A twisted duality for parabolic Kazhdan-Lusztig R-polynomials [1]

Francesco Brenti
Dipartimento di Matematica
Universitá di Roma “Tor Vergata”
Via della Ricerca Scientifica, 1
00133 Roma, Italy
brenti@mat.uniroma2.it

Abstract

We prove a duality result for the parabolic Kazhdan-Lusztig R-polynomials of a finite Coxeter system. This duality is similar to, but different from, the one obtained in [9]. As a consequence of our duality we obtain an identity between the parabolic Kazhdan-Lusztig and inverse Kazhdan-Lusztig polynomials of a finite Coxeter system. We also obtain applications to certain modules defined by Deodhar and derive a result that gives evidence in favor of Marietti's combinatorial invariance conjecture for parabolic Kazhdan-Lusztig polynomials.

1 Introduction

In [13] Kazhdan and Lusztig defined, for any Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which are now known as the Kazhdan-Lusztig polynomials of W. These polynomials play an important role in several areas, including representation theory and the geometry of Schubert varieties (see, e.g., [1], [12], and the references cited there). Kazhdan-Lusztig polynomials can be computed using another family of polynomials, usually called the R-polynomials of W. The R-polynomials also encode deep information about the structure constants of the Hecke algebra of W (see, e.g., [12, §7.4]), the geometry of intersections in flag varieties ([6]), the enumeration of reduced decompositions in W, and the

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Bruhat graph of W (see, e.g., [1, Theorems 5.3.4 and 5.3.7], or [6] and [11]). In addition to these, there are also purely combinatorial reasons to be interested in the Kazhdan-Lusztig and R-polynomials. In fact, in [15] and independently in [10], it was conjectured that if W_1 and W_2 are two Coxeter groups and if $u_1, v_1 \in W_1$, $u_2, v_2 \in W_2$ are such that $[u_1, v_1]$ and $[u_2, v_2]$ are isomorphic as posets (where $[u, v]$ denotes the interval determined by u and v in the Bruhat order of W, i.e., the set of all elements $z \in W$ such that $u \leq z \leq v$) then $P_{u_1,v_1}(q) = P_{u_2,v_2}(q)$ (where $P_{u,v}(q)$ denotes the Kazhdan-Lusztig polynomial of u and v). This conjecture is known as the *Combinatorial Invariance Conjecture* (see, e.g., [1], [16], and the references cited there) and is equivalent to the same statement with $R_{u_1,v_1}(q)$ and $R_{u_2,v_2}(q)$ in place of $P_{u_1,v_1}(q)$ and $P_{u_2,v_2}(q)$ (where $R_{u,v}(q)$ denotes the R-polynomial of u and v).

In 1987 Deodhar [7] introduced parabolic analogues $P^{J,x}_{u,v}(q)$ and $R^{J,x}_{u,v}(q)$ of these polynomials. These parabolic polynomials are indexed by pairs of elements u, v in a parabolic quotient W^J of W, determined by some subset J of the canonical generating set S of W, and by a parameter $x \in \{-1, q\}$, and reduce to the ordinary ones if $J = \emptyset$. The natural analogue of the Combinatorial Invariance Conjecture (where $[u, v]$ is replaced by $[u, v] \cap W^J$) has recently been shown to be false (see [5]), but a parabolic generalization of it has recently been proposed by Marietti in [16]. More precisely, Marietti conjectures that if W_1 and W_2 are two Coxeter groups, J_1 and J_2 are two subsets of their generating sets S_1 and S_2, and if $u_1, v_1 \in (W_1)^{J_1}$, $u_2, v_2 \in (W_2)^{J_2}$ are such that there is a poset isomorphism $\phi : [u_1, v_1] \mapsto [u_2, v_2]$ such that $\phi([u_1, v_1] \cap (W_1)^{J_1}) = [u_2, v_2] \cap (W_2)^{J_2}$ then $P^{J_1,x}_{u_1,v_1}(q) = P^{J_2,x}_{u_2,v_2}(q)$ (equivalently, $R^{J_1,x}_{u_1,v_1}(q) = R^{J_2,x}_{u_2,v_2}(q)$). Clearly, this conjecture reduces to the Combinatorial Invariance Conjecture if $J_1 = J_2 = \emptyset$.

The purpose of this note is to prove a new duality for the parabolic R-polynomials of a finite Coxeter group. This duality is similar to, but different from, the one obtained by Douglass in [9]. More precisely, if (W, S) is a finite Coxeter system and we denote by $w_0(J)$ the element of maximal length in W_J, where $J \subseteq S$, then we prove that $R^{J,x}_{u,v} = R^{w_0 J w_0, x}_{w_0 w_0(J) v, w_0 w_0(J) u}$ for all $u, v \in W^J$ and all $x \in \{-1, q\}$, where $w_0 = w_0(S)$. As an application of this we obtain a corresponding duality between the parabolic Kazhdan-Lusztig and inverse parabolic Kazhdan-Lusztig polynomials $Q^{J,x}_{u,v}$ (see §2 for definitions), namely that $P^{J,x}_{u,v} = Q^{w_0 J w_0, q-1-x}_{w_0 w_0(J) v, w_0 w_0(J) u}$ for all $u, v \in W^J$ and all $x \in \{-1, q\}$. As an application of this identity, and the one by Douglass, we obtain that $P^{J,x}_{u,v} = P^{w_0 J w_0, x}_{w_0 u w_0, w_0 v w_0}$ for all u, v, x as above, which gives evidence in favor of Marietti's conjecture on the combinatorial invariance of parabolic Kazhdan-Lusztig polynomials. We also obtain applications of our duality to certain modules defined by Deodhar in [7].

The organization of this note is as follows. In the next section we recall some notation, definitions, and results, that are used in the sequel. In section 3 we prove our duality result. As a consequence of it we obtain an identity between the parabolic Kazhdan-Lusztig polynomials and the inverse parabolic Kazhdan-Lusztig polynomials of any finite Coxeter group. In section 4 we derive applications of our main result to certain modules defined by Deodhar and we obtain a result that gives evidence in favor of Marietti's conjecture on the combinatorial invariance of parabolic Kazhdan-Lusztig polynomials ([16, Conj. 1.3]).

2 Preliminaries

In this section we recall some notation, definitions, and results that are used in the sequel.

We let $\mathbb{P} := \{1, 2, \ldots\}$ be the set of positive integers and $\mathbb{N} := \mathbb{P} \cup \{0\}$. For all $m, n \in \mathbb{Z}$, $m \leq n$ we let $[m,n] := \{m, m+1, \ldots, n\}$ and $[n] := [1, n]$. Given a set I we denote by $|I|$ its cardinality.

We follow [17, Chap.3] for notation and terminology concerning posets. In particular, if P is a locally finite poset then we denote by $I(P;\mathbb{Z}[q])$ the incidence algebra of P with coefficients in $\mathbb{Z}[q]$, by δ its identity, so δ is the Kronecker delta, and we let $Int(P) := \{(u,v) \in P^2 : u \leq v\}$. The following result is the analogue of a well known result (see, e.g., [17, Prop. 3.6.2]), and its verification is omitted.

Proposition 2.1 *Let P be a locally finite poset, and $f \in I(P;\, \mathbb{Z}[q])$. Then f has a two-sided inverse if and only if $f_{u,u} \in \{1, -1\}$ for all $u \in P$.*

We then denote by f^{-1} the two-sided inverse of f. We refer the reader to [17, §3.6] for further information about incidence algebras.

Suppose now that P is graded, with $\hat{0}$, and let ρ be its rank function. We let $\tilde{I}(P;\mathbb{Z}[q]) := \{f \in I(P;\mathbb{Z}[q]) : \ deg(f_{u,v}) \leq \rho(v) - \rho(u)$ for all $(u,v) \in Int(P)\}$. It is easy to see that $\tilde{I}(P;\, \mathbb{Z}[q])$ is a subalgebra of $I(P;\, \mathbb{Z}[q])$. Given $f \in \tilde{I}(P;\, \mathbb{Z}[q])$ we let

$$\overline{f}_{u,v}(q) := q^{\rho(v)-\rho(u)}\ f_{u,v}(q^{-1})$$

for all $(u,v) \in Int(P)$. It is clear that $\overline{f} \in \tilde{I}(P;\mathbb{Z}[q])$ and that, if f is invertible, $(\overline{f})^{-1} = \overline{(f^{-1})}$. If $f \in I(P;\ \mathbb{Z}[q])$ then we let $\tilde{f}_{u,v}(q) := (-1)^{\rho(v)-\rho(u)}\ f_{u,v}(q)$ for all $(u,v) \in Int(P)$. Note that if $f,g \in I(P;\mathbb{Z}[q])$ then $\widetilde{(f\,g)} = \tilde{f}\,\tilde{g}$ and so $(\tilde{f})^{-1} = \widetilde{(f^{-1})}$ if f is invertible, while if $f \in \tilde{I}(P;\mathbb{Z}[q])$ then $\tilde{\overline{\overline{f}}} = \tilde{\overline{f}}$.

We follow [1] and [12] for general Coxeter groups notation and terminology (see also [2]). In particular, given a Coxeter system (W,S) and $u \in W$ we denote by $\ell(u)$ the length of u in W, with respect to S, and we let $\ell(u,v) := \ell(v) - \ell(u)$ for all $u,v \in W$. We let

$$D_R(u) := \{s \in S |\, \ell(us) < \ell(u)\}$$

be the set of (right) descents of u and we denote by e the identity of W. Given $J \subseteq S$ we let W_J be the parabolic subgroup generated by J and

$$W^J := \{u \in W |\, \ell(su) > \ell(u) \text{ for all } s \in J\}. \tag{1}$$

Note that $W^{\emptyset} = W$. It is well known (see, e.g., [1, Proposition 2.4.4]) that if $w \in W$ then there exist unique elements $w^J \in W^J$ and $w_J \in W_J$ such that $w = w_J\, w^J$ and that, furthermore $\ell(w) = \ell(w_J) + \ell(w^J)$. If W_J is finite, then we denote by $w_0(J)$ its longest element and we let, for brevity, $w_0 := w_0(S)$. We always assume that W^J is partially ordered by *Bruhat order* (see, e.g., [1, §2.1]). Given $u,v \in W^J$, $u \leq v$, we let

$$[u,v]^J := \{w \in W^J | u \leq w \leq v\},$$

and $[u,v] := [u,v]^{\emptyset}$.

The next two results are due to Deodhar, and we refer the reader to [7, §§2-3] for their proofs.

Theorem 2.2 *Let (W,S) be a Coxeter system, and $J \subseteq S$. Then, for each $x \in \{-1,q\}$, there is a unique family of polynomials $\{R^{J,x}_{u,v}\}_{u,v \in W^J} \subseteq \mathbb{Z}[q]$ such that, for all $u,v \in W^J$:*

i) $R^{J,x}_{u,v}(q) = 0$ *if $u \not\leq v$;*

ii) $R^{J,x}_{u,u}(q) = 1$*;*

iii) *if $u < v$ and $s \in D_R(v)$ then*

$$R^{J,x}_{u,v}(q) = \begin{cases} R^{J,x}_{us,vs}(q), & \text{if } us < u, \\ (q-1)R^{J,x}_{u,vs}(q) + qR^{J,x}_{us,vs}(q), & \text{if } u < us \in W^J, \\ (q-1-x)R^{J,x}_{u,vs}(q), & \text{if } u < us \notin W^J. \end{cases}$$

Theorem 2.3 *Let (W,S) be a Coxeter system, and $J \subseteq S$. Then, for each $x \in \{-1,q\}$, there is a unique family of polynomials $\{P^{J,x}_{u,v}(q)\}_{u,v\in W^J} \subseteq \mathbb{Z}[q]$, such that, for all $u,v \in W^J$:*

i) $P^{J,x}_{u,v}(q) = 0$ *if* $u \not\leq v$*;*

ii) $P^{J,x}_{u,u}(q) = 1$*;*

iii) *deg*$(P^{J,x}_{u,v}(q)) < \frac{1}{2}\,\ell(u,v)$ *if* $u < v$*;*

iv)

$$q^{\ell(u,v)}\,P^{J,x}_{u,v}(q^{-1}) = \sum_{z\in[u,v]^J} R^{J,x}_{u,z}(q)\,P^{J,x}_{z,v}(q)$$

if $u \leq v$*.*

The polynomials $R^{J,x}_{u,v}(q)$ and $P^{J,x}_{u,v}(q)$, whose existence is guaranteed by the two previous theorems, are called the *parabolic R-polynomials* and *parabolic Kazhdan-Lusztig polynomials* (respectively) of W^J of type x. It follows immediately from Theorems 2.2 and 2.3 and from well known facts (see, e.g., [12, §7.5] and [12, §§7.9-11]) that $R^{\emptyset,-1}_{u,v}$ $(= R^{\emptyset,q}_{u,v})$ and $P^{\emptyset,-1}_{u,v}$ $(= P^{\emptyset,q}_{u,v})$ are the (ordinary) *R-polynomials* and *Kazhdan-Lusztig polynomials* of W which we denote simply by $R_{u,v}$ and $P_{u,v}$, as customary.

The parabolic and ordinary Kazhdan-Lusztig and R-polynomials have the following properties. Let (W,S) be a Coxeter system, $J \subseteq S$, and $x \in \{-1,q\}$. Then

1.
$$q^{\ell(u,v)} R^{J,q}_{u,\,v}(q^{-1}) = (-1)^{\ell(u,v)} R^{J,-1}_{u,\,v}(q) \tag{2}$$
for all $u,v \in W^J$.

2.
$$R_{u,\,v} = R_{u^{-1},\,v^{-1}} \tag{3}$$
for all $u,v \in W$.

3. If (W,S) is a finite Coxeter system then
$$R_{u,\,v} = R_{w_0 v, w_0 u} \tag{4}$$
for all $u,v \in W$.

4.
$$R^{J,x}_{u,v} = \sum_{w \in W_J} (-x)^{\ell(w)} R_{wu,v}, \tag{5}$$
$$P^{J,q}_{u,v} = \sum_{w \in W_J} (-1)^{\ell(w)} P_{wu,v},$$
and, if W_J is finite,
$$P^{J,-1}_{u,v} = P_{w_0(J)u, w_0(J)v},$$
for all $u, v \in W^J$.

Proofs of these properties can be found in [8, Cor. 2.2], [1, Chap.5, Ex. 10] (see also [12, Prop. 7.6]) and [7] (see Propositions 2.12 and 3.4, and Remark 3.8).

Given $J \subseteq S$ and $x \in \{-1, q\}$ we define, following [4], a family of polynomials $\{Q^{J,x}_{u,v}\}_{u,v \in W^J} \subseteq \mathbb{Z}[q]$ by
$$\sum_{z \in [u,v]^J} (-1)^{\ell(u,z)} P^{J,x}_{u,z}\, Q^{J,x}_{z,v} = \delta_{u,v} \tag{6}$$
for all $u, v \in W^J$, $u \leq v$. We also let $Q^{J,x}_{u,v} := 0$ if $u \not\leq v$. Following [4] we call the polynomials $\{Q^{J,x}_{u,v}\}_{u,v \in W^J}$ the *inverse parabolic Kazhdan-Lusztig polynomials* of W^J of type x. It follows immediately from this definition that $Q^{\emptyset,-1}_{u,v}$ ($= Q^{\emptyset,q}_{u,v}$) are the *inverse Kazhdan-Lusztig polynomials* of W as defined in [14, p. 190].

The next result is probably known. However, for lack of an adequate reference, and for completeness, we prove it here.

Proposition 2.4 *Let (W, S) be a Coxeter system, $J \subseteq S$, and $x \in \{-1, q\}$. Then, for all $u, v \in W^J$:*

i) $Q^{J,x}_{u,u}(q) = 1$*;*

ii) *deg*$(Q^{J,x}_{u,v}(q)) < \frac{1}{2}\,\ell(u,v)$ *if $u < v$;*

iii)
$$q^{\ell(u,v)}\, Q^{J,x}_{u,v}(q^{-1}) = \sum_{z \in [u,v]^J} Q^{J,x}_{u,z}(q)\, R^{J,\,q-1-x}_{z,v}(q)$$
if $u \leq v$.

Proof. The first two statements follow easily from the definition (6). For the third one let $I := I(W^J; \mathbb{Z}[q])$ be the incidence algebra of W^J with coefficients in

$\mathbb{Z}[q]$. The families of polynomials $R^{J,x}_{u,v}$, $P^{J,x}_{u,v}$, and $Q^{J,x}_{u,v}$ naturally define elements in I that will be denoted by $\mathcal{R}^{J,x}$, $\mathcal{P}^{J,x}$, and $\mathcal{Q}^{J,x}$, respectively. Thus, $\mathcal{R}^{J,x}_{u,v} := R^{J,x}_{u,v}$ for $(u,v) \in Int(W^J)$, and similarly for $\mathcal{P}^{J,x}_{u,v}$ and $\mathcal{Q}^{J,x}_{u,v}$. Then by our definition (6), (2), and Theorem 2.3, we have that

$$\begin{aligned} \widetilde{\mathcal{P}}^{J,x}\, \mathcal{Q}^{J,x} &= \delta, && (7)\\ \overline{\mathcal{R}^{J,x}} &= \widetilde{\mathcal{R}}^{J,\,q-1-x}, && (8) \end{aligned}$$

and

$$\mathcal{R}^{J,x}\, \mathcal{P}^{J,x} = \overline{\mathcal{P}^{J,x}}.$$

Therefore $(\mathcal{P}^{J,x})^{-1} = (\widetilde{\mathcal{Q}^{J,x}})$. Hence

$$\mathcal{R}^{J,x} = \overline{\mathcal{P}^{J,x}}\, (\mathcal{P}^{J,x})^{-1} = \overline{\mathcal{P}^{J,x}}\, (\widetilde{\mathcal{Q}^{J,x}})$$

so

$$\widetilde{\mathcal{R}}^{J,q-1-x} = \overline{\mathcal{R}^{J,x}} = \mathcal{P}^{J,x}\, \overline{(\widetilde{\mathcal{Q}}^{J,x})}.$$

Therefore

$$\widetilde{\mathcal{Q}}^{J,x}\, \widetilde{\mathcal{R}}^{J,q-1-x} = (\widetilde{\mathcal{Q}}^{J,x})\, \mathcal{P}^{J,x} \overline{(\widetilde{\mathcal{Q}}^{J,x})} = \overline{(\widetilde{\mathcal{Q}}^{J,x})}$$

and iii) follows. $\square$

3 Main result

In this section we prove the main result of this note. Namely we prove a duality formula for the parabolic Kazhdan-Lusztig R-polynomials of a finite Coxeter group. As a consequence of it we obtain a relation between the parabolic and inverse parabolic Kazhdan-Lusztig polynomials of any finite Coxeter group.

Note first that, by well known results (see, e.g., [1, Proposition 2.3.2], $w_0 J w_0 \subseteq S$ if $J \subseteq S$. The following result is probably known, so we omit its proof.

Lemma 3.1 *Let (W,S) be a finite Coxeter system and $J \subseteq S$. Then $W_{w_0 J w_0} = w_0\, W_J\, w_0$, $w_0(w_0\, J w_0) = w_0\, w_0(J) w_0$, and $W^{w_0 J w_0} = w_0 w_0(J) W^J$.*

The next result is the crucial technical fact needed in the proof of our duality result.

Lemma 3.2 *Let (W,S) be a finite Coxeter system and $J \subseteq S$. Then*

$$R_{w_0(J)u,\, w\, w_0(J)v} = R_{w_0(J)w^{-1}w_0(J)u,\, v}$$

for all $u, v \in W^J$ and $w \in W_J$.

Proof. Note that $w_0(J)u = ww_0(J)w_0(J)w^{-1}w_0(J)u$, and that $\ell(ww_0(J)) = \ell(w_0(J)) - \ell(w)$ and $\ell(w_0(J)w^{-1}w_0(J)) = \ell(w)$. Let $s_{i_1} \cdots s_{i_h}$ $(h = \ell(w))$ and $s_{j_1} \cdots s_{j_{m-h}}$ $(m = \ell(w_0(J)))$ be reduced expressions for $w_0(J)\, w^{-1}w_0(J)$ and $w\, w_0(J)$, respectively. Then $s_{j_1} \cdots s_{j_{m-h}}\, s_{i_1} \cdots s_{i_h}$ is a reduced expression for $w_0(J)$. Hence, since $u, v \in W^J$ and $s_{j_1}, \ldots, s_{j_{m-h}}, s_{i_1}, \ldots, s_{i_h} \in J$, $\ell(s_{j_i} \cdots s_{j_{m-h}}\, s_{i_1} \cdots s_{i_h} u) = \ell(u) + h + (m-h) - i + 1$ and $\ell(s_{j_i} \cdots s_{j_{m-h}} v) = \ell(v) + m - h - i + 1$ for all $i = 1, \ldots, m-h+1$. Hence, by Theorem 2.2 and (3)

$$\begin{aligned} R_{w_0(J)u,ww_0(J)v} &= R_{s_{j_1} \cdots s_{j_{m-h}}\, s_{i_1} \cdots s_{i_h} u, s_{j_1} \cdots s_{j_{m-h}} v} \\ &= R_{s_{i_1} \cdots s_{i_h} u, v}, \end{aligned}$$

as desired. $\square$

We can now prove our first main result.

Theorem 3.3 *Let (W,S) be a finite Coxeter system and $J \subseteq S$. Then*

$$R^{J,x}_{u,v} = R^{w_0 J w_0,\, x}_{w_0 w_0(J) v, w_0 w_0(J) u}$$

for all $u, v \in W^J$ and all $x \in \{-1, q\}$.

Proof. We have from (5), (4), and Lemmas 3.1 and 3.2, that

$$\begin{aligned} R^{w_0 J w_0, x}_{w_0 w_0(J)v,\, w_0 w_0(J)u} &= \sum_{w \in w_0 W_J w_0} (-x)^{\ell(w)}\, R_{w w_0 w_0(J) v,\, w_0 w_0(J) u} \\ &= \sum_{w \in W_J} (-x)^{\ell(w)}\, R_{w_0 w w_0(J) v,\, w_0 w_0(J) u} \\ &= \sum_{w \in W_J} (-x)^{\ell(w)}\, R_{w_0(J) u,\, w w_0(J) v} \\ &= \sum_{w \in W_J} (-x)^{\ell(w)}\, R_{w_0(J) w^{-1} w_0(J) u, v} \\ &= \sum_{w \in W_J} (-x)^{\ell(w_0(J) w^{-1} w_0(J))}\, R_{w_0(J) w^{-1} w_0(J) u, v} \\ &= \sum_{z \in W_J} (-x)^{\ell(z)}\, R_{zu,v} \\ &= R^{J,x}_{u,v}, \end{aligned}$$

again by (5), as desired. $\square$

The next result can be deduced from Theorem 3.3, (5), and some well known properties of the R-polynomials, and we omit its proof.

Lemma 3.4 *Let (W,S) be a finite Coxeter system, $J \subseteq S$ and $u,v \in W^J$, $u \leq v$. Then the map $x \mapsto w_0\, w_0(J)x$ is a poset anti-isomorphism between $[u,v]^J$ and $[w_0\, w_0(J)v, w_0\, w_0(J)u]^{w_0\, J w_0}$.*

We can now prove our second main result.

Corollary 3.5 *Let (W,S) be a finite Coxeter system, $J \subseteq S$, and $x \in \{-1, q\}$. Then*

$$P^{J,\,x}_{u,\,v} = Q^{w_0\, J w_0,\, q-1-x}_{w_0\, w_0(J)v,\, w_0\, w_0(J)u}$$

for all $u,v \in W^J$, $u \leq v$.

Proof. We proceed by induction on $\ell(u,v)$, the result being clear if $u=v$. So assume that $\ell(u,v) \geq 1$. Then we have, by Theorems 2.3 and 3.3, our induction hypothesis, Lemma 3.4 and Proposition 2.4, that

$$\begin{aligned}
q^{\ell(u,v)}\, P^{J,x}_{u,v}(q^{-1}) - P^{J,x}_{u,v}(q) &= \sum_{z \in (u,v]^J} R^{J,x}_{u,z}(q)\, P^{J,x}_{z,v}(q) \\
&= \sum_{z \in (u,v]^J} R^{\tilde{J},x}_{w_0\, w_0(J)z,\, w_0\, w_0(J)u}(q)\, Q^{\tilde{J},q-1-x}_{w_0\, w_0(J)v,\, w_0\, w_0(J)z}(q) \\
&= \sum_{w \in [w_0\, w_0(J)v,\, w_0\, w_0(J)u)^{\tilde{J}}} R^{\tilde{J},x}_{w, w_0\, w_0(J)u}(q)\, Q^{\tilde{J},q-1-x}_{w_0\, w_0(J)v, w}(q) \\
&= q^{\ell(w_0\, w_0(J)v,\, w_0\, w_0(J)u)}\, Q^{\tilde{J},q-1-x}_{w_0\, w_0(J)v,\, w_0\, w_0(J)u}(q^{-1}) \\
&\quad - Q^{\tilde{J},q-1-x}_{w_0\, w_0(J)v, w_0\, w_0(J)u}(q),
\end{aligned}$$

where $\tilde{J} := w_0\, J w_0$, and the result follows from Theorem 2.3 and Proposition 2.4 since $\ell(w_0\, w_0(J)v, w_0\, w_0(J)u) = \ell(u,v)$. $\square$

4 Applications

In this section we derive some applications of our main result to certain modules defined by Deodhar and to Marietti's combinatorial invariance conjecture for parabolic Kazhdan-Lusztig's polynomials.

Let (W,S) be a Coxeter system and $J \subseteq S$. Following Deodhar [7] we let M^J be the free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-module having $\{m_u^J\}_{u \in W^J}$ as a basis. Let $x \in \{-1, q\}$. We denote by $^-$ the involution on M^J defined in [7, §2] so

$$\overline{m_v^J} = q^{-\ell(v)} \sum_{u \in W^J} \varepsilon_u \, \varepsilon_v \, R_{u,v}^{J,x} \, m_u^J$$

for all $v \in W^J$.

For $u \in W^J$ let

$$\widetilde{m_u^J} := q^{\ell(u)} \sum_{v \in W^J} \varepsilon_v \, \varepsilon_u \, R_{u,v}^{J,x} \, m_v^J .$$

Extend $\sim$ to all of M^J by linear extension and by letting $\widetilde{q^{1/2}} := q^{-1/2}$. Note that $\sim$ is an involution of M^J, as it is easy to check.

Suppose now that (W,S) is finite. Let $\psi_J : \ M^J \to M^{w_0 J w_0}$ be defined by

$$\psi_J(m_u^J) := m_{w_0 w_0(J)\, u}^{w_0 J w_0} \tag{9}$$

for all $u \in W^J$, and $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-linearity.

Corollary 4.1 *Let (W,S) be a finite Coxeter system, $x \in \{-1, q\}$, and $J \subseteq S$. Then*

$$\widetilde{\psi_J(m_v^J)} = q^{\ell(w_0 w_0(J))} \, \psi_J(\overline{m_v^J})$$

for all $v \in W^J$.

Proof. We have that

$$\widetilde{\psi_J(m_v^J)} = \widetilde{m_{w_0 w_0(J)\, v}^{w_0 J w_0}} = q^{\ell(w_0 w_0(J) v)} \sum_{y \in W^{w_0 J w_0}} \varepsilon_{w_0 w_0(J) v} \, \varepsilon_y \, R_{w_0 w_0(J) v, y}^{w_0 J w_0} \, m_y^{w_0 J w_0} .$$

On the other hand, by (9)

$$\begin{aligned} q^{\ell(w_0 w_0(J))} \, \psi_J(\overline{m_v^J}) &= q^{\ell(w_0 \, w_0(J))} \psi_J \left(q^{-\ell(v)} \sum_{u \in W^J} \varepsilon_u \varepsilon_v R_{u,v}^J \, m_u^J \right) \\ &= q^{\ell(w_0 \, w_0(J)) - \ell(v)} \sum_{u \in W^J} \varepsilon_u \varepsilon_v R_{u,v}^J \, m_{w_0 w_0(J) u}^{w_0 J w_0} \\ &= q^{\ell(w_0 w_0(J) v)} \sum_{y \in W^{w_0 J w_0}} \varepsilon_{w_0(J) w_0 y} \varepsilon_v R_{w_0(J) w_0 y, v}^J \, m_y^{w_0 J w_0} . \end{aligned}$$

so the result follows from Theorem 3.3. $\square$

Another consequence of Theorem 3.3 is the following curious identity.

Corollary 4.2 *Let (W,S) be a finite Coxeter system, $J \subseteq S$, $u, v \in W^J$ and $s \in S$ be such that $v < vs \notin W^J$, $u < us \in W^J$. Then*

$$R^{J,x}_{u,v}(q) = (q-1-x)\, R^{J,x}_{us,v}(q).$$

Proof. Since $u < us \in W^J$ we have that $w_0 w_0(J) u > w_0 w_0(J) us$. Also, by Lemma 3.1, $w_0 w_0(J) vs \notin W^{w_0 J w_0}$ and so $w_0 w_0(J) v < w_0 w_0(J) vs$. Therefore, by Theorems 2.2 and 3.3 we have that

$$\begin{aligned} R^{J,x}_{u,v} &= R^{w_0 J w_0, x}_{w_0 w_0(J) v, w_0 w_0(J) u} = (q-1-x) R^{w_0 J w_0, x}_{w_0 w_0(J) v, w_0 w_0(J) us} \\ &= (q-1-x) R^{J,x}_{us,v}, \end{aligned}$$

as claimed. $\Box$

As an application of Corollary 3.5, and a result of Douglass ([9, Theorem 4.6]), we obtain the following consequence.

Corollary 4.3 *Let (W,S) be a finite Coxeter system, $J \subseteq S$, and $x \in \{-1, q\}$. Then*

$$P^{J,\,x}_{u,\,v} = P^{w_0\,Jw_0,\,x}_{w_0 u\, w_0,\, w_0 v\, w_0}$$

for all $u, v \in W^J$, $u \leq v$.

Proof. From our definition (6) and Theorem 4.6 of [9] we have that

$$Q^{J,\,x}_{u,\,v} = P^{J,\,q-1-x}_{w_0(J) v\, w_0,\, w_0(J) u\, w_0}.$$

Since $w_0(J) W^J w_0 = W^J$ this may be rewritten as

$$P^{J,\,x}_{u,\,v} = Q^{J,\,q-1-x}_{w_0(J) v\, w_0,\, w_0(J) u\, w_0}.$$

But by Corollary 3.5 (applied to $w_0\, J w_0 \subseteq S$ and to $w_0 u w_0$, $w_0 v w_0 \in W^{w_0\, J w_0}$) we have that

$$P^{\tilde{J},\,x}_{w_0 u w_0,\, w_0 v w_0} = Q^{J,\,q-1-x}_{w_0\, w_0(\tilde{J}) w_0 v w_0,\, w_0\, w_0(\tilde{J}) w_0 u w_0},$$

where $\tilde{J} := w_0\, J w_0$, and the result follows from Lemma 3.1. $\Box$

For $J = \emptyset$ the previous corollary reduces to Corollary 4.3 of [3]. Corollary 4.3 gives evidence in favor of Marietti's combinatorial invariance conjecture for parabolic Kazhdan-Lusztig polynomials [16, Conj. 1.3]. As noted by the referee, alternate proofs of Corollary 4.3 are possible.

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