From discrete to continuous variational problems: an introduction

Andrea Braides
Dipartimento di Matematica
Università di Roma 'Tor Vergata'
via della Ricerca Scientifica, 00133 Rome, Italy

Introduction

In this notes we treat the problem of the description of variational limits of discrete lattice systems. Consider a fixed reference open set $\Omega \subset \mathbb{R}^N$, and given $\varepsilon > 0$ the reference lattice $\varepsilon\mathbb{Z}^N$. We consider energies defined on the discrete functions $i \mapsto u_i$ for $\{i \in \mathbb{Z}^N : i\varepsilon \in \Omega\}$, of the general form

$$E_\varepsilon(\{u_i\}) = \sum_k \sum_i \psi_\varepsilon^k(u_{i+k} - u_i),$$

where the sum is performed for $k \in \mathbb{Z}^N$ and on indices $i \in \mathbb{Z}^N$ such that $i\varepsilon \in \Omega$ and $(i+k)\varepsilon \in \Omega$. If we picture the lattice $\varepsilon\mathbb{Z}^n \cap \Omega$ as the reference configuration of a set of material points interacting through some forces, and u_i represents the displacement of the i-th point, then ψ_ε^k can be thought as the energy density of the interaction of points with distance $k\varepsilon$ in the reference lattice. Note that the only assumption we make is that ψ_ε^k depends on $\{u_i\}$ through the differences $u_{i+k} - u_i$. It is usually more convenient to make change in the notation and set

$$\varphi_\varepsilon^j(z) = \varepsilon^{-N}\psi_\varepsilon^j(j\varepsilon z);$$

In such a way we write

$$E_\varepsilon(\{u_i\}) = \sum_k \sum_i \varepsilon^N \varphi_\varepsilon^k\Big(\frac{u_{i+k} - u_i}{k\varepsilon}\Big),$$

to highlight the dependence of E_ε on 'discrete difference quotients'.

Our goal is to describe the behaviour of problems of the form

$$\min\Big\{E_n(\{u_i\}) - \sum_i \varepsilon^N u_i f_i : \ \{u_i\} = \phi \text{ on } \partial\Omega\Big\}$$

(and similar), where the boundary conditions are given appropriately, and to show that for a quite general class of energies these problems have a limit continuous counterpart. Here $\{f_i\}$ represents the external forces. More general problems can be also examined. To make this asymptotic analysis precise, we use the notation and methods of De Giorgi's Γ-convergence (see [3], [8]). We will show that, under some growth conditions of superlinear growth and upon suitably identifying the discrete functions $\{u_i\}$ with some interpolations, the free energies E_ε '*Γ-converge*' to a limit energy F. In the simplest case, the limit functional F is defined on a Sobolev space and takes the form

$$F(u) = \int_\Omega \psi(\nabla u)\,dx.$$

This usually follows from the 'superlinear growth' of some interactions. As a consequence we obtain that minimizers of the problem above are 'very close' to minimizers of a classical problem of the Calculus of Variations

$$\min\Big\{\int_0^L \big(\psi(\nabla u) - fu\big)\,dx :\ u = \phi \text{ on } \partial\Omega\Big\}.$$

If the growth conditions of superlinear growth fail the identification is more complex and the limit problem involves energies defined on functions of bounded variation with a bulk and surface part (i.e., in the terminology of De Giorgi the limit problem is a 'free-discontinuity problem'). In the notation of the SBV spaces of Ambrosio and De Giorgi (see [1]) the limit functional will be in this case of the form

$$F(u) = \int_\Omega \psi(\nabla u)\,dx + \int_{S(u)} g(u^+ - u^-, \nu_u)\,d\mathcal{H}^{N-1},$$

and the limit problem must be changed accordingly.

The energy densities ψ and g can be explicitly identified by a series of operations on the functions ψ_ε^k, which follow some general 'principles'. In order to describe those principles, we start with the one-dimensional case and the limit is defined on a Sobolev space.

1. *Nearest-neighbour superlinear interaction: a convexification principle.* The case when only nearest-neighbour interactions are taken into account (i.e., $\psi_\varepsilon^k = 0$ if $k \neq 1$) and at least one of the interactions is uniformly of superlinear type the limit energy density is given by the limit of the convex envelopes of the functions $\varphi_\varepsilon^1(z)$, which exists up to subsequences.

If the limit is defined on a subspace of BV then we have the appearance of the interfacial energy.

2. *Nearest-neighbour (sub)linear interactions: a separation of scales principle.* Still in the case when only nearest-neighbour interactions are taken into account, if all interactions are uniformly of at most linear type then an interfacial energy

appears, whose energy density g can be computed by examining the behaviour of the (subadditive envelopes of the) scaled energy densities $\varphi^1_\varepsilon(\varepsilon z)/\varepsilon$.

When more interactions are taken into account we have to describe their mutual interference.

3. *Long-range interactions: a clustering principle.* The description of the limit energy gets more complex when not only nearest-neighbour interactions come into play. In the case when interactions up to a fixed order K are taken into account (i.e., $\psi^j_\varepsilon = 0$ if $|j| > K$), the main idea is to show that (upon some controllable errors) we can find a lattice spacing η (possibly much larger than ε) such that E_ε is 'equivalent' (as Γ-convergence is concerned) to a nearest-neighbour interaction energy on a lattice of step size η, of the form

$$\overline{E}_\eta(\{u_j\}) = \sum_j \eta\overline{\varphi}_\eta\Big(\frac{u_{j+1}-u_j}{\eta}\Big),$$

and to which then the recipe above can be applied.

The crucial points are the computation of $\overline{\varphi}_\eta$ and the choice of the scaling η. In the case of *next-to-nearest neighbours* this computation is particularly simple, as it consists in choosing $\eta = 2\varepsilon$ and in 'integrating out the contribution of first neighbours': in formula,

$$\overline{\varphi}_{2\varepsilon}(z) = \varphi^2_\varepsilon(z) + \frac{1}{2}\min\{\varphi^1_\varepsilon(z_1) + \varphi^1_\varepsilon(z_2) : z_1 + z_2 = 2z\}.$$

In a sense this is a formula of *relaxation type*. If $K > 2$ then the formula giving $\overline{\psi}_n$ resembles more a *homogenization formula*, and we have to choose $\eta_n = K_n\lambda_n$ with K_n large. In this case the reasoning that leads from E_n to $\overline{E}_n$ is that the overall behaviour of a system of interacting point will behave as *clusters* of large arrays of neighbouring points interacting through their 'extremities'.

4. *Unbounded-range interactions: non local terms.* When the number of interaction orders we consider is not bounded the description becomes more complex. In particular, additional *non-local* terms may appear in F through some type of (possibly noninear) Dirichlet form.

We will show how all these principles carry also to higher dimension (to some extent) upon suitably modifying the representation formulas in the spirit of the homogenization of free-discontinuity problems in nonlinear elasticity.

References

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford University Press, Oxford, 2000.

[2] A. Braides, Non-local variational limits of discrete systems, *Comm. Contemporary Math.* **2** (2000), 285–297.

[3] A. Braides, *Γ-convergence for Beginners.* Oxford University Press, Oxford, to appear.

[4] A. Braides, G. Dal Maso and A. Garroni, Variational formulation of softening phenomena in fracture mechanics: the one-dimensional case, *Arch. Rational Mech. Anal.* **146** (1999), 23-58.

[5] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

[6] A. Braides and M.S. Gelli, Continuum limits of discrete systems without convexity hypotheses, *Math. Mech. Solids*, to appear.

[7] A. Braides and M.S. Gelli, Limits of discrete systems with long-range interactions, to appear.

[8] G. Dal Maso, *An Introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[9] S. Pagano and R. Paroni, A simple model for phase transitions: from the discrete to the continuum problem, Preprint, Oxford, 2000.

[10] G. Puglisi and L. Truskinovsky, Mechanics of a discrete chain with bi-stable elements. *J. Mech. Phys. Solids* **48**(2000), 1-27.

[11] L. Truskinovsky, Fracture as a phase transition, *Contemporary research in the mechanics and mathematics of materials* (R.C. Batra and M.F. Beatty eds.) CIMNE, Barcelona, 1996, 322–332.

Contents

Chapter 1

Introduction

1.1 Limits of discrete problems

We face the description of variational limits of discrete problems (for the sake of brevity in a one-dimensional setting). Given $n \in \mathbf{N}$ and points $x_i^n = i\lambda_n$ ($\lambda_n = L/n$ is the *lattice spacing*, which plays the role of the small parameter ε) we consider energies of the general form

$$E_n(\{u_i\}) = \sum_{j=1}^{n} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big(\frac{u_{i+j} - u_i}{j\lambda_n} \Big).$$

If we picture the set $\{x_i^n\}$ as the reference configuration of an array of material points interacting through some forces, and u_i represents the displacement of the i-th point, then ψ_n^j can be thought as the energy density of the interaction of points with distance $j\lambda_n$ (j lattice spacings) in the reference lattice. Note that the only assumption we make is that ψ_n^j depends on $\{u_i\}$ through the differences $u_{i+j} - u_i$, but we find it more convenient to highlight its dependence on 'discrete difference quotients'. For a quite general class of energies it is possible to describe the behaviour of problems of the form

$$\min \Big\{ E_n(\{u_i\}) - \sum_{i=0}^{n} \lambda_n u_i f_i : \ u_0 = U_0, \ u_n = U_L \Big\}$$

(and similar), and to show that these problems have a limit continuous counterpart. Here $\{f_i\}$ represents the external forces and U_0, U_L are the boundary conditions at the endpoints of the interval $(0, L)$. More general statement and different problems can be also obtained. Under some growth conditions, minimizers of the problem above are 'very close' to minimizers of a classical problem of the Calculus of Variations

$$\min \Big\{ \int_0^L \big(\psi(u') - fu \big)\, dt : \ u(0) = U_0, \ u(L) = U_L \Big\}.$$

The energy densities ψ can be explicitly identified by a series of operations on the functions ψ_n^j. The case when only *nearest-neighbour interactions* are taken into account,

$$E_n(\{u_i\}) = \sum_{i=0}^{n-1} \lambda_n \psi_n\Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big),$$

is particularly simple. In this case, the limit energy density is given by the limit of the convex envelopes of the functions $\psi_n(z)$, which exists up to subsequences. The description of the limit energy gets more complex when not only nearest-neighbour interactions come into play. In the case when interactions up to a fixed order K are taken into account:

$$E_n(\{u_i\}) = \sum_{j=1}^{K}\sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j}-u_i}{j\lambda_n}\Big)$$

(or, equivalently, $\psi_n^j = 0$ if $j > K$), the main idea is to show that (upon some controllable errors) we can find a lattice spacing η_n (possibly much larger than λ_n) such that E_n is 'equivalent' to a nearest-neighbour interaction energy on a lattice of step size η_n, of the form

$$\overline{E}_n(\{u_i\}) = \sum_{i=0}^{m-1} \eta_n \overline{\psi}_n\Big(\frac{u_{i+1}-u_i}{\eta_n}\Big),$$

and to which then the recipe above can be applied. The crucial points are the computation of $\overline{\psi}_n$ and the choice of the scaling η_n. In the case of *next-to-nearest neighbours* this computation is particularly simple, as it consists in choosing $\eta_n = 2\lambda_n$ and in 'integrating out the contribution of first neighbours': in formula,

$$\overline{\psi}_n(z) = \psi_n^2(z) + \frac{1}{2}\min\{\psi_n^1(z_1) + \psi_n^1(z_2) : z_1 + z_2 = 2z\}.$$

If $K > 2$ then the formula giving $\overline{\psi}_n$ resembles more a *homogenization* process, and we have to choose $\eta_n = K_n\lambda_n$ with K_n large. In this case the reasoning that leads from E_n to $\overline{E}_n$ is that the overall behaviour of a system of interacting point will behave as *clusters* of large arrays of neighbouring points interacting through their 'extremities'.

1.2 Γ-convergence

The questions risen above can be placed in a variational framework. The general problem amounts to studying the asymptotic behaviour of a family of minimum problems depending on a parameter; in an abstract notation,

$$\min\{F_\varepsilon(u) : \; u \in X_\varepsilon\}. \tag{1.1}$$

An answer to this problem is provided by substituting such a family by an 'effective problem' (not depending on ε)

$$\min\{F(u) : \ u \in X\}, \tag{1.2}$$

which captures the relevant behaviour of minimizers. This notion of 'convergence of problems' must be sensible, as it must include cases where the limit problem is set on a space X completely different from all X_ε, and even when X is the same it may be very different from pointwise convergence. Furthermore, it must not rely on any *a priori ansatz* on the asymptotic form of minimizers, and it should in a sense itself suggest the precise meaning of this asymptotic question, as this could not be supplied by problems (1.1).

Γ-convergence is a convergence on functionals which loosely speaking amounts to requiring the convergence of minimizers of problems (1.1) and of their continuous perturbations. The fact that this convergence is given in terms of convergence of minimizers assures precisely that the limit 'theory' can be considered as an effective theory, whose solutions capture the important properties of the theories at level ε. We now derive the desired definition of convergence for functionals from the requirements that it implies the convergence of minimizers and minimum values (under suitable assumptions), that it is stable under continuous perturbations, and that it is given in *local* terms (i.e., we can also speak of convergence 'at one point'). For the sake of simplicity, from here onwards all our problems will be set on metric spaces, so that the topology is described by just using sequences. The starting point will be the examination of the so-called *direct methods of the calculus of variations.* The idea is very simple: in order to prove the existence of a minimizer of a problem of the form

$$\min\{F(u) : \ u \in X\}, \tag{1.3}$$

we examine the behaviour of a *minimizing sequence*; i.e., a sequence $(\overline{u}_j)$ such that

$$\lim_j F(\overline{u}_j) = \inf\{F(u) : \ u \in X\}, \tag{1.4}$$

which clearly always exists.

Such a sequence, in general might lead nowhere. The first thing to check is then that we may find a *converging* minimizing sequence. This property may be at times checked by hand, but it is often more convenient to check that an *arbitrary* minimizing sequence lies in a *compact* subset K of X (i.e., since X is metric, that for any sequence (u_j) in K we can extract a subsequence (u_{j_k}) converging to some $u \in K$). This property is clearly stronger than requiring that there exists *one* converging minimizing sequence, but its verification often may rely on a number of characterization of compact sets in different spaces. In its turn this compactness requirement can be directly made on the functional F by asking that it be *coercive*; i.e., that for all t its sub-level sets $\{F < t\} = \{u \in X : F(u) < t\}$ are *pre-compact* (this means that for fixed t there exists a compact set K_t containing $\{F < t\}$, or, equivalently, in terms of sequences, that for all sequences (u_j) with $\sup_j F(u_j) < +\infty$ there exists a converging subsequence).

Again, this is an even stronger requirement, but it may be derived directly from the form of the functional F and not from special properties of minimizing sequences. Once some compactness properties of a minimizing sequence are established, we may extract a (minimizing) subsequence, that we still denote by $(\overline{u}_j)$, converging to some $\overline{u}$.

At this stage, the point $\overline{u}$ is a candidate to be a minimizer of F; we have to prove that indeed

$$F(\overline{u}) = \inf\{F(u) : \ u \in X\}. \tag{1.5}$$

One inequality is trivial, since $\overline{u}$ can be used as a test function in (1.5) to obtain an *upper inequality* for $\inf F$

$$\inf\{F(u) : \ u \in X\} \leq F(\overline{u}). \tag{1.6}$$

To obtain a *lower inequality* we have to link the value at $\overline{u}$ to those computed at $\overline{u}_j$, to obtain the right inequality

$$F(\overline{u}) \leq \lim_j F(\overline{u}_j) = \inf\{F(u) : \ u \in X\}. \tag{1.7}$$

Since we do not want to rely on special properties of $\overline{u}$ or of the approximating sequence $(\overline{u}_j)$, but instead we would like to isolate properties of the functional F, we require that *for all* $u \in X$ and *for all* sequences (u_j) tending to u we have the inequality

$$F(u) \leq \liminf_j F(u_j). \tag{1.8}$$

This property is called the *lower semicontinuity* of F. It is much stronger than requiring (1.7), but it is much stabler under perturbations, and it may be interpreted as a structure condition on F and often derived from general considerations.

At this point we have not only proven that F admits a minimum, but we have also found a minimizer $\overline{u}$ by following a minimizing sequence. We may condensate the reasoning above in the following formula

$$\text{coerciveness} \quad + \quad \text{lower semicontinuity} \quad \Rightarrow \quad \text{existence of minimizers,} \tag{1.9}$$

which summarizes the direct methods of the calculus of variations. It is worth noticing that the coerciveness of F is easier to verify if we have *many* converging sequences, while the lower semicontinuity of F is more easily satisfied if we have *few* converging sequences. These two opposite requirements will result in a balanced choice of the metric on X, which is in general not given a priory, but in a sense forms a part of the problem.

We now turn our attention to the problem not of proving the existence of a minimum for a single problem but of describing the behaviour of a family of minimum problems depending on a parameter. In order to simplify the notation we deal with the case of a sequence of problems

$$\inf\{F_j(u) : \ u \in X_j\} \tag{1.10}$$

depending on a discrete parameter $j \in \mathbf{N}$; the case of a family depending on a continuous parameter ε introduces only a little extra complexity in the notation.

As j increases we would like these problems to be approximated by a 'limit theory' described by a problem of the form

$$\min\{F(u):\ u\in X\}. \tag{1.11}$$

In order to make this notion of 'convergence' precise we try to follow closely the direct approach outlined above. In this case we start by examining a minimizing sequence for the family F_j; i.e., a sequence $(\overline{u}_j)$ such that

$$\lim_j \Big(F_j(\overline{u}_j) - \inf\{F_j(u):\ u\in X_j\}\Big) = 0, \tag{1.12}$$

and try to follow this sequence.

In many problems the space X_j indeed varies with j, so that now we have to face a preliminary problem of defining the convergence of a sequence of functions which belong to different spaces. This is usually done by choosing X large enough so that it contains the domain of the candidate limit and all X_j. We can always consider all functionals F_j as defined on this space X by identifying them with the functionals

$$\widetilde{F}_j(u) = \begin{cases} F_j(u) & \text{if } u\in X_j\\ +\infty & \text{if } u\in X\setminus X_j. \end{cases} \tag{1.13}$$

This type of identification is customary in dealing with minimum problems and is very useful to include constraints directly in the functional. We may therefore suppose that all $X_j = X$. If one is not used to dealing with functionals which take the value $+\infty$, one may regard this as a technical tool; if the limit functional is not finite on the whole X it will always be possible to restrict it to its domain $\operatorname{dom} F = \{u\in X:\ F(u) < +\infty\}$.

As in the case of a minimizing sequence for a single problem, it is necessary to find a converging minimizing (sub)sequence. In general it will be possible to find a minimizing sequence lying in a compact set of X as before, or prove that the functional themselves satisfy an *equi-coerciveness* property: for all t there exists a compact K_t such that for all j we have $\{F_j < t\}\subset K_t$.

If a compactness property as above is satisfied, then from the sequence $(\overline{u}_j)$ we can therefore extract a converging subsequence $(\overline{u}_{j_k})$. In this presentation we may suppose that the whole sequence $(\overline{u}_j)$ converges to some $\overline{u}$ (this is a technical point that will be made clear in the next section).

First, we want to obtain an *upper bound* for the limit behaviour of the sequence of minima, of the form

$$\limsup_j \inf\{F_j(u):\ u\in X\} \le \inf\{F(u):\ u\in X\} \le F(\overline{u}). \tag{1.14}$$

The second inequality is trivially true; the first inequality means that for all $u\in X$ we have

$$\limsup_j \inf\{F_j(v):\ v\in X\} \le F(u). \tag{1.15}$$

This is a requirement of global type; we can 'localize' it in the neighbourhood of the point u by requiring a stronger condition: that for all $\delta>0$ we have

$$\limsup_j \inf\{F_j(v):\ d(u,v)<\delta\} \le F(u). \tag{1.16}$$

By the arbitrariness of δ we can rephrase this condition as a condition on sequences converging to u as:

(*limsup inequality*) for all $u \in X$ there exists a sequence (u_j) converging to u such that

$$\limsup_j F_j(u_j) \le F(u). \tag{1.17}$$

This condition can be considered as a local version of (1.14); it clearly implies all conditions above and (1.14) in particular.

Next, we want to obtain a *lower bound* for the limit behaviour of the sequence of minima of the form

$$F(\overline{u}) \le \liminf_j F_j(\overline{u}_j) \tag{1.18}$$

As we do not want to rely on particular properties of minimizers we regard $\overline{u}$ as an arbitrary point in X and $(\overline{u}_j)$ as any converging sequence; hence, condition (1.18) can be deduced from the more general requirement:

(*liminf inequality*) for all $u \in X$ and for all sequences (u_j) converging to u we have

$$F(u) \le \liminf_j F_j(u_j). \tag{1.19}$$

This condition is the analog of the lower semicontinuity hypothesis in the case of a single functional.

From the considerations above, if we can find a functional F such that the liminf and limsup inequalities are satisfied and if we have a converging sequence of minimizers, from (1.18) and (1.14) we deduce the chain of inequalities

$$\begin{aligned} \limsup_j \inf\{F_j(u):\ u \in X\} &\le \inf\{F(u):\ u \in X\} \le F(\overline{u}) \\ &\le \liminf_j F_j(\overline{u}_j) = \liminf_j \inf\{F_j(u):\ u \in X\}. \end{aligned} \tag{1.20}$$

As the last term is clearly not greater than the first, all inequalities are indeed equalities; i.e., we deduce that

(i) (*existence*) the limit problem $\min\{F(u):\ u \in X\}$ admits a solution,

(ii) (*convergence of minimum values*) the sequence of infima $\inf\{F_j(u):\ u \in X\}$ converges to this minimum value,

(ii) (*convergence of minimizers*) up to subsequences, the minimizing sequence for (F_j) converges to a minimizer of F on X.

Therefore, if we define the Γ-convergence of (F_j) to F as the requirement that the limsup and the liminf inequalities above both hold, then we may summarize the considerations above in the formula

$$\text{equi-coerciveness} \quad + \quad \Gamma\text{-convergence} \quad \Rightarrow \quad \text{convergence of minimum problems.} \tag{1.21}$$

As in the case of the application of the direct methods, a crucial role will be played by the type of metric we choose on X. In this case, again, it will be a matter of balance between the convenience of a stronger notion of convergence, that will make the liminf inequality easier to verify, and a weaker one, which would be more convenient both to satisfy an equi-coerciveness condition and to find sequences satisfying the limsup inequality.

Chapter 2

A quick guide to Γ-convergence

This chapter is a quick summary of the main properties of Γ-convergence.

We recall the definition of Γ-convergence, and make some first remarks.

Definition 2.1 (Γ-convergence) *We say that a sequence $f_j : X \to \overline{\mathbf{R}}$* Γ-converges *in X to $f_\infty : X \to \overline{\mathbf{R}}$ if for all $x \in X$ we have*

(i) (lim inf inequality) *for every sequence (x_j) converging to x*

$$f_\infty(x) \leq \liminf_j f_j(x_j); \tag{2.1}$$

(ii) (lim sup inequality) *there exists a sequence (x_j) converging to x such that*

$$f_\infty(x) \geq \limsup_j f_j(x_j). \tag{2.2}$$

The function f_∞ is called the Γ-limit *of (f_j), and we write $f_\infty = \Gamma\text{-}\lim_j f_j$.*

Pointwise definition *The definition above can be also given at a fixed point $x \in X$: we say that (f_j) Γ-converges* at *x to the value $f_\infty(x)$ if* (i), (ii) *above hold; in this case we write $f_\infty(x) = \Gamma\text{-}\lim_j f_j(x)$. In this notation, f_j Γ-converges to f_∞ if and only if $f_\infty(x) = \Gamma\text{-}\lim_j f_j(x)$ at all $x \in X$.*

If we want to highlight the role of the metric, we can add the dependence on the distance d, and write $\Gamma(d)\text{-}\lim_j$, $\Gamma(d)$-convergence, and so on.

Remark 2.2 (Γ-convergence as an equality of upper and lower bounds) The liminf inequality (i) can be rewritten as

$$f_\infty(x) \leq \inf\{\liminf_j f_j(x_j) : x_j \to x\}.$$

Trivially, we always have

$$\inf\{\liminf_j f_j(x_j) : x_j \to x\} \leq \inf\{\limsup_j f_j(x_j) : x_j \to x\},$$

and, if $(\overline{x}_j)$ is a recovery sequence for (ii) we have

$$\inf\{\limsup_j f_j(x_j) : x_j \to x\} \le \limsup_j f_j(\overline{x}_j) \le f_\infty(x),$$

so that (i) and (ii) imply that we have

$$f_\infty(x) = \min\{\liminf_j f_j(x_j) : x_j \to x\} = \min\{\limsup_j f_j(x_j) : x_j \to x\} \quad (2.3)$$

(and actually both minima are obtained as limits along a recovery sequence). *It is important to keep in mind this characterization as many properties of the Γ-limit will be easily explained from it.*

It is sometimes convenient to state the equality in (2.3) as an equality of *infima*:

$$f_\infty(x) = \inf\{\liminf_j f_j(x_j) : x_j \to x\} = \inf\{\limsup_j f_j(x_j) : x_j \to x\}. \quad (2.4)$$

This equality is indeed equivalent to the definition of Γ-limit; i.e., the Γ-limit exists if and only if the two infima in (2.4) are equal. This characterization will be important in that in this way the existence of the Γ-limit (which not always exists) is expressed as the equality of two quantities which are always defined, and which can (and will) be studied separately. The first quantity can be thought as a *lower bound* for the Γ-limit, the second as an *upper bound*.

By (2.4) we obtain in particular that the Γ-limit, if it exists, is unique.

Remark 2.3 (Different ways of writing the limsup inequality) Note that if (x_j) satisfies the limsup inequality, then by (2.1) we have

$$f_\infty(x) \le \liminf_j f_j(x_j) \le \limsup_j f_j(x_j) \le f_\infty(x),$$

so that indeed $f_\infty(x) = \lim_j f_j(x_j)$; hence, (ii) can be substituted by

(ii)′ *(existence of a recovery sequence)* there exists a sequence (x_j) converging to x such that

$$f_\infty(x) = \lim_j f_j(x_j). \quad (2.5)$$

On the other hand, sometimes it is more convenient to prove (ii) with a small error and then deduce its validity by an approximation argument; i.e., (ii) can be replaced by

(ii)″ *(approximate limsup inequality)* for all $\varepsilon > 0$ there exists a sequence (x_j) converging to x such that

$$f_\infty(x) \ge \limsup_j f_j(x_j) - \varepsilon. \quad (2.6)$$

In the following (and in the literature) all conditions (ii), (ii)′ and (ii)″ are equally referred to as the limsup inequality or as the existence of a recovery sequence.

Remark 2.4 (Stability under continuous perturbations) An important property of Γ-convergence is its stability under continuous perturbations: if (f_j) Γ-converges to f_∞ and $g : X \to [-\infty, +\infty]$ is a d-continuous function then $(f_j + g)$ Γ-converges to $f_\infty + g$. This is an immediate consequence of the definition, since if (i) holds then for all $x \in X$ and $x_j \to x$ we get

$$f_\infty(x) + g(x) \le \liminf_j f_j(x_j) + \lim_j g(x_j) = \liminf_j (f_j(x_j) + g(x_j)),$$

while if (ii)$'$ above holds then we get

$$f_\infty(x) + g(x) = \lim_j f_j(x_j) + \lim_j g(x_j) = \lim_j (f_j(x_j) + g(x_j)),$$

and (x_j) is a recovery sequence also for $f_\infty + g$.

Remark 2.5 (Γ-limit of a constant sequence) Consider the simplest case $f_j = f$ for all $j \in \mathbf{N}$. In this case it will be easily seen that (f_j) Γ-converges. By the liminf inequality, the limit f_∞ must satisfy

$$f_\infty(x) \le \liminf_j f(x_j)$$

for all x and $x_j \to x$. If f is *not* lower semicontinuous then there exists $\overline{x}$ and a sequence $\overline{x}_j \to \overline{x}$ such that

$$\liminf_j f(\overline{x}_j) < f(\overline{x});$$

hence, in particular $f_\infty(\overline{x}) \ne f(\overline{x})$. This shows that Γ-convergence *does not satisfy* the requirement that a constant sequence $f_j = f$ converges to f (if f is not lower semicontinuous). We will see however that this holds true in the family of lower semicontinuous functions (see Remark 2.7 below).

Remark 2.6 (Dependence on the metric) The choice of the metric on X is clearly a fundamental step in problems involving Γ-limits. In general, even when two distances d and d' are comparable; i.e.,

$$\lim_j d'(x_j, x) = 0 \qquad \Longrightarrow \qquad \lim_j d(x_j, x) = 0, \tag{2.7}$$

the existence of the Γ-limit in one metric does not imply the existence of the Γ-limit in the second (see the examples in the next section). However, in this situation, if both Γ-limits exist then we have

$$\Gamma(d)\text{-}\lim_j f_j \le \Gamma(d')\text{-}\lim_j f_j.$$

This is clear for example from the characterization (2.3) since the set of converging sequences for d is larger that that for d'.

Remark 2.7 (Comparison with pointwise and uniform limits) As a very particular case, we can consider the metric d' of the *discrete topology* (where the only converging sequences are constant sequences). In this case the Γ-limit coincides with the pointwise limit. If d is any other metric then (2.7) holds trivially, so that we obtain

$$\Gamma(d)\text{-}\lim_j f_j \le \lim_j f_j$$

as a particular case of the previous remark.

If f_j converge *uniformly* to a f on an open set U (in particular if $f_j = f$) and f is l.s.c. then we have also that f_j Γ-converge to f. Indeed, the limsup inequality is obtained by the constant sequence, while the liminf inequality is immediately verified once we remark that if $x_j \to x \in U$ then $x_j \in U$ for j large enough, so that $\liminf_j f_j(x_j) = \lim_j(f_j(x_j) - f(x_j)) + \liminf_j f(x_j) \geq f(x)$.

2.1 Some examples on the real line

In this section we will compute some simple Γ-limits of functions defined on the real line (equipped with the usual euclidean distance), and we will also make some comparisons with the pointwise convergence (which can be thought of as a Γ-limit with respect to the discrete metric, as explained in Remark 2.7).

The computations in these examples will be quite straightforward, but nevertheless will allow us to highlight the different roles of the limsup and liminf inequalities. The first inequality is more constructive, as it amounts to finding the optimal approximating sequence for a fixed target point x, while the second one is more technical, and amounts to proving that the bound given by the recovery sequence is indeed optimal.

Example 2.8 We have seen that a constant sequence $f_j = f$ Γ-converges to f if and only if f is lower semicontinuous; hence, if f is not l.s.c. the pointwise limit and the Γ-limit are different. Now we construct an example where these two limits differ even if the pointwise limit is lower semicontinuous. Take $f_j(t) = f_1(jt)$, where $f_1(t) = \sqrt{2}\, te^{-(2t^2-1)/2}$ or

$$f_1(t) = \begin{cases} \pm 1 & \text{if } t = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_j \to 0$ pointwise, but $\Gamma\text{-}\lim_j f_j = f$, where

$$f(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ -1 & \text{if } t = 0. \end{cases}$$

Indeed, the sequence f_j converges locally uniformly (and hence also Γ-converges) to 0 in $\mathbf{R} \setminus \{0\}$, while clearly the optimal sequence for $x = 0$ is $x_j = -1/j$, for which $f_j(x_j) = -1$. In this case the pointwise and Γ-limits both exist and are different at one point.

Example 2.9 Take $f_j(t) = -f_1(jt)$, where f_1 is as in the previous example. Clearly, the Γ-limit remains unchanged. This shows that in general

$$\Gamma\text{-}\lim(-f_j) \neq -\Gamma\text{-}\lim_j f_j,$$

$$\Gamma\text{-}\lim(f_j + g_j) \neq \Gamma\text{-}\lim_j f_j + \Gamma\text{-}\lim_j g_j$$

(taking in the example $g_j = -f_j$) even if all functions are continuous.

Example 2.10 The pointwise and Γ-limits may exist and be different at *every* point. Take $f_j = g_j$, where

$$g_j(t) = \begin{cases} 0 & \text{if } t \notin \mathbf{Q} \text{ or } t = \frac{k}{n}, \text{ with } k \in \mathbf{Z} \text{ and } n \in \{1, \dots, j\}, \\ \\ -1 & \text{otherwise.} \end{cases}$$

We then have $f_j \to 0$ pointwise, but $\Gamma\text{-}\lim_j f_j = -1$ The liminf inequality is trivial, and the limsup inequality is easily obtained by remarking that $\{g_j = -1\}$ is dense for all $j \in \mathbf{N}$.

Example 2.11 There may be no pointwise converging subsequence of (f_j) but the $\Gamma\text{-}\lim_j f_j$ may exist all the same. Take for example $f_j(t) = -\cos(jt)$. In this case $\Gamma\text{-}\lim_j f_j = -1$. Again, the liminf inequality is trivial, while the limsup inequality is easily obtained by taking for example $x_j = [jx/2\pi]2\pi/j$ ($[t]$ the integer part of t).

Example 2.12 The sequence f_j may be converging pointwise, but may not Γ-converge. Take for example $f_j = (-1)^j g_j$ with g_j the function of Example 2.10. In this case $f_j \to 0$ pointwise, but the $\Gamma\text{-}\lim_j f_j$ does not exist at any point.

2.2 The many definitions of Γ-convergence

In this section we state different equivalent definitions of Γ-convergence, which will be useful in different contexts. Some of the different ways to state the limsup inequality have been already pinpointed above. Before this overview we recall the definition of compact set.

Definition 2.13 *By a* compact *subset of X we mean a sequentially compact set $K \subset X$; i.e., such that all sequences in K admit a subsequence converging to some point in K. In formula,*

$$\forall (x_j) \subset K \ \exists x \in K, \ \exists (x_{j_k}): \ x_{j_k} \to x.$$

A set $K \subset X$ is called precompact *if its closure is compact; i.e., all sequences in K admit a converging subsequence (but its limit may also be outside K). In formula,*

$$\forall (x_j) \subset K \ \exists (x_{j_k}): \ x_{j_k} \textit{ converges in } X.$$

Topological definitions Sometimes, it is useful to have the definition of Γ-limit directly expressed in terms of the topology of X, and not only through the convergence of sequences. In this case it is easily seen that we can rewrite the equality in (2.4)

$$f_\infty(x) = \sup_{U \in \mathcal{N}(x)} \liminf_j \inf_{y \in U} f_j(y) = \sup_{U \in \mathcal{N}(x)} \limsup_j \inf_{y \in U} f_j(y). \qquad (2.8)$$

This definition makes sense in any topological space and in the case of arbitrary topological spaces (in particular if X is not metric) is taken as *the* definition of

Γ-convergence (while Definition 2.1 above is called *sequential* Γ-convergence). However, we will always be able to stick to metric spaces. A suggestive observation is that equivalently to (2.8), we may also write

$$f_\infty(x) = \sup_{U\in\mathcal{N}(x)} \sup_{k\in\mathbf{N}} \inf_{j\geq k} \inf_{y\in U} f_j(y) = \sup_{U\in\mathcal{N}(x)} \inf_{k\in\mathbf{N}} \sup_{j\geq k} \inf_{y\in U} f_j(y); \tag{2.9}$$

in this way, Γ-limits may be interpreted as compositions of the 'elementary operators' of the type inf/sup.

Note that in (2.8) we can substitute $\mathcal{N}(x)$ by a suitable family of neighbourhoods generating the topology of X; e.g., in the metric case a family of open balls with center in x. For example we can require equivalently that

$$\begin{aligned} f_\infty(x) &= \sup_{n\in\mathbf{N}} \liminf_j \inf\{f_j(y) : d(y,x) < 1/n\} \\ &= \sup_{n\in\mathbf{N}} \limsup_j \inf\{f_j(y) : d(y,x) < 1/n\} \end{aligned}$$

or

$$\begin{aligned} f_\infty(x) &= \sup_{\delta>0} \liminf_j \inf\{f_j(y) : d(y,x) < \delta\} \\ &= \sup_{\delta>0} \limsup_j \inf\{f_j(y) : d(y,x) < \delta\}. \end{aligned}$$

A definition in terms of the convergence of minima Γ-convergence is designed so that it implies the convergence of 'compact' minimum problems. In turn, starting from the topological definition above, a definition of Γ-convergence can be expressed in terms of the asymptotic behaviour of (local) minimum problems: from the second equality in (2.8) we have

$$\inf_U f_\infty \geq \limsup_j \inf_U f_j \tag{2.10}$$

for all open sets U, while requiring that

$$\inf_K f_\infty \leq \sup\{\liminf_j \inf_U f_j : \ U\supset K,\ U \text{ open}\} \tag{2.11}$$

for all compact sets K implies the first equality in (2.8) by choosing $K = \{x\}$.

Finally, starting from (2.10), back to the case of metric spaces, we can substitute the problems on balls by unconstrained problems, where we penalize the distance from the point x. For example, if all f_j are non negative, we have that an equivalent definition is that for some $p > 0$

$$\begin{aligned} f_\infty(x) &= \sup_{\lambda\geq 0} \liminf_j \inf_{y\in X} \{f_j(y) + \lambda d(x,y)^p\} \\ &= \sup_{\lambda\geq 0} \limsup_j \inf_{y\in X} \{f_j(y) + \lambda d(x,y)^p\} \end{aligned} \tag{2.12}$$

for all $x \in X$. Note that in this case the Γ-convergence is determined by looking at a family of particular problems, which sometimes can be solved easily.

We can explicitly state the equivalence of all the definitions above in the following theorem.

Theorem 2.14 *Let $f_j, f_\infty : X \to [-\infty, +\infty]$. Then the following conditions are equivalent:*

(i) $f_\infty = \Gamma\text{-}\lim_j f_j$ *in X as in Definition* 2.1*;*

(ii) *for every $x \in X$* (2.3) *holds;*

(iii) *the liminf inequality in Definition* 2.1(i) *and the* approximate limsup inequality (ii)″ *hold;*

(iv) *for every $x \in X$* (2.4) *holds;*

(v) *for every $x \in X$* (2.8) *holds;*

(vi) *inequality* (2.10) *holds for all open sets U and inequality* (2.11) *holds for all compact sets K.*

Furthermore, if $f_j(x) \geq -c(1 + d(x, x_0)^p)$ for some $p > 0$ and $x_0 \in X$, then each of the conditions above is equivalent to

(vii) *we have* (2.12) *for all $x \in X$.*

The proof of the equivalence of (i)–(vi) is essentially contained in the considerations made hitherto and details are left to the reader; point (vii) will be analysed in Proposition 2.23.

Note that the asymmetry of Definition 2.1 is reflected in the different roles of the sup and inf operators in the equivalent conditions above. Of course, this comes from the fact that Γ-convergence is designed to study *minimum* problems (and not maximum problems!).

2.3 Convergence of minima

We first observe that some requirements on the behaviour of sequences of the form $(f_j(x_j))$ give some information on the behaviour of minimum problems.

Proposition 2.15 *Let $f_j, f : X \to [-\infty, +\infty]$ be functions. Then we have*

(i) *if Definition* 2.1(i) *is satisfied for all $x \in X$ then we have*

$$\inf_K f_\infty \leq \liminf_j \inf_K f_j$$

for all compact sets $K \subset X$;

(ii) *if Definition* 2.1(ii) *is satisfied for all $x \in X$ then we have*

$$\inf_U f_\infty \geq \limsup_j \inf_U f_j$$

for all open sets $U \subset X$.

Proof (i) Let $(\tilde{x}_j)$ be such that $\liminf_j \inf_K f_j = \liminf_j f_j(\tilde{x}_j)$. After extracting a subsequence we obtain $(\tilde{x}_{j_k})$ such that

$$\lim_k f_{j_k}(\tilde{x}_{j_k}) = \liminf_j \inf_K f_j,$$

and $x_{j_k} \to \overline{x} \in K$. If $x_j = \begin{cases} \tilde{x}_{j_k} & \text{if } j = j_k \\ \overline{x} & \text{if } j \neq j_k \text{ for all } k, \end{cases}$ then

$$\begin{aligned} \inf_K f_\infty &\leq f_\infty(\overline{x}) \leq \liminf_j f_j(x_j) \\ &\leq \liminf_k f_{j_k}(x_{j_k}) = \lim_k f_{j_k}(\tilde{x}_{j_k}) = \liminf_j \inf_K f_j, \end{aligned} \qquad (2.13)$$

as required.

(ii) With fixed $\delta > 0$, let $x \in U$ be such that $f_\infty(x) \le \inf_U f_\infty + \delta$. Then, if (x_j) is a recovery sequence for x we have

$$\inf_U f_\infty + \delta \ge f_\infty(x) \ge \limsup_j f_j(x_j) \ge \limsup_j \inf_U f_j, \tag{2.14}$$

and the thesis follows by the arbitrariness of δ. □

Definition 2.16 *A function $f : X \to \overline{\mathbf{R}}$ is* coercive *if for all $t \in \mathbf{R}$ the set $\{f \le t\}$ is precompact. A function $f : X \to \overline{\mathbf{R}}$ is* mildly coercive *if there exists a non-empty compact set $K \subset X$ such that $\inf_X f = \inf_K f$.*

Remark 2.17 If f is coercive then it is mildly coercive. In fact, if f is not identically $+\infty$ (in which case we take K as any compact subset of X), then there exists $t \in \mathbf{R}$ such that $\{f \le t\}$ is not empty, and we take K as the closure of $\{f \le t\}$ in X. An example of a non-coercive, mildly coercive function is given by any periodic function $f : \mathbf{R}^n \to \mathbf{R}$.

An intermediate condition between coerciveness and mild coerciveness is the requirement that there exists $t \in \mathbf{R}$ such that $\{f \le t\}$ is not empty and precompact.

We immediately obtain the required convergence result as follows.

Theorem 2.18 *Let (X, d) be a metric space, let (f_j) be a sequence of equi-mildly coercive functions on X, and let $f_\infty = \Gamma\text{-}\lim_j f_j$; then*

$$\exists \min_X f_\infty = \lim_j \inf_X f_j. \tag{2.15}$$

Moreover, if (x_j) is a precompact sequence such that $\lim_j f_j(x_j) = \lim_j \inf_X f_j$, then every limit of a subsequence of (x_j) is a minimum point for f_∞.

Proof The proof follows immediately from Proposition 2.15. In fact, if $\overline{x}$ is as in the proof of Proposition 2.15(i) (in particular we can take $\overline{x} = \lim_k x_{j_k}$ if (x_{j_k}) is a converging subsequence such that $\lim_k f_j(x_{j_k}) = \lim_j \inf_X f_j$) then by (2.13) and (2.14) with $U = X$, and by the equi-mild coerciveness we get

$$\begin{aligned} \inf_X f_\infty &\le \inf_K f_\infty \le f_\infty(\overline{x}) \le \liminf_j \inf_K f_j \\ &= \liminf_j \inf_X f_j \le \limsup_j \inf_X f_j \le \inf_X f_\infty. \end{aligned}$$

As the first and last terms coincide, we easily get the thesis. □

Remark 2.19 (Γ-convergence as a choice criterion) If in the theorem above all functions f_j admit a minimizer x_j then, up to subsequences, x_j converge to a minimum point of f_∞. The converse is clearly not true: we may have minimizers of f_∞ which are not limits of minimizers of f_j. A trivial example is $f_j(t) = \frac{1}{j}t^2$ on the real line. This situation is not exceptional; on the contrary: we may often view some functional as a Γ-limit of some particular perturbations, and single out from its minima those chosen as limits of minimizers.

2.4 Upper and lower Γ-limits

Condition (iv) in Theorem 2.14 justifies the following definition.

Definition 2.20 *Let $f_j : X \to \overline{\mathbf{R}}$ and let $x \in X$. The quantity*

$$\Gamma\text{-}\liminf_j f_j(x) = \inf\{\liminf_j f_j(x_j) :\ x_j \to x\} \tag{2.16}$$

is called the Γ-lower limit *of the sequence (f_j) at x. The quantity*

$$\Gamma\text{-}\limsup_j f_j(x) = \inf\{\limsup_j f_j(x_j) :\ x_j \to x\} \tag{2.17}$$

is called the Γ-upper limit *of the sequence (f_j) at x. If we have the equality*

$$\Gamma\text{-}\liminf_j f_j(x) = \lambda = \Gamma\text{-}\limsup_j f_j(x) \tag{2.18}$$

for some $\lambda \in [-\infty, +\infty]$, then we write

$$\lambda = \Gamma\text{-}\lim_j f_j(x), \tag{2.19}$$

and we say that λ is the Γ-limit *of the sequence (f_j)* at x. *Again, if we need to highlight the dependence on the metric d we may add it in the notation.*

Remark 2.21 Clearly, the Γ-lower limit and the Γ-upper limit exist at every point $x \in X$. Definition 2.20 is in agreement with Definition 2.1, and we can say that a sequence (f_j) Γ-converges to f_∞ if and only if for fixed $x \in X$ the Γ-limit exists and we have $\lambda = f_\infty(x)$ in (2.19).

Remark 2.22 It can be easily checked, as we did for the Γ-limit, that we have

$$\begin{aligned}\Gamma\text{-}\liminf_j f_j(x) &= \min\{\liminf_j f_j(x_j) : x_j \to x\}\\ &= \sup_{U\in\mathcal{N}(x)} \liminf_j \inf_{y\in U} f_j(y), \qquad (2.20)\\ \Gamma\text{-}\limsup_j f_j(x) &= \min\{\limsup_j f_j(x_j) : x_j \to x\}\\ &= \sup_{U\in\mathcal{N}(x)} \limsup_j \inf_{y\in U} f_j(y) \qquad (2.21)\end{aligned}$$

The reader is encouraged to fill the details of the proof of this statement.

We also have the following characterization of upper and lower Γ-limits, which proves in particular the last statement of Theorem 2.14.

Proposition 2.23 *Let $f_j : X \to [-\infty, +\infty]$ be a sequence of functions satisfying $f_j(x) \geq -c(1 + d(x, x_0)^p)$ for some $p > 0$ and $x_0 \in X$; then we have*

$$\Gamma\text{-}\liminf_j f_j(x) = \sup_{\lambda\geq 0} \liminf_j \inf_{y\in X} \{f_j(y) + \lambda d(x, y)^p\}, \tag{2.22}$$

$$\Gamma\text{-}\limsup_j f_j(x) = \sup_{\lambda\geq 0} \limsup_j \inf_{y\in X} \{f_j(y) + \lambda d(x, y)^p\} \tag{2.23}$$

for all $x \in X$.

2.5 Lower semicontinuity and Γ-limits

The notion of Γ-convergence does not have the property that a constant sequence $f_j = f$ converges to its constant term f. Conversely, this property is true for lower semicontinuous functions. Moreover, on the family of lower semicontinuous functions, Γ-convergence enjoys more interesting and useful properties.

2.5.1 Lower semicontinuity of Γ-limits

Proposition 2.24 *The* Γ*-upper and lower limits of a sequence* (f_j) *are* d*-lower semicontinuous functions.*

Remark 2.25 (Proof of the limsup inequality by density) The lower semicontinuity of the Γ-limsup in the proposition above is sometimes useful for the estimate of Γ-limits as follows. Let d' be a distance on X inducing a topology which is not weaker than that induced by d; i.e., $d'(x_j, x) \to 0$ implies $d(x_j, x) \to 0$. Suppose D is a dense subset of X for d' and that we have $\Gamma\text{-}\limsup_j f_j(x) \leq f(x)$ on D, where f is a function which is continuos with respect to d. Then we have $\Gamma\text{-}\limsup_j f_j \leq f$ on X. In fact, if $d'(x_j, x) \to 0$ and $x_j \in D$ then

$$\begin{aligned}\Gamma\text{-}\limsup_j f_j(x) &\leq \liminf_k \Big(\Gamma\text{-}\limsup_j f_j(x_k)\Big) \\ &\leq \liminf_k f(x_k) = f(x).\end{aligned}$$

2.5.2 The lower-semicontinuous envelope

If f is not lower semicontinuous, it is sometimes interesting to compute the lower semicontinuous envelope of f.

Definition 2.26 *Let* $f : X \to \overline{\mathbf{R}}$ *be a function. Its* lower-semicontinuous envelope $\mathrm{sc}f$ *is the greatest lower-semicontinuous function not greater than* f*, i.e. for every* $x \in X$

$$\mathrm{sc}f(x) = \sup\{g(x) : \ g \text{ l.s.c.}, \ g \leq f\}. \tag{2.24}$$

Note that $\mathrm{sc}f$ is l.s.c. as the supremum of a family of l.s.c. functions. Moreover, if $f_1 \leq f_2$ then $scf_1 \leq scf_2$.

Proposition 2.27 *We have* $\mathrm{sc}f(x) = \Gamma\text{-}\lim_j f(x) = \liminf_{y\to x} f(y)$.

Proposition 2.28 *We have*

$$\Gamma\text{-}\liminf_j f_j = \Gamma\text{-}\liminf_j \mathrm{sc}f_j, \qquad \Gamma\text{-}\limsup_j f_j = \Gamma\text{-}\limsup_j \mathrm{sc}f_j. \tag{2.25}$$

2.5.3 The direct method

Combined lower semicontinuity and coerciveness ensure the existence of minimum points, as specified by the following version of a well-known theorem.

Theorem 2.29 (Weierstrass' Theorem) *If $f : X \to \overline{\mathbf{R}}$ is mildly coercive, then there exists the minimum value* $\min\{\mathrm{sc}f(x) : x \in X\}$*, and it equals the infimum* $\inf\{f(x) : x \in X\}$*. Moreover, the minimum points for* $\mathrm{sc}f$ *are exactly all the limits of converging sequences* (x_j) *such that* $\lim_j f(x_j) = \inf_X f$*.*

Proof The theorem is a particular case of Theorem 2.18. The only thing to notice is that if $\overline{x}$ is a minimum point for $\mathrm{sc}f$, we can find a sequence (x_j) converging to $\overline{x}$ such that $\lim_j f(x_j) = \mathrm{sc}f(\overline{x}) = \inf_X f$. □

Remark 2.30 The previous theorem gives, of course, that if f is l.s.c. and mildly coercive then the problem $\min_X f$ has a solution.

The application of Theorem 2.29, and in particular of the remark above, to prove the existence of solutions of minimum problems is usually referred to as the 'direct method' of the calculus of variations.

2.6 More properties of Γ-limits

From the definitions of Γ-convergence we immediately obtain the following properties.

Remark 2.31 If (f_{j_k}) is a subsequence of (f_j) then

$$\Gamma\text{-}\liminf_j f_j \le \Gamma\text{-}\liminf_k f_{j_k}, \qquad \Gamma\text{-}\limsup_k f_{j_k} \le \Gamma\text{-}\limsup_j f_j.$$

In particular, if $f_\infty = \Gamma\text{-}\lim_j f_j$ exists then for every increasing sequence of integers (j_k) $f_\infty = \Gamma\text{-}\lim_k f_{j_k}$.

Remark 2.32 If g is a continuous function then $f_\infty + g = \Gamma\text{-}\lim_j (f_j + g)$; more in general, if $g_j \to g$ uniformly, and g is continuous then $f_\infty + g = \Gamma\text{-}\lim_j (f_j + g_j)$. In particular, if $f_j \to f$ uniformly on an open set U, then

$$\Gamma\text{-}\lim_j f_j = \mathrm{sc}f \tag{2.26}$$

on U.

Remark 2.33 If $f_j \to f$ pointwise then $\Gamma\text{-}\limsup_j f_j \le f$, and hence, also $\Gamma\text{-}\limsup_j f_j \le \mathrm{sc}f$.

2.6.1 Γ-limits of monotone sequences

We can state some simple but important cases when the Γ-limit does exist, and it is easily computed.

Remark 2.34 (i) If $f_{j+1} \leq f_j$ for all $j \in \mathbf{N}$, then

$$\Gamma\text{-}\lim_j f_j = \mathrm{sc}(\inf_j f_j) = \mathrm{sc}(\lim_j f_j). \tag{2.27}$$

In fact as $f_j \to \inf_k f_k$ pointwise, by Remark 2.33 we have $\Gamma\text{-}\limsup_j f_j \leq \mathrm{sc}(\inf_k f_k)$, while the other inequality comes trivially from the inequality $\mathrm{sc}(\inf_k f_k) \leq \inf_k f_k \leq f_j$;

(ii) if $f_j \leq f_{j+1}$ for all $j \in \mathbf{N}$, then

$$\Gamma\text{-}\lim_j f_j = \sup_j \mathrm{sc} f_j = \lim_j \mathrm{sc} f_j; \tag{2.28}$$

in particular if f_j is l.s.c. for every $j \in \mathbf{N}$, then

$$\Gamma\text{-}\lim_j f_j = \lim_j f_j. \tag{2.29}$$

In fact, since $\mathrm{sc} f_j \to \sup_k \mathrm{sc} f_k$ pointwise,

$$\Gamma\text{-}\limsup_j f_j = \Gamma\text{-}\limsup_j \mathrm{sc} f_j \leq \sup_k \mathrm{sc} f_k$$

by Remark 2.33. On the other hand $\mathrm{sc} f_k \leq f_j$ for all $j \geq k$ so that the converse inequality easily follows.

Remark 2.35 By Remark 2.34(ii), if f_j is a equi-mildly coercive non-decreasing sequence of l.s.c. functions then $\sup_j \min_X f_j = \min_X \sup_j f_j$.

2.6.2 Compactness of Γ-convergence

Proposition 2.36 *Let (X, d) be a separable metric space, and for all $j \in \mathbf{N}$ let $f_j : X \to \overline{\mathbf{R}}$ be a function. Then there is an increasing sequence of integers (j_k) such that the $\Gamma\text{-}\lim_k f_{j_k}(x)$ exists for all $x \in X$.*

Remark 2.37 If (X, d) is not separable, then Proposition 2.36 fails. As an example, we can take $X = \{-1, 1\}^{\mathbf{N}}$ equipped with the discrete topology. X is metrizable, and Γ-convergence on X is equivalent to pointwise convergence. We take the sequence $f_j : X \to \{-1, 1\}$ defined by $f_j(\mathbf{x}) = x_j$ if $\mathbf{x} = (x_0, x_1, \ldots)$. If (f_{j_k}) is a subsequence of (f_j) and we define $\mathbf{x}$ by $x_{j_k} = (-1)^k$, and $x_j = 1$ if $j \notin \{j_k : \ k \in \mathbf{N}\}$, then the limit $\lim_k f_{j_k}(\mathbf{x})$ does not exist. Hence no subsequence of (f_j) Γ-converges.

2.6.3 Stability of Γ-convergence by subsequences

Proposition 2.38 *We have $\Gamma\text{-}\lim_j f_j = f_\infty$ if and only if for every subsequence (f_{j_k}) there exists a further subsequence which Γ-converges to f_∞.*

2.7 Γ-limits indexed by a continuous parameter

In applications, our energies will often depend on a continuous parameter $\varepsilon > 0$, so that we will have a family of functions $f_\varepsilon : X \to \overline{\mathbf{R}}$. It is necessary then to make precise the definition of Γ-limit in this case, as follows.

Definition 2.39 *We say that f_ε Γ-converges to f_0 if for all sequences (ε_j) converging to 0 we have $\Gamma\text{-}\lim_j f_{\varepsilon_j} = f_0$.*

Remark 2.40 It can be easily checked that all the characterizations and properties of the Γ-limits, as well as the definitions of Γ- upper and lower limits, can be still obtained in this case with the due changes. We usually prefer to stick to sequences, as in the proof it is more convenient to extract subsequences.

2.8 Development by Γ-convergence

As already remarked, the process of Γ-limit enatails a choice in the minimizers of the limit problem by minimizers of the approximating ones. In the case that this 'choice' is still not unique, we may proceed further to a 'Γ-limit of higher order'.

Theorem 2.41 (development by Γ-convergence) . *Let $f_\varepsilon : X \to \overline{\mathbf{R}}$ be a family of d-equi-coercive functions and let $f^0 = \Gamma(d)\text{-}\lim_{\varepsilon\to 0} f_\varepsilon$. Let $m_\varepsilon = \inf f_\varepsilon$ and $m^0 = \min f^0$. Suppose that for some $\alpha > 0$ there exists the Γ-limit*

$$f^\alpha = \Gamma(d')\text{-}\lim_{\varepsilon\to 0} \frac{f_\varepsilon - m^0}{\varepsilon^\alpha}, \tag{2.30}$$

and that the sequence $f_\varepsilon^\alpha = (f_\varepsilon - m^0)/\varepsilon^\alpha$ is d'-equi-coercive for a metric d' which is not weaker than d. Define $m^\alpha = \min f^\alpha$ and suppose that $m^\alpha \neq +\infty$; then we have that

$$m_\varepsilon = m^0 + \varepsilon^\alpha m^\alpha + o(\varepsilon^\alpha) \tag{2.31}$$

and from all sequences (x_ε) such that $f_\varepsilon(x_\varepsilon) - m_\varepsilon = o(\varepsilon^\alpha)$ (in particular this holds for minimizers) there exists a subsequence converging in (X, d') to a point x which minimizes both f^0 and f^1.

Proof The proof is a simple refinement of that of Theorem 2.18. Since we have

$$m^\alpha = \lim_{\varepsilon\to 0} \min f_\varepsilon^\alpha = \lim_{\varepsilon\to 0} \frac{m_\varepsilon - m^0}{\varepsilon^\alpha}$$

we deduce immediately (2.31). Let (x_ε) be such that $f_\varepsilon(x_\varepsilon) = m_\varepsilon + o(\varepsilon^\alpha)$. By the equi-coerciveness of f_ε^α we can assume that x_ε converges to some x in (X, d') and hence also in (X, d), upon extracting a subsequence. By Theorem 2.18 applied to f_ε x is a minimizer of f^0. From (2.31) we get that $\min f_\varepsilon^\alpha = (m_\varepsilon - m^0)/\varepsilon^\alpha = f_\varepsilon^\alpha(x_\varepsilon) + o(1)$, so that we can also apply Theorem 2.18 to f_ε^α and obtain that x is a minimizer of f^1. □

2.9 Γ-development with respect a family of data

We may refine the notion of development by Γ-convergence when our problems depend also on some *data* (which we denote by the letter d). Suppose that functionals as above are given, a set D of data, and spaces X^d_ε that form a partition of the space X_ε. We say that the family f_ε has a *development* $f^0+\varepsilon^\alpha f^\alpha$ *with respect to the family* D, if there exists a partition X^d_0 of X_0 and functionals f^0_d, f^α_d each defined on X^d_0 such that

$$f^0_d = \Gamma\text{-}\lim_\varepsilon f^d_\varepsilon, \qquad f^0_d = f^0 \text{ on } X^0_d \tag{2.32}$$

and, having set

$$f^{\alpha,d}_\varepsilon(u) = \frac{f_\varepsilon(u) - \min f^0_d}{\varepsilon^\alpha} \text{ for } u \in X^d_\varepsilon, \tag{2.33}$$

we have

$$f^\alpha_d = \Gamma\text{-}\lim_\varepsilon f^{\alpha,d}_\varepsilon, \qquad f^\alpha_d = f^\alpha \text{ on } X^0_d. \tag{2.34}$$

In particular this implies that, upon setting

$$m_\varepsilon(d) = \min\{f_\varepsilon(u) : u \in X^d_\varepsilon\}, \tag{2.35}$$

$$m^0(d) = \min\{f^0(u) : u \in X^0_d\}, \tag{2.36}$$

$$m^\alpha(d) = \min\{f^\alpha(u) : u \in X^0_d\}, \tag{2.37}$$

we obtain that for all $d \in D$

$$m_\varepsilon(d) = m^0(d) + \varepsilon^\alpha m^\alpha(d) + o(\varepsilon^\alpha). \tag{2.38}$$

Example 2.42 Take $f_\varepsilon(u) = \int_I (W(u) + \varepsilon^2|u'|^2)\,dt$, $\qquad X_\varepsilon = H^1(I)$, where $W(u) = (u^2-1)^2$, $X^d_\varepsilon = \left\{u \in H^1(I) : \fint_I u\,dt = d\right\}$ and $D = \mathbf{R}$. In this case we have

$$f_0(u) = \int_I W^{**}(u)\,dt, \qquad X^0 = L^2(I).$$

Take $\alpha = 1$; if $-1 \le d \le 1$ then

$$f^1_d(u) = c\#(S(u))\,dt, \qquad X^1_d = \left\{u \in BV(I;\{-1,1\}) : \fint_I u\,dt = d\right\}$$

with $c = \int_{-1}^1 2\sqrt{W(s)}ds$; if $|d| > 1$ then

$$f^1_d(u) = \begin{cases} 0 & \text{if } u = d \text{ a.e.} \\ +\infty & \text{otherwise.}\end{cases}$$

Definition 2.43 *The previous definition suggests an equivalence relation among families of functionals based on development by Γ-convergence: we say that two families (f_ε) and $(\tilde f_\varepsilon)$ are equivalent up to order α with respect to the data D if they have the same development in the sense as above. Note that this definition does not require that f_ε and $\tilde f_\varepsilon$ be defined on the same space.*

Chapter 3

Discrete systems in Sobolev spaces

We will consider the limit of energies defined on one-dimensional discrete systems of n points as n tends to $+\infty$. In order to define a limit energy on a continuum we parameterize these points as a subset of a single interval $(0, L)$. Set

$$\lambda_n = \frac{L}{n}, \qquad x_i^n = \frac{i}{n} L = i\lambda_n, \qquad i = 0, 1, \ldots, n. \tag{3.1}$$

We denote $I_n = \{x_0^n, \ldots, x_n^n\}$ and by $\mathcal{A}_n(0, L)$ the set of functions $u : I_n \to \mathbf{R}$. If n is fixed and $u \in \mathcal{A}_n(0, L)$ we equivalently denote

$$u_i = u(x_i^n).$$

Given $K \in \mathbf{N}$ with $1 \leq K \leq n$ and functions $f^j : \mathbf{R} \to [0, +\infty]$, with $j = 1, \ldots, K$, we will consider the related functional $E : \mathcal{A}_n(0, L) \to [0, +\infty]$ given by

$$E(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} f^j(u_{i+j} - u_i). \tag{3.2}$$

Note that E can be viewed simply as a function $E : \mathbf{R}^n \to [0, +\infty]$.

An interpretation with a physical flavour of the energy E is as the internal interaction energy of a chain of $n + 1$ material points each one interacting with its K-nearest neighbours, under the assumption that the interaction energy densities depend only on the order j of the interaction and on the distance between the two points $u_{i+j} - u_i$ in the reference configuration. If $K = 1$ then each point interacts with its nearest neighbour only, while if $K = n$ then each pair of points interacts.

Remark 3.1 From elementary calculus we have that E is lower semicontinuous if each f^j is lower semicontinuous, and that E is coercive on bounded sets of $\mathcal{A}_n(0, L)$.

We will describe the limit as $n \to +\infty$ of sequences (E_n) with $E_n : \mathcal{A}_n(0,L) \to [0,+\infty]$ of the general form

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} f_n^j(u_{i+j} - u_i), \tag{3.3}$$

and show that it provides an energy defined on a Sobolev space. We use standard notation for Sobolev and Lebesgue spaces. The letter c denotes a generic strictly positive constant.

Since each functional E_n is defined on a different space, the first step is to identify each $\mathcal{A}_n(0,L)$ with a subspace of a common space of functions defined on $(0,L)$. In order to identify each discrete function with a continuous counterpart, we extend u by $\tilde{u} : (0,L) \to \mathbf{R}$ as the piecewise-affine function defined by

$$\tilde{u}(s) = u_{i-1} + \frac{u_i - u_{i-1}}{\lambda_n}(s - x_{i-1}) \qquad \text{if } s \in (x_{i-1}, x_i). \tag{3.4}$$

In this case, $\mathcal{A}_n(0,L)$ is identified with those continuous $u \in \mathrm{W}^{1,1}(0,L)$ (actually, in $\mathrm{W}^{1,\infty}(0,L)$) such that u is affine on each interval (x_{i-1}, x_i). Note moreover that we have

$$\tilde{u}' = \frac{u_i - u_{i-1}}{\lambda_n} \tag{3.5}$$

on (x_{i-1}, x_i). If no confusion is possible, we will simply write u in place of $\tilde{u}$.

As we will treat limit functionals defined on Sobolev spaces, it is convenient to rewrite the dependence of the energy densities in (4.14) with respect to difference quotients rather than the differences $u_{i+j} - u_i$. We then write

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big), \tag{3.6}$$

where

$$\psi_n^j(z) = \frac{1}{\lambda_n} f_n^j(j\lambda_n z).$$

With the identification of u with $\tilde{u}$, E_n may be viewed as an integral functional defined on $\mathrm{W}^{1,1}(0,L)$ by the equality

$$E_n(u) = F_n(\tilde{u}), \tag{3.7}$$

where

$$F_n(v) = \begin{cases} \displaystyle\sum_{j=1}^{K_n} \sum_{l=0}^{j-1} \frac{1}{j} \int_{l\lambda_n}^{L-(j-1-l)\lambda_n} \psi_n^j\Big(\frac{1}{j} \sum_{k=-l}^{j-1-l} v'(x + k\lambda_n)\Big) dx & \text{if } v \in \mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.8}$$

Note that in the particular case $K_n = 1$ we have (set $\psi_n = \psi_n^1$)

$$F_n(v) = \begin{cases} \int_0^L \psi_n(v')dx & \text{if } v \in \mathcal{A}_n(0, L) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.9}$$

Definition 3.2 (Convergence of discrete functions and energies) With the identifications above we will say that u_n *converges to* u (respectively, in L^1, in measure, in $\mathrm{W}^{1,1}$, etc.) if $\tilde{u}_n$ converge to u (respectively, in L^1, in measure, weakly in $\mathrm{W}^{1,1}$, etc.), and we will say that E_n Γ-*converges to* F (respectively, with respect to the convergence in L^1, in measure, weakly in $\mathrm{W}^{1,1}$, etc.) if F_n Γ-converges to F (respectively, with respect to the convergence in L^1, in measure, weakly in $\mathrm{W}^{1,1}$, etc.); i.e., if for all u
(i) (*liminf inequality*) $F(u) \le \liminf_n F_n(u_n)$ for all u_n converging to u;
(ii) (*limsup inequality*) there exists u_n converging to u such that $\limsup_n F_n(u) \le F(u)$.

We recall that since our functionals will always be equi-coercive then Γ-convergence entails the desired convergence of minimum problems.

We will often use the following simple Γ-convergence result.

Theorem 3.3 *Let ψ_n be locally equi-bounded convex functions and let $\psi = \lim_n \psi_n$. Then the functionals defined on $W^{1,1}(0, L)$ by*

$$\int_0^L \psi_n(u')\, dt$$

Γ-*converge with respect to the weak convergence in $W^{1,1}(0, L)$ to the functional defined on $W^{1,1}(0, L)$ by*

$$\int_0^L \psi(u')\, dt.$$

3.1 Convex energies

First we briefly recall the case when the energies ψ_n^j are convex. We will see that in the case of nearest neighbours, the limit is obtained by simply replacing sums by integrals, while in the case of long-range interactions a superposition principle holds.

3.1.1 Nearest-neighbour interactions: an identification principle

We start by considering the case $K = 1$, so that the functionals E_n are given by

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi_n\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big). \tag{3.10}$$

The integral counterpart of E_n is given by

$$F_n(v)=\begin{cases}\displaystyle\int_0^L \psi_n(v')dx & \text{if } v\in\mathcal{A}_n(0,L)\\ +\infty & \text{otherwise.}\end{cases}\tag{3.11}$$

The following result states that as n approaches ∞ the identification of E_n with its continuous analog is complete.

Theorem 3.4 *Let $\psi_n:\mathbf{R}\to[0,+\infty)$ be convex and locally equi-bounded. Let E_n be given by* (3.10)*and let $\psi=\lim_n\psi_n$ (note that it is not restrictive to suppose that such limit exists upon extraction a subsequence).*

(i) *The Γ-limit of E_n with respect to the weak convergence in $\mathrm{W}^{1,1}(0,L)$ is given by F defined by*

$$F(u)=\int_{(0,L)}\psi(u')\,dx.\tag{3.12}$$

(ii) *If*

$$\lim_{|z|\to\infty}\frac{\psi(z)}{|z|}=+\infty\tag{3.13}$$

then the Γ-limit of E_n with respect to the convergence in $L^1(0,L)$ is given by F defined by

$$F(u)=\begin{cases}\displaystyle\int_{(0,L)}\psi(u')\,dx & \textit{if } u\in\mathrm{W}^{1,1}(0,L)\\ +\infty & \textit{otherwise}\end{cases}\tag{3.14}$$

on $L^1(0,L)$.

This theorem can be seen as a particular case of many results. However, since its proof is particularly simple we include it for the reader's convenience.

Proof (i) By Theorem 3.3 we have Γ-$\liminf_j F_j(u)\geq F(u)$. Conversely, fixed $u\in\mathrm{W}^{1,\infty}(0,L)$ let $u_n\in\mathcal{A}_n(0,L)$ be such that $u_n(x_i^n)=u(x_i^n)$. By convexity we have

$$\int_{x_i^n}^{x_{i+1}^n}\psi(u')\,dt\geq\lambda_n\psi\Big(\frac{1}{\lambda_n}\int_{x_i^n}^{x_{i+1}^n}u'\,dt\Big)=\lambda_n\psi\Big(\frac{u(x_{i+1}^n)-u(x_i^n)}{\lambda_n}\Big);$$

hence, summing up, letting $n\to+\infty$ and using the pointwise convergence of ψ_n to ψ, we get

$$\int_0^L\psi(u')\,dt=\lim_n\int_0^L\psi_n(u')\,dt\geq\limsup_n E_n(u_n).$$

By a density argument we recover the same inequality on the whole $\mathrm{W}^{1,1}(0,L)$.

(ii) If (3.13) holds then the sequence (E_n) is equi-coercive on bounded sets of $L^1(0,L)$ with respect to the weak convergence in $\mathrm{W}^{1,1}(0,L)$, from which the thesis is easily deduced. $\square$

3.1.2 Long-range interactions: a superposition principle

Let now $K \in N$ be fixed. The energies E_n take the form

$$E_n(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big(\frac{u_{i+j} - u_i}{j\lambda_n} \Big). \tag{3.15}$$

In this case E_n can be seen as the superposition of K nearest-neighbour functionals to which we can apply the result above. The theorem below can be easily proven and is a particular case of the results in thye sequel.

Theorem 3.5 *Let $\psi_n^j : \mathbf{R} \to [0, +\infty)$ be convex and locally equi-bounded. Let E_n be given by (3.15) and for all j let $\psi^j = \lim_n \psi_n^j$ (note that it is not restrictive to suppose that such limit exists upon extraction a subsequence). Let ψ^1 satisfy*

$$\lim_{|z|\to\infty} \frac{\psi^1(z)}{|z|} = +\infty \tag{3.16}$$

then the Γ-limit of E_n with respect to the convergence in $L^1(0,L)$ is given by F defined by

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \textit{if } u \in \mathrm{W}^{1,1}(0,L) \\ +\infty & \textit{otherwise} \end{cases} \tag{3.17}$$

on $L^1(0,L)$, where

$$\psi = \sum_{j=1}^{K} \psi^j. \tag{3.18}$$

3.2 Non-convex energies

We now investigate the effects of the lack of convexity.

3.2.1 Nearest-neighbour interactions: a convexification principle

In the case $K = 1$ the only effect of the passage from the discrete setting to the continuum is a convexification of the integrand.

Theorem 3.6 *Let $\psi_n : \mathbf{R} \to [0, +\infty)$ be locally equi-bounded Borel functions satisfying (3.13), and suppose that $\psi = \lim_n \psi_n^{**}$. Let E_n be given by (3.10); then the Γ-limit of E_n with respect to the convergence in $L^1(0,L)$ is given by F defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \textit{if } u \in \mathrm{W}^{1,1}(0,L) \\ +\infty & \textit{otherwise} \end{cases} \tag{3.19}$$

on $L^1(0,L)$. In particular if $\psi_n = \widehat{\psi}$ then $\psi = \widehat{\psi}^{**}$

Proof The proof is a particular case of the results in the sequel. □

3.2.2 Next-to-nearest neighbour interactions: non-convex relaxation

In the non-convex setting, the case $K = 2$ offers an interesting way of describing the two-level interactions between first and second neighbours. Such description is more difficult in the case $K \geq 3$. Essentially, the way the limit continuum theory is obtained is by first integrating-out the contribution due to nearest neighbours by means of an inf-convolution procedure and then by applying the previous results to the resulting functional.

Theorem 3.7 *Let $\psi_n^1, \psi_n^2 : \mathbf{R} \to [0, +\infty)$ be locally equi-bounded Borel functions such that*

$$\lim_{|z|\to\infty} \frac{\psi_n^1(z)}{|z|} = +\infty, \tag{3.20}$$

uniformly in n, and let $E_n(u) : \mathcal{A}_n(0, L) \to [0, +\infty)$ be given by

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi_n^1\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big) + \sum_{i=0}^{n-2} \lambda_n \psi_n^2\Big(\frac{u_{i+2} - u_i}{2\lambda_n}\Big) \tag{3.21}$$

Let $\tilde{\psi}_n : \mathbf{R} \to [0, +\infty)$ be defined by

$$\begin{aligned} \tilde{\psi}_n(z) &= \psi_n^2(z) + \frac{1}{2}\inf\{\psi_n^1(z_1) + \psi_n^1(z_2)) : z_1 + z_2 = 2z\} \\ &= \inf\Big\{\psi_n^2(z) + \frac{1}{2}(\psi_n^1(z_1) + \psi_n^1(z_2)) : z_1 + z_2 = 2z\Big\}, \end{aligned} \tag{3.22}$$

and suppose that

$$\psi = \lim_n \tilde{\psi}_n^{**}. \tag{3.23}$$

Then the Γ-limit of E_n with respect to the convergence in $L^1(0, L)$ is given by F defined by

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \text{if } u \in \mathrm{W}^{1,1}(0, L) \\ +\infty & \text{otherwise} \end{cases} \tag{3.24}$$

on $L^1(0, L)$.

Remark 3.8 (i) The growth conditions on ψ_n^2 can be weakened, by requiring that $\psi_n^2 : \mathbf{R} \to \mathbf{R}$ and

$$-c_1\psi_n^1 \leq \psi_n^2 \leq c_2(1 + \psi_n^1)$$

provided that we still have

$$\lim_{|z|\to\infty} \frac{\psi(z)}{|z|} = +\infty.$$

(ii) If ψ_n^1 is convex then $\tilde{\psi}_n = \psi_n^1 + \psi_n^2$. If also ψ_n^2 is convex then we recover a particular case of Theorem 3.5.

Proof Let $u \in \mathcal{A}_n(0,L)$. We have, regrouping the terms in the summation,

$$\begin{aligned}
E_n(u) &= \sum_{\substack{i=0\\ i \text{ even}}}^{n-2} \lambda_n\Big(\psi_n^2\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)+\frac{1}{2}\psi_n^1\Big(\frac{u_{i+2}-u_{i+1}}{\lambda_n}\Big)+\frac{1}{2}\psi_n^1\Big(\frac{u_{i+2}-u_{i+1}}{\lambda_n}\Big)\Big)\\
&\quad+\sum_{\substack{i=0\\ i \text{ odd}}}^{n-2} \lambda_n\Big(\psi_n^2\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)+\frac{1}{2}\psi_n^1\Big(\frac{u_{i+2}-u_{i+1}}{\lambda_n}\Big)+\frac{1}{2}\psi_n^1\Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big)\Big)\\
&\quad+\frac{\lambda_n}{2}\psi_n^1\Big(\frac{u_n-u_{n-1}}{\lambda_n}\Big)+\frac{1}{2}\psi_n^1\Big(\frac{u_1-u_0}{\lambda_n}\Big)\\
&\geq \frac{1}{2}\left(\sum_{\substack{i=0\\ i \text{ even}}}^{n-2} 2\lambda_n\tilde\psi_n\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)+\sum_{\substack{i=0\\ i \text{ odd}}}^{n-2} 2\lambda_n\tilde\psi_n\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)\right)\\
&\geq \frac{1}{2}\left(\sum_{\substack{i=0\\ i \text{ even}}}^{n-2} 2\lambda_n\tilde\psi_n^{**}\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)+\sum_{\substack{i=0\\ i \text{ odd}}}^{n-2} 2\lambda_n\tilde\psi_n^{**}\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)\right)\\
&= \frac{1}{2}\Big(\int_0^{2\lambda_n[n/2]}\tilde\psi_n^{**}(\tilde u_1')\,dt+\int_{\lambda_n}^{(1+2[n-1/2])\lambda_n}\tilde\psi_n^{**}(\tilde u_2')\,dt\Big),
\end{aligned}\tag{3.25}$$

where $\tilde u_k$, respectively, with $k=1,2$, are the continuous piecewise-affine functions such that

$$\tilde u_k' = \frac{u_{i+2}-u_i}{2\lambda_n} \qquad \text{on } (x_i^n, x_{i+2}^n) \tag{3.26}$$

for i, respectively, even or odd.

Let now $u_n \to u$ in $L^1(0,L)$ and $\sup_n E_n(u_n) < +\infty$; then $u_n \rightharpoonup u$ in $\mathrm{W}^{1,1}(0,L)$. Let $u_{k,n}$ be defined as in (3.26); we then deduce that $u_{k,n} \rightharpoonup u$ as $n \to +\infty$, for $k=1,2$. For every fixed $\eta>0$ by (3.25) we obtain

$$\begin{aligned}
\liminf_n E_n(u_n) &\geq \frac{1}{2}\left(\liminf_n \int_\eta^{L-\eta}\tilde\psi_n^{**}(u_{1,n}')\,dt+\liminf_n\int_\eta^{L-\eta}\tilde\psi_n^{**}(u_{2,n}')\,dt\right)\\
&\geq \int_\eta^{L-\eta}\psi(u')\,dt
\end{aligned}$$

by Theorem 3.3, and the liminf inequality follows by the arbitrariness of $\eta>0$.

Now we prove the limsup inequality. By an easy relaxation argument, it suffices to treat the case when $\tilde\psi_n$ is lower semicontinuous, $u(x)=zx$ and $\psi(z)=\lim_n\tilde\psi_n(z)$. With fixed $\eta>0$ let z_1^n, z_2^n be such that $z_1^n+z_2^n=2z$ and

$$\psi_n^2(z)+\frac{1}{2}(\psi_n^1(z_1^n)+\psi_n^2(z_2^n)) \leq \tilde\psi(z)+\eta$$

for all n sufficiently large. We define the recovery sequence u_n as

$$u_n(x_i^n) = \begin{cases} zx_i^n & \text{if } i \text{ is even}\\ z(i-1)\lambda_n + z_1^n\lambda_n & \text{if } i \text{ is odd.}\end{cases}$$

We then have

$$\begin{aligned} E_n(u_n) &= \sum_{i=0}^{n-1} \lambda_n \psi_n^1 \Big(\frac{u_n(x_{i+1}^n) - u_n(x_i^n)}{\lambda_n} \Big) + \sum_{i=0}^{n-2} \lambda_n \psi_n^2 \Big(\frac{u_n(x_{i+2}^n) - u_n(x_i^n)}{2\lambda_n} \Big) \\ &\le \frac{L}{2}(\psi_n^1(z_1^n) + \psi_n^1(z_2^n)) + L\psi_n^2(z) \le L\psi(z) + L\eta = F(u) + L\eta, \end{aligned}$$

and the limsup inequality follows by the arbitrariness of η. □

Remark 3.9 (Multiple-scale effects) The formula defining ψ highlights a double-scale effect. The operation of inf-convolution highlights oscillations on the scale λ_n, while the convexification of $\tilde{\psi}$ acts at a much larger scale.

3.2.3 Long-range interactions: a 'clustering' principle

We consider now the case of a general $K \ge 1$. In this case the effective energy density will be given by a *homogenization* formula. As the statement of the general result (which is postponed to the next section) will be quite complex, we begin by treating the case of energy densities independent of n. We suppose for the sake of simplicity that $\psi^j : \mathbf{R} \to [0, +\infty)$ are lower semicontinuous and there exists $p > 1$ such that

$$\psi^1(z) \ge c_0(|z|^p - 1), \qquad \psi^j(z) \le c_j(1 + |z|^p). \tag{3.27}$$

for all $j = 1, \ldots, K$. Before stating the convergence result we define some energy densities.

Let $N \in \mathbf{N}$. We define $\psi_N : \mathbf{R} \to [0, +\infty)$ as follows:

$$\begin{aligned} \psi_N(z) &= \min\Big\{ \frac{1}{N} \sum_{j=1}^{K} \sum_{i=0}^{N-j} \psi^j \Big(\frac{u(i+j) - u(i)}{j} \Big) \\ &\qquad u : \{0, \ldots, N\} \to \mathbf{R},\ u(i) = zi \text{ for } i \le K \text{ or } i \ge N - K \Big\}. \end{aligned} \tag{3.28}$$

Proposition 3.10 *For all $z \in \mathbf{R}$ there exists the limit $\psi(z) = \lim_N \psi_N(z)$.*

Proof With fixed $z \in \mathbf{R}$, let $N, M \in \mathbf{N}$ with $M > N$, and let u_N be a minimizer for $\psi_N(z)$. We define $u_M : \{0, \ldots, M\} \to \mathbf{R}$ as follows:

$$u_M(i) = \begin{cases} u_N(i - lN) + lNz & \text{if } lN \le i \le (l+1)N \ (0 \le l \le \frac{M}{N} - 1) \\ zi & \text{otherwise.} \end{cases}$$

Then we can estimate

$$\begin{aligned} \psi_M(z) &\le \frac{1}{M} \sum_{j=1}^{K} \sum_{i=0}^{M-j} \psi^j \Big(\frac{u_M(i+j) - u_M(i)}{j} \Big) \\ &\le \frac{1}{N} \sum_{j=1}^{K} \sum_{i=0}^{N-j} \psi^j \Big(\frac{u_N(i+j) - u_N(i)}{j} \Big) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{N}\sum_{j=1}^{K}(2K-j)\psi^j(z)+\sum_{j=1}^{K}\frac{M-[M/N]N+K-j}{M}\psi^j(z)\\
\leq\quad &\psi_N(z)+\frac{2K}{N}\sum_{j=1}^{K}\psi_j(z)+\frac{N+K}{M}\sum_{j=1}^{K}\psi_j(z)\\
\leq\quad &\psi_N(z)+c\Big(\frac{2K}{N}+\frac{N+K}{M}\Big)(1+|z|^p). \qquad (3.29)
\end{aligned}
$$

Taking first the limsup in M and then the liminf in N we deduce that

$$\limsup_M \psi_M(z) \leq \liminf_N \psi_N(z)$$

as desired □

Remark 3.11 (i) $c_0(|z|^p-1)\leq\psi^1(z)\leq\psi(z)\leq c(1+|z|^p)$;
(ii) ψ is lower semicontinuous;
(iii) ψ is convex;
(iv) for all $N\in\mathbf{N}$ we have $\psi(z)\leq\psi_N(z)+\frac{c}{N}(1+|z|^p)$.

We can state the convergence theorem.

Theorem 3.12 *Let ψ^j be as above and let E_n be defined by* (3.15)*. Then the Γ-limit of E_n with respect to the convergence in $L^1(0,L)$ is given by F defined by*

$$F(u)=\begin{cases}\displaystyle\int_{(0,L)}\psi(u')\,dx & \textit{if } u\in \mathrm{W}^{1,p}(0,L)\\ +\infty & \textit{otherwise}\end{cases} \qquad (3.30)$$

on $L^1(0,L)$, where ψ is given by Proposition 3.10.

Proof We begin by establishing the liminf inequality. Let $u_n\to u$ in $L^1(0,L)$ be such that $\sup_n E_n(u_n)<+\infty$. Note that this implies that

$$\sup_n\int_0^L|u_n'|^p\,dt<+\infty,$$

so that indeed $u_n\rightharpoonup u$ weakly in $\mathrm{W}^{1,p}(0,L)$ and hence also $u_n\to u$ in $L^\infty(0,L)$.

For all $k\in\{0,\dots,N-1\}$ let

$$\Phi_n(k)=\sum_{l\in\mathbf{N}}\int_{((k+Nl-2K)\lambda_n,(k+Nl+2K)\lambda_n)\cap(0,L)}|u_n'|^p\,dt.$$

We have

$$\sum_{k=0}^{N-1}\Phi_n(k)\leq 2K\int_0^L|u_n'|^p\,dt\leq c,$$

so that, upon choosing a subsequence if necessary, there exists k such that

$$\Phi_n(k) \leq \frac{c}{N}.$$

For the sake of notational simplicity we will suppose that this holds with $k = 0$, and also that $n = MN$ with $M \in \mathbf{N}$, so that the inequality above reads

$$\sum_{l=0}^{M-1} \int_{((Nl-2K)\lambda_n,(Nl+2K)\lambda_n)\cap(0,L)} |u_n'|^p \, dt \leq \frac{c}{N}. \tag{3.31}$$

We may always suppose so, upon first reasoning in slightly smaller intervals than $(0, L)$ and then let those intervals invade $(0, L)$.

Let v_n^N be the piecewise-affine function defined on $(0, L)$ such that

$$\begin{aligned} & v_n^N(0) = u_n(0) \\ & (v_n^N)' = u_n' \qquad \text{on } (x_i^n, x_{i+1}^n),\ Nl + K \leq i \leq N(l+1) - K - 1 \\ & (v_n^N)' = \frac{u_n((Nl + N - K)\lambda_n) - u_n((Nl + K)\lambda_n)}{(N - 2K)\lambda_n} =: z_{n,l}^N \\ & \qquad\qquad \text{on } (Nl\lambda_n, (Nl + K)\lambda_n) \cup ((N(l+1) - K - 1)\lambda_n, N(l+1)\lambda_n). \end{aligned}$$

The construction of v_n^N deserves some words of explanation. The function v_n^N is constructed on each interval $(Nl\lambda_n, (N + 1)\lambda_n)$ as equal to the function u_n (up to an additive constant) in the middle interval $((Nl+K)\lambda_n, (N(l+1)-K)\lambda_n)$, and as the affine function of slope $z_{n,l}^N$ in the remaining two intervals. Note that the construction implies that the function

$$v_{n,l}^N : \{0, \dots, N\} \to \mathbf{R}$$

defined by

$$v_{n,l}^N(i) = \frac{1}{\lambda_n} v_n^N((lN + i)\lambda_n)$$

is a test function for the minimum problem defining $\psi_N(z_{n,l}^N)$, and that

$$\begin{aligned} & \sum_{j=1}^{K} \sum_{i=Nl}^{N(l+1)-j} \lambda_n \psi^j \Big(\frac{v_n^N(x_{i+j}^n) - v_n^N(x_i^n)}{j\lambda_n} \Big) \\ = \quad & \sum_{j=1}^{K} \sum_{i=0}^{N-j} \lambda_n \psi^j \Big(\frac{v_{n,l}^N((i+j)) - v_{n,l}^N(i)}{j} \Big) \geq N\lambda_n \psi_N(z_{n,l}^N). \end{aligned} \tag{3.32}$$

Moreover, note that, by Hölder's inequality, we have

$$\int_{(0,L)} |(v_n^N)' - u_n'| \, dt \leq \Big(\frac{2K}{N} L \Big)^{1-1/p} \|u_n'\|_{L^p(0,L)} + \frac{2K}{N - 2K} \|u_n'\|_{L^1(0,L)},$$

so that, since $u_n(0) = v_n^N(0)$ we have a uniform bound

$$\|v_n^N - u_n\|_{L^\infty(0,L)} \leq \frac{c}{N}. \tag{3.33}$$

We have that

$$\begin{aligned}
E_n(u_n) &\geq \sum_{l=0}^{M-1}\sum_{j=1}^{K}\sum_{i=Nl+K}^{N(l+1)-K-j}\lambda_n\psi^j\Big(\frac{u_n(x_{i+j}^n)-u_n(x_i^n)}{j\lambda_n}\Big)\\
&= \sum_{l=0}^{M-1}\sum_{j=1}^{K}\sum_{i=Nl+K}^{N(l+1)-K-j}\lambda_n\psi^j\Big(\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big)\\
&= \sum_{l=0}^{M-1}\sum_{j=1}^{K}\sum_{i=Nl}^{N(l+1)-j}\lambda_n\psi^j\Big(\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big)\\
&\qquad -\sum_{l=0}^{M-1}\sum_{j=1}^{K}\sum_{i=Nl}^{Nl+K}\lambda_n\psi^j\Big(\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big)\\
&\qquad -\sum_{l=1}^{M}\sum_{j=1}^{K}\sum_{i=Nl-K-j}^{Nl-j}\lambda_n\psi^j\Big(\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big)\\
&=: \sum_{l=0}^{M-1}\sum_{j=1}^{K}\sum_{i=Nl}^{N(l+1)-j}\lambda_n\psi^j\Big(\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big)-I_n^1-I_n^2\\
&\geq \sum_{l=0}^{M-1}\sum_{j=1}^{K}N\lambda_n\psi_N(z_{n,l}^N)-I_n^1-I_n^2, \qquad (3.34)
\end{aligned}$$

the last estimate being given by (3.32).

We give an estimate of the term I_n^1; the term I_n^2 can be dealt with similarly. Let $i < Nl + K \leq i + j$; by the growth conditions on ψ^j and the convexity of $z \mapsto |z|^p$ we have

$$\begin{aligned}
&\psi^j\Big(\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big)\\
&\leq c\Big(1+\Big|\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big|^p\Big)\\
&\leq c\Big(1+\frac{1}{j}\sum_{k=i}^{i+j-1}\Big|\frac{v_n^N(x_{k+1}^n)-v_n^N(x_k^n)}{\lambda_n}\Big|^p\Big)\\
&\leq c\Big(1+K|z_{n,l}^N|^p+\frac{1}{\lambda_n}\int_{((Nl-2K)\lambda_n,(Nl+2K)\lambda_n)\cap(0,L)}|u_n'|^p\,dt\Big)
\end{aligned}$$

By (3.31) and the fact that $|z|^p \leq c(1+\psi_N(z))$, we deduce that

$$\begin{aligned}
I_n^1 &\leq \sum_{l=0}^{M-1}\sum_{j=1}^{K}\sum_{i=Nl}^{Nl+K}\lambda_n c\Big(1+\psi_N(z_{n,l}^N)+\frac{1}{\lambda_n}\int_{((Nl-2K)\lambda_n,(Nl+2K)\lambda_n)}|u_n'|^p\,dt\Big)\\
&\leq \frac{c}{N}+\frac{c}{N}\sum_{l=0}^{M-1}N\lambda_n\psi_N(z_{n,l}^N). \qquad (3.35)
\end{aligned}$$

Plugging this estimate and the analog for I_n^2 into (3.34) we get

$$E_n(u_n) \quad \geq \quad \Big(1 - \frac{c}{N}\Big) \sum_{l=0}^{M-1} N\lambda_n \psi_N(z_{n,l}^N) - \frac{c}{N}. \tag{3.36}$$

By Remark 3.11(iv) we have

$$\psi_N(z) \geq \psi(z) - \frac{c}{N}(1 + |z|^p) \geq \Big(1 - \frac{c}{N}\Big)\psi(z) - \frac{c}{N}.$$

From (3.36) we then have

$$E_n(u_n) \geq \Big(1 - \frac{c}{N}\Big) \sum_{l=0}^{M-1} N\lambda_n \psi(z_{n,l}^N) - \frac{c}{N} \tag{3.37}$$

Now, note that the piecewise-affine functions u_n^N defined by

$$u_n^N(0) = u_n(0) \qquad \text{and} \qquad (u_n^N)' = z_{n,l}^N \text{ on } (Nl\lambda_n, N(l+1)\lambda_n)$$

are weakly precompact in $W^{1,p}(0, L)$, so that we may suppose that $u_n^N \rightharpoonup u^N$. Then by Theorem 3.4 we have

$$\liminf_n \sum_{l=0}^{M-1} N\lambda_n \psi(z_{n,l}^N) = \liminf_n \int_0^L \psi((u_n^N)')\, dt \geq \int_0^L \psi((u^N)')\, dt, \tag{3.38}$$

so that

$$\liminf_n E_n(u_n) \geq \Big(1 - \frac{c}{N}\Big) \int_0^L \psi((u^N)')\, dt - \frac{c}{N} \tag{3.39}$$

By (3.33) and the uniform convergence of u_n to u we have

$$\|u^N - u\|_{L^\infty(0,L)} \leq \frac{c}{N}. \tag{3.40}$$

By letting $N \to +\infty$ we then obtain the thesis by the lower semicontinuity of $\int \psi(u')\, dt$.

To prove the limsup inequality it suffices to deal with the case $u(x) = zx$ since from this construction we easily obtain a recovery sequence for piecewise-affine functions and then reason by density. To exhibit a recovery sequence for such u it suffices to fix $N \in \mathbf{N}$, consider v^N a minimum point for the problem defining $\psi_N(z)$ and define

$$u_n(x_i^n) = v^N(i - Nl)\lambda_n + zNl\lambda_n \qquad \text{if } Nl \leq i \leq N(l+1).$$

We then have

$$\limsup_n E_n(u_n) \leq L\,\psi_N(z) + \frac{c}{N} \sum_{j=1}^{K} \psi^j(z),$$

and the thesis follows by the arbitrariness of N. $\square$

3.2.4 The general convergence theorem

By slightly modifying the proof of Theorem 3.12 we can easily state a general Γ-convergence result, allowing a dependence also on n for the energy densities.

Theorem 3.13 *Let $K \geq 1$. Let $\psi^j_n : \mathbf{R} \to [0, +\infty)$ be lower semicontinuous functions and let $p > 1$ exists such that*

$$\psi^1_n(z) \geq c_0(|z|^p - 1), \qquad \psi^j_n(z) \leq c_j(1 + |z|^p). \tag{3.41}$$

for all $j \in \{1, \dots, K\}$ and $n \in \mathbf{N}$. For all $N, n \in \mathbf{N}$ let $\psi_{N,n} : \mathbf{R} \to [0, +\infty)$ be defined by

$$\begin{aligned} \psi_{N,n}(z) \quad &= \quad \min\Big\{ \frac{1}{N} \sum_{j=1}^{K} \sum_{i=0}^{N-j} \psi^j_n\Big(\frac{u(i+j) - u(i)}{j}\Big) \\ &\qquad u : \{0, \dots, N\} \to \mathbf{R},\ u(i) = zi \text{ for } i \leq K \text{ or } i \geq N - K \Big\} \end{aligned} \tag{3.42}$$

Suppose that $\psi : \mathbf{R} \to [0, +\infty)$ exists such that

$$\psi(z) = \lim_N \lim_n \psi^{**}_{N,n}(z) \qquad \text{for all } z \in \mathbf{R} \tag{3.43}$$

(note that this is not restrictive upon passing to a subsequence of n and N). Let E_n be defined on $\mathcal{A}_n(0, L)$ by

$$E_n(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi^j_n\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big). \tag{3.44}$$

Then the Γ-limit of E_n with respect to the convergence in $L^1(0, L)$ is given by F defined by

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\, dx & \text{if } u \in \mathrm{W}^{1,p}(0, L) \\ +\infty & \text{otherwise} \end{cases} \tag{3.45}$$

on $L^1(0, L)$.

Proof Let $u_n \to u$ in $L^1(0, L)$. We can repeat the proof for the liminf inequality for Theorem 3.12, substituting ψ^j by ψ^j_n and ψ_N by $\psi_{N,n}$. We then deduce as in (3.38)–(3.39) that

$$\begin{aligned} \liminf_n E_n(u_n) \quad &\geq \quad \Big(1 - \frac{c}{N}\Big) \liminf_n \int_0^L \psi_{N,n}((u^N_n)')\, dt - \frac{c}{N} \\ &\geq \quad \Big(1 - \frac{c}{N}\Big) \int_0^L \psi_N((u^N)')\, dt - \frac{c}{N}, \end{aligned}$$

where $\psi_N = \lim_n \psi^{**}_{N,n}$ and the thesis by letting $N \to +\infty$.

To prove the limsup inequality it suffices to deal with the case $u(x) = zx$ since from this construction we easily obtain a recovery sequence for piecewise-affine functions and then reason by density. To exhibit a recovery sequence for such u it suffices to fix $N \in \mathbf{N}$, consider $z_{1,n}, z_{2,n}$ and $\eta_n \in [0,1]$ such that

$$\psi^{**}_{N,n}(z) = \eta_n \psi_{N,n}(z_{1,n}) + (1-\eta_n)\psi_{N,n}(z_{2,n}), \qquad z = \eta_n z_{1,n} + (1-\eta_n) z_{2,n}.$$

Let $v^N_{1,n}$, $v^N_{2,n}$ be minimum points for the problem defining $\psi_{N,n}(z_{1,n})$, $\psi_{N,n}(z_{2,n})$, respectively. For the sake of simplicity assume that there exists m such that $mN\eta_n \in \mathbf{N}$ for all n. Define

$$u_n(x^n_i) = \begin{cases} v^N_{1,n}(i - Nl)\lambda_n + zmNl\lambda_n & \text{if } mNl \le i \le mNl + mN\eta_n \\ \\ v^N_{2,n}(i - Nl - mN\eta_n)\lambda_n + zmNl + z_{1,n} mN\eta_n\lambda_n & \\ & \text{if } mNl + mN\eta_n \le i \le mN(l+1). \end{cases}$$

By the growth conditions on ψ^j_n it is easily seen that $(z_{k,n})$ are equi bounded and that

$$\sup\{v^N_{k,n}(i) - z_{k,n} i : \ i \in \{0, \dots, N\}, n \in \mathbf{N}\} < +\infty,$$

so that u_n converges to zx uniformly. We then have

$$\limsup_n E_n(u_n) \quad \le \quad L \limsup_n \psi^{**}_{N,n}(z)$$

and the thesis follows by the arbitrariness of N. □

3.2.5 Convergence of minimum problems

From Theorem 3.13 we immediately deduce the following theorem.

Theorem 3.14 *Let E_n and F be given by Theorem* 3.13*, let $f \in L^1(0,L)$ and $d > 0$. Then the minimum values*

$$m_n = \min\Big\{ E_n(u) + \int_0^L fu\,dt : \ u(0) = 0, \ u(L) = d \Big\} \tag{3.46}$$

converge to

$$m = \min\Big\{ F(u) + \int_0^L fu\,dt : \ u(0) = 0, \ u(L) = d \Big\}, \tag{3.47}$$

and from each sequence of minimizers of (3.46) *we can extract a subsequence converging to a minimizer of* (3.47)*.*

Proof Since the sequence of functionals (E_n) is equi-coercive, it suffices to show that the boundary conditions do not change the form of the Γ-limit; i.e., that for all $u \in \mathrm{W}^{1,p}(0,L)$ such that $u(0) = 0$ and $u(L) = d$ and for all $\varepsilon > 0$ there exists a sequence u_n such that $u_n(0) = 0$, $u_n(L) = d$ and $\limsup_n E_n(u_n) \le F(u) + \varepsilon$.

Let $v_n \to u$ in $L^\infty(0,L)$ be such that $\lim_n E_n(v_n) = F(u)$. With fixed $\eta > 0$ and $N \in \mathbf{N}$ let $K_n \in \mathbf{N}$ be such that

$$\lim_n K_n \lambda_n = \frac{\eta}{N}.$$

For all $l \in \{1, \dots, N\}$ let $\phi_n^{N,l} : [0,L] \to [0,1]$ be the piecewise-affine function defined by $\phi_n^{N,l}(0) = 0$,

$$\phi_n^{N,l\prime} = \begin{cases} 1/(K_n\lambda_n) & \text{on } ((l-1)K_n\lambda_n, lK_n\lambda_n) \\ -1/(K_n\lambda_n) & \text{on } ((n-lK_n)\lambda_n, (n-lK_n+K_n)\lambda_n) \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$u_n^{N,l} = \phi_n^{N,l} v_n + (1 - \phi_n^{N,l}) u.$$

We have

$$\begin{aligned} E_n(u_n^{N,l}) \quad \leq \quad & E_n(u_n) + c\Big(\int_0^{\eta + K\lambda_n} (1 + |u'|^p)\, dt + \int_{L-\eta-K\lambda_n}^L (1 + |u'|^p)\, dt\Big) \\ & + c\Big(\int_{((l-1)K_n - K)\lambda_n, (lK_n + K)\lambda_n) \cap (0,L)} |u_n'|^p \, dt \\ & + \int_{((n - lK_n - K)\lambda_n, (n - lK_n + K_n + K)\lambda_n) \cap (0,L)} |v_n'|^p \, dt \\ & + \int_0^L \frac{1}{(K_n\lambda_n)^p} |v_n - u|^p \Big) \\ \leq \quad & E_n(u_n) + c\Big(\int_0^{2\eta} (1 + |u'|^p)\, dt + \int_{L-2\eta}^L (1 + |u'|^p)\, dt\Big) \\ & + c\Big(\int_{(((l-2)\eta/N, ((l+1)\eta/N) \cup (L - (l+1)\eta/N, L - (l-2)\eta/N)) \cap (0,L)} |v_n'|^p \, dt\Big) \\ & + c\frac{N^p}{\eta^p} \|v_n - u\|^p_{L^\infty(0,L)} \end{aligned}$$

for n large enough. Since

$$\begin{aligned} & \sum_{l=1}^N \int_{((l-2)\eta/N, ((l+1)\eta/N) \cup (L - (l+1)\eta/N, L - (l-2)\eta/N) \cap (0,L)} |u_n'|^p \, dt \\ \leq \quad & 2\int_0^L (1 + |v_n'|^p)\, dt \leq c, \end{aligned}$$

for all n there exists $l_n \in \{1, \dots, N\}$ such that

$$\begin{aligned} E_n(u_n^{N,l_n}) \quad \leq \quad & E_n(v_n) + c\Big(\int_0^{2\eta} (1 + |u'|^p)\, dt + \int_{L-2\eta}^L (1 + |u'|^p)\, dt\Big) \\ & + \frac{c}{N} + c\frac{N^p}{\eta^p} \|v_n - u\|^p_{L^\infty(0,L)} \end{aligned}$$

Setting $u_n = u_n^{N,l_n}$ we then have

$$\limsup_n E_n(u_n) \le F(u) + c\Big(\int_0^{2\eta} (1 + |u'|^p)\, dt + \int_{L-2\eta}^L (1 + |u'|^p)\, dt\Big) + \frac{c}{N},$$

and the desired inequality by the arbitrariness of η and N. □

3.3 Infinite-range interactions

We now treat an example where interactions at all length must be taken into account, giving in the limit a non-local term (Dirichlet form). In order to highlight this effect, isolating it from all non-convex behaviour we will treta the quadratic case only. For the sake of simplicity we replace λ_n by a continuous parameter ε.

For all $\varepsilon > 0$ let $\rho_\varepsilon : \varepsilon\mathbf{Z} \to [0, +\infty)$. With fixed a bouded open interval (a, b), consider the discrete energies

$$\sum_{\substack{x,y\in\varepsilon\mathbf{Z}\cap(a,b)\\ x\neq y}} \varepsilon\, \rho_\varepsilon(x-y) \Big(\frac{u(x)-u(y)}{x-y}\Big)^2 \tag{3.48}$$

defined for $u : \varepsilon\mathbf{Z} \to \mathbf{R}$. Note that we may assume that ρ_ε is an even function, upon replacing $\rho_\varepsilon(z)$ by $\tilde\rho_\varepsilon(z) = (1/2)(\rho_\varepsilon(z) + \rho_\varepsilon(-z))$. We will tacitly make this simplifying assumption in the sequel.

We will consider the following hypotheses on ρ_ε:

(H1) (*equi-coerciveness of nearest-neighbour interactions*) $\inf_\varepsilon \rho_\varepsilon(\varepsilon) > 0$;

(H2) (*local uniform summability of ρ_ε*) for all $T > 0$ we have

$$\sup_\varepsilon \sum_{x\in\varepsilon\mathbf{Z}\cap(0,T)} \rho_\varepsilon(x) < +\infty.$$

Remark 3.15 Note that (H2) can be rephrased as a local uniform integrability property for $\varepsilon\rho_\varepsilon$ on $\mathbf{R}^2$: for all $T > 0$

$$\sup_\varepsilon \sum_{\substack{x,y\in\varepsilon\mathbf{Z}\\ x\neq y, |x|,|y|\le T}} \varepsilon\rho_\varepsilon(x-y) < +\infty.$$

As a consequence, if (H2) holds then, up to a subsequence, we can assume that the Radon measures

$$\mu_\varepsilon = \sum_{x,y\in\varepsilon\mathbf{Z},\ x\neq y} \varepsilon\rho_\varepsilon(x-y)\delta_{(x,y)}$$

(δ_z denotes the Dirac mass at z) locally converge weakly in $\mathbf{R}^2$ to a Radon measure μ_0, and that the Radon measures

$$\lambda_\varepsilon = \sum_{z\in\varepsilon\mathbf{Z}} \rho_\varepsilon(z)\delta_z$$

locally converge weakly in $\mathbf{R}$ to a Radon measure λ_0. These two limit measures are linked by the relation

$$\mu_0(A) = \frac{1}{\sqrt{2}} \int_{\mathbf{R}} |A_s| d\lambda_0(s), \tag{3.49}$$

where $|A_s|$ is the Lebesgue measure of the set

$$A_s = \{t \in \mathbf{R} : (s(e_1 - e_2) + t(e_1 + e_2))/\sqrt{2} \in A\}.$$

If (H1) holds then we have the orthogonal decomposition

$$\lambda_0 = \lambda_1 + c_1 \delta_0, \tag{3.50}$$

for some $c_1 > 0$ and a Radon measure λ_1 on $\mathbf{R}$. We also denote

$$\mu = \mu_0 \llcorner (\mathbf{R}^2 \setminus \Delta) \tag{3.51}$$

(the restriction of μ_0 to $\mathbf{R}^2 \setminus \Delta$, where $\Delta = \{(x, x) : x \in \mathbf{R}\}$. By the decomposition above, we have

$$\mu_0 = \mu + \frac{1}{\sqrt{2}} c_1 \mathcal{H}^1 \llcorner \Delta,$$

where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure.

Each function $u : \varepsilon\mathbf{Z} \to \mathbf{R}$ will be identified, upon slightly abusing notation, with its extension to a function $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R})$ which is continuous on $\mathbf{R}$ and affine on each interval $(i\varepsilon, (i+1)\varepsilon)$. We denote by $\mathcal{A}_\varepsilon$ the set of such functions. The energy (3.48) is extended to an equivalent functional defined on $\mathrm{L}^1(a, b)$ by setting

$$F_\varepsilon(u) = \begin{cases} \displaystyle\sum_{x,y \in \varepsilon\mathbf{Z} \cap (a,b),\ x \neq y} \varepsilon\, \rho_\varepsilon(x-y) \Big(\frac{u(x) - u(y)}{x - y}\Big)^2 & \text{if } u \in \mathcal{A}_\varepsilon \\ +\infty & \text{otherwise.} \end{cases} \tag{3.52}$$

We will investigate the Γ-limit of F_ε.

Theorem 3.16 (Compactness and representation) *If conditions* (H1) *and* (H2) *hold, then there exist a subsequence (not relabelled) of* $\{\varepsilon\}$ *converging to* 0*, a Radon measure* μ *on* $\mathbf{R}^2$ *and a constant* $c_1 > 0$ *such that the energies* F_ε Γ*-converge to the energy* F *defined on* $\mathrm{L}^1(a, b)$ *by*

$$F(u) = \begin{cases} c_1 \displaystyle\int_{(a,b)} |u'|^2 \, dt + \int_{(a,b)^2} \Big(\frac{u(x) - u(y)}{x - y}\Big)^2 d\mu(x, y) \\ \qquad\qquad\qquad\qquad \text{if } u \in W^{1,2}a, b) \\ +\infty \qquad\qquad\qquad \text{otherwise,} \end{cases} \tag{3.53}$$

where the measure μ *and* c_1 *are given by* (3.51) *and* (3.50)*, respectively.*

Remark 3.17 By taking (3.49) into account, we can also write (3.53) in the form

$$c_1 \int |u'|^2 \, dt + \int\!\!\int (u(t+s) - u(t))^2 \, d\lambda(s) \, dt, \tag{3.54}$$

with $\lambda = \sqrt{2}\, s^2 \lambda_1$ and λ_1 given by (3.50).

Proof Upon passing to a subsequence we may also assume that the measures μ_ε in Remark 3.15 converge to μ_0. Then, μ and c_1 given by (3.51) and (3.50) are well defined as well. Hence, it suffices to prove the representation for the Γ-limit along this sequence.

We begin by proving the liminf inequality. Let $u_\varepsilon \to u$ in $\mathrm{L}^1(a,b)$ be such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. By hypothesis (H1) the sequence u_ε converges weakly in $W^{1,2}(a,b)$. Note moreover that for all $\eta > 0$ the convergence

$$\frac{u_\varepsilon(x) - u_\varepsilon(y)}{x-y} \longrightarrow \frac{u(x)-u(y)}{x-y}$$

as $\varepsilon \to 0$ is uniform on $(a,b)^2 \setminus \Delta_\eta)$, where $\Delta_\eta = \{(x,y) \in \mathbf{R}^2 : \ |x-y| > \eta\}$.

With fixed $m \in \mathbf{N}$, we have the inequality

$$\begin{aligned} F_\varepsilon(u_\varepsilon) \quad \geq \quad & \sum_{\substack{x,y \in \varepsilon\mathbf{Z} \cap (a,b) \\ |x-y| \leq 4/m,\, x \neq y}} \rho_\varepsilon(x-y)\varepsilon \Big(\frac{u_\varepsilon(x) - u_\varepsilon(y)}{x-y}\Big)^2 \\ & + \sum_{\substack{x,y \in \varepsilon\mathbf{Z} \cap (a,b) \\ |x-y| > 4/m}} \varepsilon\, \rho_\varepsilon(x-y) \Big(\frac{u_\varepsilon(x) - u_\varepsilon(y)}{x-y}\Big)^2 \\ =: \quad & I^1_\varepsilon(u_\varepsilon) + I^2_\varepsilon(u_\varepsilon). \end{aligned} \tag{3.55}$$

We now estimate these two terms separately.

As for the first term, there exist positive α_ε converging to 0 as $\varepsilon \to 0$ such that

$$\lim_\varepsilon 2 \sum_{k=1}^{[\alpha_\varepsilon/\varepsilon]} \rho_\varepsilon(\varepsilon k) \geq c_1 - \frac{1}{m}.$$

Let $(a', b') \subset (a,b)$. For all $N \in \mathbf{N}$ and ε small enough we then have

$$\begin{aligned} & \sum_{\substack{x,y \in \varepsilon\mathbf{Z} \cap (a',b') \\ |x-y| \leq \alpha_\varepsilon,\ x \neq y}} \rho_\varepsilon(x-y)\varepsilon \Big(\frac{u_\varepsilon(x) - u_\varepsilon(y)}{x-y}\Big)^2 \\ & \geq \sum_{i=1}^N 2 \sum_{k=1}^{[\alpha_\varepsilon/\varepsilon]} \sum_{\substack{x,y \in \varepsilon\mathbf{Z} \cap (y_{i-1}, y_i) \\ |x-y| = \varepsilon k}} \varepsilon \rho(\varepsilon k) \Big(\frac{u(x) - u(y)}{x-y}\Big)^2 \\ & \geq \sum_{i=1}^N 2 \sum_{k=1}^{[\alpha_\varepsilon/\varepsilon]} \frac{(b'-a')}{N} \rho(\varepsilon k) \Big(\frac{u(y_i) - u(y_{i-1})}{y_i - y_{i-1}}\Big)^2 + o(1) \end{aligned}$$

as $\varepsilon \to 0$, where we have set

$$y_i = a' + \frac{i}{N}(b' - a'),$$

we have used the fact that $u_\varepsilon \to u$ uniformly and the convexity of $z \mapsto z^2$. The same reasoning applied to a set $I \subset (a, b)$ which can be written as a finite union of open intervals shows that

$$\liminf_\varepsilon I^1_\varepsilon(u_\varepsilon) \geq \Big(c_1 - \frac{1}{m}\Big) \int_I |u'|^2\, dt.$$

From this inequality and the arbitrariness of I, we easily obtain that

$$\liminf_\varepsilon I^1_\varepsilon(u_\varepsilon) \geq \Big(c_1 - \frac{1}{m}\Big) \int_{(a,b)} |u'|^2\, dt.$$

As for the second term, it suffices to remark that for all $\eta > 0$

$$\lim_\varepsilon \Big(\frac{u_\varepsilon(x) - u_\varepsilon(y)}{x - y}\Big)^2 = \Big(\frac{u(x) - u(y)}{x - y}\Big)^2$$

uniformly on $(a, b)^2 \setminus \Delta_\eta$ as $\varepsilon \to 0$, so that, by the weak convergence of μ_ε we have

$$\liminf_\varepsilon I^2_\varepsilon(u_\varepsilon) \geq \int_{(a,b)^2 \setminus \Delta_{4/m}} \Big(\frac{u(x) - u(y)}{x - y}\Big)^2 d\mu(x, y).$$

By summing up all these inequalities and letting $m \to +\infty$ we eventually get

$$\begin{aligned}\liminf_\varepsilon F_\varepsilon(u_\varepsilon) \quad &\geq \quad c_1 \int_{(a,b)} |u'|^2\, dt \\ &\quad + \int_{(a,b)^2} \Big(\frac{u(x) - u(y)}{x - y}\Big)^2 d\mu(x, y).\end{aligned}$$

The limsup inequality for smooth functions easily follows by comparing with the pointwise limit, once we remark that by (3.49) the limit measure μ does not charge $\partial(a, b)^2$. The proof is concluded by a density argument. □

Chapter 4

Discrete systems and free-discontinuity problems

We now consider dyscrete systems leading to minimization problems for functionals whose natural domains are sets of functions which admit a finite number of discontinuities. The set of these discontinuities will be an unknown of the problems, and for this reason the latter will be called 'free-discontinuity problems'.

4.1 Piecewise-Sobolev functions

To have a precise statement of free-discontinuity problems, it will be useful to define some spaces of piecewise weakly-differentiable functions.

Definition 4.1 *We say that a function* $u : (a, b) \to \mathbf{R}$ *is* piecewise constant on (a, b) *if there exist points* $a = t_0 < t_1 < \cdots < t_N < t_{N+1} = b$ *such that*

$$u(t) \text{ is constant a.e. on } (t_{i-1}, t_i) \text{ for all } i = 1, \ldots, N + 1. \tag{4.1}$$

The subspace of $\mathrm{L}^\infty(a, b)$ *of all such* u *is denoted by* $\mathrm{PC}(a, b)$*. If* $u \in \mathrm{PC}(a, b)$ *we define* $S(u)$ *as the minimal set* $\{t_1, \ldots, t_N\} \subset (a, b)$ *such that* (4.1) *holds.*

At all points $t \in (a, b)$ *we define the values* $u(t+)$ *and* $u(t-)$ *as the values taken a.e. by* u *on* $(t, t + \varepsilon)$ *and* $(t - \varepsilon, t)$*, respectively, for* ε *small enough, or, equivalently,*

$$u(t+) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} u(s)\, ds, \qquad u(t-) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} u(s)\, ds.$$

In the same way we define $u(a+)$ *and* $u(b-)$*. We finally define the functions* $u^+ : [a, b) \to \mathbf{R}$ *and* $u^- : (a, b] \to \mathbf{R}$ *as* $u^\pm(t) = u(t\pm)$*.*

Definition 4.2 *Let* $1 \leq p \leq +\infty$. *We define the space* $P\text{-}\mathrm{W}^{1,p}(a,b)$ *of* piecewise-$\mathrm{W}^{1,p}$ functions *on the bounded interval* (a,b) *as the direct sum*

$$P\text{-}\mathrm{W}^{1,p}(a,b) = \mathrm{W}^{1,p}(a,b) + \mathrm{PC}(a,b), \tag{4.2}$$

i.e. $u \in P\text{-}\mathrm{W}^{1,p}(a,b)$ *if and only if* $v \in \mathrm{W}^{1,p}(a,b)$ *and* $w \in \mathrm{PC}(a,b)$ *exist such that* $u = v + w$. *Note that* $\mathrm{W}^{1,p}(a,b) \cap \mathrm{PC}(a,b)$ *equals the set of constant functions, so that* u *and* v *are uniquely determined up to an additive constant. The function* u *inherits the notation valid for* v *and* w*; namely, we define the* jump set *of* u *and the* weak derivative *of* u *as*

$$S(u) = S(w) \qquad \text{and} \qquad u' = v', \tag{4.3}$$

respectively. Moreover, the left and right-hand side values *of* u *are defined by*

$$u^{\pm}(x) = v(x) + w^{\pm}(x). \tag{4.4}$$

Remark 4.3 Clearly, $u \in P\text{-}\mathrm{W}^{1,p}(a,b)$ if and only if there exist $a = t_0 < t_1 < \ldots < t_N = b$ such that $u \in \mathrm{W}^{1,p}(t_{i-1}, t_i)$ for all $i = 1, \ldots, N$. With this definition $S(u)$ is interpreted as the minimal of such sets of points, and $u \in L^2(a,b)$ is defined piecewise on $(a,b) \setminus S(u)$.

4.2 Some model problems

Even though the treatment of minimization problems for functionals defined on $P\text{-}\mathrm{W}^{1,p}(a,b)$ with $p > 1$ will be easily dealt with by combining the results that we have already proved for functionals defined on Sobolev functions and on piecewise-constant functions we illustrate their importance with two examples.

4.2.1 Signal reconstruction: the Mumford-Shah functional

As for functionals defined on piecewise-constant functions a model for signal reconstruction can be proposed using piecewise-Sobolev functions. Mumford and Shah proposed a model which can be translated in dimension one in the following: Given a datum g (the *distorted signal*) recover the original piecewise-smooth signal u by solving the problem

$$\min\Big\{c_1 \int_{(a,b)} |u'|^2\,dt + c_2\#(S(u)) + c_3 \int_{(a,b)} |u-g|^2\,dt : \ u \in P\text{-}W^{1,2}(a,b)\Big\}. \tag{4.5}$$

The parameters c_1, c_2, c_3 are *tuning parameters.* A large c_1 penalizes high gradients, a large c_2 forbids the introduction of too many discontinuity points, and c_3 controls the distance of u to g.

4.2.2 Fracture mechanics: the Griffith functional

A simple approach to some problems in the mechanics of brittle solids is that proposed by Griffith, which can be stated more or less like this: Each time a crack is created, an energy is spent proportional to the area of the fracture site. We consider as an example that of a brittle elastic bar subject to a forced displacement at its ends, so that volume integrals become line integrals and surface discontinuities turn into jumps. In this case, if g denotes the external body forces acting on the bar, the deformation u of the bar at equilibrium will solve the following problem:

$$\min\Big\{\int_{(a,b)} f(u')\,dt + \lambda\#(S(u)) - \int_{(a,b)} gu\,dt \\ :\ u(a)=u_a,\ u(b)=u_b,\ u^+ > u^- \text{ on } S(u)\Big\}. \tag{4.6}$$

on the space of functions $u \in P\text{-}\mathrm{W}^{1,p}(a,b)$, for some $p > 1$. The function f represents the elastic response of the bar in the unfractured region, while the condition $u^+ > u^-$ derives from the inpenetrability of matter.

4.3 Functionals on piecewise-Sobolev functions

We consider energies on $P\text{-}\mathrm{W}^{1,p}(a,b)$ of the form

$$F(u) = \int_{(a,b)} f(u')\,dt + \sum_{S(u)} \vartheta(u^+ - u^-). \tag{4.7}$$

Lower-semicontinuity and coerciveness properties for such functionals will easily follow from the corresponding properties on $\mathrm{W}^{1,p}(a,b)$ and $\mathrm{PC}(a,b)$.

Theorem 4.4 *Let $p > 1$.*

(i) (Coerciveness) *If (u_j) is a sequence in $P\text{-}\mathrm{W}^{1,p}(a,b)$ such that*

$$\sup_j\Big(\int_{(a,b)} |u_j'|^p\,dt + \#(S(u_j))\Big) < +\infty \tag{4.8}$$

and for all open sets $I \subset (a,b)$ we have $\liminf_j \inf_I |u_j| < +\infty$, then there exists a subsequence of (u_j) (not relabeled) converging in measure to some $u \in P\text{-}\mathrm{W}^{1,p}(a,b)$. Moreover, we can write $u_j = v_j + w_j$ with $v_j \in \mathrm{W}^{1,p}(a,b)$ and $w_j \in \mathrm{PC}(a,b)$, with v_j weakly converging in $\mathrm{W}^{1,p}(a,b)$ and w_j converging in measure.

(ii) (Lower semicontinuity) *If $f : \mathbf{R} \to [0+\infty]$ is a convex and lower semicontinuous function, and if $\vartheta : \mathbf{R} \to [0+\infty]$ is a subadditive and lower semicontinuous function then the functional F defined in* (4.7) *is lower semicontinuous on $P\text{-}\mathrm{W}^{1,p}(a,b)$ with respect to convergence in measure along sequences (u_j) satisfying* (4.8).

Proof (i) Let $v_j \in \mathrm{W}^{1,p}(a,b)$ be defined by

$$v_j(t) = \int_a^t u_j'(s)\,ds.$$

Since $v_j(a) = 0$ for all j, the sequence (v_j) is bounded in $\mathrm{W}^{1,p}(a,b)$ by Poincarè inequality, and hence we can extract a weakly converging subsequence (that we still denote by (v_j)) that weakly converges to some v in $\mathrm{W}^{1,p}(a,b)$. Now, set $w_j = u_j - v_j \in \mathrm{PC}(a,b)$. Since $v_j \to v$ in $\mathrm{L}^\infty(a,b)$, upon extracting a subsequence, (w_j) converges in measure to some $w \in \mathrm{PC}(a,b)$. The sequence (u_j) satisfies the required properties with $u = v + w$.

(ii) Let (u_j) satisfy (4.8) and $u_j \to u$ in measure. Then by (i) we can write $u_j = v_j + w_j$ with $v_j \in \mathrm{W}^{1,p}(a,b)$ and $w_j \in \mathrm{PC}(a,b)$, $w_j \to w$ weakly in $\mathrm{W}^{1,p}(a,b)$ and $v_j \to v \in \mathrm{PC}(a,b)$ in measure. We then get

$$F(u) = F(v) + F(w) \le \liminf_j F(v_j) + \liminf_j F(w_j) \le \liminf_j F(u_j)$$

as desired. □

Corollary 4.5 *Let $f, \vartheta : \mathbf{R} \to [0,+\infty]$ be functions satisfying*

$$c|z|^p \le f(z) \qquad \text{and} \qquad c \le \vartheta(z) \tag{4.9}$$

for all $z \in \mathbf{R}$, then the functional F defined in (4.7) *is lower semicontinuous on P-$\mathrm{W}^{1,p}(a,b)$ with respect to convergence in measure if and only if f is convex and lower semicontinuous and ϑ is subadditive and lower semicontinuous.*

Proof Let F be lower semicontinuous. Then also its restrictions to $\mathrm{W}^{1,p}(a,b)$ and to $\mathrm{PC}(a,b)$ are lower semicontinuous; hence, we deduce that f is convex and lower semicontinuous and ϑ is subadditive and lower semicontinuous. The converse is a immediate consequence of Theorem 4.4. □

4.4 Examples of existence results

As examples of an application of the lower semicontinuity theorems on the space P-$\mathrm{W}^{1,p}(a,b)$ we prove the existence of solutions for the problems outlined in Section 4.2.

Example 4.6 (Existence in Image Reconstruction problems) We use the notation of Section 4.2.1. Let $g \in \mathrm{L}^2(a,b)$ and let (u_j) be a minimizing sequence for the problem

$$m = \inf\Big\{F(u) + c_3 \int_{(a,b)} |u-g|^2\,dt :\ u \in P\text{-}W^{1,2}(a,b)\Big\}, \tag{4.10}$$

where

$$F(u) = c_1 \int_{(a,b)} |u'|^2\,dt + c_2 \#(S(u)).$$

By taking $u = 0$ as a test function, we get that $m \le \int_{(a,b)} |g|^2\,dt$. Moreover, we immediately get that (u_j) is bounded in $\mathrm{L}^2(a,b)$; hence, it satisfies the hypotheses of Theorem 4.4(i). We can thus suppose that $u_j \to u \in P$-$\mathrm{W}^{1,p}(a,b)$

in measure and a.e., so that by Theorem 4.4(ii) (with $p = 2$, $f(z) = |z|^2$ and $\vartheta(z) = 1$) $F(u) \le \liminf_j F(u_j)$, and by Fatou's Lemma

$$\int_{(a,b)} |u-g|^2\,dt \le \liminf_j \int_{(a,b)} |u_j-g|^2\,dt,$$

so that u is a minimum point for (4.10).

Example 4.7 (Existence in Fracture Mechanics problems) We use the notation of Section 4.2.2. In this case we may have to specify the boundary conditions better, as $S(u)$ may tend to a or b; i.e, the elastic bar may break at its ends. The minimization problem with relaxed boundary condition takes the form

$$m = \inf\Big\{F(u) - \int_{(a,b)} gu\,dt + \vartheta(u(a+)-u_a) + \vartheta(u_b - u(b-)) : \ u \in P\text{-}\mathrm{W}^{1,p}(a,b)\Big\}, \tag{4.11}$$

where

$$F(u) = \int_{(a,b)} f(u')\,dt + \sum_{S(u)} \vartheta(u^+ - u^-),$$

f is some convex function, which we suppose satisfies $f(z) \ge |z|^p - c$, and ϑ is defined by

$$\vartheta(z) = \begin{cases} +\infty & \text{if } z < 0 \\ 0 & \text{if } z = 0 \\ 1 & \text{if } z > 0. \end{cases}$$

Note that ϑ takes care of the inpenetrability condition, which needs not be repeated in the statement of the minimum problem in the form (4.11).

We deal with the case $u_b > u_a$, and suppose $f(0) = 0$ and $\lambda = 1$. We may use $u = (u_a + u_b)/2$ as a test function, obtaining

$$m \le F(u) + \vartheta(u(a+)-u_a) + \vartheta(u_b - u(b-)) = 2\,\vartheta\Big(\frac{u_b - u_a}{2}\Big) = 2.$$

Let (u_j) be a minimizing sequence for (4.11). We set $u_j = v_j + w_j$ with $v_j \in \mathrm{W}^{1,p}(a,b)$, $w_j \in \mathrm{PC}(a,b)$ and $v_j(a) = 0$. By the Poincarè inequality and the continuous imbedding of $\mathrm{W}^{1,p}(a,b)$ into $\mathrm{L}^\infty(a,b)$ we obtain that

$$\|v_j\|_{\mathrm{L}^\infty(a,b)} \le c\|v_j'\|_{\mathrm{L}^p(a,b)}. \tag{4.12}$$

Note that the condition $u_j^+ > u_j^-$ implies that w_j is increasing, so that

$$\|w_j\|_{\mathrm{L}^\infty(a,b)} \le |u_a| + |u_b| + c\|v_j'\|_{\mathrm{L}^p(a,b)}. \tag{4.13}$$

From the condition

$$\int_{(a,b)} f(u_j')\,dt + \sum_{S(u)} \vartheta(u_j^+ - u_j^-) - \int_{(a,b)} gu_j\,dt \le c$$

we then get in particular

$$\int_{(a,b)} |v_j'|^p \, dt - \int_{(a,b)} g v_j \, dt - \int_{(a,b)} g w_j \, dt \le c,$$

from which we deduce by (4.12)-(4.13)

$$\int_{(a,b)} |v_j'|^p \, dt - c\|v_j\|_{\mathrm{L}^\infty(a,b)} - c\|w_j\|_{\mathrm{L}^\infty(a,b)} \le c$$

and, from the inequalities above, eventually

$$\int_{(a,b)} |v_j'|^p \, dt \le c.$$

Hence, we may assume that v_j weakly converge in $\mathrm{W}^{1,p}(a,b)$, and by (4.13) we obtain that (w_j) is a bounded sequence in $\mathrm{L}^\infty(a,b)$. Hence (u_j) satisfies the assumptions of Theorem 4.4(i), so that we may assume that it converges to u in measure. Moreover, we may assume that w_j converges a.e. and in $\mathrm{L}^1(a,b)$, so that we get that u is a minimum point for (4.11) by using Theorem 4.4(ii).

4.5 Discrete systems with (sub)linear growth

We now study discrete systems with integrands satisfying a growth condition of the form

$$\psi(z) \le C(1+|z|);$$

in particular we will deal with integrands of strictly-sublinear growth. It is clear that convexification arguments in this case cannot apply since they would give trivial results (the convex envelope of such functions is constant). In this case a further principle of **separation of scales** applies. We will describe it through a series of simple examples.

4.5.1 Discretization of the Mumford Shah functional

A simple case of functionals E_n $E_n : \mathcal{A}_n(a,b) \to [0,+\infty]$ of the form

$$E_n(u) = \sum_{i=1}^{n} \lambda_n f_n(u_i - u_{i-1}). \tag{4.14}$$

is obtained by taking

$$\psi_n(z) = \frac{1}{\lambda_n} \min\{\lambda_n z^2, 1\} =: \frac{1}{\lambda_n}\psi(\lambda_n z^2). \tag{4.15}$$

Note that we have

$$\lim_n \psi_n(z) = z^2, \qquad \lim_n \lambda_n \psi_n\Big(\frac{z}{\lambda_n}\Big) = 1 \tag{4.16}$$

for all $z \in \mathbf{R}$ and for all $z \neq 0$, respectively.

Theorem 4.8 *If ψ_n are as above then the functionals E_n Γ-converge to the functional*

$$F(u) = \int_a^b |u'|^2\, dt + \#(S(u)) \tag{4.17}$$

on P-$W^{1,2}(a,b)$.

Proof Note that

$$F_n(\tilde u_n) = \int_a^b |\tilde u_n'|^2\, dt + \#(S(\tilde u_n)).$$

If $\sup_n E(u_n) < +\infty$ and $u_n \to u$ then $\tilde u_n \to u$ in measure so that $u \in P$-$W^{1,2}(a,b)$ and

$$\liminf_n E_n(u_n) = \liminf_n F_n(\tilde u_n) = \liminf_n F(\tilde u_n) \geq F(u)$$

by Theorem 4.4.

Vice versa, a recovery sequence for a function in P-$W^{1,2}(a,b)$ is easily obtained by taking $u_n(x_i) = u(x_i)$ on I_n. □

Remark 4.9 The same proof as above shows that if we take

$$\psi_n(z) = \begin{cases} \frac{1}{\lambda_n}\min\{\lambda_n c z^2, \alpha\} & \text{if } z \geq 0 \\ \frac{1}{\lambda_n}\min\{\lambda_n c z^2, \beta\} & \text{if } z \leq 0, \end{cases} \tag{4.18}$$

then the limit is

$$F(u) = c\int_a^b |u'|^2\, dt + \alpha\#(\{t \in S(u) : [u] > 0\}) + \beta\#(\{t \in S(u) : [u] < 0\}) \tag{4.19}$$

on P-$W^{1,2}(a,b)$.

4.6 Fracture as a phase transition

We now deal with the case of functionals E_n with $\psi_n(z) = J(z/\lambda_n)$; i.e,

$$E_n(u) = \sum_{i=1}^n \lambda_n J\Big(\frac{u_i - u_{i-1}}{\lambda_n}\Big)$$

Let $J : \mathbf{R} \to [0,+\infty]$ satisfy the following conditions: $J = +\infty$ on $(-\infty, 0]$, J is continuous on $[0,+\infty)$, there exists $C > 1$ such that J is convex on $(0, C]$ with minimum in 1 and concave on $[C, +\infty)$, and there exists the limit $J(+\infty) \in \mathbf{R}$.

Theorem 4.10 *Under the hypotheses above the functionals defined on $\mathcal{A}_n(a,b)$ by E_n Γ-converge to the functional F defined by*

$$F(u) = \int_{(a,b)} f(u')\, dt$$

on P-$\mathrm{W}^{1,1}(a,b)$*, with* $f(z) = J(z \wedge 1)$*, and the functionals*

$$E_n^{(1)}(u) = \frac{1}{\lambda_n}(E_n(u) - \min F))$$

Γ*-converge to the functional* $E^{(1)}$ *given by*

$$E^{(1)}(u) = \begin{cases} (J(+\infty) - J(1))\,\#(S(u)) & \text{if } u \text{ is piecewise affine on } (a,b) \\ & u' = 1 \text{ and } u^+ > u^- \text{ on } S(u) \\ \\ +\infty & \text{otherwise} \end{cases}$$

on P-$\mathrm{W}^{1,1}(a,b)$*. This functional is the first-order* Γ*-limit of* (E_n)*.*

Proof The existence of the 'zero-order' Γ-limit F and its representation follow immediately by a comparison argument since $F \le E_n$ for all n.

Note that $\min F = J(1) = \min J$ so that we can suppose that

$$E_n^{(1)}(u) = \sum_{i=1}^{n} \Big(J\Big(\frac{u(x_i) - u(x_{i-1})}{\lambda_n}\Big) - J(1)\Big).$$

We check now the first-order Γ-limit. We first give an estimate from below by comparison. Let $\gamma > 0$ be given such that

$$J(1) + \gamma(z-1)^2 \le J(z) \text{ for } z \le 1,$$

$$J(1) + \min\{\gamma(z-1)^2, J(+\infty) - \gamma\} \le J(z) \text{ for } z \ge 1.$$

We then have

$$E_n^{(1)}(u) \ge \sum_{i=1}^{n} \psi_n\Big(\frac{u(x_i) - u(x_{i-1})}{\lambda_n}\Big),$$

whenever n is large enough and ψ_n is given by (4.18), where $\alpha = J(+\infty) - \gamma - J(1)$, and c and β are arbitrary.

Upon changing variables and considering $v(t) = u(t) - t$, we can apply Remark 4.9 to estimate from below the Γ-limit by

$$\begin{aligned} F(u) \quad = \quad & c\int_a^b |u' - 1|^2\,dt + (J(+\infty) - \gamma - J(1))\#(\{t \in S(u) : [u] > 0\}) \\ & +\beta\#(\{t \in S(u) : [u] < 0\}). \end{aligned} \tag{4.20}$$

Since c, β and γ are arbitrary positive numbers we obtain the desired estimate from below.

To complete the proof it suffices to exhibit a recovery sequence for such a u. Let u_n be defined simply by $u_n(x_i) = u(x_i)$. It suffices to consider the case of a single jump: $S(u) = \{x_0\}$, with $u(x_0+) = z$, $u(x_0-) = 0$. In this case we trivially have

$$\lim_n E^{(1)}(u_n) = \lim_n \Big(J\Big(\frac{z}{\lambda_n}\Big) - J(1)\Big) = J(+\infty) - J(1),$$

and the proof is concluded. $\square$

Chapter 5

Discrete systems leading to phase transitions

If the function ψ giving the limit energy density in Theorem 3.13 is not strictly convex, converging sequences of minimizers of problems of the type (3.46) may converge to particular minimizers of (3.47). This happens in the case of next-to-nearest interactions, where the formula giving ψ is of particular help.

5.1 Equivalence with phase transitions

We consider next-to-nearest-neighbour energies of the form

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi^1\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big) + \sum_{i=0}^{n-2} \lambda_n \psi^2\Big(\frac{u_{i+2} - u_i}{2\lambda_n}\Big), \tag{5.1}$$

and $\tilde{\psi} : \mathbf{R} \to [0, +\infty)$ defined by

$$\begin{aligned} \tilde{\psi}(z) &= \psi^2(z) + \frac{1}{2} \inf\{\psi^1(z_1) + \psi^1(z_2)) : z_1 + z_2 = 2z\} \\ &= \inf\Big\{\psi^2(z) + \frac{1}{2}(\psi^1_n(z_1) + \psi^1(z_2)) : z_1 + z_2 = 2z\Big\}, \end{aligned} \tag{5.2}$$

We show that for some type of second-neighbour interactions the energies E_n are *equivalent* (in the sense of Definition 2.43) to functionals of the form

$$F_n(u) = \int_0^L (W(u') + c_1 \lambda_n^2 |u''|^2)\, dt + c_2$$

with respect to Dirichlet boundary data. From Theorem 3.7 we may take

$$W = \tilde{\psi}.$$

In the following section we characterize c_1 and c_2, that are given the interpretation of the effect of surface tension and boundary layers, respectively.

5.2 Study of minimum problems

We examine the case when $\tilde{\psi}$ in (3.22) is not convex and of minimum problems (3.46) with $f = 0$. Upon some change of coordinates it is not restrictive to examine problems of the form

$$m_n = \min\{E_n(u) : \ u(0) = 0, \ u(L) = d\}, \tag{5.3}$$

. We will treat in detail only the case $d = 0$. We may also suppose

(H1) we have

$$\min \tilde{\psi} = \tilde{\psi}(1) = \tilde{\psi}(-1). \tag{5.4}$$

For the sake of simplicity we make the additional assumptions

(H2) we have

$$\tilde{\psi}(z) > 0 \text{ if } |z| \neq 1; \tag{5.5}$$

(H3) there exist unique z_1^+, z_2^+ and z_1^-, z_2^- such that

$$\psi^2(\pm 1) + \frac{1}{2}\Big(\psi^1(z_1^\pm) + \psi^1(z_2^\pm)\Big) = \min \tilde{\psi}, \qquad z_1^\pm, z_2^\pm = \pm 2;$$

We set

$$\mathbf{M}^+ = \{(z_1^+, z_2^+), (z_2^+, z_1^+)\}, \qquad \mathbf{M}^- = \{(z_1^-, z_2^-), (z_2^-, z_1^-)\} \tag{5.6}$$

$$\mathbf{M} = \mathbf{M}^+ \cup \mathbf{M}^-. \tag{5.7}$$

(H4) we have $z_i^+ \neq z_j^-$ for all $i, j \in \{1, 2\}$;

(H5) all functions are C^1.

Under hypotheses (H1)–(H2) Theorem 3.13 simply gives that $m_n \to 0$ and that the limits u of minimizers satisfy $|u'| \leq 1$ a.e. We will see that indeed they are 'extremal' solutions to the problem

$$\min\{F(u) : \ u(0) = 0, \ u(L) = 0\}. \tag{5.8}$$

The effect of the non validity of hypotheses (H3)–(H5) is explained in Remark 5.5.

The key idea is that it is energetically convenient for discrete minimizer to remain close to the two states minimizing $\tilde{\psi}$, and that every time we have a transition from one of the two minimal configurations to the other a fixed amount of energy is spent (independent of n). To exactly quantify this fact we introduce some functions and quantities.

Definition 5.1 (Minimal energy configurations) *Let* $\mathbf{z} = (z_1, z_2) \in \mathbf{M}$*; we define* $u^{\mathbf{z}} : \mathbf{Z} \to \mathbf{R}$ *by*

$$u^{\mathbf{z}}(i) = \Big[\frac{i}{2}\Big] z_2 + \Big(i - \Big[\frac{i}{2}\Big]\Big) z_1, \tag{5.9}$$

and $u_n^{\mathbf{z}} : \lambda_n \mathbf{Z} \to \mathbf{R}$ *by*

$$u_n^{\mathbf{z}}(x_i^n) = u^{\mathbf{z}}(i)\lambda_n \tag{5.10}$$

Definition 5.2 (Crease and boundary-layer energies) Let $v : \mathbf{Z} \to \mathbf{R}$. The *right-hand side boundary layer energy of* v is

$$\begin{aligned} B_+(v) \quad = \quad & \inf_{N\in\mathbf{N}} \min\Big\{ \sum_{i\geq 0} \Big(\psi^2\Big(\frac{u(i+2)-u(i)}{2}\Big) + \psi^1(u(i+1)-u(i)) - \min\tilde{\psi}\Big) \\ & : u : \mathbf{N}\cup\{0\} \to \mathbf{R},\ u(i) = v(i) \text{ if } i \geq N \Big\}, \end{aligned}$$

The *left-hand side boundary layer energy of* v is

$$\begin{aligned} B_-(v) \quad = \quad & \inf_{N\in\mathbf{N}} \min\Big\{ \sum_{i\leq 0} \Big(\psi^2\Big(\frac{u(i)-u(i-2)}{2}\Big) + \psi^1(u(i)-u(i-1)) - \min\tilde{\psi}\Big) \\ & : u : -\mathbf{N}\cup\{0\} \to \mathbf{R},\ u(i) = v(i) \text{ if } i \leq -N \Big\}, \end{aligned}$$

Let $v^\pm : \mathbf{Z} \to \mathbf{R}$. The *transition energy between* v^- *and* v^+ is

$$\begin{aligned} C(v^-, v^+) \quad = \quad & \inf_{N\in\mathbf{N}} \min\Big\{ \sum_{i\in\mathbf{Z}} \Big(\psi^2\Big(\frac{u(i+2)-u(i)}{2}\Big) + \psi^1(u(i+1)-u(i)) - \min\tilde{\psi}\Big) \\ & : u : \mathbf{Z} \to \mathbf{R},\ c^\pm \in \mathbf{R}, u(i) = v^\pm(i) + c^\pm \text{ if } \pm i \geq N \Big\}. \end{aligned}$$

Remark 5.3 Condition (H4) implies that

$$C(u^{\mathbf{z}^+}, u^{\mathbf{z}^-}) > 0, \qquad C(u^{\mathbf{z}^-}, u^{\mathbf{z}^+}) > 0$$

if $\mathbf{z}^\pm \in \mathbf{M}^\pm$.

We can now describe the behaviour of minimizing sequences for (3.46).

Theorem 5.4 *Suppose that* (H1)–(H5) *hold. We then have:*

(Case n even) *The minimizers* (u_n) *of* (3.46) *for* n *even converge, up to subsequences, to one of the functions*

$$\overline{u}_+(x) = \begin{cases} x & \text{if } 0 \leq x \leq L/2 \\ L-x & \text{if } L/2 \leq x \leq L, \end{cases} \qquad \overline{u}_-(x) = \begin{cases} -x & \text{if } 0 \leq x \leq L/2 \\ -(L-x) & \text{if } L/2 \leq x \leq L \end{cases}.$$

Let

$$\begin{aligned} D \quad := \quad \min\Big\{ & B_+(u^{\mathbf{z}^+}) + C(u^{\mathbf{z}^+}, u^{\mathbf{z}^-}) + B_-(u^{\mathbf{z}^-}), \\ & B_+(u^{\mathbf{z}^-}) + C(u^{\mathbf{z}^-}, u^{\mathbf{z}^+}) + B_-(u^{\mathbf{z}^+}) :\ \mathbf{z}^+ \in \mathbf{M}^+,\ \mathbf{z}^- \in \mathbf{M}^- \}. \end{aligned}$$

If (u_n) *converges (up to subsequences) to* $\overline{u}_\pm$ *then there exist* $\mathbf{z}^+ \in \mathbf{M}^+$, *and* $\mathbf{z}^- \in \mathbf{M}^-$ *such that*

$$D = B_+(u^{\mathbf{z}^+}) + C(u^{\mathbf{z}^+}, u^{\mathbf{z}^-}) + B_-(u^{\mathbf{z}^-}) \tag{5.11}$$

and

$$E_n(u_n) = D\,\lambda_n + o(\lambda_n). \tag{5.12}$$

(Case n odd) *In the case n odd the same conclusions hold, upon substituting terms of the form*

$$B_+(u^{\mathbf{z}^\pm}) + C(u^{\mathbf{z}^\pm}, u^{\mathbf{z}^\mp}) + B_-(u^{\mathbf{z}^\mp})$$

by terms of the form

$$B_+(u^{\mathbf{z}^\pm}) + C(u^{\mathbf{z}^\pm}, u^{\mathbf{z}^\mp}) + B_-(u^{\overline{\mathbf{z}^\mp}}),$$

where we have set $\overline{(z_1, z_2)} = (z_2, z_1)$.

Proof We only deal with the case n even, as the case n odd is dealt with similarly.

Let u_n be a minimizer for (3.46). We may assume that u_n converge in $\mathrm{W}^{1,p}(0, L)$ and uniformly. By comparison with $E_n(\overline{u})$ we have

$$E_n(u_n) \le L \min \tilde{\psi} + c\lambda_n. \tag{5.13}$$

We can consider the scaled energies

$$E_n^1(u) = \frac{1}{\lambda_n}(E_n(u) - L \min \tilde{\psi}). \tag{5.14}$$

Note that we have

$$\begin{aligned} E_n^1(u) \quad = \quad & \sum_{i=0}^{n-2} \Big(\psi^2 \Big(\frac{u_{i+2} - u_i}{2\lambda_n} \Big) \\ & + \frac{1}{2} \Big(\psi^1 \Big(\frac{u_{i+2} - u_{i+1}}{\lambda_n} \Big) + \psi^1 \Big(\frac{u_{i+1} - u_i}{\lambda_n} \Big) \Big) - \min \tilde{\psi} \Big) \\ & + \frac{1}{2} \Big(\psi^1 \Big(\frac{u_n - u_{n-1}}{\lambda_n} \Big) + \psi^1 \Big(\frac{u_1 - u_0}{\lambda_n} \Big) \Big) - \min \tilde{\psi}. \end{aligned} \tag{5.15}$$

From (5.13) and (5.15) we deduce that

$$\begin{aligned} & \sum_{i=0}^{n-2} \Big(\psi^2 \Big(\frac{u_n(x_{i+2}^n) - u_n(x_i^n)}{2\lambda_n} \Big) \\ & + \frac{1}{2} \Big(\psi^1 \Big(\frac{u_n(x_{i+2}^n) - u_n(x_{i+1}^n)}{\lambda_n} \Big) + \psi^1 \Big(\frac{u_n(x_{i+1}^n) - u_n(x_i^n)}{\lambda_n} \Big) \Big) - \min \tilde{\psi} \Big) \le c. \end{aligned}$$

We infer that for every $\eta > 0$ we have that if we denote by $I_n(\eta)$ the set of indices i such that

$$\begin{aligned} & \psi^2 \Big(\frac{u_n(x_{i+2}^n) - u_n(x_i^n)}{2\lambda_n} \Big) \\ & + \frac{1}{2} \Big(\psi^1 \Big(\frac{u_n(x_{i+2}^n) - u_n(x_{i+1}^n)}{\lambda_n} \Big) + \psi^1 \Big(\frac{u_n(x_{i+1}^n) - u_n(x_i^n)}{\lambda_n} \Big) \Big) \le \min \tilde{\psi} + \eta \end{aligned}$$

then

$$\sup_n I_n(\eta) < +\infty.$$

Let $\varepsilon = \varepsilon(\eta)$ be defined so that if

$$\psi^2\Big(\frac{z_1+z_2}{2}\Big)+\frac{1}{2}\Big(\psi^1(z_1)+\psi^1(z_2)\Big)-\min\tilde{\psi}\le\eta$$

then

$$\mathrm{dist}\,((z_1,z_2),\mathbf{M})\le\varepsilon(\eta).$$

Choose $\eta>0$ so that

$$2\varepsilon(\eta)<\min\{|\mathbf{z}^+-\mathbf{z}^-|,\ \mathbf{z}^+\in\mathbf{M}^+,\ \mathbf{z}^-\in\mathbf{M}^-\}.$$

We then deduce that if $i-1,i\notin I_n(\eta)$ then there exists $\mathbf{z}\in\mathbf{M}$ such that

$$\Big|\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_i)}{\lambda_n},\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)-\mathbf{z}\Big|\le\varepsilon$$

and

$$\Big|\Big(\frac{u_n(x^n_{i})-u_n(x^n_{i-1})}{\lambda_n},\frac{u_n(x^n_{i+1})-u_n(x^n_{i})}{\lambda_n}\Big)-\overline{\mathbf{z}}\Big|\le\varepsilon$$

Hence, there exist a finite number of indices $0=i_0<i_1<i_2<\cdots<i_{N_n}=n$ such that for all $j=1,\ldots,N_n$ there exists $\mathbf{z}^n_j\in\mathbf{M}$ such that for all $i\in\{i_{j-1}+1,\ldots,i_j-1\}$ we have

$$\Big|\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_i)}{\lambda_n},\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)-\mathbf{z}^n_j\Big|\le\varepsilon.$$

Let $\{j_0,j_1,\ldots,j_{M_n}\}$ be the maximal subset of $\{i_0,i_1,\ldots,i_{N_n}\}$ defined by the requirement that if $z^n_{j_k}\in\mathbf{M}^\pm$ then $z^n_{j_k+1}\in\mathbf{M}^\mp$. Note that in this case we deduce that $E_n(u_n)\ge cM_n$, so that M_n are equi-bounded. Upon choosing a subsequence we may then suppose $M_n=M$ independent of n, and also that $x^n_{j_k}\to x_k\in[0,L]$ and $\mathbf{z}^n_{j_k}=\mathbf{z}_k$. By the arbitrariness of η we deduce that $\lim_n u_n=u$, and u is characterized by $u(0)=u(L)=L$ and $u'=\pm1$ on (x_{k-1},x_k), the sign determined by whether $\mathbf{z}_k\in\mathbf{M}^+$ or $\mathbf{z}_k\in\mathbf{M}^+$. Let $y_0=0,y_1,\ldots,y_N=L$ be distinct ordered points such that $\{y_i\}=\{x_k\}$ (the set of indices may be different if $x_k=x_{k+1}$ for some k). Choose indices $k_1,\ldots,k_N$ such that $x^n_{k_j}\to(y_{j-1}+y_j)/2$. Let $\mathbf{z}_j$ be the limit of $\mathbf{z}^n_{j_k}$ related to the interval (y_j,y_{j+1}). We then have, for a suitable continuous $\omega:[0,+\infty)\to[0,+\infty)$,

$$\begin{aligned}&\sum_{i=0}^{k_1-2}\Big(\psi^2\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_i)}{2\lambda_n}\Big)\\&+\frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)+\psi^1\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_i)}{\lambda_n}\Big)\Big)-\min\tilde{\psi}\Big)\\\ge\ &B_+(u^{\mathbf{z}_1})-\omega(\varepsilon)\end{aligned}$$

$$\sum_{i=k_j}^{k_{j+1}-2}\Big(\psi^2\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_i)}{2\lambda_n}\Big)$$

$$
\begin{aligned}
&+\frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)+\psi^1\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_{i})}{\lambda_n}\Big)\Big)-\min\tilde\psi\Big)\\
\ge\quad & C(u^{\mathbf{z}_j},u^{\mathbf{z}_{j+1}})-\omega(\varepsilon)\text{ for all } j\in\{1,\dots,N-1\},
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=k_N}^{n-2}\Big(\psi^2\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i})}{2\lambda_n}\Big)\\
&+\frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)+\psi^1\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_{i})}{\lambda_n}\Big)\Big)-\min\tilde\psi\Big)\\
\ge\quad & B_-(u^{\mathbf{z}_N})-\omega(\varepsilon).
\end{aligned}
$$

By the arbitrariness of ε and the definition of D we easily get $\liminf_n E^1_n(u_n)\ge D$, and by Remark 5.3 that if $u\ne\overline{u}_\pm$ then $\liminf_n E^1_n(u_n)>D$.

It remains to show that $\limsup_n E^1_n(u_n)\le D$; i.e., for every fixed $\eta>0$ to exhibit a sequence $\overline{u}_n$ such that $\overline{u}_n(0)=\overline{u}_n(L)=0$ and $\limsup_n E^1_n(\overline{u}_n)\le D+c\eta$. Suppose that

$$
D=B_+(u^{\mathbf{z}^+})+C(u^{\mathbf{z}^+},u^{\mathbf{z}^-})+B_-(u^{\mathbf{z}^-}),
$$

with $\mathbf{z}^+=(z_1^+,z_2^+)$, $\mathbf{z}^-=(z_1^-,z_2^-)$, the other cases being dealt with in the same way. Let $\eta>0$ be fixed and let $N\in\mathbf{N}$, $v_+,v_-,v:\mathbf{Z}\to\mathbf{R}$ be such that

$$
v_+(i)=u^{\mathbf{z}^+}(i)\qquad\text{for } i\ge N,
$$

$$
v_-(i)=u^{\mathbf{z}^-}(i)\qquad\text{for } i\le -N,
$$

$$
v(i)=\begin{cases} u^{\mathbf{z}^+}(i) & \text{for } i\le -N\\ u^{\mathbf{z}^-}(i) & \text{for } i\ge N\end{cases},
$$

and

$$
\sum_{i\ge 0}\Big(\psi^2\Big(\frac{v_+(i+2)-v_+(i)}{2}\Big)+\psi^1(u(i+1)-u(i))-\min\tilde\psi\Big)\le B_+(u^{\mathbf{z}^+})+\eta
$$

$$
\sum_{i\le 0}\Big(\psi^2\Big(\frac{v_-(i)-v_-(i-2)}{2}\Big)+\psi^1(u(i)-u(i-1))-\min\tilde\psi\Big)\le B_-(u^{\mathbf{z}^-})+\eta
$$

$$
\sum_{i\in\mathbf{Z}}\Big(\psi^2\Big(\frac{v(i+2)-v(i)}{2}\Big)+\psi^1(v(i+1)-v(i))-\min\tilde\psi\Big)\le C(u^{\mathbf{z}^+},u^{\mathbf{z}^-})+\eta.
$$

We then set

$$\overline{u}(x_i^n) = \begin{cases} (v_+(i) - v_+(0))\lambda_n & \text{if } i \le N \\ u_n^{\mathbf{z}^+}(x_i^n) - v_+(0)\lambda_n + z_n^1(x_i^n - x_N^n) & \\ & \text{if } N \le i \le \frac{n}{2} - N \\ v\Big(i - \frac{n}{2}\Big)\lambda_n - \frac{L}{2} & \text{if } \frac{n}{2} - N \le i \le \frac{n}{2} + N \\ u_n^{\mathbf{z}^-}(x_{n-i}^n) - v_-(0)\lambda_n + z_n^2(x_i^n - x_{n-N}^n) & \\ & \text{if } \frac{n}{2} + N \le i \le n - N \\ (v_-(n-i) - v_-(0))\lambda_n & \text{if } n - N \le i \le n, \end{cases}$$

where

$$z_n^1 = \frac{u^{\mathbf{z}^+}\big(\frac{n}{2}\big)\lambda_n - \frac{L}{2} + v_+(0)\lambda_n}{\big(\frac{n}{2} - 2N\big)\lambda_n}$$

$$z_n^2 = \frac{u^{\mathbf{z}^-}\big(\frac{n}{2}\big)\lambda_n + \frac{L}{2} + v_-(0)\lambda_n}{\big(\frac{n}{2} - 2N\big)\lambda_n}.$$

Note that $\lim_n z_n^1 = \lim_n z_n^2 = 0$. Using (H5) we easily get the desired inequality. □

Remark 5.5 From the proof above it can be easily seen that hypotheses (H3)–(H5) may be relaxed at the expense of a heavier notation and some changes in the results. Clearly, if (H3) does not hold then the sets of minimal pairs $\mathbf{M}^+$, $\mathbf{M}^-$ are larger, and the definition of D must be changed accordingly, possibly taking into account also more than one transition.

If hypothesis (H4) does not hold then $C(u^{\mathbf{z}^+}, u^{\mathbf{z}^-}) = C(u^{\mathbf{z}^-}, u^{\mathbf{z}^+}) = 0$ for some $\mathbf{z}^+ \in \mathbf{M}^+$, $\mathbf{z}^- \in \mathbf{M}^-$. In this case the energetic analysis of E_n^1 is not sufficient to characterize the minimizers, as we have no control on the number of transitions between $u' = 1$ and $u' = -1$.

Condition (H5) has been used to construct the recovery sequence $(\overline{u}_n)$. It can be relaxed to assuming that $\tilde{\psi}$ is smooth at ± 1; more precisely, it suffices to suppose that

$$\lim_{z \to \pm 1} \frac{\tilde{\psi}(z) - \min \tilde{\psi}}{|z \mp 1|} = 0. \tag{5.16}$$

If this condition does not hold the value D is given by a more complex formula, where we take into account also the values at 0 of the solutions of the boundary layer terms.

The proof of Theorem 5.4 easily yields the corresponding Γ-limit result for E_n^1. We leave the details to the reader.

Bibliography

[1] R. Alicandro, M. Focardi and M.S. Gelli, Finite-difference approximation of energies in fracture mechanics. *Ann. Scuola Norm. Pisa*, to appear.

[2] M. Amar and A. Braides, A characterization of variational convergence for segmentation problems, *Discrete Continuous Dynamical Systems* **1** (1995), 347-369.

[3] M. Amar and A. Braides, Γ-convergence of nonconvex functionals defined on measures, *Nonlinear Anal. TMA* **34** (1998), 953-978.

[4] L. Ambrosio, Existence theory for a new class of variational problems, *Arch. Rational Mech. Anal.* **111** (1990), 291-322.

[5] L. Ambrosio and A. Braides, Energies in SBV and variational models in fracture mechanics. In *Homogenization and Applications to Material Sciences*, (D. Cioranescu, A. Damlamian, P. Donato eds.), GAKUTO, Gakkōtosho, Tokio, Japan, 1997, p. 1-22.

[6] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, Oxford, 2000.

[7] Attouch, H. *Variational Convergence for Functions and Operators.* Pitman, Boston, 1984.

[8] G.I. Barenblatt, The mathematical theory of equilibrium cracks in brittle fracture, *Adv. Appl. Mech.* **7** (1962), 55-129.

[9] A. Blake and A. Zisserman, *Visual Reconstruction*, MIT Press, Cambridge, Massachusetts, 1987.

[10] A. Braides, *Approximation of Free-Discontinuity Problems*, Lecture Notes in Mathematics **1694**, Springer Verlag, Berlin, 1998.

[11] A. Braides, Γ-*convergence for Beginners*, Oxford University Press, Oxford, 20012.

[12] A. Braides, G. Dal Maso and A. Garroni, Variational formulation of softening phenomena in fracture mechanics: the one-dimensional case, *Arch. Rational Mech. Anal.* **146** (1999), 23-58.

[13] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

[14] A. Braides and M.S. Gelli, Continuum limits of discrete systems without convexity hypotheses. *Math. Mech. Solids*, to appear.

[15] A. Braides and M.S. Gelli, Limits of discrete systems with long-range interactions. Preprint SISSA, 1999.

[16] G. Buttazzo, *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations*, Longman, Harlow, 1989.

[17] A. Chambolle Un theoreme de Γ-convergence pour la segmentation des signaux. *C. R. Acad. Sci., Paris, Ser. I* **314** (1992), 191-196.

[18] G. Dal Maso, *An Introduction to* Γ*-convergence*, Birkhäuser, Boston, 1993.

[19] E. De Giorgi and L. Ambrosio, Un nuovo funzionale del calcolo delle variazioni, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199-210.

[20] G. Del Piero and L. Truskinovsky, A one-dimensional model for localized and distributed failure, *Journal de Physique IV France 8*, **8** (1998), 95-102.

[21] M. Gobbino, Finite difference approximation of the Mumford-Shah functional, *Comm. Pure Appl. Math.* **51** (1998), 197-228.

[22] A. A. Griffith, The phenomenon of rupture and flow in solids, *Phil Trans. Royal Soc. London A* **221** (1920), 163–198.

[23] B. Houchmandzadeh, J. Lajzerowicz and E. Salje, Relaxation near surfaces and interfaces for first-, second- and third-neighbour interactions: theory and applications to polytypism, *J. Phys.: Condens. Matter* **4** (1992), 9779-9794.

[24] L. Modica and S. Mortola, Un esempio di Γ-convergenza, *Boll. Un. Mat. It. B* **14** (1977), 285-299.

[25] J. M. Morel and S. Solimini, *Variational Models in Image Segmentation*, Birkhäuser, Boston, 1995.

[26] D. Mumford and J. Shah, Optimal approximation by piecewise smooth functions and associated variational problems, *Comm. Pure Appl. Math.* **17** (1989), 577-685.

[27] J. Pouget, Dynamics of patterns in ferroelastic-martensitic transformations, *Phys. Rev. B* **43** (1991), 3575-3581.

[28] L. Truskinovsky, Fracture as a phase transition, *Contemporary research in the mechanics and mathematics of materials* (R.C. Batra and M.F. Beatty eds.) CIMNE, Barcelona, 1996, 322-332.