ a.e.})$$

Therefore

$$\begin{aligned} \int_0^T |Au_f|^2 dt &\leqslant \int_{-\infty}^{+\infty} |AU|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |A\widehat{U}|^2 d\tau \\ &\leqslant \frac{(M+1)^2}{2\pi} \int_{-\infty}^{+\infty} |\widehat{F}|^2 d\tau = (M+1)^2 \int_0^T |f|^2 dt. \end{aligned}$$

Finally, in order to remove the extra assumption $f \in L^2(0,T;D(A))$, let $f_n(t) := nR(n,A)f(t)$ for all $n > 0$. Then

$$u_n := u_{f_n} \in W^{1,2}(0,T;X) \cap L^2(0,T;D(A))$$

and the above inequality yields

$$\|A(u_n - u_m)\|_2 \leqslant (M+1)\|f_n - f_m\|_2$$

and

$$\|u_n' - u_m'\|_2 \leqslant (M+2)\|f_n - f_m\|_2$$

for all $n, m \geqslant 1$. Thus

$$\{u_n\} \quad \text{is a Cauchy sequence in} \quad W^{1,2}(0,T;X) \cap L^2(0,T;D(A)).$$

Since $\{u_n\}$ converges to u we conclude that u is the strict solution of (4.3.1). Estimate (4.3.10) follows from the analogous inequality for u_n. □

In order to obtain similar regularity results for(4.2.2), let us set

$$[D(A), X]_{1/2} = \Big\{ x \in X \ : \ \int_0^\infty |Ae^{tA}x|^2 dt < \infty \Big\}. \tag{4.3.11}$$

It is easy to see that $[D(A), X]_{1/2}$ is a subspace of X containing $D(A)$. The following risult is a direct consequence of Theorem 16 and definition (4.3.11).

Corollary 8 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of an analytic semigroup with negative growth bound. If*

$$x \in [D(A), X]_{1/2} \quad \textit{and} \quad f \in L^2(0,T;X)$$

then the strong solution u of (4.2.2) is strict.

Example 20 On $X = L^2(0,\pi)$ let $A : D(A) \subset X \to X$ be the operator (studied in Exercise 8)

$$\begin{cases} D(A) = H^2(0,\pi) \cap H_0^1(0,\pi) \\ Af(x) = f''(x) & x \in (0,\pi) \text{ a.e.} \end{cases}$$

We know that A is self-adjoint and dissipative. Moreover, A is the infinitesimal generator of and analytic semigroup of negative type (Example 13). We now show that

$$[D(A), X]_{1/2} = H_0^1(0,\pi). \tag{4.3.12}$$

Let us fix $f \in H_0^1(0,\pi)$ and consider its the Fourier series

$$f(x) = \sum_{n=1}^\infty f_n \sin(nx) \qquad (x \in [0,\pi]).$$

By Parseval's identity we have

$$\sum_{n=1}^\infty n^2 |f_n|^2 = \frac{2}{\pi} \int_0^\pi |f'(x)|^2 dx.$$

Moreover

$$Ae^{tA}f(x) = -\sum_{n=1}^{\infty} n^2 e^{-n^2 t} f_n \sin(nx) \qquad (x \in [0,\pi]).$$

Therefore

$$\begin{aligned}\int_0^\infty |Ae^{tA}f|^2 dt &= \frac{\pi}{2}\sum_{n=1}^{\infty}\int_0^\infty n^4 |f_n|^2 e^{-2n^2 t}\,dt \\ &= \frac{\pi}{4}\sum_{n=1}^{\infty} n^2|f_n|^2 = \frac{1}{2}\int_0^\pi |f'(x)|^2 dx < \infty.\end{aligned}$$

This identity implies $H_0^1(0,\pi) \subset [D(A), X]_{1/2}$ as well as the converse inclusion.

We can use (4.3.12) to the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t,x) & (t,x) \in (0,T)\times(0,\pi) \text{ a.e.} \\ u(t,0) = 0 = u(t,\pi) & t \in (0,T) \\ u(0,x) = u_0(x) & x \in (0,\pi). \end{cases} \tag{4.3.13}$$

Since

$$L^2\big(0,T;L^2(0,\pi)\big) = L^2\big((0,T)\times(0,\pi)\big),$$

by Corollary 8 we conclude that for all

$$f \in L^2\big((0,T)\times(0,\pi)\big) \quad \text{and} \quad u_0 \in H_0^1(0,\pi)$$

problem (4.3.13) has a unique strict solution u such that

$$\frac{\partial u}{\partial t},\ \frac{\partial^2 u}{\partial x^2} \in L^2\big((0,T)\times(0,\pi)\big).$$

4.4 Problems

1. Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X of negative type. Prove that

   $$|x|_{D(A)} = |Ax| \qquad \forall x \in D(A)$$

   is a norm on $D(A)$, equivalent to the graph norm.

2. Give an example to show that (4.1.5) is not a compact embedding.

3. Generalize Corollary 8 removing the assumption $\omega_0(e^{tA}) < 0$.

5 Notation

- $\mathbb{R} = (-\infty,\infty)$ stands for the real line, $\mathbb{R}_+$ for $[0,\infty)$, and $\mathbb{R}_+^*$ for $(0,\infty)$

- $\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, \dots\}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\} = \{\pm 1, \pm 2, \dots\}$
- For any $\lambda \in \mathbb{C}$, $\Re \lambda$ and $\Im \lambda$ denote the real and imaginary parts of λ, respectively
- $|\cdot|$ stands for the norm of a Banach space X, as well as for the absolute value of a real number or the modulus of a complex number
- $\mathcal{L}(X)$ is the Banach space of all bounded linear operators $\Lambda : X \to X$ equipped with the norm $\|\Lambda\| = \sup_{|x| \leqslant 1} |\Lambda x|$
- $\mathcal{K}(X)$ is the closed subspace of $\mathcal{L}(X)$ of all compact operators $\Lambda : X \to X$
- $\omega_0(S)$ denotes the growth bound of a $\mathcal{C}_0$-semigroup of bounded linear operators on X (Definition 4)
- $s(A)$ denotes the spectral bound of a closed operator $A : D(A) \subset X \to X$ (Definition 7)
- $\Pi_\omega = \big\{\lambda \in \mathbb{C} \; : \; \Re \lambda > \omega\big\}$ for any $\omega \in \mathbb{R}$
- $\Sigma_{\omega,\theta} = \big\{\lambda \in \mathbb{C} \setminus \{\omega\} \; : \; \big|\arg(\lambda - \omega)\big| < \theta\big\}$ for any $\omega \in \mathbb{R}$ and $\theta \in (0, \pi]$
- $C(z_0, r) = \big\{z \in \mathbb{C} \; : \; |z - z_0| < r\big\}$ for any $z_0 \in \mathbb{C}$ and $r > 0$

Bibliography

[1] Fatiha Alabau-Boussouira and Piermarco Cannarsa. A constructive proof of Gibson's stability theorem. *Discrete Contin. Dyn. Syst. Ser. S*, 6(3):611–617, 2013.

[2] Alain Bensoussan, Giuseppe Da Prato, Michel C. Delfour, and Sanjoy K. Mitter. *Representation and control of infinite dimensional systems. 2nd ed.* Boston, MA: Birkhäuser, 2nd ed. edition, 2007.

[3] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* New York, NY: Springer, 2011.

[4] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[5] J. S. Gibson. A note on stabilization of infinite-dimensional linear oscillators by compact linear feedback. *SIAM J. Control Optim.*, 18(3):311–316, 1980.

[6] A. Pazy. Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 44. New York etc.: Springer-Verlag. VIII, 279 p. DM 88.00; $ 34.20 (1983)., 1983.

[7] R. Triggiani. Lack of uniform stabilization for noncontractive semigroups under compact perturbation. *Proc. Amer. Math. Soc.*, 105(2):375–383, 1989.

[8] Roberto Triggiani. On the stabilizability problem in Banach space. *J. Math. Anal. Appl.*, 52:383–403, 1975.