Stanley's work on unimodality

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Abstract. This paper surveys Stanley's work on unimodality, and its impact. It also poses some open problems that arise naturally from his work in this area.

1. Introduction

In this article I (try to) survey Stanley's work on unimodality, with particular emphasis on the results reprinted in the present volume, and its impact. Stanley's work on unimodality bears the hallmark of most of his mathematical work, namely that of applying ideas, results, and techniques from other areas of mathematics (mainly algebra and geometry, but not only) to the solution of combinatorial (in this case, unimodality) problems. Another hallmark of Stanley's work on unimodality is the extreme generality of his results, which is particularly striking when compared with the great majority of the results obtained in this area.

I have been asked specifically to comment on the two papers [**50**] and [**52**], reprinted in this volume. Since [**52**] is a survey paper, commenting on it means to comment also on the great majority of Stanley's work on unimodality.

I do not recall the basic definitions since they appear on the first page of [**52**], but I add two more here. A sequence of real numbers $\{a_i\}_{i=0,\ldots,n}$ is said to be *ultra log-concave* if $\{a_i/\binom{n}{i}\}_{i=0,\ldots,n}$ is log-concave. A symmetric unimodal sequence $\{a_i\}_{i=0,\ldots,n}$ is said to be *strictly unimodal* if $a_1 < a_2 < \cdots < a_{\lfloor \frac{n}{2} \rfloor} = a_{\lceil \frac{n}{2} \rceil} > \cdots > a_{n-2} > a_{n-1}$.

2. Unimodality and Lie Superalgebras

In [**50**] Stanley uses the theory of Lie superalgebras to prove the unimodality of some sequences. In §2 of [**50**] Stanley recalls precisely how the representation theory of $s\ell_2(\mathbb{C})$ is used to show that certain sequences are unimodal. Essentially, the result here (Thm. 2.1) states that given any representation ϕ of $s\ell_2(\mathbb{C})$ its character (which may be regarded as a Laurent polynomial in one variable q) has the property that the sequence of coefficients of the even (respectively, odd) powers of q are both (integer) unimodal sequences, symmetric about 0. In fact, even though Stanley does

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not state this explicitly, the reasoning given in §2 shows that Thm. 2.1 is actually an if and only if. In §3 Stanley uses Thm. 2.1 to prove the result (Thm. 3.2) that given any symmetric unimodal polynomial $f(q) \in \mathbb{N}[q]$ the Pólya composition of the generalized cycle index of S_m with respect to any character χ of S_m with $f(q)$ is again symmetric and unimodal (the *Pólya composition* of a multivariate polynomial $C(x_1, x_2, \ldots)$ with a univariate polynomial $f(q)$ is $C(f(q), f(q^2), f(q^3), \ldots))$. This result includes as a special case the well known and celebrated result (Thm. 3.1) that the principal specialization of any Schur function is a symmetric unimodal polynomial.

Stanley then obtains analogues of these results for the orthosymplectic Lie superalgebra $osp\,(1,2)$ [**37**]. More precisely, the result (Cor. 4.3) states that a Laurent polynomial in $\mathbb{N}[q, q^{-1}]$ is unimodal and symmetric about 0 if and only if it is the character (which may be viewed as a Laurent polynomial in q) of some representation of $osp\,(1,2)$ (to be more precise, the "only if" part of this statement is not stated in Cor. 4.3, but follows from the proof of Stanley). As applications of this result he shows (Thm. 6.1) that the "principal specialization" $s_\lambda(1, q^2, \ldots, q^{2n}/q, q^3, \ldots, q^{2n-1})$ of any super-Schur function is a symmetric unimodal polynomial, and that (Thm. 7.1) given any symmetric unimodal polynomial $f(q) \in \mathbb{N}[q]$ with an even symmetry, and any character χ of S_m, $C(f(q), f(-q^2), f(q^3), f(-q^4), \ldots))$ is again symmetric and unimodal, where $C(x_1, \ldots, x_m)$ is the generalized cycle index of S_m with respect to the character χ. In the last section Stanley obtains a characterization of Peck posets of even rank in terms of the representation theory of $osp\,(1,2)$ (Thm. 8.3), which is the "superanalogue" of a result of Proctor and himself (Thm. 8.1).

Interesting applications of Lie algebras and superalgebras to unimodality have been obtained by Hughes and Van der Jeugt in [**24**] and by Reiner and Stanton in [**42**]. An analogous result for log-concave sequences has been obtained by Wagner in [**59**].

3. Log-concave and unimodal sequences in algebra, combinatorics, and geometry

In the paper "Log concave and unimodal sequences in algebra, combinatorics, and geometry" [**52**] Stanley surveys the state of the art in the field of unimodality. This is the first survey ever written on this topic and has definitely been enormously influential both for the open problems that it posed, and for pointing out the many different techniques that could be used to prove the unimodality of sequences. This survey showed to the mathematical community that unimodal sequences are ubiquitous in mathematics and that to prove that a sequence is unimodal is very often an extremely difficult task. These two facts are now considered well known truths, but this was not so in 1986, when unimodality was considered a "weird" property to look at and study (I remember an eminent mathematician telling me, when I was a graduate student, that unimodality is nice but "then, what do you do with it?"). It must be mentioned that many of the techniques surveyed in [**52**] were introduced and pioneered by Stanley himself, particularly the use of Lie superalgebras, commutative algebra, and algebraic geometry.

In §2 of [**52**] Stanley collects unimodality results that are proved in an inductive or combinatorial way (i.e., by establishing appropriate injections), as well as some operations on sequences that preserve the property of being log-concave or unimodal. Many results using these techniques have been obtained, too numerous to be cited here, since [**52**]. As a sample, we mention an inductive proof of the unimodality of the principal specialization of any Schur function ([**23**], see also [**16**]) and injective proofs of unimodality and log-concavity of the Eulerian polynomials [**22**]. Also, other linear and bilinear transformations preserving log-concavity or real-rootedness have been studied, e.g., in [**14**], [**58**], [**11**], and [**33**].

In §3 Stanley describes the method of polynomials with only real zeros, which is based on the fact, Thm. 2, that if a real polynomial has only real roots then its sequence of coefficients is ultra log-concave (hence log-concave and unimodal). Stanley describes two general ways of proving that a polynomial (or sequence of polynomials) has only real roots, namely Sturm sequences and characteristic polynomials of real symmetric matrices (Prop. 4), giving some combinatorial applications to matching polynomials (Ex. 1) and spanning trees (Prop. 6) of graphs. In this section Stanley also gives a sufficient condition on the roots of a polynomial for its coefficients to be log-concave (Prop. 7). A stronger result has later been obtained in [**15**, Thm. 1.3].

In this section Stanley also states the first three conjectures mentioned in [**52**]. The first one is Conj. 1 on p. 506. Let $P = ([n], \preceq)$ be a finite labeled partially ordered set (or poset, for short). Assume that P is naturally labeled (i.e., $i \preceq j$ implies $i < j$ for all $i, j \in [n]$). For each $j \in \mathbb{N}$ let $e_j(P)$ (resp., $\bar{e}_j(P)$)be the number of surjective order preserving (resp., strictly order preserving) maps $f : P \mapsto [j]$ (where $[j] := \{1, \ldots, j\}$ has the natural ordering). A permutation $\sigma \in S_n$ is a linear extension of P if $i \preceq j$ implies $\sigma^{-1}(i) \leq \sigma^{-1}(j)$ for all $i, j \in [n]$. Let $\mathcal{L}(P)$ be the set of all linear extensions of P and $E_P(q) := \sum_{j=0}^{|P|} e_j(P)q^j$, $\bar{E}_P(q) := \sum_{j=0}^{|P|} \bar{e}_j(P)q^j$, and $W(P) := \sum_{u \in \mathcal{L}(P)} q^{d(u)+1}$, where $d(u)$ denotes the number of descents of u. There is a simple relationship between these three polynomials ([**52**, p. 505]) which implies that one of them has only real roots if and only if all of them do. Then Conj. 1 states that the polynomial $E_P(q)$ (equiv., $\bar{E}_P(q)$, $W_P(q)$) has only real zeros. In particular, the conjecture implies that all these polynomials are log-concave and unimodal.

The second conjecture stated in [**52**] is Conj. 2 on p. 507. Let $G = (V, E)$ be a simple finite graph on vertex set V and edge set E (i.e., G has no loops nor multiple edges). For $i \in \mathbb{N}$ let b_i be the number of spanning forests of G having i edges. So $b_0 = 1$, and $b_i = 0$ if $i > V - c$ where c is the number of connected components of G. Then Conj. 2 states that the sequence $(b_0, \ldots, b_r)$, where $r \stackrel{\text{def}}{=} |V| - c$, is ultra log-concave. Hence, in particular, log-concave and unimodal, both statements being open at the time of [**52**].

The third conjecture stated in §3 is Conj. 3 on p. 508. This collects some of the most famous open problems in unimodality. Namely the conjectures that if $\chi_G(x) = \sum_{j=0}^{p}(-1)^{p-i}a_i x^i$ is the chromatic polynomial of a graph G on p vertices then the sequence $(a_0, \ldots, a_p)$ is log-concave, and hence unimodal, and that if M is a matroid of rank n and W_i is the number of flats of rank i in M, for $i = 0, \ldots, n$, then the sequence $(W_0, \ldots, W_n)$ is log-concave, and hence unimodal.

This section of [**52**] stimulated a lot of research activity, particularly on the once classical topic of polynomials with only real roots, among combinatorialists, which has led to spectacular successes, including the solution of problems posed by analysts more then a century ago ([**8**], [**9**]). Probably the most powerful results obtained in this direction are the Chudnovsky-Seymour Theorem ([**19**], see also

[**13**]) and the Borcea-Brändén Theorem which characterizes all the linear operators on $\mathbb{R}[x_1, \dots, x_n]$ that send stable polynomials to stable polynomials and includes a characterization of all the linear operators on $\mathbb{R}[x]$ that send polynomials with only real roots to polynomials with only real roots ([**8**], see also [**13**]). The concept of stability is the right generalization of the property of "having only real roots" to multivariate polynomials and we refer the reader to [**13**], [**41**] or [**60**] for further information about it. For combinatorial applications to characterize all linear operators $\varphi : \mathbb{R}[x] \to \mathbb{R}[x]$ such that $\varphi(f)$ has nonnegative coefficients and only real roots if $f \in \mathbb{R}[x]$ does would be more useful. This more difficult problem is open. Some of the most common linear transformations arising in combinatorics have been studied from this point of view in [**14**].

At the time of [**52**] only these four concepts were considered, namely "only real roots", "ultra log-concavity", "log-concavity" and "unimodality", each one being stronger then the following one for positive sequences. Now there are two infinite chains of concepts each one implied by the previous one, between "only real roots" and "log-concavity". One of them is obtained through the theory of total positivity (more precisely, Pólya frequency sequences of order k), and one is obtained through the concept of k-log-concavity. We refer the reader to, e.g., [**14**, §§2.2 and 2.5], [**28**, Chap. 8] and [**34**] for further information about these concepts.

Regarding the conjectures, Conj. 1 was disproved by Stembridge in 2007 [**56**]. Probably the most general positive result known about Conj. 1 was obtained by Reiner and Welker in 2005 [**43**] where they show that the polynomial $W_P(q)$ is (symmetric and) unimodal if the poset P is graded. Apart from its generality, the result is interesting also for its method of proof, which itself shows the legacy and influence of Stanley's work in unimodality and more generally in combinatorics, which uses geometry, namely a unimodular triangulation of a convex polytope associated to the poset P. This result was then further generalized by Athanasiadis in [**2**] in the context of convex polytopes. A simpler proof of a stronger result was shortly after obtained by Bränden in [**10**] (see also [**12**]) where he shows that the polynomial $W_P(q)$ has a nonnegative γ-vector (a property which immediately implies symmetry and unimodality) if P is graded (we refer the reader to [**13**] for the theory of γ-vectors, that is one of the most important developments in the theory of unimodality in the last 10 years). There are many interesting consequences of Conj. 1 that are still open. In particular, it is open whether $W_P(q)$ (and hence $E(q)$ and $\bar{E}(q)$) has only real zeros if P is graded, and whether $W_P(q)$, $E(q)$, or $\bar{E}(q)$ are always log-concave (it is known [**14**, Thm. 2.5.8 and p. 3] that log-concavity of $W_P(q)$, implies that of $E(q)$ and $\bar{E}(q)$).

The first conjecture in Conj. 3, and hence the second, were recently proved by Huh in [**25**]. More precisely, Huh proves that if $\chi_M(x) = \sum_{i=0}^{n} (-1)^{n-i} b_i x^i$ is the characteristic polynomial of a matroid M of rank n representable over a field of characteristic zero then the sequence $(b_0, \dots, b_n)$ is log-concave and unimodal. The conjectures then follow by applying Huh's result to graphic matroids. The proof of Huh's result uses deep results from algebraic geometry and includes a characterization of log-concave sequences in terms of homology classes of the cartesian product of two projective spaces (see [**25**, Thm. 21]) and mixed volumes, both tools that Stanley discusses in [**52**] in one of the sections following Conj. 3. Huh's result was then extended by Huh and Katz in [**26**] to all representable matroids. The more general conjecture (mentioned in Conj. 3 of [**52**]) that the characteristic polynomial of any finite matroid M (not necessarily representable over a field) has the stated log-concavity and unimodality properties is open. The third and fourth conjectures made in Conj. 3 (dealing with the enumeration of flats by rank in a finite matroid) are still open. Probably the most general result on these conjectures is still the one obtained by Seymour in [**45**] where he shows that $W_2^2 \geq W_1 W_3$ for matroids such that any line has at most 4 points. Other partial results have been obtained by Kung [**31**].

Conjecture 2 is still open. However, it had been remarked by Brylawsky in [**17**], that Conj. 3 implies the log-concavity statement of Conj. 2 (this fact was independently rediscovered by Lenz in [**32**]). Hence, by Huh's result the sequence $(b_0, \ldots, b_r)$ is log-concave, and hence unimodal.

In §4, "Analytic techniques", Stanley describes the method of using complex analytic techniques (such as contour integrals) on the generating function of a combinatorial sequence to prove unimodality of the sequence, and mentions some applications of this method to the number of partitions into a given number of parts (and distinct parts) and to some apparently "simple" products.

In §5 (p. 510) Stanley describes the Aleksandrov-Fenchel inequalities arising in the theory of mixed volumes, which produce a log-concave sequence starting from any two convex subsets K and L of Euclidean space (Thm. 4 on p. 511), which interpolates between the volumes of K and L. He then describes some applications of this result to order preserving bijections from finite posets to chains and to sequences arising from commutative algebra. This section also contains the fourth conjecture posed in [**52**], namely Conj. 4 on p. 512. Let R be a standard graded algebra over a field K (so R is a commutative ring such that $K \subseteq R$ and there are K-subspaces $R_0, R_1, \ldots$ of R such that $R = R_0 \oplus R_1 \oplus \cdots$ as a K-vector space, $R_0 = K$, $R_i R_j \subseteq R_{i+j}$ for all i, j and R is finitely generated as a K-algebra by elements of R_1). It is then well known (see, e.g., [**3**, Thm. 11.1], or [**49**, Thm. 8]) that one has that

$$\sum_{m \geq 0} H(R, m) x^m = \frac{\sum_{i=0}^{s} h_i x^i}{(1-x)^d}$$

as formal power series in $\mathbb{Z}[[x]]$ where d is the Krull dimension of R, and $H(R, m) \stackrel{\text{def}}{=} dim_K(R_m)$ is the Hilbert function of R. The first part of Conj. 4 states that, if R is a Cohen-Macaulay integral domain, then $(h_0, \ldots, h_s)$ is log-concave (and hence unimodal). The second part states that if A is a regular local ring with residue field K and I is an ideal of A such that A/I is Cohen-Macaulay then the sequence of Betti numbers $(\beta_0, \ldots, \beta_{n-d})$ of A/I as an A-module (so $\beta_i \stackrel{\text{def}}{=} dim_K Tor_i^A(A/I, K)$) is log-concave, where $n = dim(A)$ and $d = dim\,(A/I)$.

No new combinatorial applications of the Aleksandrov-Fenchel inequalities seem to have appeared since the publication of [**52**], except that they are used in the remarkable proof of Conj. 3 on p. 508 by Huh, which I have already described. On the other hand, looking at the unimodality of sequences arising from commutative algebra is now a large area of research (see, e.g., [**36**], and the references cited there). Regarding the conjectures, both statements in the first part of Conj. 4, have been disproved by Niesi and Robbiano in [**39**] (see Examples 2.3 and 2.4). However, Stanley already mentioned in [**52**] that maybe one should take the stronger hypothesis that R is a Gorenstein (rather than Cohen-Macaulay) domain. In this case it is known (see [**46**, Thm. 4.4]) that the sequence $(h_0, \ldots, h_s)$ is symmetric.

This weaker conjecture, namely, that if R is a Gorenstein standard graded domain then $(h_0, \ldots, h_s)$ is log-concave, is still open even for the unimodality statement. The unimodality statement is known to be true if $s \leq 5$ ([**54**, Prop. 3.4]), if $h_1 \leq 3$ or if $h_1 = 4$ and $h_4 \leq 33$ ([**46**, Thm. 4.2], [**35**]), and for the semigroup algebras of certain integer convex polytopes ([**2**, Cor. 4.2]). The second part of Conj. 4 has been disproved by Boij in 1999 [**7**] even for the unimodality statement.

In the sixth section (p. 513) Stanley discusses the use of linear algebra for proving the unimodality of (usually symmetric) combinatorial sequences. The basic idea is that of finding, rather than an injection between the sets counted by two numbers in the sequence, an injective linear transformation between the vector spaces formally spanned by the two sets. Stanley calls this the "Linear Algebra Paradigm" (or "LAP") and also gives two slight variations of it. Stanley then illustrates the LAP by giving (modulo details) a linear algebra proof of the unimodality of the number of submultisets of a multiset M enumerated by size (Prop. 8, p. 514). The naturality and simplicity of the relevant linear transformations well illustrate the power of the method. Stanley then enriches the LAP by adding an action of a group G on each of the vector spaces involved, and observing that, if the relevant linear transformations are G-equivariant, then they restrict to the G-invariant subspaces, thus giving rise to another symmetric unimodal sequence (the sequence of dimensions of the G-invariant subspaces) (Prop. 9, p. 515). Also, in this case, it follows from Schur's Lemma that, for any irreducible representation ρ of G, the multiplicities of ρ in each one of the vector spaces involved also form a symmetric unimodal sequence (p. 517, not stated explicitly as a result). Stanley then remarks that if G acts by permutation representations on each one of the vector spaces then by Burnside's Lemma the dimension of each invariant subspace is just the number of orbits of G on the basis of the vector space, thus making more explicit and concrete the permutation representation special case of the "G-equivariant LAP". Stanley then gives some combinatorial applications of these techniques to the unimodality of the sequence counting non-isomorphic n-vertex graphs by number of edges (Prop. 10, p. 516), of the q-binomial coefficients (Thm. 11, p. 516), i.e., of the sequence counting the order ideals of the direct product of two chains by size, and of the principal specialization of the Schur functions $s_\lambda(1, q, q^2, \ldots, q^{\ell-1})$ (Thm. 13, p. 518). From that he deduces the unimodality of the sequence enumerating the order ideals of the direct product of three chains by size (Cor. 3, p. 519). This leads him naturally to the fifth conjecture stated in [**52**] namely Conj. 5 on p. 519, which states that the lattice of order ideals of a product of chains is rank unimodal.

Stanley then concludes the section by producing, for any subgroup G of a symmetric group, and any character of G, a transformation, on polynomials of one variable, that preserves the property of being symmetric and unimodal, a result that follows from a result that I have already described in the previous section (see [**50**, Cor. 3.3]).

Conjecture 5 is still open even in the case of four chains, or of chains of size 2 (i.e., for the lattice of order ideals of a Boolean algebra) and no new partial results on it seem to be known. As just described, the conjecture is known to be true for the product of at most three chains. Note that since an order ideal in a Boolean algebra is just an abstract simplicial complex, Conj. 5 in the case of chains of size 2 implies that for any $n \in \mathbb{P}$, the polynomial $\sum_\Delta x^{f(\Delta)}$ is unimodal where the sum is over all abstract simplicial complexes Δ whose vertex set is contained in $[n]$ and where $f(\Delta) := \sum_{i=0}^{n} f_{i-1}(\Delta)$ and $(f_{-1}(\Delta), \ldots, f_{n-1}(\Delta))$ is the f-vector of Δ (so $f_{i-1}(\Delta) \stackrel{\text{def}}{=} |\{A \in \Delta : \ |A| = i\}|$).

In the next section "Representations of $s\ell_2(\mathbb{C})$" (p. 520) Stanley surveys the use of the Lie algebra $s\ell_2(\mathbb{C})$ to prove unimodality of a sequence. The first result in this section (Thm. 15, p. 521) is Thm. 2.1 of [**50**] which I have already commented in the previous section. The second result (Thm. 16, p. 522) is devoted to an interesting application of Thm. 15. This is obtained by noting that any finite-dimensional complex semisimple Lie algebra $\mathcal{G}$ contains a principal three-dimensional subalgebra which is isomorphic to $s\ell_2(\mathbb{C})$. Hence any irreducible representation of $\mathcal{G}$ restricts to a representation of $s\ell_2(\mathbb{C})$ and hence gives rise to two symmetric unimodal sequences (one of which is always trivial, as it turns out). The computation of the other unimodal sequence (i.e., of the corresponding character of $s\ell_2(\mathbb{C})$) had been carried out by Dynkin in [**21**] and an exposition of this result was given by Stanley in [**47**]. The description can be given in terms of root systems. Let Φ be a root system of rank n, $\{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots for Φ and Φ^+ be the set of positive roots (we refer the reader to, e.g., [**27**, §§1.2-1.3] for the theory of root systems). One defines a polynomial $P_\Phi(x_1, \ldots, x_n)$ by

$$P_\Phi(x_1, \ldots, x_n) := \prod_{\beta \in \Phi^+} (1 - x_1^{c_1} \cdots x_n^{c_n})$$

where $\beta = \sum_{i=1}^{n} c_i \alpha_i$ (note that there is an obvious misprint in the definition of $P_\Phi(x_1, \ldots, x_n)$ on p. 522 of [**52**]). The result obtained (Thm. 16, p.522) is then that the quotient

$$\frac{P_\Phi(q^{m_1}, \ldots, q^{m_n})}{P_\Phi(q, \ldots, q)}$$

is a symmetric unimodal polynomial with nonnegative integer coefficients for any $m_1, \ldots, m_n \in \mathbb{P}$ and any root system Φ. Specializing this to the root systems of types A, C, and B then yields the symmetry and unimodality of the principal specialization of any Schur function (already proved by Stanley in the previous section in Thm. 13, p. 518), of a "simple" product already considered in §4, p. 509, and of a curious q-analogue of the Catalan numbers. The section concludes by briefly explaining how the ideas and techniques illustrated in this section can be generalized to Lie superalgebras. This is a very brief survey of [**50**] on which I have already commented in the previous section.

The eight section "The Hard Lefschetz Theorem" on p. 524, is about a particularly powerful application of the representation theory of $s\ell_2(\mathbb{C})$. More precisely, if X is an irreducible complex projective V-variety (that is, the singularities of X all look like $\mathbb{C}^n$ modulo the action of a finite subgroup of $GL_n(\mathbb{C})$), then there is an action ψ of $s\ell_2(\mathbb{C})$ on the singular cohomology ring of X over $\mathbb{C}$, $H^*(X)$, so that the character of ψ is, essentially, the Poincarè polynomial of $H^*(X)$. So one obtains (Thm. 18, p. 525) that under these hypotheses the two sequences $(\beta_0(X), \beta_2(X), \ldots, \beta_{2n}(X))$ and $(\beta_1(X), \beta_3(X), \ldots, \beta_{2n-1}(X))$ are symmetric and unimodal, where $\beta_i(X)$ is the i-th Betti number of X (so $\beta_i(X) := dim(H^i(X))$, where $H^*(X) = H^0(X) \oplus \cdots \oplus H^{2n}(X)$) and $n := dim_{\mathbb{C}}(X)$. The result can also be deduced from the Hard Lefschetz Theorem for X. By taking X to be the generalized flag manifold G/P where G is any complex connected semisimple Lie group and P is any parabolic subgroup, Stanley then deduces (Thm. 19, p. 525) the symmetry and unimodality of the polynomial enumerating any quotient W^J of any

Weyl group W by length (we refer the reader to, e.g., [**6**], for the theory of Weyl groups). It turns out that these polynomials have a very nice product formula, namely

$$\sum_{w \in W^J} q^{\ell(w)} = \frac{\prod_{i=1}^{m}(1+q+q^2+\ldots+q^{e_i})}{\prod_{i=1}^{h}(1+q+q^2+\ldots+q^{f_i})}$$

where $e_1, \ldots, e_m$ and $f_1, \ldots, f_h$ are the exponents of W and W_J, respectively. If W is the symmetric group S_n and $|J| = n-2$ then this result reduces to the symmetry and unimodality of the q-binomial coefficients, already shown by Stanley in Thm. 11, p. 516. Stanley also deduces Thm. 10 on p. 515 as a special case of Thm. 19. It is natural to wonder for which complex projective V-varieties X the two sequences $(\beta_0(X), \beta_2(X), \ldots, \beta_{2n}(X))$ and $(\beta_1(X), \beta_3(X), \ldots, \beta_{2n-1}(X))$ are log-concave, or strictly unimodal. This would be interesting even in the special case of Thm. 19. In this case, strict unimodality has been shown to hold for $W = S_n$, $|J| = n-2$ and $n \geq 16$ by Pak and Panova in [**40**]. Another interesting problem is that of enumerating any quotient W^J of any Weyl group W by descents (see, e.g., [**6**, p. 17]). In the case $J = \emptyset$ these polynomials have recently been shown to have always real roots (see, e.g., [**13**]).

Certainly the most famous application of Thm. 18 is the proof of the Generalized Lower Bound Conjecture (or GLBC) for simplicial polytopes obtained by Stanley in [**48**] and which he surveys next. Essentially, Thm. 18 can be applied because if P is a d-dimensional simplicial (i.e., all the faces of P are simplices) convex polytope (note that a simplicial polytope is in particular a simplicial complex, so has a well defined h-vector, see (3.1) below), then one can associate to P a complex projective (toric) V-variety $X(P)$ (a construction and result due to Demazure) such that the even Betti numbers of $X(P)$ coincide with the h-vector of P (a result of Danilov and Jurciewicz [**20**, Thm. 10.8], the odd Betti numbers all vanish). So one obtains (Thm. 20, p. 526) that the h-vector of a simplicial polytope is symmetric and unimodal. It is not clear at this point why this statement should be called a "generalized lower bound" but it is, as briefly explained by Stanley on p. 526. Stanley follows Thm. 20 by posing a conjecture that generalizes it, and then by a generalization of Thm. 18.

The conjecture is Conj. 6 on p. 527. Let Δ be an abstract simplicial complex of dimension $n-1$ and $(f_{-1}(\Delta), \ldots, f_{n-1}(\Delta))$ be its f-vector. Define the h-vector of Δ by $h(\Delta) \stackrel{\text{def}}{=} (h_0(\Delta), \ldots, h_n(\Delta))$ where

$$\sum_{i=0}^{n} h_i(\Delta) x^{n-i} \stackrel{\text{def}}{=} \sum_{i=0}^{n} f_{i-1}(\Delta)(x-1)^{n-i}. \tag{3.1}$$

Then Conj. 6 states that if the geometric realization of Δ is a sphere then the sequence $(h_0(\Delta), \ldots, h_n(\Delta))$ is unimodal. It is known that the sequence is symmetric (see, e.g., [**55**, Thm. 3.16.9] or [**53**, Prop. 4.4]). A generalization of Conj. 6 has been proposed by Nevo in [**38**, Conj. 1.5].

The generalization of Thm. 18 on p. 525 comes about by considering, for any complex irreducible projective variety X, the (middle perversity) intersection homology of X. This is a graded vector space $IH^*(X) = IH^0(X) \oplus \cdots \oplus IH^{2n}(X)$ that reduces to the ordinary singular cohomology of X if X is smooth (we refer the reader to, e.g., [**30**] for the theory of intersection homology). Then one has ([**4**, Thm. 5.4.10]) that the sequences $(\tilde{\beta}_0(X), \tilde{\beta}_2(X), \ldots, \tilde{\beta}_{2n}(X))$ and

$(\tilde{\beta}_1(X), \tilde{\beta}_3(X), \ldots, \tilde{\beta}_{2n-1}(X))$ are always symmetric and unimodal, where $\tilde{\beta}_i(X) := dim IH^i(X)$ (Thm. 21, p. 517). This leads Stanley to a generalization of Thm. 20. It is clear how to define the f-vector of any convex polytope P. However, simple examples show that the corresponding h-vector defined by (3.1) is in general not even symmetric. If P has vertices all of whose coordinates are rational (i.e., if P is rational) then one can still define the toric variety $X(P)$ (it is known that a simplicial polytope is always combinatorially equivalent to a rational polytope) except that, if P is not simplicial, $X(P)$ is in general not a V-variety. This leads Stanley to *define* the h-vector $h(P) = (h_0(P), \ldots, h_n(P))$ of a rational polytope P by $h_i(P) := \tilde{\beta}_{2i}(X(P))$ (this is nowadays usually called the *toric* (or *generalized*) *h-vector* of P, it is known that $\tilde{\beta}_{2i-1}(X(P)) = 0$ for all i). Thus one has the result (not stated as a theorem in [**52**], but in [**51**, Cor. 3.2]) that the sequence $(h_0(P), \ldots, h_n(P))$ is symmetric and unimodal for any rational polytope P.

Stanley concludes the section by considering the case of a complex projective V-variety X of dimension n on which a finite group G acts. In this case there is an action of G on $H^i(X)$ and one obtains two symmetric unimodal sequences $(\beta_0(\rho, X), \beta_2(\rho, X), \ldots, \beta_{2n}(\rho, X))$ and $(\beta_1(\rho, X), \beta_3(\rho, X), \ldots, \beta_{2n-1}(\rho, X))$ for each irreducible character ρ of G, where $\beta_i(\rho, X)$ is the multiplicity of ρ in the representation of G on $H^i(X)$. Stanley then concludes the section with some combinatorial applications of this technique to h-vectors of centrally symmetric simplicial polytopes (Thm. 22 on p. 528) and to certain polynomials that refine the symmetry and unimodality of the Eulerian polynomials (Props. 12 and 14) and of the polynomials enumerating the derangements of S_n by excedances (Prop.13).

The ideas and results in this section have had an impact on many areas of mathematics (including geometric combinatorics, commutative algebra, and algebraic geometry) which is hard to overestimate. Thm. 20 is without doubt one of the most spectacular results ever proved in the area of unimodality. While using techniques from other areas of mathematics to solve combinatorial problems is now considered standard, this was revolutionary at the time, and is standard now exactly because of Stanley's work and successes. The generalization of Thm. 18 stated on p. 528 also led to further developments. As stated in [**52**, p. 528], several mathematicians (including Khovanskii, Bernstein, and MacPherson) had computed $\tilde{\beta}_i(X(P))$ from the combinatorics of P. The result is a fairly complicated recursion relation that takes place on the subintervals of the face lattice of P. It turns out that this recursion can be formally carried out over any Eulerian poset P (the face lattice of a convex polytope is an Eulerian poset). Thus Stanley defines in [**51**] the toric h-vector of *any* Eulerian poset P in this way (see, [**55**, §3.16] for this definition). While this toric h-vector is in general not unimodal (or even nonnegative), it is always symmetric (see [**55**, Thm. 3.16.9] or [**51**, Thm. 2.4]) and Stanley conjectured in 1987 [**51**, Conj. 4.2] (so, one year after [**52**]) in particular that if P is a Gorenstein* Eulerian lattice then $h(P)$ is unimodal. This conjecture is still open. Also, the toric h-vector is defined in particular for any convex polytope and it is a natural question to wonder (as Stanley does in [**51**, p. 197]) whether the toric h-vector of any polytope (not necessarily equivalent to a rational one) is unimodal. This question has led to the development of combinatorial intersection homology (see, e.g., [**30**], and the references cited there) which in turn led to its positive solution in 2004 by Karu in [**29**]. More precisely, Karu shows that the toric h-vector of any polytope is (symmetric and) unimodal. Thm. 22 was generalized in 1995 by Adin in [**1**]. Many extensions and analogues of Props. 12, 13, and 14 have been obtained see, e.g., [**18**], [**57**], and the references cited there. Conjecture 6 is still open and no new results about it have been obtained. The most general positive result on it is probably the result by Stanley described, namely Thm. 20.

Stanley concludes the survey by noting that the f-vector of a convex polytope is not always unimodal, as had previously been conjectured, to caution the reader against easy optimism. Even though the results obtained in unimodality since the publication of [**52**] are much more on the positive side than on the negative one, negative results do continue to be found and published. As a sample we mention here that the sequence enumerating the independent sets of a bipartite graph by size is not in general unimodal [**5**] and that the square of a unimodal polynomial can have arbitrarily many modes [**44**].

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