Convergence of Dirichlet Forms on Fractals

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1. Introduction.

The purpose of these notes is to introduce the theory of the convergence, and specially, to discuss the Γ-convergence, of Dirichlet forms on fractals. A Dirichlet form is a sort of "energy", in some sense it is a generalization of the Dirichlet integral $u \mapsto \int |\text{grad}\, u|^2$ defined for u in open regions in $\mathbb{R}^n$. The investigation of Dirichlet forms on fractals began in connection with the construction of a Brownian motion on a fractal. It is also closely related to the construction of a Laplacian or in other words to the definition of harmonic functions on a fractal. In some sense, in fact, we can say that, on fractals, Dirichlet forms, Brownian motion, and harmonic functions are three different points of view of the same problem. In the following I will discuss the notion of Dirichlet form, and also the notion of harmonic function, as we will need it in relation with some properties of Dirichlet forms. On the contrary, I will not discuss the notion of Brownian motion. However, as we will see in the following, we cannot define those notions on fractals as in the case of open regions in $\mathbb{R}^n$, mainly as fractals have usually an empty interior. The class of fractals which I discuss here is that of finitely ramified fractals. More or less we are in the following situation. We are given finitely many similarities in $\mathbb{R}^\nu$ with $\nu \geq 1$. Let us denote them by $\psi_1, ..., \psi_k$. Suppose they are contractions, i.e., their factors $r_1, ..., r_k$ are < 1. Then, by a theorem of Hutchinson [4], there exists a unique nonempty compact K in $\mathbb{R}^\nu$ such that $K = \bigcup_{i=1}^{k} \psi_i(K)$. We will call such a set K the fractal generated by the set of similarities. The finite ramification means, more or less that different copies $\psi_i(K)$ of K can have at most finitely many (in general at most one) common points. This class of fractals includes for example the Gasket, but not the Carpet. The construction of a Dirichlet form on fractals which are not finitely ramified is much more complicated and will be not discussed in these notes. A finitely ramified fractal can also be seen as the closure of the union of an increasing sequence of finite sets $V^{(0)}, ..., V^{(n)}, ...$. If we want to define on the fractal a Dirichlet form, we meet the problem that usually, K has an empty interior, hence we cannot define the gradient on it, at least in the usual sense. So, the usual way of constructing a Dirichlet form on K, consists in defining discrete Dirichlet forms on $V^{(n)}$ and then in taking the limit. There is the problem of the existence of such a limit form. However, if the initial form E is an eigenvector of a particular nonlinear minimization operator, also called renormalization, in our language if E is an eigenform, then for every function v defined on K, the sequence $E^{\Sigma}_{(n)}$ of discrete forms is increasing and thus has a limit which we will denote by $E^{\Sigma}_{(\infty)}$. Hence, we have defined a Dirichlet form on K. In Sections 2 and 3, I describe the construction of a Dirichlet form on finitely ramified fractals. For the general theory of Dirichlet forms the reader can refer to [2]. The construction of Dirichlet forms on finitely ramified fractals is described for example in [1], [8], [6]. [1] and [8] specially stress the probabilistic point of view. [6] deals with Dirichlet forms and Laplacian in a very general class of fractals, called p.c.f. self-similar sets, introduced by J. Kigami in [5]. Note however, that, in order to simplify the presentation, I have preferred not to give the general definition of Dirichlet form, but only to discuss a particular kind of Dirichlet form. Moreover, I have thought better to start, in Section 2, with a specific example, the Sierpinki Gasket. This, as I think that the Gasket is an example that permits us to understand well what are the problems of the construction of a Dirichlet form in a simple case. In Section 3, I first give a general definition of finitely ramified fractal, and then I discuss in such a class the construction of a Dirichlet form. Such a construction is more difficult in this general case, in which in the case of Gasket, by symmetry we had an explicit eigenform. In finitely ramified fractals, the situation is not so simple. There are finitely ramified fractals having no eigenforms, and however, the existence of an eigenform is a nontrivial problem. However, many important examples of finitely ramified fractals, in particular, the nested fractals introduced by T. Lindstrøm [9], have at least an eigenform, and we will only consider fractals having at least an eigenform. Section 4 is devoted to the investigation of the properties of the renormalization operator, and of the properties of the related notion of harmonic extension. When E is not an eigenform we cannot construct, at least in a direct way, a Dirichlet form starting with E, like in the case of an eigenform, for the corresponding sequence of discrete forms is no longer increasing. It could be proved that nevertheless, the discrete forms pointwise converge to a Dirichlet form defined on K, but the proof is much more complicated. In case of E not eigenform, we will study the Γ-convergence of the sequence of discrete forms. We will prove that in fact $E^{\Sigma}_{(n)}$ Γ-converges to the Dirichlet form on K associated to an eigenform $\tilde{E}$. We will obtain $\tilde{E}$ as the limit of a sequence of forms $E_{(n)}$ on $V^{(0)}$, defined as $\tilde{M}_1^n(E)$ where $\tilde{M}_1$ is the renormalization operator divided by the eigenvalue ρ. The convergence of $\tilde{M}_1^n(E)$ is the most delicate point in the proof of Γ-convergence. It is, in some sense, a problem of convergence of the iterated of a map once we know that the map has a fixed point. In the present case, the fixed point is the eigenform. In section 5, I discuss the Γ-convergence on the Gasket. Due to the symmetry of the Gasket, the proof of the convergence of $\tilde{M}_1^n(E)$ is much simpler than in the general case. There are many different proofs of this result in the case of the Gasket. The proof presented here is probably not the simplest, but suggests the way of proceeding in the general case. Then, following an argument due to S. Kozlov

[7], we deduce the Γ-convergence result. In section 6, I prove the convergence of $\tilde{M}_1^n(E)$ in the general case. I restrict, in fact, the class of fractals in order to simplify the argument, avoiding some technical difficulties. At the end of the section, I merely hint the idea in the most general case. The class of fractals considered in the actual proof in Section 6, however, includes most of the usual fractals, thus it is sufficient general for many purposes. The idea of the proof consists in proving that, with respect to a special metric, related to Hilbert's projective metric, the iterated $\tilde{M}_1^n(E)$ get closer and closer. Note that in general $\tilde{M}_1$ is not a contraction and in fact can have different fixed points. Once the convergence of $\tilde{M}_1^n(E)$ is proved, the Γ-convergence result follows as in the case of the Gasket. As these notes are not intended for specialists of fractals, I have tried to stress the general ideas rather than the details. So, in some cases, for example for the notion of finitely ramified fractal, I have not used the most general definitions. For the same reason, I tried to give proofs which are not necessarily the shortest, but requiring no non usually known results, like as the general theory of Hilbert's projective metric, in order to make these notes as self-contained as possible.

I now fix some notation for the following. When we are in $\mathbb{R}^n$, we will denote by d the euclidean distance, and unless specified otherwise, we will denote by $\| \ \|$ the euclidean norm. Sometimes, we will use $\mathbb{R}^A$ where A is a set. In such a case, of course, on $\mathbb{R}^A$, we put the norm and the metric, obtained by identifying $\mathbb{R}^A$ with $\mathbb{R}^M$ where $M = \#A$. If f is a map from a set X into itself, we will denote by f^n the n^{th}-iterated of f, i.e., the composition of n maps equal to f. If X is a topological space, we will denote by $C(X)$ the set of the continuous functions from X into $\mathbb{R}$. If A is a subset of a euclidean space, we will denote by $\mathrm{co}A$ the convex hull of A.

2. Construction of an Energy on the Gasket.

In this section we will introduce an "energy" on a specific example of fractal, the Sierpinski Gasket. Energy here, means an analogous of the Dirichlet integral, $u \mapsto \int |\mathrm{grad}\, u|^2$, on fractals. Probably, the reader has at least a rough idea of the notion of fractal. It is in some sense, a set that contains copies of itself, at arbitrarily small scales. In order to make these notions clear, before introducing the theory on general fractals, I prefer to describe a specific example of fractal, the Sierpinski Gasket. I later will give a precise notion of fractal (or self-similar set). The Gasket, in some sense, can be constructed like the Cantor set, but starting from a triangle, instead of from a segment-line. More precisely, start with an equilateral triangle T, whose vertices are denoted by P_1, P_2, P_3, and consider the three similarities ψ_i, $i = 1, 2, 3$, in $\mathbb{R}^2$, that are contractions with factor $\frac{1}{2}$ and have P_i as fixed points, in formula $\psi_i(x) = P_i + \frac{1}{2}(x - P_i)$. Then the (Sierpinski) Gasket is the set K defined by

$$K_0 = T,\ K_{n+1} = \bigcup_{i=1}^{3} \psi_i(K_n), \quad K = \bigcap_{n=0}^{\infty} K_n\,, \tag{2.1}$$

in other words, we split the initial triangle T into four similar triangles and remove the central one. Then, we remove the central triangle in each of the three remaining triangles. Then, we repeat the same process on each of the nine remaining triangles, and so on. In Figure 1, we picture K_0, K_1 and K_2.

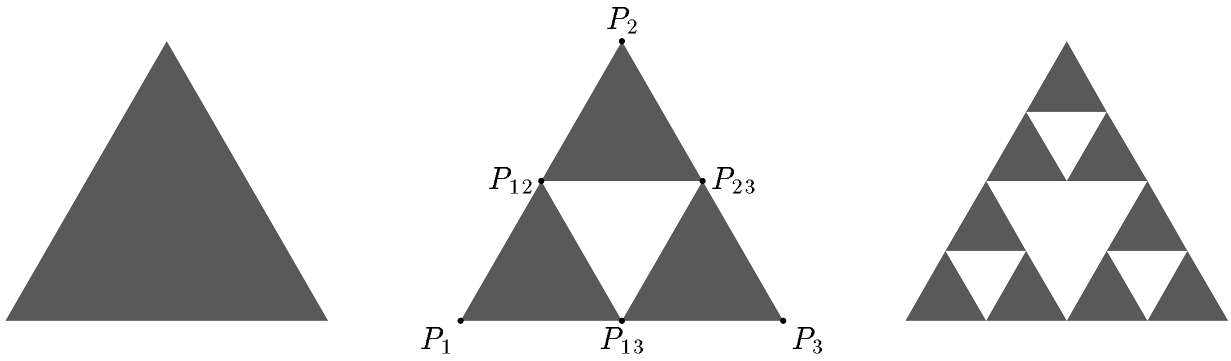

Figure 1. The Sierpinski Gasket

In previous construction, we can fix the vertices, e.g., $P_1 = (0,0)$, $P_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $P_3 = (1,0)$, in order to have a precise triangle. Put now

$$V = V^{(0)} = \{P_1, P_2, P_3\}, \quad V_{i_1,\dots,i_n} = \psi_{i_1,\dots,i_n}(V^{(0)})$$

for $i_1, ..., i_n = 1,2,3$, where $\psi_{i_1,\dots,i_n}$ is an abbreviation for $\psi_{i_1} \circ ... \circ \psi_{i_n}$, and put

$$V^{(n)} = \bigcup_{i_1,..,i_n=1}^{3} V_{i_1,\dots,i_n}, \quad V^{(\infty)} = \bigcup_{n=0}^{\infty} V^{(n)}.$$

The sets $V_{i_1,\dots,i_n}$ are called n-cells and are, in some sense, small copies of $V^{(0)}$, more precisely, the copies of $V^{(0)}$ at the n^{th} step. They can, of course, also be interpreted as the sets of the vertices of the triangles $\psi_{i_1,\dots,i_n}(T)$, which are copies of T at the n^{th} step. More generally, put $A_{i_1,\dots,i_n} = \psi_{i_1,\dots,i_n}(A)$ for every $A \subseteq \mathbb{R}^2$. I will call $A_{i_1,\dots,i_n}$ an ($n^{th} A$) copy. Clearly, we have

$$V \subseteq K \subseteq T. \tag{2.2}$$

Now, we want to define an "energy" on K. For the moment, I do give no precise notion of what we mean by energy. The precise definition will be given when we treat the case of general fractals. We try to define an object that in some sense resembles the Dirichlet integral. Note however, that, clearly, the Gasket has an empty interior, so it is not possible to define the gradient on it. The way of constructing an energy is based on a finite-difference scheme that I will now illustrate. Since it is easy to see that $V^{(n)} \subseteq V^{(n+1)}$, and, as we will see in the following, K is the closure of $V^{(\infty)}$, the idea consists in defining first an energy E on $V^{(0)}$, then on $V^{(n)}$ as the sum of E on all n-cells, and finally in taking a sort of limit. More precisely, let

$$E(u) = \sum_{1 \leq j_1 < j_2 \leq 3} (u(P_{j_1}) - u(P_{j_2}))^2$$

for all $u : V^{(0)} \to \mathbb{R}$. Note that $E(u) \geq 0$, and the equality holds if and only if u is constant on $V^{(0)}$. This is a trivial but crucial remark. Let

$$S_0(E) = E$$

$$S_n(E)(v) = \sum_{i_1,\ldots,i_n=1}^{3} E(v \circ \psi_{i_1,\ldots,i_n}) \;\text{ for } v \in \mathbb{R}^{V^{(n)}}, n \geq 1\,.$$

We can also write

$$S_n(E)(v) = \sum \left(v(Q) - v(Q')\right)^2 \;\text{ for } v \in \mathbb{R}^{V^{(n)}}, n \geq 1,$$

where the sum is extended over all pairs $(Q, Q') \in V^{(n)} \times V^{(n)}$ which are *close* in the sense that they lie in the same n-cell. Now, we would like to define an energy $\mathcal{E}$ for functions defined on K as the limit of $S_n(E)$, but we meet some problems. First, this limit may not exist; moreover, it is possible to see that when it exists, it is 0 for too many functions, for example it is 0 on all the linear functions; we on the contrary require that it is 0 only for the constant functions. The way for overcoming such problems consists in introducing a renormalization factor ρ, i.e., we take the limit of $\frac{1}{\rho^n} S_n(E)$. This is a natural device in the sense that in classical cases, such as e.g., the Dirichlet integral we have to introduce a renormalization factor. In the present case, however, the value of ρ is not much expected. As we will see later, we have, in fact, $\rho = \frac{3}{5}$. In order to find the value of ρ, we now introduce a minimization operator. Namely, for $u : \mathbb{R}^{V^{(0)}} \to \mathbb{R}$, let

$$M_n(E)(u) = \inf\{S_n(E)(v) : v \in \mathcal{L}(n, u)\} \;\forall\, u \in \mathbb{R}^{V^{(0)}} \tag{2.3}$$

where $\mathcal{L}(n, u) = \{v \in \mathbb{R}^{V^{(n)}} : v = u \text{ on } V^{(0)}\}$. We will see in the following that the infimum in (2.3) is in fact a mimimum. In terms of potential theory, $M_n(E)$ is the *trace* of $S_n(E)$ on $V^{(0)}$. For the moment let us study only M_1. Put $P_{12} = \psi_1(P_2) = \psi_2(P_1)$, $P_{13} = \psi_1(P_3) = \psi_3(P_1)$, $P_{23} = \psi_2(P_3) = \psi_3(P_2)$. We so have

$$S_1(E)(v) = (v(P_1) - v(P_{12}))^2 + (v(P_1) - v(P_{13}))^2 + (v(P_{12}) - v(P_{13}))^2$$
$$+ (v(P_2) - v(P_{12}))^2 + (v(P_2) - v(P_{23}))^2 + (v(P_{12}) - v(P_{23}))^2$$

$$+(v(P_3)-v(P_{13}))^2+(v(P_3)-v(P_{23}))^2+(v(P_{13})-v(P_{23}))^2$$

and $M_1(E)(u)$ is the infimum of

$$(u(P_1)-x)^2+(u(P_1)-y)^2+(x-y)^2$$
$$+(u(P_2)-x)^2+(u(P_2)-z)^2+(x-z)^2$$
$$+(u(P_3)-y)^2+(u(P_3)-z)^2+(y-z)^2$$

for $x,y,z\in\mathbb{R}$. As the function to minimize is convex, it attains its minimum at the points at which its gradient is 0. Hence, (x,y,z) is a minimum point if and only if

$$\begin{cases} 4x=y+z+u(P_1)+u(P_2)\\ 4y=x+z+u(P_1)+u(P_3)\\ 4z=x+y+u(P_2)+u(P_3)\end{cases}\tag{2.4}$$

A simple calculation yields

$$\begin{cases} x=\dfrac{2u(P_1)+2u(P_2)+u(P_3)}{5}\\ y=\dfrac{2u(P_1)+2u(P_3)+u(P_2)}{5}\\ z=\dfrac{2u(P_2)+2u(P_3)+u(P_1)}{5}\end{cases}.$$

We will denote by $H_{(1;E)}(u):V^{(1)}\to\mathbb{R}$ the so obtained solution of (2.4), i.e., the unique function $v\in\mathcal{L}(1,u)$ such that $M_1(E)(u)=S_1(E)(v)$. By substituting the value of $v=H_{(1;E)}(u)$, we get

$$M_1(E)(u)=\frac{3}{5}E(u),$$

thus $M_1(E)$ is a multiple of E. Is this a lucky case or not? To answer this question, observe that $H_{(1;E)}$ is linear, so $M_1(E)(u)$ is quadratic in u, in other words it is a linear combination of terms of the form $(u(P_j))^2$ and of terms of the form $u(P_{j_1})u(P_{j_2})$. Moreover, since, clearly $H_{(1;E)}(u+c)=H_{(1;E)}(u)+c$, we can replace $u(P_j)$ by $u(P_j)-u(P_1)$, and thus we have that

$$M_1(E)(u)=a(u(P_2)-u(P_1))^2+b(u(P_3)-u(P_1))^2+d(u(P_2)-u(P_1))(u(P_3)-u(P_1));$$

now by the well-known formula $\alpha\beta=\frac{1}{2}(\alpha^2+\beta^2-(\alpha-\beta)^2)$ with $\alpha=u(P_2)-u(P_1)$, $\beta=u(P_3)-u(P_1)$, we have

$$M_1(E)(u)=a'(u(P_2)-u(P_1))^2+b'(u(P_3)-u(P_1))^2+d'(u(P_2)-u(P_3))^2$$

for some a', b', d' and by the symmetry of the Gasket we must have $a' = b' = d'$ (I here do not prove completely the last assertion, but a formal proof of it due to symmetry can be easily given). In conclusion, there is a natural reason for which $M_1(E)$ is a multiple of E, and the previous calculation can be used to the only aim of evaluating the factor $\frac{3}{5}$. Note that, by the definition of $M_1(E)$, we have

$$S_1(E)(v) \geq M_1(E)(v)$$

for every $v : V^{(1)} \to \mathbb{R}$ where here, and similarly in other cases, we identify a function $v : V^{(1)} \to \mathbb{R}$ with its restriction on $V^{(0)}$. Hence, putting $\rho = \frac{3}{5}$ we have

$$\frac{S_1(E)(v)}{\rho} \geq \frac{M_1(E)(v)}{\rho} = E(v) = \frac{S_0(E)(v)}{\rho^0}. \tag{2.5}$$

Suppose now $v : V^{(\infty)} \to \mathbb{R}$. We are going to prove that, more generally, the sequence $\frac{S_n(E)(v)}{\rho^n}$ is increasing in n, so it has a limit, which will be the energy on K. To see this, note that

$$\begin{aligned}
\frac{1}{\rho^{n+1}} S_{n+1}(E)(v) &= \frac{1}{\rho^n} \sum_{i_1,\ldots,i_n=1}^{3} \Big(\frac{1}{\rho} \sum_{i_{n+1}=1}^{3} E(v \circ \psi_{i_1,\ldots,i_n,i_{n+1}})\Big) \\
&= \frac{1}{\rho^n} \sum_{i_1,\ldots,i_n=1}^{3} \Big(\frac{1}{\rho} S_1(E)(v \circ \psi_{i_1,\ldots,i_n})\Big) \\
&\geq \frac{1}{\rho^n} \sum_{i_1,\ldots,i_n=1}^{3} E(v \circ \psi_{i_1,\ldots,i_n}) = \frac{1}{\rho^n} S_n(E)(v) \,.
\end{aligned}$$

In the above inequality we have used (2.5). Now put

$$E^{\Sigma}_{(n)}(v) = \frac{1}{\rho^n} S_n(E)(v), \quad E^{\Sigma}_{(\infty)}(v) = \lim_{n \to \infty} E^{\Sigma}_{(n)}(v) \,,$$

for $v : K \to \mathbb{R}$. We have thus constructed an energy on K. Note that of course $E^{\Sigma}_{(\infty)}(v)$ can amount to ∞. An unexpected fact is that the linear nonconstant functions have infinite energy. In fact, let $v : K \to \mathbb{R}$ be linear (or more generally affine). We easily get that $E(v \circ \psi_{i_1,\ldots,i_n}) = (\frac{1}{4})^n E(v)$, hence $S_n(E)(v) = (\frac{3}{4})^n E(v)$, and $E^{\Sigma}_{(n)}(v) = (\frac{5}{4})^n E(v)$. It follows that $E^{\Sigma}_{(\infty)}(v) = +\infty$ unless $E(v) = 0$, i.e., v is constant. Thus the set of the functions with finite energy is in some sense completely different from that in the case of a region in $\mathbb{R}^n$. So, one could even suspect that $E^{\Sigma}_{(\infty)}$ is identically $+\infty$. In the rest of this section we prove that this is not so. More precisely, we will prove the following two properties of $E^{\Sigma}_{(\infty)}$.

a) $E^{\Sigma}_{(\infty)}(v) = 0$ if and only if v is constant.

b) The set of $v \in C(K)$ such that $E^{\Sigma}_{(\infty)}(v) < +\infty$ is dense in $C(K)$.

Thus, $E^{\Sigma}_{(\infty)}$ attains a finite, but strictly positive value, at many functions. In order to prove a), we need a Lemma.

Lemma 2.1. *If $v : V^{(n)} \to \mathbb{R}$ and $E^{\Sigma}_{(n)}(v) = 0$, then v is constant on $V^{(n)}$.*

Proof. By the definition of $V^{(n)}$, we have $V^{(m+1)} = \bigcup_{i=1}^{3} \psi_i(V^{(m)})$ for all $m \in \mathbb{N}$. Moreover, $\psi_i(V^{(m)}) \cap \psi_j(V^{(m)}) \supseteq \psi_i(V^{(0)}) \cap \psi_j(V^{(0)}) \neq \emptyset$. Thus, if v is nonconstant on $V^{(n)}$, it is also nonconstant on $\psi_i(V^{(n-1)})$, or in other words, $v \circ \psi_i$ is nonconstant on $V^{(n-1)}$, for some $i = 1, 2, 3$, and by a recursive argument, $v \circ \psi_{i_1, i_2, \ldots, i_n}$ is nonconstant on $V^{(0)}$ for some $i_1, i_2, \ldots, i_n = 1, 2, 3$. Hence by the definition of $E^{\Sigma}_{(n)}$, we have $E^{\Sigma}_{(n)}(v) > 0$. ∎

Theorem 2.2. *If $v \in C(K)$ and $E^{\Sigma}_{(\infty)}(v) = 0$, then v is constant on K.*

Proof. If v is continuous and nonconstant on K, since we know that the union of all $V^{(n)}$ is dense in K, it is nonconstant on $V^{(n)}$ for some n. So, using previous lemma, $E^{\Sigma}_{(\infty)}(v) \geq E^{\Sigma}_{(n)}(v) > 0$. ∎

We are now going to prove b). To do this, it is useful to consider the notion of harmonic extension on $V^{(1)}$. Given $u : V^{(0)} \to \mathbb{R}$ we know that the function $v := H_{(1;E)}(u)$ can be characterized as the unique solution of the system (2.4) where $v(P_{12}) = x, v(P_{13}) = y, v(P_{23}) = z$. Such a function v is called *harmonic extension* of u on $V^{(1)}$. Note that (2.4) says that the value of v at a point Q of $V^{(1)} \setminus V^{(0)}$ is the mean value of the values at the points of $V^{(1)}$ close to Q; this is an analogous property to the mean property of harmonic functions in regions in $\mathbb{R}^n$. Another analogy is that harmonic functions also are a sort of minimum of the Dirichlet integral. Note that we have

$$E^{\Sigma}_{(1)}(H_{(1;E)}(u)) = E(u) \tag{2.6}$$

We will use harmonic extension to construct a function v defined on all K, by extending harmonically u on $V^{(1)}$, then applying this process on every 1-cell to obtain a function defined on $V^{(2)}$, then applying the same process on every 2-cell and so on. However, in order to know that in this way we have in fact defined a continuous function, we have to prove that the oscillation on n-cells tends to 0. The harmonic extension satisfies the following maximum principle:

Lemma 2.3. *For every $u \in \mathbb{R}^{V^{(0)}}$ we have*

$$\min_{V^{(0)}} u \leq H_{(1;E)}(u)(Q) \leq \max_{V^{(0)}} u \quad \forall\, Q \in V^{(1)} \setminus V^{(0)}.$$

(weak maximum principle). Moreover, the inequalities are strict unless u is constant (strong maximum principle).

Proof. It suffices to prove the first inequality, the proof of the second being analogous. Put $v = H_{(1;E)}(u)$. If v does not attain its minimum at any point of $V^{(1)} \setminus V^{(0)}$, then we have $\min v = v(Q) = u(Q) \geq \min u$ for some $Q \in V^{(0)}$, and the first inequality is trivial. Suppose, on the contrary, there exists $Q \in V^{(1)} \setminus V^{(0)}$ at which v attains its minimum. By virtue of (2.4), since $v(P) \geq v(Q) =: m$ for each point P close to Q, we have in fact $v(P) = v(Q)$ for each point P close to Q. In particular, $v(P) = m$ for all $P \in V^{(1)} \setminus V^{(0)}$, thus we can repeat the same argument at all $P \in V^{(1)} \setminus V^{(0)}$. But every point of $V^{(0)}$ is close to some point in $V^{(1)} \setminus V^{(0)}$. Thus, $v = m$ on $V^{(1)}$. In particular, since $v = u$ on $V^{(0)}$, u is constant. ∎

We will now prove as a simple consequence of the maximum principle, that the oscillation of the harmonic extension of u on every 1-cell is estimated by a constant times the oscillation of u on $V^{(0)}$. Let

$$\mathcal{S} := \left\{ u \in \mathbb{R}^{V^{(0)}} : u(P_1) = 0, ||u|| = 1 \right\}. \tag{2.7}$$

and, given a function f from a nonempty set A to $\mathbb{R}$ let

$$\mathrm{Osc}_A f = \sup_{x,y \in A} |f(x) - f(y)| = \sup_A f - \inf_A f\,.$$

Corollary 2.4. *There exists* $\gamma \in]0,1[$ *such that* $\sup_{i=1,2,3} \mathrm{Osc}_{V_i}(H_{(1;E)}(u)) \leq \gamma\, \mathrm{Osc}_{V^{(0)}}(u)$ *for all* $v : V^{(0)} \to \mathbb{R}$.

Proof. Suppose $u \in \mathbb{R}^{V^{(0)}}$, u nonconstant. Then, by the strong maximum principle, for every $i = 1,2,3$ and for every $P \in V^{(0)}$ we have $\min_{V^{(0)}} u \leq H_{(1;E)}(u)(\psi_i(P)) \leq \max_{V^{(0)}} u$ and the inequalities are strict if $P \neq P_i$. Hence, at least one of the two inequalities is strict for every $P \in V^{(0)}$, and $\mathrm{Osc}_{V_i}(H_{(1;E)}(u)) < \mathrm{Osc}_{V^{(0)}}(u)$ for $i = 1,2,3$. Thus, putting

$$\alpha(u) = \frac{\sup_{i=1,2,3} \mathrm{Osc}_{V_i}(H_{(1;E)}(u))}{\mathrm{Osc}_{V^{(0)}}(u)}\,,$$

we have $\alpha(u) < 1$. Since α is continuous it has a maximum $\gamma < 1$ on $\mathcal{S}$. Since $\alpha(u + c) = \alpha(u)$ for every $c \in \mathbb{R}$, and α is 0-homogeneous, we have

$$\alpha(u) = \alpha\Big(\frac{u - u(P_1)}{||u - u(P_1)||}\Big) \leq \gamma$$

for every nonconstant $u \in \mathbb{R}^{V^{(0)}}$. ∎

Given a function $u : V^{(0)} \to \mathbb{R}$, as hinted above, we want to extend it to a function v on $V^{(\infty)}$ in the following way. Put $v = u$ on $V^{(0)}$, then we define v recursively on $V^{(n+1)} \setminus V^{(n)}$ for $n \geq 0$. For $n = 0$, v is the harmonic extension of u on $V^{(1)}$. More generally, suppose v is already defined on $V^{(n)}$ and extend v on $V^{(n+1)}$. Note that $V^{(n+1)} = \bigcup_{i_1,\dots,i_n=1}^{3} \psi_{i_1,\dots,i_n}(V^{(1)})$. So, we define v separately on every $\psi_{i_1,\dots,i_n}(V^{(1)})$. Let

$$v\big(\psi_{i_1,\dots,i_{n+1}}(P)\big) = H_{1;E}\big(v\circ\psi_{i_1,\dots,i_n}\big)\big(\psi_{i_{n+1}}(P)\big) \quad \forall P\in V^{(0)},\ \forall i_1,\dots,i_n = 1,2,3\,, \tag{2.8}$$

in other words, $v\circ\psi_{i_1,\dots,i_{n+1}} = H_{1;E}\big(v\circ\psi_{i_1,\dots,i_n}\big)\circ\psi_{i_{n+1}}$ on $V^{(0)}$. The right-hand side makes sense, as we have already defined v on $V^{(n)}$. In some sense, by identifying a function on $\psi_{i_1,\dots,i_n}(V^{(1)})$ with a function on $V^{(1)}$, (2.8) defines v on $\psi_{i_1,\dots,i_n}(V^{(1)})$ as the harmonic extension of the restriction of v on $\psi_{i_1,\dots,i_n}(V^{(0)})$. The problem in defining v on $V^{(n+1)}$ by (2.8), is that we could suspect that a point Q can be represented as a point of $V^{(n+1)}$ in different ways, for example

$$Q = \psi_{i_1,\dots,i_{n+1}}(P) = \psi_{i'_1,\dots,i'_{n+1}}(P'), \tag{2.9}$$

and that the definition of $v(Q)$ via (2.8) can depend on such a representation. We are now going to prove that this is not the case, so that (2.8) provides an actual definition of a function v. The geometrical reason for this is that we see that different ($n^{th}T$) copies can have as common points only their vertices, so that the points Q in $V^{(n+1)}\setminus V^{(n)}$ can be represented in only one way as in (2.9). Such an argument can be considered as sufficiently persuasive. However, I will give a formal proof. We have to prove that if (2.9) holds then

$$H_{1;E}\big(v\circ\psi_{i_1,\dots,i_n}\big)\big(\psi_{i_{n+1}}(P)\big) = H_{1;E}\big(v\circ\psi_{i'_1,\dots,i'_n}\big)\big(\psi_{i'_{n+1}}(P')\big)\,. \tag{2.10}$$

If $(i_1,\dots,i_n) = (i'_1,\dots,i'_n)$ then, as every map ψ_i is one-to-one, we must have by (2.9), $\psi_{i_{n+1}}(P) = \psi_{i'_{n+1}}(P')$, thus (2.10) holds. Suppose now $(i_1,\dots,i_n)\neq(i'_1,\dots,i'_n)$. Then, it suffices to prove that

$$\psi_{i_{n+1}}(P), \psi_{i'_{n+1}}(P') \in V^{(0)}\,, \tag{2.11}$$

so that by definition of harmonic extension, we have

$$\begin{aligned} H_{1;E}\big(v\circ\psi_{i_1,\dots,i_n}\big)\big(\psi_{i_{n+1}}(P)\big) &= v\Big(\psi_{i_1,\dots,i_n}\big(\psi_{i_{n+1}}(P)\big)\Big) \\ &= v(Q) = H_{1;E}\big(v\circ\psi_{i'_1,\dots,i'_n}\big)\big(\psi_{i'_{n+1}}(P')\big) \end{aligned}$$

and (2.10) holds. In order to prove (2.11), we need the following lemma.

Lemma 2.5. *Suppose* $(i_1,\dots,i_n)\neq(i'_1,\dots,i'_n)$. *Then,*

$$T_{i_1,\dots,i_n}\cap T_{i'_1,\dots,i'_n} \subseteq V_{i_1,\dots,i_n}\cap V_{i'_1,\dots,i'_n}\,.$$

Proof. By the definition of ψ_i, we have $\psi_i(T)\cap\psi_{i'}(T)\subseteq\psi_i(V^{(0)})\cap\psi_{i'}(V^{(0)})$ if $i,i'=1,2,3$, $i\neq i'$. Thus, if $i_m\neq i'_m$, and $i_l = i'_l$ for all $l<m$, in view of the trivial fact that every ψ_i maps T into itself, we have

$$\psi_{i_m,\dots,i_n}(T)\cap\psi_{i'_m,\dots,i'_n}(T)\subseteq\psi_{i_m}(T)\cap\psi_{i'_m}(T)\subseteq\psi_{i_m}(V^{(0)})\cap\psi_{i'_m}(V^{(0)}).$$

Clearly, every ψ_i maps $T\setminus V^{(0)}$ into itself, hence so does $\psi_{i_{m+1},\dots,i_n}$. It follows that

$$\psi_{i_{m+1},\dots,i_n}(T)\cap V^{(0)}\subseteq\psi_{i_{m+1},\dots,i_n}(V^{(0)})$$

hence, using also the fact that every ψ_i is one-to-one,

$$\psi_{i_m,\dots,i_n}(T)\cap\psi_{i'_m,\dots,i'_n}(T)\subseteq\psi_{i_m,\dots,i_n}(T)\cap\psi_{i_m}(V^{(0)})$$
$$=\psi_{i_m}\left(\psi_{i_{m+1},\dots,i_n}(T)\cap V^{(0)}\right)\subseteq\psi_{i_m,\dots,i_n}(V^{(0)}),$$

$$\psi_{i_1,\dots,i_n}(T)\cap\psi_{i'_1,\dots,i'_n}(T)=\psi_{i_1,\dots,i_{m-1}}\big(\psi_{i_m,\dots,i_n}(T)\big)\cap\psi_{i_1,\dots,i_{m-1}}\big(\psi_{i'_m,\dots,i'_n}(T)\big)$$
$$\subseteq\psi_{i_1,\dots,i_n}(V^{(0)}).$$

The same argument is valid for $i'_1,\dots,i'_n$ in place of $i_1,\dots,i_n$, so we have proved the lemma. ■

Now, since we have supposed $(i_1,\dots,i_n)\neq(i'_1,\dots,i'_n)$, and the ψ_i are one-to-one, in view of (2.9) and Lemma 2.5, (2.11) follows at once, and so (2.8) defines a function on $V^{(\infty)}$. We now want to prove that v is uniformly continuous on $V^{(\infty)}$, and hence it can be extended continuously on K. We will get this by proving that the oscillation of v on the n-cells tends to 0 as n tends to ∞. Let us give the following definition. For $f:V^{(\infty)}\to\mathbb{R}$, define $\mathrm{Osc}_{,n}f=\sup_{i_1,..,i_n=1,2,3}\mathrm{Osc}_{V^{(\infty)}_{i_1,\dots,i_n}}f$. We have

Lemma 2.6. $\mathrm{Osc}_{,n}(v)\underset{n\to+\infty}{\longrightarrow}0$.

Proof. By (2.8) and Corollary 2.4, we have

$$\mathrm{Osc}_{V_{i_1,\dots,i_{n+1}}}(v)\leq\gamma\mathrm{Osc}_{V_{i_1,\dots,i_n}}(v)$$

for all $n\in\mathbb{N}$, $i_1,\dots,i_{n+1}=1,2,3$, thus, for all $i_1,\dots,i_n=1,2,3$ we have

$$\mathrm{Osc}_{V_{i_1,\dots,i_n}}(v)\leq\gamma^n\mathrm{Osc}_{V^{(0)}}(u).$$

Now, if $P\in V^{(\infty)}_{i_1,\dots,i_n}$, we have $P=\psi_{i_1,\dots,i_n,i_{n+1},\dots,i_h}(Q)$ for some $Q\in V^{(0)}$ and $i_{n+1},\dots,i_h=1,2,3$. Thus, using (2.8) again and the maximum principle, a recursive argument yields

$$\min_{V_{i_1,\dots,i_n}}v\leq\ v(P)\leq\max_{V_{i_1,\dots,i_n}}v$$

and thus

$$\mathrm{Osc}_{V^{(\infty)}_{i_1,\dots,i_n}}(v) \le \mathrm{Osc}_{V_{i_1,\dots,i_n}}(v) \le \gamma^n \mathrm{Osc}_{V^{(0)}}(u)\,,$$

Since this holds for every $i_1, ..., i_n$, we have proved the lemma. ■

Corollary 2.7. *v is uniformly continuous on $V^{(\infty)}$.*

Proof. Let $\varepsilon > 0$ be given. Let n be such that $\mathrm{Osc}_{,n}(v) < \frac{\varepsilon}{2}$. Since

$$V \subseteq V^{(\infty)} \subseteq T\,,$$

using Lemma 2.5 we get that for every $i_1, ..., i_n$, $i'_1, ..., i'_n = 1, 2, 3$, either $V^{(\infty)}_{i_1,\dots,i_n}$ and $V^{(\infty)}_{i'_1,\dots,i'_n}$ have nonempty intersection, or $T_{i_1,\dots,i_n}$ and $T_{i'_1,\dots,i'_n}$ are disjoint, and so, they being compact, they have a positive minimum distance. Let $\delta > 0$ be such that δ is less than the minimum of the distance of disjoint $T_{i_1,\dots,i_n}$ and $T_{i'_1,\dots,i'_n}$. Now, if $P, P' \in V^{(\infty)}$ and $d(P, P') < \delta$, we have $P \in V^{(\infty)}_{i_1,\dots,i_n}$, $P' \in V^{(\infty)}_{i'_1,\dots,i'_n}$ for some $i_1, ..., i_n$, $i'_1, ..., i'_n$ and there exists $Q \in V^{(\infty)}_{i_1,\dots,i_n} \cap V^{(\infty)}_{i'_1,\dots,i'_n}$. Hence, $|v(P) - v(Q)| < \frac{\varepsilon}{2}$ and $|v(P') - v(Q)| < \frac{\varepsilon}{2}$, thus $|v(P) - v(P')| < \varepsilon$. ■

We will call the continuous extension on K of v defined by (2.8), the *harmonic extension* of u on K, and we will denote it by $H_{(\infty;E)}(u)$. I explicitly state the following lemma which is implicit in the proof of Lemma 2.6.

Lemma 2.8. *For every $u : V^{(0)} \to \mathbb{R}$ we have*

$$\min u \le H_{(\infty;E)}(u) \le \max u\,.$$

Proof. Put $v = H_{(\infty;E)}(u)$. By (2.8) and the maximum principle we have

$$\min_{V_{i_1,\dots,i_n}} v \le \min_{V_{i_1,\dots,i_{n+1}}} v \le \max_{V_{i_1,\dots,i_{n+1}}} v \le \max_{V_{i_1,\dots,i_n}} v$$

so that, by a recursive argument we get

$$\min_{V^{(0)}} u = \min_{V^{(0)}} v \le \min_{V_{i_1,\dots,i_n}} v \le \max_{V_{i_1,\dots,i_n}} v \le \max_{V^{(0)}} v = \max_{V^{(0)}} u\,,$$

hence, $\min u \le v(P) \le \max u$, for every $P \in V^{(\infty)}$, thus by continuity for every $P \in K$. ■

The use of harmonic extension is illustrated in the following theorem.

Theorem 2.9. *For every $u \in \mathbb{R}^{V^{(0)}}$, we have*

$$E^{\Sigma}_{(\infty)}(H_{(\infty;E)}(u)) = E(u)\,.$$

Proof. Put $H_{(\infty;E)}(u) =: v$. Using also (2.6), we get

$$E^{\Sigma}_{(n+1)}(v) = \frac{1}{\rho^{n+1}} \sum_{i_1,\dots,i_{n+1}=1}^{3} E(v \circ \psi_{i_1,\dots,i_{n+1}})$$

$$= \frac{1}{\rho^{n+1}} \sum_{i_1,\dots,i_n=1}^{3} \Big(\sum_{i_{n+1}=1}^{3} E\big(H_{(1;E)}(v \circ \psi_{i_1,\dots,i_n}) \circ \psi_{i_{n+1}}\big)\Big)$$

$$= \frac{1}{\rho^{n}} \sum_{i_1,\dots,i_n=1}^{3} E^{\Sigma}_{(1)}\big(H_{(1;E)}(v \circ \psi_{i_1,\dots,i_n})\big)$$

$$= \frac{1}{\rho^{n}} \sum_{i_1,\dots,i_n=1}^{3} E(v \circ \psi_{i_1,\dots,i_n}) = E^{\Sigma}_{(n)}(v)\,,$$

for all $n \in \mathbb{N}$, thus $E^{\Sigma}_{(n)}(H_{(\infty;E)}(u)) = E(u)$ for all $n \in \mathbb{N}$. ■

Note that by Theorem 2.9 we see that $H_{(\infty;E)}(u)$ minimizes $E^{\Sigma}_{(\infty)}$ among the functions defined on K which amount to u on $V^{(0)}$. We need the following simple lemma.

Lemma 2.10. *We have*

$$K = \bigcup_{i_1,\dots,i_n=1}^{3} \psi_{i_1,\dots,i_n}(K)\,.$$

Proof. We have $V^{(\infty)} = \bigcup_{i_1,\dots,i_n=1}^{3} \psi_{i_1,\dots,i_n}(V^{(\infty)})$. It now suffices to observe that $K = \overline{V^{(\infty)}}$ and that $\psi_{i_1,\dots,i_n}$ are continuous. ■

Given a continuous function v on K, we can now construct for every $m \in \mathbb{N}$, a function denoted by $v_{(m;E)}$, which is in some sense the harmonic extension of the restriction of v on $V^{(m)}$. Namely, for every $P \in K$, put

$$v_{(m;E)}\big(\psi_{i_1,\dots,i_m}(P)\big) = H_{(\infty;E)}(v \circ \psi_{i_1,\dots,i_m})(P)\,.$$

Thanks to Lemma 2.5, such a definition is correct, and the function $v_{(m;E)}$ is continuous, its restriction on every $K_{i_1,\dots,i_m}$ being continuous and the sets $K_{i_1,\dots,i_m}$ being closed subsets of K covering K. Moreover, using an argument like that in Theorem 2.9, we have

$$E^{\Sigma}_{(\infty)}(v_{(m;E)}) = E^{\Sigma}_{(m)}(v) < +\infty,$$

so that $v_{(m;E)}$ has finite energy. Also, we are going to prove that $v_{(n;E)} \underset{n\to\infty}{\longrightarrow} v$ uniformly. In fact, for every $Q \in K$ let $i_1, ..., i_n = 1, 2, 3$, $P \in K$ be such that $Q = \psi_{i_1,...,i_n}(P)$. By the definition of $v_{(n;E)}$ and Lemma 2.8, we have $v_{(n;E)}(Q), v(Q) \in [\inf_{K_{i_1,...,i_n}} v, \sup_{K_{i_1,...,i_n}} v]$. Since

$$\text{diam} K_{i_1,...,i_n} \le (\frac{1}{2})^n \text{diam} K \,, \tag{2.12}$$

(we have in fact the equality in 2.12) and v is uniformly continuous, $v_{(n;E)} \underset{n\to\infty}{\longrightarrow} v$ uniformly, as claimed. In conclusion,

Theorem 2.11. *The set of functions with finite energy is dense in $C(K)$.* ∎

3. Dirichlet Forms on Finitely Ramified Fractals

In this section we want to extend the construction described in previous section from the case of the Gasket to the case of more general fractals. First, I describe other examples of fractals. The Vicsek set can be constructed analogously to the Gasket, by putting K_0 to be a square, P_1, P_2, P_3, P_4 its vertices, and P_5 its centre, and $\psi_i(x) = P_i + \frac{1}{3}(x - P_i)$, and $K_{n+1} = \bigcup_{i=1}^{5} \psi_i(K_n)$, then using (2.1). The Sierpinski Carpet is also constructed starting from a square K_0, then splitting it into nine small squares of edge $\frac{1}{3}$, and considering the eight similarities carrying it into the small squares but the central one. The Snowflake is obtained from a hexagon, and seven similarities, six of them having for fixed points the vertices of the hexagon, and the other one the center of the hexagon. The tree-like Gasket is similar to the Gasket, but two of the three small triangles are disjoint. I finally recall also the celebrated Cantor set, where the initial set K_0 is a segment-line, and there are two similarities having the end points of K_0 for fixed points and of factor $\frac{1}{3}$. What could be a general definition of fractal? In all previous cases the set in $\mathbb{R}^\nu$ is obtained using an initial set and a finite number of contractive (i.e, having a factor < 1) similarities. By a slightly deeper investigation we realize that the initial set is not essential in this construction, as, for example in the case of the Gasket, we can start with $V^{(0)}$ instead of with T. In Figures 2, 3, 4, 5, we describe the previous fractals, by picturing K_1.

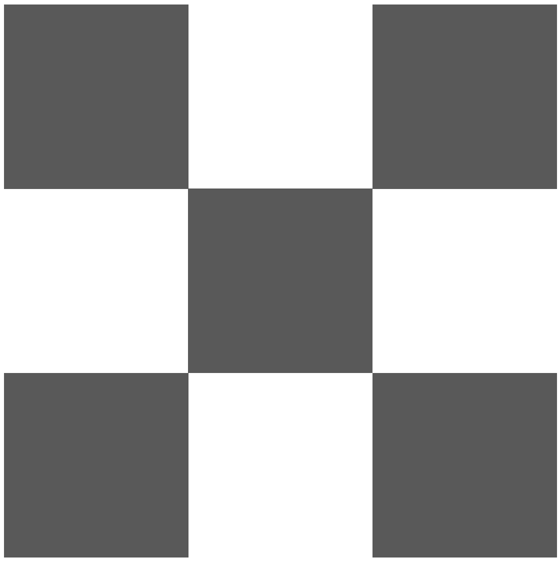

Figure 2. The Vicsek set

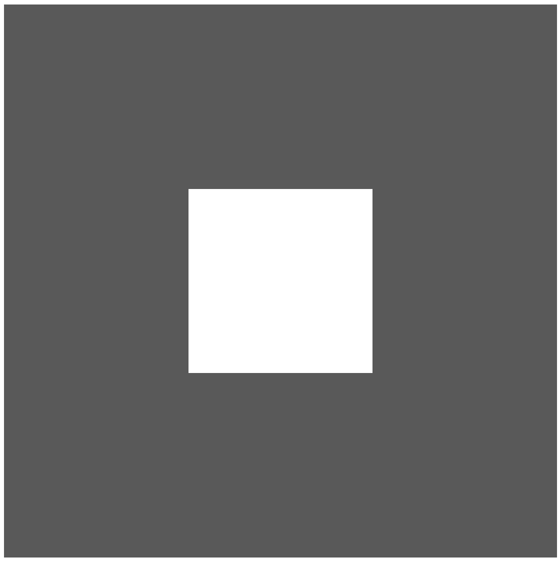

Figure 3. The Sierpinski Carpet

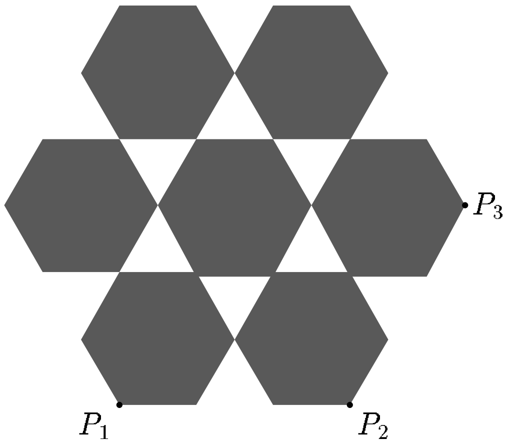

Figure 4. The Snowflake

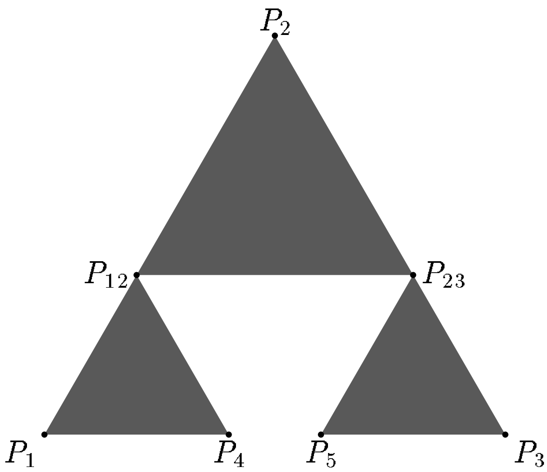

Figure 5. The tree-like Gasket

What is the relationship between the similarities and the fractal obtained? The answer is: if $\psi_1, ..., \psi_k$ are the similarities, then K is the unique nonempty compact subset of $\mathbb{R}^\nu$ such that $K = \bigcup_{i=1}^{k} \psi_i(K)$. In order to realize such a construction, we need some preliminary considerations. Fix $\nu = 1, 2, 3, ...$ in the following. Let $\mathcal{K}$ be the set of nonempty compact subsets of $\mathbb{R}^\nu$. We equip $\mathcal{K}$ with the so-called Hausdorff distance, namely $d_H(C_1, C_2) = \sup\{d(x, C_2) : x \in C_1, d(y, C_1) : y \in C_2\}$. Such a distance in some sense measures how much close are two sets. It is easy to prove that it is in fact a metric on $\mathcal{K}$. We now prove that it is complete.

Theorem 3.1. *$(\mathcal{K}, d_H)$ is complete.*

Proof. Suppose C_n is a Cauchy sequence in $\mathcal{K}$, and prove that it has a limit. There exists a subsequence of C_n, that we will call D_n such that $d_H(D_n, D_{n+1}) < \frac{1}{2^n}$ for all n. We will prove that D_n has a limit and this suffices to conclude the proof. Let

$$D = \{x \in \mathbb{R}^\nu : \exists\, x_n \in D_n \text{ with } x = \lim_{n\to\infty} x_n\}.$$

We will prove that D is nonempty and compact and that $D = \lim_{n\to\infty} D_n$. In order to achieve this, observe that for all $n = 1, 2, 3, ...$ we have

$$\forall\, x \in D_n\ \exists\, y \in D : d(x, y) \le \frac{1}{2^{n-1}}, \tag{3.1}$$

$$\forall\, x \in D\ \exists\, x_n \in D_n : d(x, x_n) \le \frac{3}{2^n}. \tag{3.2}$$

In order to prove (3.1), note that by the definition of d_H, there exists a sequence $(x_1, ..., x_m, ...)$ so that $x = x_n$, and for all $m = 1, 2, 3, ...,$ $x_m \in D_m$ and $d(x_m, x_{m+1}) < \frac{1}{2^m}$. Then x_m is a Cauchy sequence and its limit y belongs to D. Clearly, by the triangular inequality, $d(x_n, x_m) < \frac{1}{2^{n-1}}$ for all $m \ge n$, hence $d(x, y) \le \frac{1}{2^{n-1}}$. Let us now prove (3.2). Let $x \in D$, $y_m \in D_m$ with $x = \lim_{m\to\infty} y_m$, and let $m > n$ be such that $d(x, y_m) \le \frac{1}{2^n}$. As above we find $x_n \in D_n$ such that $d(x_n, y_m) < \frac{1}{2^{n-1}}$, and (3.2) follows at once. From (3.1) we see, in particular, that D nonempty. Since D_n are bounded, in view of (3.2), so is D. In order to prove that D is closed, suppose $x_n \in D$ and $x = \lim_{n\to\infty} x_n$, and prove that $x \in D$. Using (3.2) we find $y_n \in D_n : d(y_n, x_n) \le \frac{3}{2^n}$. Hence, $x = \lim_{n\to\infty} y_n$, and $x \in D$. From (3.1) and (3.2) it immediately follows that $D = \lim_{n\to\infty} D_n$. ■

Suppose now ψ_i are contractive similarities in $\mathbb{R}^\nu$, for $i = 1, ..., k$, with factors $r_i \in]0, 1[$, i.e. we have

$$||\psi_i(x) - \psi_i(y)|| = r_i ||x - y|| \quad \forall\, x, y \in \mathbb{R}^\nu.$$

Let $\Phi : \mathcal{K} \to \mathcal{K}$ be defined as

$$\Phi(C) = \bigcup_{i=1}^{k} \psi_i(C).$$

We are searching for a fixed point of Φ. We have the following

Theorem 3.2. *Φ is a contraction of $(\mathcal{K}, d_H)$, hence it has a unique fixed point K, and moreover, $\Phi^n(A) \underset{n\to\infty}{\longrightarrow} K$ for every $A \in \mathcal{K}$.*

Proof. Let $r = \max r_i$. Then, clearly, for every $A, B \in \mathcal{K}$ we have $d_H(\psi_i(A), \psi_i(B)) \leq r\ d_H(A, B)$ for each $i = 1, ..., k$. From the definition of d_H it easily follows that Φ is a contraction with factor r. ∎

In some important particular cases, we can also give a better characterization of $\Phi^n(A)$. For example, if $T \in \mathcal{K}$ is such that $\Phi(T) \subseteq T$, we easily see by recursion that $\Phi^n(T)$ is a decreasing sequence of sets. It easily follows that $\bigcap_{n=0}^{\infty} \Phi^n(T) = K$. In case of Gasket if T is the triangle, we see that $\Phi^n(T) = K_n$ defined as in (2.1), so we find again $K = \bigcap_{n=0}^{\infty} T_n$. On the other hand, if we denote by F the set of the fixed points of ψ_i if $\emptyset \neq V \subseteq F$, we see that $V \subseteq \Phi(V)$, so $\Phi^n(V) =: V^{(n)}$ is an increasing sequence, and we easily get $\overline{\bigcup_{n=0}^{\infty} V^{(n)}} = K$. I stated this in the case of Gasket. Note, however, that in general, we do not require that V is the set of the fixed points of *all* ψ_i but of some ψ_i. This is important for the following. We will say that the set K as in Theorem 3.2 is the fractal (or self-similar set) generated by the set $\Psi = \{\psi_i : i = 1, ..., k\}$ of contracting similarities. Note that a fractal can be generated by different sets of similarities. However, for the following we fix a set K as above with the associated similarities $\psi_1, ..., \psi_k$, and call P_i the fixed point of ψ_i for $i = 1, ..., k$, and put $F = \{P_i : i = 1, ..., k\}$. We have so explained what we mean by fractal. The next problem is how to construct an energy on it. We will do that by imitating the construction on the Gasket. However, as we will see, that kind of construction is not possible on every fractal, but we have in fact to require some additional hypotheses. Before analyzing what properties we need, we are going to give a more precise notion of the so far vague word *energy*. We require of course that an energy is a functional defined on $\mathcal{K}$ which is nonnegative, quadratic, and which takes the value 0 on the constant functions. It appears also to be natural to assume that it takes the value 0 only on constant functions and that it is finite on a dense set of continuous functions. Moreover, it seems to be natural to require a property of compatibility with the fractal structure. The need of giving a notion of energy generalizing the Dirichlet integral has lead to the notion of *Dirichlet form*. Usually, a Dirichlet form is defined on an L^2 space. Here, in order to avoid problems related to the construction of measures on the fractal, I prefer to define it on $C(K)$, although this is, in some sense, less natural.

Definition 3.3. *We say that a functional $\mathcal{E}$ is a good Dirichlet form if it satisfies the following properties*
a) $\mathcal{E}$ is a quadratic form from $C(K)$ to $[0,+\infty]$ in the sense that there exists a linear subspace Z of $C(K)$ such that $\mathcal{E}(v) < +\infty$ if and only if $v \in Z$, and there exists $\hat{\mathcal{E}}$: $Z \times Z \to \mathbb{R}$ bilinear and symmetric such that

$$\mathcal{E}(v) = \hat{\mathcal{E}}(v,v) \geq 0$$

for all $v \in Z$.
b) $\mathcal{E}((u \wedge 1) \vee 0) \leq \mathcal{E}(v) \quad \forall\, v \in C(K)$ (Markov property).
c) $\mathcal{E}$ is lower semicontinuous (with respect to the L^∞ topology).
d) $\mathcal{E}(v + c) = \mathcal{E}(v) \quad \forall\, v \in C(K), \forall\, c \in \mathbb{R}$, (thus, $\mathcal{E}$ is 0 on all constant functions).
e) $\mathcal{E}$ is irreducible, i.e., $\mathcal{E}(v) = 0 \Rightarrow v$ constant.
f) There exists a set $\mathcal{H} \subseteq C(K)$ such that $\mathcal{H}$ is dense in $C(K)$ with respect to the L^∞ topology, and $\mathcal{E}(v) < +\infty$ for every $v \in \mathcal{H}$.
g) $\exists\, \rho > 0 : \mathcal{E}(v) = \frac{1}{\rho} \sum_{i=1}^{k} \mathcal{E}(v \circ \psi_i)$.

Properties a), b), c) characterizes the Dirichlet forms, d), e), f) are in some sense properties of regularity of a Dirichlet form, and g) is the self-similarity property. The functional $E_{(\infty)}^{\Sigma}$ constructed on the Gasket in Section 2 is a good Dirichlet form. Indeed, a) is trivial, b) easily follows from the analogous property of the function $(x, y) \mapsto (x - y)^2$ (i.e. if $s(t) = ((t \wedge 1) \vee 0)$, then $(s(x) - s(y))^2 \leq (x - y)^2$); c) follows from the fact that $E_{(\infty)}^{\Sigma}$ is the sup of the continuous functionals $E_{(n)}^{\Sigma}$, d) is trivial, e) and f) have been proved in section 2, and g) can be easily proved with $\rho = \frac{3}{5}$ on the base of the definition of $E_{(n)}^{\Sigma}$. We now try to imitate the construction of $E_{(\infty)}^{\Sigma}$ on a general fractal. To do this, we have to restrict the class of fractals considered. We will see what properties of the Gasket we used in the construction of $E_{(\infty)}^{\Sigma}$, and we will require that analogous properties hold in our fractals. Here, in order to simplify the presentation, we will not try to define the widest class of fractals suitable for these considerations, but we will restrict ourselves to consider a class of fractals which is at the same time simpler to define and sufficient to include the most usual cases. In the case of Gasket, we used the fact that $\psi_i(T \setminus V) \subseteq T \setminus V$. Such a property was crucial in the construction of the harmonic extension on K of a function defined in V, which in turn allowed us to prove property f) of Def. 3.3. Also, we implicitly used the fact that the points P_j are different from each other, and that the points $\psi_i(P_j)$ are not in $V^{(0)}$ unless $i = j$. So, we are lead to require the following property which includes the previous two:

Definition 3.4. *We say that K has the (strong) nesting property if $P_{j_1} \neq P_{j_2}$ when $j_1 \neq j_2$, and there exists a set $V = V^{(0)} \subseteq F$, $V = \{P_1, ..., P_N\}$, $2 \leq N \leq k$, and, putting $T = \mathrm{co}V$, we have $\psi_i(T) \subseteq T$ and $\psi_i(x) \notin V^{(0)}$ if $x \in T \setminus \{P_i\}$, (thus, in particular, $\psi_i(T \setminus V^{(0)}) \subseteq T \setminus V^{(0)}$), for all $i = 1, ..., k$, and in addition*

$$\psi_i(T) \cap \psi_{i'}(T) = \psi_i(V) \cap \psi_{i'}(V) \text{ if } i, i' = 1, ..., k, i \neq i' . \tag{3.3}$$

Note that we used (3.3) in Lemma 2.5. In the case of the Gasket, V and T are as in Section 2. In absence of the nesting property, we cannot even conclude that $E^{\Sigma}_{(\infty)}$ is finite for some nonconstant function. Note that the nesting property implies in particular that every n-cell contains at most one point of $V^{(0)}$ for $n > 0$. In addition,

$$P_i = \psi_h(P_j), \quad i, j = 1, ..., N, \ h = 1, ...k \ \Rightarrow \ i = j = h .$$

This means that every $P_j \in V^{(0)}$ belongs to precisely one 1-cell, i.e., V_j. The phrase *strong nesting property* is more appropriate, as usually nesting property has a weaker meaning, and properly is a weaker version of (3.3), but in the following we conventionally omit the word strongly. In some sense, in Def. 3.4, (3.3) is the most characterizing property, in which it distinguishes the most usual fractals. In fact, the Vicsek set, the tree-like Gasket and the Snowflake have the nesting property, but the Carpet has not, as (3.3) does not hold. In our examples we have $N = k = 3$ in the Gasket and in the tree-like Gasket, $N = 4$, $k = 5$ in the Vicsek set, $N = 6$, $k = 7$ in the Snowflake, so that we can in fact have $N < k$. Another property we have to require is suggested by the Cantor set. There, if we try to imitate the definition of $M_1(E)$ we easily see that we obtain 0, as, for every function defined on V, which in this case is the set of the end-points of the segment-line, it can be extended on $V^{(1)}$ by a function which is constant on each 1-cell. The reason is that the Cantor set is too much disconnected, not so in a topological sense but in the combinatorial sense that the 1-cells are disjoint. In order to give a precise notion of connectedness, we recall the following definitions about graphs.

A graph is a pair (V, W) where V is a nonempty set and W is a subset of the set of the subsets of V having precisely two elements. The elements of W will be called the edges of the graph. We will say that $P, Q \in V$ are close (in (V, W)) if $\{P, Q\}$ is an edge. We will say that $P, Q \in V$ are connected (in (V, W)) if there exist $n = 1, 2..., P_1, ..., P_n \in V$ such that $P_1 = P$, $P_n = Q$, and $\{P_i, P_{i+1}\} \in W$ for $i = 1, ..., n-1$. In such a case, we will say that $(P_1, ..., P_n)$ is a path that connects P and Q (in (V, W)) and has length m. We will say that (V, W) is connected, if any two points in V are connected. When V is clear from the context we can identify the graph with W, and say for example that W is connected.

Now define $V_{i_1,...,i_n} = \psi_{i_1,...,i_n}(V^{(0)})$ for $i_1, ..., i_n = 1, ..., k$, where $\psi_{i_1,...,i_n}$ is an abbreviation for $\psi_{i_1} \circ ... \circ \psi_{i_n}$, and put $V^{(n)} = \bigcup_{i_1,..,i_n=1}^{k} V_{i_1,...,i_n}$, $V^{(\infty)} = \bigcup_{n=0}^{\infty} V^{(n)}$ as in the case of Gasket. Enumerate $V = \{P_1, ..., P_N\}$. Then we say that our fractal is connected if the following graph is connected: $\mathcal{G}_1 = (V^{(1)}, W)$ where W is the set of $\{\psi_i(P_{j_1}), \psi_i(P_{j_2})\}$ for $i = 1, ..., k$, $j_1, j_2 = 1, ..., N$, $j_1 \neq j_2$. In other words, we require that for every $P, Q \in V^{(1)}$ there exists a finite sequence $P_1, ..., P_m$ of points in $V^{(1)}$ such that, $P = P_1$, $Q = P_m$, and for every $r = 1, ..., m-1$, P_r, P_{r+1} belong to some 1-cell, or also, that any two points in $V^{(1)}$ can be connected by a path in $V^{(1)}$ whose edges are contained in some 1-cell (depending on the edge). It can be easily verified that the Vicsek set, the tree-like Gasket and the Snowflake are connected. We will say that a fractal is (strongly) finitely ramified if both it has the nesting property and it is connected. As for the nesting property, we will omit the word strongly for sake of simplicity, although, the usual definition of finitely ramified fractal is more general. We will construct a good Dirichlet form on every finitely ramified fractal. At first glance, in the construction in Section 2 we used other properties of the Gasket. For example, in the proof of the maximum principle, we used the fact, that every point in $V^{(1)} \setminus V^{(0)}$ is close to a point close to a fixed point in $V^{(0)}$. However, this tells in other words that a point in $V^{(1)} \setminus V^{(0)}$ and a point in $V^{(0)}$ are connected in $\mathcal{G}_1$ by a path of length ≤ 3, and the connectedness can well replace this assumption. A more serious difficulty is that, when we proved that $M_1(E)$ is a multiple of E we heavily used the very strong symmetry of the Gasket, for example in the Snowflake there is no reason that the coefficients of $(u(P_1) - u(P_2))^2$ and of $(u(P_1) - u(P_3))^2$ are equal, as P_1 and P_3 are connected in $V^{(1)}$ in an essentially different way from P_1 and P_3. In order to avoid such a problem, we will consider more general quadratic forms on $V^{(0)}$. We will now describe the construction in detail. Suppose then K is a finitely ramified fractal with similarities $\psi_1, ..., \psi_k$. We define $\mathcal{D}$ to be the set of functionals from $\mathbb{R}^{V^{(0)}}$ into $\mathbb{R}$ satisfying the following property:

there exist $c_{j_1,j_2}(E)(= c_{j_1,j_2}) \geq 0$ ($j_1 \neq j_2$) with $c_{j_1,j_2} = c_{j_2,j_1}$ such that

$$E(u) = \sum_{1 \leq j_1 < j_2 \leq N} c_{j_1,j_2}(u(P_{j_1}) - u(P_{j_2}))^2 \tag{3.4}$$

for all $u : V^{(0)} \to \mathbb{R}$.

Moreover, we define $\tilde{\mathcal{D}}$ to be the set of those $E \in \mathcal{D}$ which are irreducible, i.e., $E(u) = 0$ if and only if u is constant. Regarding the previous definitions, we are only interested in $\tilde{\mathcal{D}}$. However, in some cases, we will also need to consider the set $\mathcal{D}$ which has for example the advantage that it is in some sense a closed set. The difference with respect to the case of the Gasket is that in this case we consider forms with possibly different coefficients. Now, we define $S_n(E)$ and $M_n(E)$ as in Section 2, with the obvious variant that the indices in the sum in the definition of $S_n(E)$ vary from 1 to k instead of from 1 to 3. In order to imitate the construction in Section 2, we need an $E \in \tilde{\mathcal{D}}$ such that there exists $\rho > 0$ with $M_1(E) = \rho E$, in other words, we have to prove the existence of an eigenvector for the operator $M_1 : \tilde{\mathcal{D}} \to \tilde{\mathcal{D}}$, which as we will see, is in general, nonlinear. Before discussing this problem, however, we need a more detailed analysis of the previous notions. For example, I have not proved that M_1 maps in fact $\tilde{\mathcal{D}}$ into $\tilde{\mathcal{D}}$. First of all, we note that the coefficients of E are unique (this enables us to use the notation $c_{j_1,j_2}(E)$). This is a consequence of the following remark.

Remark 3.5. Given $j_1, j_2 = 1, ..., k$ let u_{j_1,j_2}, v_{j_1,j_2} be the functions in $\mathbb{R}^{V^{(0)}}$ that take the value 0 at every P different from P_{j_1}, P_{j_2}, and such that

$$u_{j_1,j_2}(P_{j_1}) = u_{j_1,j_2}(P_{j_2}) = v_{j_1,j_2}(P_{j_1}) = 1, \quad v_{j_1,j_2}(P_{j_2}) = -1\,.$$

Then, if E is defined as in (3.4) (*not necessarily* $c_{j_1,j_2} \geq 0$), we must have $c_{j_1,j_2} = \frac{1}{4}\big(E(v_{j_1,j_2}) - E(u_{j_1,j_2})\big)$, as can be easily verified. ∎

We now want to prove that M_1 maps in fact $\tilde{\mathcal{D}}$ into $\tilde{\mathcal{D}}$. In the case of Gasket, we evaluated precisely $M_1(E)$. We did this by solving explicitly the system in (2.4). Clearly, this is not possible in the general case. However, it is not necessary to solve explicitly such a kind of system, but it is sufficient to prove that it has a unique solution. For in such a case it depends linearly on u and we can proceed as in Section 2. We need some further considerations on graphs in $V^{(1)}$.

Remark 3.6. It is easy to see that the irreducibility condition for $E \in \mathcal{D}$ amounts to the fact that the graph on $V^{(0)}$, whose edges are the sets $\{P_{j_1}, P_{j_2}\}$ such that $c_{j_1,j_2} > 0$, is connected. We will denote such a graph by $\mathcal{G}(E)$. Note that, roughly speaking, the irreducibility condition amounts to the fact that there are not too many coefficients equal to 0. For example, if $N = 3$, this means that at most one of the coefficients $c_{1,2}, c_{1,3}, c_{2,3}$ is 0. ∎

Fix $E \in \tilde{\mathcal{D}}$. We call $\mathcal{G}_1(E)$ the graph in $V^{(1)}$ whose edges are the sets of the form $\{\psi_i(P_{j_1}), \psi_i(P_{j_2})\}$ with $i = 1, ..., k$, $j_1, j_2 = 1, ..., N$, $j_1 \neq j_2$ and $c_{j_1,j_2} > 0$. We say that two points Q and Q' in $V^{(1)}$ are close (resp. E-close) if they are close in $\mathcal{G}_1$ (resp. in $\mathcal{G}_1(E)$). So, two points are close if they lie in the same cell. We say that Q and Q' are connected (resp. E-connected) if they are connected in $\mathcal{G}_1$ (resp. in $\mathcal{G}_1(E)$, i.e., if there exists a path $(Q_1, ..., Q_m)$ with $Q_1, ..., Q_m \in V^{(1)}$, $m \geq 1$, and $Q_1 = Q$, $Q_m = Q'$, and Q_i and Q_{i+1} are close (resp. E-close). In such a case we say that the path connects (resp. E-connects) Q to Q'. We say that Q and Q' are strongly connected (resp. strongly E-connected), and that the path strongly connects (resp. strongly E-connects) Q to Q', if we can also assume $Q_2, ..., Q_{m-1} \notin V^{(0)}$. Note however, that by our assumptions any two points in $V^{(1)}$ are connected.

Remark 3.7. It easily follows from Remark 3.6 that any two points that lie in the same 1-cell V_i are E-connected by a path $(P_1, ..., P_m)$ with $P_h \in V_i$ for each h. It follows that, if $i > N$, so that V_i contains no points of $V^{(0)}$, then any two points in V_i are strongly E-connected. If, instead, $i \leq N$, so that P_i is the unique point in $V^{(0)} \cap V_i$, then any point in V_i is strongly E-connected to P_i. ∎

Lemma 3.8. *Every point $Q \in V^{(1)}$ is strongly E-connected with some point of $V^{(0)}$.*

Proof. Fix $Q' \in V^{(0)}$. Since the fractal is connected, there exist $Q_1, ..., Q_m \in V^{(1)}$ such that $Q_1 = Q$, $Q_m = Q'$ and for every $h = 1, ..., m-1$, Q_h and Q_{h+1} are close. Thus, by

Remark 3.7 Q_h and Q_{h+1} are E-connected, and therefore Q and Q' are E-connected. In a path $(\tilde{Q}_1, ..., \tilde{Q}_l)$ E-connecting Q and Q', let Q'' be the first element $\tilde{Q}_h \in V^{(0)}$. Then it is easy to see that Q and Q'' are strongly E-connected. ∎

Note that, unlike the case of the Gasket, in general we cannot conclude that Q is E-connected with *any* point of $V^{(0)}$. For example in the tree-like Gasket, if $c_{j_1,j_3} = 0$ the point P_1 is not E-connected with P_3. As in Section 2, we define $\mathcal{L}(n,u) = \{v \in \mathbb{R}^{V^{(n)}} : v = u \text{ on } V^{(0)}\}$. We can characterize the minimum point of $M_1(E)$ in $\mathcal{L}(1,u)$ like in section 2.

Lemma 3.9. *If $u \in \mathbb{R}^{V^{(0)}}$, then a function $v \in \mathcal{L}(1,u)$ satisfies $M_1(E)(u) = S_1(E)(v)$ if and only if*

$$\sum c_{j_1,j_2}\Big(v(\psi_i(P_{j_1})) - v(\psi_i(P_{j_2}))\Big) = 0, \quad \forall P \in V^{(1)} \setminus V^{(0)} \tag{3.5}$$

where the sum is extended over all $i = 1, ..., k$, $j_1, j_2 = 1, ..., N$ such that $j_1 \neq j_2$ and $P = \psi_i(P_{j_1})$. ∎

Lemma 3.10. *Suppose $u \in \mathbb{R}^{V^{(0)}}$ and $v \in \mathcal{L}(1,u)$ satisfies $M_1(E)(u) = S_1(E)(v)$. Suppose $P \in V^{(1)} \setminus V^{(0)}$ and $v(P) = \max v$ or $v(P) = \min v$. Then, we have $v(Q) = v(P)$ whenever $Q \in V^{(1)}$ is strongly E-connected to P.*

Proof. Suppose for example $v(P) = \max v =: M$. It follows from the hypothesis that $v(P') = v(P)$ if P' is E-close to P, for, in the contrary case, the left-hand side in (3.5) would be strictly positive. Now, let $(P_1, ..., P_m)$ be a path strongly E-connecting P to Q. By recursion, $v(P_r) = M$ for all $r = 1, ..., m$, and in particular, $v(Q) = M$. ∎

Proposition 3.11. *If $u \in \mathbb{R}^{V^{(0)}}$ and $v \in \mathcal{L}(1,u)$ satisfies $M_1(E)(u) = S_1(E)(v)$, then v satisfies the maximum principle: for every $P \in V^{(1)}$*

$$\min u \leq v(P) \leq \max u\,.$$

Proof. We prove for example the second inequality. Suppose that $v(P) = \max v =: M$ for some $P \in V^{(1)} \setminus V^{(0)}$. Using Lemma 3.8 and Lemma 3.10, we conclude that $v(Q)(= u(Q)) = M$ for some $Q \in V^{(0)}$. Thus, $M \leq \max u$. ∎

Remark 3.12. The strong maximum principle in general does not hold. For example, in the tree-like Gasket if E is the form defined before Lemma 3.9, and $u(P_1) = u(P_2) = 0$, then $H_{(1;E)}(u) = 0$ on the 1-cell V_1. ∎

Theorem 3.13. *For every $u \in \mathbb{R}^{V^{(0)}}$ there exists a unique $v \in \mathcal{L}(1,u)$ satisfying $M_1(E)(u) = S_1(E)(v)$.*

Proof. Clearly, $v \in \mathcal{L}(1,u)$ and satisfies $M_1(E)(u) = S_1(E)(v)$, if and only if v satisfies the equations in (3.5) and, in addition, the equations $v(P) = u(P)$ for $P \in V^{(0)}$. Thus, we have a linear system with N equations and N unknowns $v(P_1), ..., v(P_N)$. Therefore, we have to prove that the corresponding homogeneous system has no nontrivial solutions. But the homogeneous system is the system corresponding to $u = 0$. By the maximum principle, the unique solution to such a system is the function $v = 0$. ∎

We now define $H_{(1;E)}(u)$ as in Section 2. Clearly, $H_{(1;E)}$ has the following properties:

a) $H_{(1;E)}$ is linear, thus continuous.

b) $H_{(1;E)}(u + c) = H_{(1;E)}(u) + c$ for all $u \in \mathbb{R}^{V^{(0)}}$ and $c \in \mathbb{R}$.

We need another further lemma.

Lemma 3.14. *If $v : V^{(n)} \to \mathbb{R}$ and $S_n(E)(v) = 0$, then v is constant on $V^{(n)}$.*

Proof. We imitate the proof of Lemma 2.1. Observe that for every $m \in \mathbb{N}$, if $v : V^{(m+1)} \to \mathbb{R}$ has a constant value c_i on every $\psi_i(V^{(m)})$, $i = 1, ..., k$, then v is constant on $V^{(m+1)}$. Indeed, in particular, $v = c_i$ on the 1-cell V_i. Since the fractal is connected, c_i is independent of i, say $c_i = c$ for each i, and since $V^{(m+1)} = \bigcup_{i=1}^{k} \psi_i(V^{(m)})$, $v = c$ on $V^{(m+1)}$ as claimed. It follows that, if v is nonconstant on $V^{(n)}$, then $v \circ \psi_{i_1,i_2,...,i_n}$ is nonconstant on $V^{(0)}$ for some $i_1, i_2, ..., i_n = 1, ..., k$, thus $S_n(E)(v) > 0$. ∎

Theorem 3.15. $M_1(E) \in \tilde{\mathcal{D}}$.

Proof. We easily get, proceeding as in Section 2, that $M_1(E)$ has a representation as in (3.4) for suitable coefficients $c'_{j_1,j_2} (= c'_{j_2,j_1})$, $j_1, j_2 = 1, ..., N$, $j_1 \neq j_2$. It remains to prove

a) $M_1(E)$ is irreducible.

b) $c'_{j_1,j_2} \geq 0$.

Let us prove a). We have $M_1(E)(u) = S_1(E)(H_{(1;E)}(u))$. If u is nonconstant, so is $H_{(1;E)}(u)$. Hence, in view of Lemma 3.14, $M_1(E)(u) > 0$. Let us prove b). We use Remark 3.5. Let $j_1, j_2 = 1, ..., N$, $j_1 \neq j_2$, and u_{j_1,j_2}, v_{j_1,j_2} be as in Remark 3.5. Let $w = H_{(1;E)}(v_{j_1,j_2})$. Since $|w| \in \mathcal{L}(1, u_{j_1,j_2})$, by the definition of $M_1(E)$ we have

$$M_1(E)(u_{j_1,j_2}) \leq S_1(E)(|w|) \leq S_1(E)(w) = M_1(E)(v_{j_1,j_2}), \tag{3.6}$$

the second inequality being an immediate consequence of the formula

$$(|a| - |b|)^2 \leq (a - b)^2 . \tag{3.7}$$

Now, b) follows from Remark 3.5. ∎

Next, we discuss the problem whether there exist $E \in \tilde{\mathcal{D}}$ and $\rho > 0$ such that $M_1(E) = \rho E$. In such a case we say that E is an eigenform and ρ is its eigenvalue. In many fractals an eigenform exists.

Put $\overline{E}$ to be that form having all coefficients equal to 1.

Then, we have seen that in the Gasket $\overline{E}$ is an eigenform. Although the Vicsek set is less symmetric than the Gasket, it is not difficult to see that its symmetry properties are sufficient to guarantee that $\overline{E}$ is an eigenform (in some sense the kind of connection of two different points $P, P' \in V^{(0)}$ through $V^{(1)}$ is independent of P, P'). In the tree-like Gasket a direct calculations shows that every form E with $c_{1,3} = 0$ is an eigenform with eigenvalue $\frac{1}{2}$. In fact, let $a = c_{1,2}$, $b = c_{2,3}$. In the definition of M_1 we minimize the functional

$$a(u(P_1) - x)^2 + a(u(P_2) - x)^2 + b(u(P_3) - y)^2 + b(u(P_2) - y)^2 + b(z - x)^2 + a(t - y)^2,$$

where v being the function to minimize, we put $P_{12} = \psi_1(P_2) = \psi_2(P_1)$, $P_{23} = \psi_3(P_2) = \psi_2(P_3)$, $P_4 = \psi_1(P_3)$, $P_5 = \psi_3(P_1)$, $x = v(P_{12})$, $y = v(P_{23})$, $z = v(P_4)$, $t = v(P_5)$. Clearly, for the minimum v we must have $x = z$, $y = t$, and we have to minimize separately the functions $a(u(P_1) - x)^2 + a(u(P_2) - x)^2$ with respect to x and $b(u(P_3) - y)^2 + b(u(P_2) - y)^2$ with respect to y. Clearly, the result is

$$\frac{1}{2}a(u(P_1) - u(P_2))^2 + \frac{1}{2}b(u(P_3) - u(P_2))^2 = \frac{1}{2}E(u).$$

For the Snowflake the problem is more complicated. Here, and in general, the map M_1, considered as a map from the coefficients of E to the coefficients of $M_1(E)$, is a rational function. This, as the solution $H_{(1;E)}(u)$ of system (3.5) is given by the quotients of two determinants which are polynomial functions of the coefficients of E, and $M_1(E)(u) = S_1(E)(H_{(1;E)}(u))$. So, we cannot expect that M_1 is linear, and when ad hoc arguments such as symmetry do not work, the problem of the existence of an eigenform is not trivial. A result of Lindstrøm [9], states that in every nested fractal there exists an eigenform. A nested fractal is defined to be a fractal having properties similar to that of finite ramification, and further the following additional symmetry property:

If $j_1, j_2 = 1, ..., N$, $j_1 \neq j_2$, then the symmetry ϕ_{j_1,j_2} with respect to $W_{j_1,j_2} = \{z : ||z - P_{j_1}|| = ||z - P_{j_2}||\}$, maps n-cells to n-cells for $n \geq 0$ and any n-cell containing elements on both sides of W_{j_1,j_2} is mapped to itself.

See [9] for the precise definition. It is easy to see that the Gasket, the Vicsek set, and the Snowflake are nested, and the tree-like Gasket is not nested. In particular, on the Snowflake there exists an eigenform. It is not difficult to see that not all fractals have an eigenform. A rather general criterion for the existence of an eigenform valid also for nonnested fractals was given by C. Sabot in his doctoral thesis (1995) (cf. [18]).

From now on, we assume that in our fractal there exists an eigenform, and $\hat{E}$ will denote a fixed eigenform.

We can now repeat for $\hat{E}$ the same construction as that for $\overline{E}$ on the Gasket and use the same definitions of $\hat{E}^{\Sigma}_{(n)}$, and of harmonic extension $H_{(\infty;\hat{E})}(u)$. We get

Theorem 3.16. *For every $v : K \to \mathbb{R}$, $\hat{E}^{\Sigma}_{(n)}(v)$ is increasing with respect to n. If we put $\hat{E}^{\Sigma}_{(\infty)}(v) = \lim_{n\to\infty} \hat{E}^{\Sigma}_{(n)}(v)$, then $\hat{E}^{\Sigma}_{(\infty)}$ is a good Dirichlet form on K.*

Proof. We imitate the proof in Section 2. We have to modify slightly the proof of Corollary 2.4. When there, we stated that the number γ is less than 1, we used the strong maximum principle, which, as seen in Remark 3.12, is no longer valid in this situation. However, a little modification of that argument, using substantially the maximum principle, is still valid. We have to prove that $\mathrm{Osc}_{V_i}(v) < \mathrm{Osc}_{V^{(0)}}(u)$, whenever u is a nonconstant function from $V^{(0)}$ to $\mathbb{R}$ and $v = H_{(1;\hat{E})}(u)$, and $i = 1, ..., k$. To this aim, using the maximum principle, it is sufficient to prove that v cannot attain in the same 1-cell V_i both its maximum and its minimum on $V^{(1)}$. Suppose on the contrary, $\max v = v(\psi_i(P_{j_1}))$, $\min v = v(\psi_i(P_{j_2}))$, $j_1, j_2 = 1, ..., N$. In view of Remark 3.7, either $i > N$ and $\psi_i(P_{j_1})$ and $\psi_i(P_{j_2})$ are strongly $\hat{E}$-connected, or $i \le N$ and $\psi_i(P_{j_1})$ and $\psi_i(P_{j_2})$ are both strongly $\hat{E}$-connected to P_i. By Lemma 3.10, in the former case $v(\psi_i(P_{j_1})) = v(\psi_i(P_{j_2}))$, in the latter case $v(\psi_i(P_{j_1})) = v(P_i) = v(\psi_i(P_{j_2}))$. Hence, $\max u = \min u$, and v, so u, is constant, a contradiction. ■

Note that it is easy to prove that any positive multiple of an eigenform is an eigenform as well. So, a natural question is: is the eigenform unique up to a multiplicative constant? The answer is: in many cases yes, but not always. The first example of nonuniqueness was given by V.Metz in [10], where it was proved that in the Vicsek set, there are infinitely many eigenforms not multiple of each other. A simpler (but less symmetric example) is the tree-like Gasket, as we previously saw. We are now going to prove that, independently of the uniqueness, the eigenvalue ρ does not depend on the eigenform, thus it is related to the fractal, and it can be called the renormalization factor of the fractal. We need some preparatory considerations.

Lemma 3.17. *For any $E, E' \in \tilde{\mathcal{D}}$ the ratio*

$$A(u) = \frac{E'(u)}{E(u)}$$

has positive minimum, which I will denote by $\lambda_-(E, E')$ and maximum, which I will denote by $\lambda_+(E, E')$ on the set of nonconstant $u \in \mathbb{R}^{V^{(0)}}$. They are also, respectively, the minimum and the maximum of A on the set $\mathcal{S}$, defined in (2.7).

Proof. Since $A(u) = A(\frac{u - u(P_1)}{||u - u(P_1)||})$ for every nonconstant $u \in \mathbb{R}^{V^{(0)}}$, it suffices to observe that A is continuous, thus it has a maximum and a minimum on $\mathcal{S}$. ■

Remark 3.18. We clearly have

$$\lambda_-(E, E')E \le E' \le \lambda_+(E, E')E$$

for every $E, E' \in \tilde{\mathcal{D}}$. ■

We now state some simple properties of S_n and M_n which we will use in the following without explicit mention.

a) $S_n(aE) = aS_n(E)$, $M_n(aE) = aM_n(E)$,

b) $E \leq E' \Rightarrow\ S_n(E) \leq S_n(E'), M_n(E) \leq M_n(E')$,

where $E, E' \in \tilde{\mathcal{D}}$, $a > 0$. By the expression $E \leq E'$ we mean $E(u) \leq E'(u)$ for all $u \in \mathbb{R}^{V^{(0)}}$ and similarly for $S_n(E)$ and $M_n(E)$. Note that $E \leq E'$ does not amount to $c_{j_1,j_2}(E) \leq c_{j_1,j_2}(E')$ for every $j_1, j_2 = 1, ..., N$, $j_1 \neq j_2$; for example, if $N = 3$ and $c_{1,2}(E') = c_{1,3}(E') = 3$, $c_{2,3}(E') = 0$, and $c_{1,2}(E) = c_{1,3}(E) = c_{2,3}(E) = 1$, using the simple inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we have $E \leq E'$. We remark that in the previous considerations we did not require that the forms are eigenforms. Now, suppose we are given two eigenforms E and E' with eigenvalues respectively ρ and ρ'. We will prove that $\rho = \rho'$. In fact, we have $aE \leq E' \leq bE$ for some $a, b > 0$, by Remark 3.18. Using a) and b), we get $M_1^n(E) = \rho^n E$, $M_1^n(E') = \rho'^n E'$, and $a\rho^n E \leq \rho'^n E' \leq b\rho^n E$, for every $n \in \mathbb{N}$. Hence, if $\rho \neq \rho'$, we have that either E or E' is identically 0, contrary to the assumption $E, E' \in \tilde{\mathcal{D}}$.

4. Main Properties of Renormalization and Harmonic Extension

In previous sections, we studied the convergence of $E_{(n)}^{\Sigma}$ when E is an eigenform. We now want to study the corresponding problem when E is any element of $\tilde{\mathcal{D}}$. The first remark is that, as we know that the eigenvalue ρ is independent of the eigenform, we can well define

$$E_{(n)}^{\Sigma} = \frac{S_n(E)}{\rho^n}$$

for any $E \in \tilde{\mathcal{D}}$. We are now going to study the following problem: Is the sequence $E_{(n)}^{\Sigma}$ convergent in this more general case? In this case, unlike the case of an eigenform there is no reason that the sequence $E_{(n)}^{\Sigma}$ is increasing. So, the answer is less simple. In order to attack the problem, we need a more careful investigation of the renormalization operator $M_n(E)$ and of the harmonic extension on $V^{(n)}$, that, so far, we merely hinted. This is what we will do in this section. We will prove in particular that $M_{n+m}(E) = M_m(M_n(E))$. This will turn out to be a consequence of the nesting property, for, in order to minimize the sum of the copies of E on the $(n+m)$-cells, we can minimize for fixed values on $V^{(m)}$ independently on the different n-cells as they only overlap at points in $V^{(m)}$, and then minimize the sum of such minima among the possible values on $V^{(m)}$. The statement of Theorem 4.3 is thus, in some sense intuitive. However, I will give a complete proof of it. In order to do this, we need two Lemmas. We will use the notation $H_{(m,n;E)}$ for $H_{(m;M_n(E))}$.

Lemma 4.1. *Suppose* $(i_1, ..., i_n) \neq (i'_1, ..., i'_n)$. *Then,*

$$K_{i_1,...,i_n} \cap K_{i'_1,...,i'_n} \subseteq V_{i_1,...,i_n} \cap V_{i'_1,...,i'_n} \, .$$

Proof. Note that $K \subseteq T$, hence $K_{i_1,...,i_n} \cap K_{i'_1,...,i'_n} \subseteq T_{i_1,...,i_n} \cap T_{i'_1,...,i'_n}$. Then, the proof is exactly the same as that in Lemma 2.5. ∎

Lemma 4.2. *Suppose $n \in \mathbb{N}$. Suppose $P = \psi_{i_1,...,i_n}(Q) = \psi_{i'_1,...,i'_n}(Q')$ with $i_1, ..., i_n, i'_1, ..., i'_n = 1, ..., k$ and $Q, Q' \in K$, and $(i_1, ..., i_n, Q) \neq (i'_1, ..., i'_n, Q')$. Then, $Q, Q' \in V^{(0)}$.*

Proof. We have $(i_1, ..., i_n) \neq (i'_1, ..., i'_n)$, for in the contrary case, $Q = Q'$. Hence, by Lemma 4.1, $Q, Q' \in V^{(0)}$. ∎

Theorem 4.3.
i) For every $n \in \mathbb{N}$ and $E \in \tilde{\mathcal{D}}$ and $u \in \mathbb{R}^{V^{(0)}}$ the inf in the definition of $M_n(E)$ is in fact a minimum. It is unique and we will denote it by $H_{(n;E)}(u)$. Moreover, $M_n(E) \in \tilde{\mathcal{D}}$.
ii) For every $n, m \in \mathbb{N}$ and $E \in \tilde{\mathcal{D}}$ and $u \in \mathbb{R}^{V^{(0)}}$, we have

$$H_{(n+m;E)}(u) \circ \psi_{i_1,...,i_{n+m}} = H_{(n;E)}\big(H_{(m,n;E)}(u) \circ \psi_{i_1,...,i_m}\big) \circ \psi_{i_{m+1},...,i_{m+n}} \, .$$

iii) For every $n, m \in \mathbb{N}$ and $E \in \tilde{\mathcal{D}}$, we have

$$M_{n+m}(E) = M_m(M_n(E)) \, .$$

Proof. Suppose i) holds for n and m (for all E and u), and prove that the function $\overline{v} : V^{(n+m)} \to \mathbb{R}$ defined by

$$\overline{v}\Big(\psi_{i_1,...,i_m}(Q)\Big) = H_{(n;E)}\big(H_{(m,n;E)}(u) \circ \psi_{i_1,...,i_m}\big)(Q) \qquad Q \in V^{(n)}, \, i_1, ..., i_m = 1, ..., k$$

satisfies

$$\overline{v} \in \mathcal{L}(n+m, u), \quad S_{n+m}(E)(v) \geq M_m\big(M_n(E)\big)(u) \quad \forall v \in \mathcal{L}(n+m, u) \, , \tag{4.1}$$

and the equality holds if and only if $v = \overline{v}$. Since we already know that i) holds for $n = 1$, a recursive argument then yields i), ii) and iii). First, note that the definition of $\overline{v}$ is correct, i.e., it does not depend on the representation of $P \in V^{(n+m)}$ as $P = \psi_{i_1,...,i_m}(Q)$ with $i_1, ..., i_m = 1, ..., k$, $Q \in V^{(n)}$. If $P \in V^{(m)}$, then $P = \psi_{i'_1,...,i'_m}(Q')$ for some $i'_1, ..., i'_m = 1, ..., k$, $Q' \in V^{(0)}$. Thus, either $(i_1, ..., i_m) = (i'_1, ..., i'_m)$, then $Q = Q' \in V^{(0)}$, or $(i_1, ..., i_m) \neq (i'_1, ..., i'_m)$, and using Lemma 4.2 we have $Q \in V^{(0)}$ again. Hence, the definition of $\overline{v}$ gives $\overline{v}(P) = H_{(m,n;E)}(u)(P)$. If, on the contrary, $P \notin V^{(m)}$, in view of Lemma 4.2 the above representation of P is unique. Next, we prove that $\overline{v} \in \mathcal{L}(n+m, u)$.

We just proved that $\overline{v}$ amounts to $H_{(m,n;E)}(u)$ on $V^{(m)} \supseteq V^{(0)}$, which in turn amounts to u on $V^{(0)}$. Finally, if $v \in \mathcal{L}(n+m, u)$, we have

$$S_{n+m}(E)(v) = \sum_{i_1,\dots,i_m=1}^{k} \Big(\sum_{i_{m+1},\dots,i_{m+n}=1}^{k} E(v \circ \psi_{i_1,\dots,i_m} \circ \psi_{i_{m+1},\dots,i_{m+n}}) \Big)$$

$$= \sum_{i_1,\dots,i_m=1}^{k} S_n(E)(v \circ \psi_{i_1,\dots,i_m}) \geq \sum_{i_1,\dots,i_m=1}^{k} M_n(E)\big(v_{|V^{(m)}} \circ \psi_{i_1,\dots,i_m})\big)$$

$$= S_m(M_n(E))\big(v_{|V^{(m)}}\big) \geq M_m\big(M_n(E)\big)(u)\,,$$

and, the first inequality is in fact an equality if and only if

$$v \circ \psi_{i_1,\dots,i_m} = H_{(n;E)}\big(v_{|V^{(m)}} \circ \psi_{i_1,\dots,i_m}\big)$$

on $V^{(n)}$ for all $i_1, \dots, i_m = 1, \dots, k$, the second is an equality if and only if $v_{|V^{(m)}} = H_{(m,n;E)}(u)$, if and only if

$$v \circ \psi_{i_1,\dots,i_m} = H_{(m,n;E)}(u) \circ \psi_{i_1,\dots,i_m}$$

on $V^{(0)}$ for all $i_1, \dots, i_m = 1, \dots, k$. Hence, the equality holds in (4.1) if and only if $v = \overline{v}$. ∎

Corollary 4.4. *We have $M_n = M_1^n$.* ∎

It follows that, if E is an eigenform then $M_n(E) = \rho^n E$, so that $E_{(n)} = E$. Now, for every $E \in \tilde{\mathcal{D}}$, we put

$$\tilde{M}_n(E) := E_{(n)} := \frac{M_n(E)}{\rho^n}$$

when $E \in \tilde{\mathcal{D}}$. We easily deduce from Theorem 4.3 that

$$E_{(n+m)} = (E_{(n)})_{(m)}$$

for all $E \in \tilde{\mathcal{D}}$ and $n, m \in \mathbb{N}$, thus $\tilde{M}_n = \tilde{M}_1^n$. Clearly, $E \in \tilde{\mathcal{D}}$ is an eigenform if and only if it is a fixed point of $\tilde{M}_1$. In order to investigate the convergence of $E^{\Sigma}_{(n)}$ we need some information on the convergence of $E_{(n)}$. We will prove in fact that the sequence $E^{\Sigma}_{(n)}$ is Γ-convergent to $\tilde{E}^{\Sigma}_{(\infty)}$ where $\tilde{E}$ is the limit of $E_{(n)}$. The proof of the convergence of $E_{(n)}$ is the real problem in the proof of Γ-convergence of $E^{\Sigma}_{(n)}$ when E is not an eigenform. Clearly, when E is an eigenform, $E_{(n)} = E \underset{n\to\infty}{\longrightarrow} E$. In order to prove the convergence of $E_{(n)}$, it will be useful to study the behaviour of $H_{(n;E)}(u)$ on the single n-cells, in other words, to study $H_{(n;E)}(u) \circ \psi_{i_1,\dots,i_n}$. In this connection, another consequence of Theorem 4.3 is that we can split the map $u \mapsto H_{(n;E)}(u) \circ \psi_{i_1,\dots,i_n}$ into the composition of maps like

$u \mapsto H_{(1;E)}(u) \circ \psi_i$. We need thus a reformulation of $H_{(1;E)}(u) \circ \psi_i$ which represents it as a function of u. So, put

$$T_{i;E}(u) = H_{(1;E)}(u) \circ \psi_i$$

for every $i = 1, ..., k$; $E \in \tilde{\mathcal{D}}$; $u \in \mathbb{R}^{V^{(0)}}$. Put also $T_{i,n;E} := T_{i;M_n(E)}$. In previous definitions, we omit E when is clear from the context, and in such a case, we write $T_i, T_{i,n}$. The following properties can be easily verified. We have already discussed some of them in terms of properties of the map $u \mapsto H_{(1;E)}(u)$.

Proposition 4.5. *Let $u \in \mathbb{R}^{V^{(0)}}, E \in \tilde{\mathcal{D}}$. We have*
i) T_i is linear.
ii) $T_i(u + c) = T_i(u) + c$ if c is constant.
iii) $T_i(u)(P_i) = u(P_i)$.
iv) $T_{i;aE} = T_{i;E}$ if $a > 0$. ∎

Note that thanks to iii), T_i maps the space

$$\Pi_i = \{u \in \mathbb{R}^{V^{(0)}} : u(P_i) = 0\}$$

into itself, and by ii) T_i is completely determined by its values on $\Pi_i \simeq \mathbb{R}^{N-1}$. Thus, T_i can be considered as a linear operator from $\mathbb{R}^{N-1}$ into itself. Another trivial remark is that, by iv), we have $T_{i,M_n(E)} = T_{i,E_{(n)}}$. Putting $n = 1$ in Theorem 4.3 ii), a simple recursive argument on m yields

Lemma 4.6. *Let $u \in \mathbb{R}^{V^{(0)}}$, $E \in \tilde{\mathcal{D}}$. Then*

$$H_{(n;E)}(u) \circ \psi_{i_1,...,i_n} = T_{i_n,0} \circ T_{i_{n-1},1} \circ \cdots \circ T_{i_1,n-1}(u) \quad \text{on } V^{(0)},$$

in particular, if E is an eigenform, we have

$$H_{(n;E)}(u) \circ \psi_{i_1,...,i_n} = T_{i_n} \circ T_{i_{n-1}} \circ \cdots \circ T_{i_1}(u).$$

∎

Corollary 4.7. *Let $E \in \tilde{\mathcal{D}}$, $u \in \mathbb{R}^{V^{(0)}}$, $Q \in V^{(n)}$. Then*

$$\min_{V^{(0)}} u \le H_{(n;E)}(u) \le \max_{V^{(0)}} u.$$

Proof. We have $\min_{V^{(0)}} u \le T_{i;E}(u) \le \max_{V^{(0)}} u$ for every $i = 1, ..., k$. We conclude by a recursive argument. ∎

So far, I discussed about the convergence of sequences in $\tilde{\mathcal{D}}$, but I did not specify by what a sense I mean the convergence. It will be better to consider, a priori, the convergence on $\mathcal{D}$ instead of on $\tilde{\mathcal{D}}$, as $\mathcal{D}$ is closed. We can define a norm $|| \ ||$ on the linear space generated by $\mathcal{D}$ as $||E|| = \sup_{u\in\mathcal{S}} |E(u)|$. Note that if $||E|| = 0$, then as E is 2-homogeneous and satisfies $E(u) = E(u - u(P_1))$, it follows $E(u) = 0$ for all $u \in \mathbb{R}^{V^{(0)}}$, so that $|| \ ||$ is in fact a norm. We have:

Lemma 4.8. *Given $E_n, E \in \mathcal{D}$, the following properties are equivalent*
a) $E_n \underset{n\to\infty}{\longrightarrow} E$ pointwise,
b) $E_n \underset{n\to\infty}{\longrightarrow} E$ uniformly on compact subsets of $\mathbb{R}^{V^{(0)}}$,
c) $E_n \underset{n\to\infty}{\longrightarrow} E$ in the norm $|| \ ||$,
d) $c_{j_1,j_2}(E_n) \underset{n\to\infty}{\longrightarrow} c_{j_1,j_2}(E)$ for all $j_1, j_2 = 1, ..., N$, $j_1 \neq j_2$.

Proof. Clearly, we have $d) \Rightarrow b) \Rightarrow a)$. Also, $a) \Rightarrow d)$ by Remark 3.5. Since the set $\mathcal{S}$ is compact, we have $b) \Rightarrow c)$. Finally, as $E(u) = ||u - u(P_1)||^2 E\Big(\frac{u-u(P_1)}{||u-u(P_1)||}\Big)$ for every nonconstant $u \in \mathbb{R}^{V^{(0)}}$, $c) \Rightarrow a)$. ∎

So, the convergence in $\mathcal{D}$ (a fortiori in $\tilde{\mathcal{D}}$) will be meant to be in one of the four equivalent formulations in Lemma 4.8. Also, we will consider on $\mathcal{D}$ and on $\tilde{\mathcal{D}}$ the topology induced by such a convergence. In this way the main functions we defined on $\tilde{\mathcal{D}}$ are continuous.

Lemma 4.9. *We have*
i) The map from $\tilde{\mathcal{D}} \times \mathbb{R}^{V^{(0)}}$ to $\mathbb{R}^{V^{(1)}}$ defined by $(E, u) \mapsto H_{(1;E)}(u)$, is continuous.
ii) The map from $\tilde{\mathcal{D}} \times \mathbb{R}^{V^{(0)}}$ to $\mathbb{R}$ defined by $(E, u) \mapsto M_1(E)(u)$ is continuous.
iii) M_n is continuous from $\tilde{\mathcal{D}}$ to $\tilde{\mathcal{D}}$.
iv) λ_+ and λ_- are continuous from $\tilde{\mathcal{D}} \times \tilde{\mathcal{D}}$ to $]0, +\infty[$.

Proof. Prove i). Suppose $E_h, E \in \tilde{\mathcal{D}}$, $u_h, u \in V^{(0)}$, $E_h \underset{h\to\infty}{\longrightarrow} E$, $u_h \underset{h\to\infty}{\longrightarrow} u$, and prove that

$$H_{(1;E_h)}(u_h) \underset{h\to\infty}{\longrightarrow} H_{(1;E)}(u) \,. \tag{4.2}$$

By the maximum principle we have $\min_{V^{(0)}} u_h \le H_{(1;E_h)}(u_h) \le \max_{V^{(0)}} u_h$. Thus, the functions $H_{(1;E_h)}(u_h)$ are uniformly bounded and if (4.2) does not hold, we have

$$H_{(1;E_{h_l})}(u_{h_l}) \underset{l\to\infty}{\longrightarrow} v$$

with $v \neq H_{(1;E)}(u)$, for some strictly increasing sequence of naturals h_l. Thus, v satisfies (3.5), and $v = H_{(1;E)}(u)$, contrary to our assumption. ii) easily follows from i), in view of the formula $M_1(E)(u) = S_1(E)\big(H_{(1;E)}(u)\big)$. iii) is an immediate consequence of ii) and of the formula $M_n(E) = M_1^n(E)$. We now prove iv). Suppose $E_h \underset{h\to\infty}{\longrightarrow} E$, $E'_h \underset{h\to\infty}{\longrightarrow} E'$. Given a nonconstant $u \in \mathbb{R}^{V^{(0)}}$, let $\alpha > 0$ be such that $E(u) \geq \alpha$ and let $\overline{h} \in \mathbb{N}$ be such that $E_h(u) \geq \frac{\alpha}{2}$ for $h \geq \overline{h}$. By simple calculations, for every $h \geq \overline{h}$ and $u \in \mathcal{S}$ we get

$$\Big|\frac{E'_h(u)}{E_h(u)} - \frac{E'(u)}{E(u)}\Big| \leq \frac{2}{\alpha^2}\big|E'_h(u)E(u) - E_h(u)E'(u)\big|$$

$$\leq \frac{2}{\alpha^2}\Big(||E'_h||||E_h - E|| + ||E_h||||E'_h - E'||\Big) \underset{h\to\infty}{\longrightarrow} 0\,,$$

so that d) easily follows. ∎

Corollary 4.10. *If $E \in \tilde{\mathcal{D}}$ and $E_{(n)} \underset{n\to\infty}{\longrightarrow} E'$, then E' is an eigenform.*

Proof. Since $E' = \lim\limits_{n\to\infty} \big(\tilde{M}_1\big)^n(E)$, and $\tilde{M}_1$ is continuous, then E' is a fixed point of $\tilde{M}_1$. ∎

Lemma 4.9 and Corollary 4.10 will be used without mention in the following. Another equivalent way of expressing the convergence in $\tilde{\mathcal{D}}$ is given by the following corollary.

Corollary 4.11. *Given $E_n, E \in \tilde{\mathcal{D}}$, then $E_n \underset{n\to\infty}{\longrightarrow} E$ if and only if $\lambda_\pm(E, E_n) \underset{n\to\infty}{\longrightarrow} 1$.*

Proof. Part "if" follows from Remark 3.18. Part "only if" follows from Lemma 4.9 iv. ∎

In some sense, the functions $\lambda_\pm$ provide a way of measuring how much far two different $E, E' \in \tilde{\mathcal{D}}$ are. We are so lead to give the following definition.

Given $E, E' \in \tilde{\mathcal{D}}$ let

$$\lambda(E, E') = \ln(\lambda_+(E, E')) - \ln(\lambda_-(E, E')).$$

λ is a particular case of Hilbert's projective metric. For the theory of Hilbert's projective metric see for example [14]. We have in fact, as can be easily verified, that λ is a semimetric on $\tilde{\mathcal{D}}$, in the sense that it has all properties of a metric but the property that the distance is 0 only if the two elements are the same. More precisely, we have $\lambda(E, E') = 0$ if and only if E' is a (positive) multiple of E (or, E is a (positive) multiple of E'). In particular, $E \in \tilde{\mathcal{D}}$ is an eigenform if and only if $\lambda(E, E_{(1)}) = 0$. Also, we clearly have $\lambda(aE, bE') = \lambda(E, E')$ for every $a, b > 0$. Thus, λ induces a metric on the projective space $pr(\tilde{\mathcal{D}})$ generated by $\tilde{\mathcal{D}}$, that is the space of equivalence classes on $\tilde{\mathcal{D}}$, mod the relation where two elements are equivalent if they are multiple of each other. Moreover,

Lemma 4.12. *Given $E, E' \in \tilde{\mathcal{D}}$ we have*
i) $\lambda_-(M_1(E), M_1(E')) = \lambda_-(E_{(1)}, E'_{(1)}) \geq \lambda_-(E, E')$
ii) $\lambda_+(M_1(E), M_1(E')) = \lambda_+(E_{(1)}, E'_{(1)}) \leq \lambda_+(E, E')$
iii) $\lambda(M_1(E), M_1(E')) = \lambda(E_{(1)}, E'_{(1)}) \leq \lambda(E, E')$.

Proof. In i), ii), iii), the equalities are trivial, so we have to prove the inequalities. By Remark 3.18 we have $\lambda_-(E,E')M_1(E) \le M_1(E') \le \lambda_+(E,E')M_1(E)$, hence,

$$\lambda_-(E,E') \le \frac{M_1(E')(u)}{M_1(E)(u)} \le \lambda_+(E,E')$$

for every nonconstant $u \in \mathbb{R}^{V^{(0)}}$, and i) and ii) follow at once, and iii) is an immediate consequence of i) and ii). ∎

Now, since $E_{(1)}$ is a multiple of $M_1(E)$ for $E \in \tilde{\mathcal{D}}$, in view of Lemma 4.12 we get that the map $\tilde{M}_1$ is a weak contraction on $pr(\tilde{\mathcal{D}})$ with respect to λ. This fact suggests that a way for proving that the iterated $\tilde{M}_n = \tilde{M}_1^n$ converges could be to try to prove that the above considered map is in fact a strong contraction. Unfortunately, this is not true in general, as we saw that we can have two different eigenforms not multiple of each other, which corresponds to two different (on $pr(\tilde{\mathcal{D}})$) fixed points of $\tilde{M}_1$. However, we can modify such an argument in this way: In order to prove that the sequence $E_{(n)}$ tends to some eigenform $\tilde{E}$, since the distance λ between $\tilde{E}$ and $E_{(n)}$ is decreasing in view of previous lemma, we can try to prove that it is not eventually constant, so that we can hope that it tends to 0. We will use this argument on the Gasket in next section. We are so lead to investigate the cases in which in Lemma 4.12 iii) we have the equality. For the moment, however, let us discuss some simple consequences of Lemma 4.12. Given an eigenform $\tilde{E}$ and real numbers a, b with $b \ge a > 0$, let us put

$$U_{a,b,\tilde{E}} = \{E \in \tilde{\mathcal{D}} : a\tilde{E} \le E \le b\tilde{E}\} \Big(= \{E \in \tilde{\mathcal{D}} : a \le \lambda_-(\tilde{E},E), \lambda_+(\tilde{E},E) \le b\}\Big)$$

and write $U_{a,b}$ for $U_{a,b,\hat{E}}$. Since $\hat{E}$ is an eigenform and thus $\hat{E}_{(1)} = \hat{E}$, it immediately follows from Lemma 4.12 that $\tilde{M}_1$ maps $U_{a,b}$ into itself. Thus if $E \in U_{a,b}$, then every $E_{(n)}$ lies in $U_{a,b}$. Moreover, $U_{a,b}$ is (sequentially) compact. We have in fact:

Lemma 4.13. *Every sequence in $U_{a,b}$ has a subsequence convergent to some element of $U_{a,b}$.*

Proof. Let E_n be a sequence in $U_{a,b}$. Because of Remark 3.5, the coefficients $c_{j_1,j_2}(E_n)$ are estimated by $\frac{b}{4}\hat{E}(v_{j_1,j_2})$. Thus, E_n has a subsequence convergent to some functional E and we immediately see that $E \in \mathcal{D}$. Moreover, we clearly have $a\hat{E} \le E \le b\hat{E}$. Thus, if $u \in \mathbb{R}^{V^{(0)}}$ is nonconstant we have $E(u) \ge a\hat{E}(u) > 0$, so that $E \in \tilde{\mathcal{D}}$. Moreover, $E \in U_{a,b}$. ∎

Corollary 4.14. *For every $E \in \tilde{\mathcal{D}}$ there exists a strictly increasing sequence of naturals n_h and $E' \in \tilde{\mathcal{D}}$ such that $E_{(n_h)} \underset{h\to\infty}{\longrightarrow} E'$.* ∎

The problem is thus to prove that all the sequence $E_{(n)}$ converges to the same limit E'. The use of $U_{a,b}$ is the point in which we need the existence of an eigenform. Lemma 4.13, in fact, states that in some sense the sequence $E_{(n)}$ is bounded with respect to λ. Another consequence of Lemma 4.12 is the following.

Corollary 4.15. *Given $E, E' \in \tilde{\mathcal{D}}$, put $\lambda_{\pm,n} = \lambda_{\pm}(E_{(n)}, E'_{(n)})$, $\lambda_n = \lambda(E_{(n)}, E'_{(n)})$. Then*
i) $\lambda_{+,n}$ is decreasing and $\lambda_{-,n}$ is increasing, thus λ_n is decreasing
ii) If we set $\lambda_{\pm} = \lim_{n\to\infty} \lambda_{\pm,n}$ we have $0 < \lambda_- \le \lambda_+ < +\infty$.

Proof. Since $E_{(n+1)} = (E_{(n)})_{(1)}$, i) is an immediate consequence of Lemma 4.12, putting there $(E_{(n)})$ in place of E and $(E'_{(n)})$ in place of E'. For ii), note that $0 < \lambda_{-,0} \le \lambda_{-,n} \le \lambda_{+,n} \le \lambda_{+,0} < +\infty$. ∎

Corollary 4.16. *If $E, E' \in \tilde{\mathcal{D}}$ and E' is an eigenform and $E_{(n_h)} \underset{h\to\infty}{\longrightarrow} E'$ for some strictly increasing sequence of naturals n_h, then $E_{(n)} \underset{n\to\infty}{\longrightarrow} E'$.*

Proof. By Corollary 4.11, $\lambda_{\pm}(E', E_{(n_h)}) \underset{h\to\infty}{\longrightarrow} 1$. On the other hand, since $E'_{(n)} = E'$, there exists $\lim_{n\to\infty} \lambda_{\pm}(E', E_{(n)}) = 1$. By Corollary 4.11 again, $E_{(n)} \underset{n\to\infty}{\longrightarrow} E'$. ∎

We have just seen that $\lambda_{+,n} \le \lambda_{+,0}$ and $\lambda_{-,n} \ge \lambda_{-,0}$. In order to know whether the operator $\tilde{M}_n$ contracts the distance λ for some n, we need to know when such inequalities are in fact equalities. To this aim, we study the set of functions in $\mathbb{R}^{V^{(0)}}$, which are in some sense extrema for the ratio $\frac{E'}{E}$. For $E, E' \in \tilde{\mathcal{D}}$ put

$$(A^{\pm} =) A^{\pm}(E, E') = \left\{ u \in \mathbb{R}^{V^{(0)}} : E'(u) = \lambda_{\pm}(E, E') E(u) \right\}.$$

By definition such sets include the constant functions. Since in some cases we need to use only nonconstant functions, put $(\tilde{A}^{\pm} =) \tilde{A}^{\pm}(E, E')$ to be the set of the functions in $A^{\pm}(E, E')$ which are not constant. Put also

$$(A^{\pm,n} =) A^{\pm,n}(E, E') = A^{\pm}(M_n(E), M_n(E'))\big(= A^{\pm}(E_{(n)}, E'_{(n)})\big)\big),$$
$$(\tilde{A}^{\pm,n} =) \tilde{A}^{\pm,n}(E, E') = \tilde{A}^{\pm}(M_n(E), M_n(E'))\big(= \tilde{A}^{\pm}(E_{(n)}, E'_{(n)})\big)\big).$$

Proposition 4.17. *Let $E, E' \in \tilde{\mathcal{D}}$. Then*
i) $A^{\pm}$ is closed.
ii) $u \in A^{\pm} \Rightarrow c_1 u + c_2 \in A^{\pm} \quad \forall\, c_1, c_2 \in \mathbb{R}$.
iii) E' is a multiple of $E \iff \tilde{A}^+ \cap \tilde{A}^- \ne \emptyset$.
iv) $\tilde{A}^{\pm} \ne \emptyset$. ∎

Note that, if $\lambda_{\pm,n} = \lambda_{\pm,0}$, then

$$A^{\pm,n}(E, E') = \left\{ u \in \mathbb{R}^{V^{(0)}} : M_n(E')(u) = \lambda_{\pm,0} M_n(E)(u) \right\}.$$

Now, we can state the following result.

Lemma 4.18. *Let E and E' and $\lambda_{\pm,n}$ be as in Corollary 4.15. If we have $\lambda_{\pm,n} = \lambda_{\pm,0}$, then for every $u \in A^{\pm,n}$ we have*

$$H_{(n;E)}(u) \circ \psi_{i_1,\dots,i_n} = H_{(n;E')}(u) \circ \psi_{i_1,\dots,i_n} \in A^{\pm}$$

for all $i_1, \dots, i_n = 1, \dots, k$.

Proof. Let $u \in A^{-,n}$. Then

$$M_n(E')(u) = S_n(E')\big(H_{(n;E')}(u)\big) =$$

$$\sum_{i_1,\dots,i_n=1}^{k} E'\big(H_{(n;E')}(u) \circ \psi_{i_1,\dots,i_n}\big) \geq \sum_{i_1,\dots,i_n=1}^{k} \lambda_{-,0} E\big(H_{(n;E')}(u) \circ \psi_{i_1,\dots,i_n}\big)$$

$$= \lambda_{-,0} S_n(E)\big(H_{(n;E')}(u)\big) \geq \lambda_{-,0} M_n(E)(u) = M_n(E')(u)$$

so that the two inequalities are in fact equalities. From the fact that the first inequality is an equality we deduce $H_{(n;E')}(u) \circ \psi_{i_1,\dots,i_n} \in A^-$, and from the fact that the second inequality is an equality we deduce $H_{(n;E)}(u) \circ \psi_{i_1,\dots,i_n} = H_{(n;E')}(u) \circ \psi_{i_1,\dots,i_n}$, for all $i_1, \dots, i_n = 1, \dots, k$. We have proved the Lemma for the case where $\pm$ is $-$. We can proceed similarly in the case where $\pm$ is $+$, taking in account that $\lambda_+(E, E') = \big(\lambda_-(E', E)\big)^{-1}$. ∎

Corollary 4.19. *In the same hypotheses as in Lemma 4.18, for every m with $0 \leq m \leq n$ we have*

$$\lambda_{\pm,m} = \lambda_{\pm,0}\,, \tag{4.3}$$

and for every $u \in A^{\pm,n}$

$$H_{(n-m,m;E)}(u) \circ \psi_{i_1,\dots,i_{n-m}} = H_{(n-m,m;E')}(u) \circ \psi_{i_1,\dots,i_{n-m}} \in A^{\pm,m} \tag{4.4}$$

for all $i_1, \dots, i_{n-m} = 1, \dots, k$.

Proof. From Corollary 4.15, (4.3) follows at once. Thus, as we have

$$u \in A^{\pm}\Big(M_{n-m}\big(M_m(E)\big), M_{n-m}\big(M_m(E')\big)\Big),$$

(4.4) follows from Lemma 4.18. ∎

Note that the first equality in (4.4) for every $i_1, \dots, i_{n-m} = 1, \dots, k$, simply amounts to $H_{(n-m,m;E)}(u) = H_{(n-m,m;E')}(u)$. Roughly speaking, whenever we have $\lambda_{\pm,n} = \lambda_{\pm,0}$, every function $u \in A^{\pm,n}$ produces, via its harmonic extension, functions in $A^{\pm,m}$ for all $m \leq n$, at every $(n-m)$-cell, in particular, for $m = 0$, functions in $A^{\pm}$, at every n-cell.

Remark 4.20. Note that, given $E, E' \in \tilde{\mathcal{D}}$, if $\lambda(E_{(n)}, E'_{(n)}) = \lambda(E, E')$ then $\lambda_{\pm}(E_{(n)}, E'_{(n)}) = \lambda_{\pm}(E, E')$. ∎

5. Homogenization on the Gasket

In this section we assume that the fractal is the Gasket. We first prove that for every $E \in \tilde{\mathcal{D}}$, the sequence $E_{(n)}$ converges, then we prove that the sequence $E^{\Sigma}_{(n)}$ is Γ-convergent. We call this phenomenon *homogenization*. Analogous convergence results also hold for general fractals, but I prefer first to illustrate the process in the Gasket, because the proof is more natural and so can be better understood. A property typical of the Gasket (and of other, but not of all, fractals), is that the limit form of $E_{(n)}$ is a multiple of $\overline{E}$. The proof of the convergence of $E_{(n)}$ on the Gasket presented in this section has not been published, but the idea is sketched in the introduction of [16]. The proof in general fractals presented in Section 6 follows more or less the approach of [16] (or also, of [15]). In Section 4 we studied what happens if $\lambda_{\pm}(E_{(n)}, E'_{(n)}) = \lambda_{\pm}(E, E')$ in the general case $E, E' \in \tilde{\mathcal{D}}$. Put now $E = \overline{E}$, so that $E_{(n)} = \overline{E}$, and put E in place of E'. Since $\overline{E}$ is an eigenform, then (4.4) yields $H_{(n-m;\overline{E})}(u) \circ \psi_{i_1,\dots,i_{n-m}} \in A^{\pm,m}$ if $u \in A^{\pm,n}$. Hence, in view of Lemma 4.6, we get

Proposition 5.1. *If $E \in \tilde{\mathcal{D}}$ and $\lambda_{\pm}(\overline{E}, E_{(n)}) = \lambda_{\pm}(\overline{E}, E)$ and $u \in A^{\pm}(\overline{E}, E_{(n)})$, then $T^n_{i;\overline{E}}(u) \in A^{\pm}(\overline{E}, E)$ for every $i = 1, 2, 3$ and $n \in \mathbb{N}$.* ∎

Let $T_i := T_{i;\overline{E}}$ in the rest of this section. The plan of the proof of the convergence of $E_{(n)}$ is more or less the following. First, we will prove that if such a convergence does not take place for some $E \in \tilde{\mathcal{D}}$, then we have $\lambda_{\pm}(\overline{E}, E_{(n)}) = \lambda_{\pm}(\overline{E}, E)$ for some $E \in \tilde{\mathcal{D}}$; then, we will use Prop. 5.1 to deduce that $\tilde{A}^{+}$ and $\tilde{A}^{-}$ contain some eigenvectors of T_i, obtained as a limit of T_i^n and, finally we will prove that these eigenvectors are the same, so that by Prop. 4.18, E is a multiple of $\overline{E}$, and thus $E_{(n)} = E$. In order to prove the first step in such a plan of proof, I will use the following notation. I will say that $E \in \tilde{\mathcal{D}}$ *approaches* $\overline{E}$ if $\lambda(\overline{E}, E_{(n)}) < \lambda(\overline{E}, E)$ for some n thus for sufficiently large n. I will say that E *goes to* $\overline{E}$ if $E_{(n)}$ tends, as $n \to \infty$, to a multiple of $\overline{E}$. Then

Lemma 5.2. *If every $E \in \tilde{\mathcal{D}}$ which is not a multiple of $\overline{E}$ approaches $\overline{E}$, then every $E \in \tilde{\mathcal{D}}$ goes to $\overline{E}$.*

Proof. Let $E \in \tilde{\mathcal{D}}$. By Corollary 4.14 there exist a strictly increasing sequence of naturals n_h and $E' \in \tilde{\mathcal{D}}$ such that $E_{(n_h)} \underset{h\to\infty}{\longrightarrow} E'$. For every $m \in \mathbb{N}$ we have

$$\lambda(\overline{E}, E'_{(m)}) = \lambda\big(\overline{E}, \lim_{h\to\infty}(E_{(n_h)})_{(m)}\big) = \lambda(\overline{E}, \lim_{h\to\infty} E_{(m+n_h)})$$

$$= \lim_{h\to\infty} \lambda(\overline{E}, E_{(m+n_h)}) = \lim_{n\to\infty} \lambda(\overline{E}, E_{(n)}),$$

the last equality depending on the fact that the last limit exists. Thus, E' does not approach $\overline{E}$ and by hypothesis, $E' = a\overline{E}$ for some positive a, in particular it is an eigenform. By Corollary 4.16, $E_{(n)} \underset{n\to\infty}{\longrightarrow} E'$. ∎

In order to investigate the convergence of T_j, I first introduce the problem by some preliminary considerations. If $u \in \mathbb{R}^{V^{(0)}}$ we identify u with a vector of $\mathbb{R}^3$ by putting $u = (u(P_1), u(P_2), u(P_3))$. By this identification, T_j is a linear operator from $\mathbb{R}^3$ into itself. Suppose for example $j = 3$. Then T_3 maps $\mathbb{R}^2 \times \{0\}$ into itself. We have $T_3(x, y, 0) = \Big(H_{(\overline{E},1)}(x, y, 0)(\psi_3(P_i))_{i=1,2,3}\Big) = (\frac{2}{5}x + \frac{1}{5}y, \frac{1}{5}x + \frac{2}{5}y, 0)$ (see solution of (2.4)). Hence, putting $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, we get $T_3(e_1) = (\frac{2}{5}, \frac{1}{5}, 0)$, $T_3(e_2) = (\frac{1}{5}, \frac{2}{5}, 0)$. The important point in these formulas is that

$$T_3(e_1) = (a, b, 0), T_3(e_2) = (b, a, 0) \quad \text{with } a, b > 0\,. \tag{5.1}$$

This can be deduced by the symmetry of P_1 and P_2 and by the strong maximum principle, with no explicit calculations. On the base of the form of T_3 we can see that T_3 maps the positive cone $D_3 := \{v \in \mathbb{R}^3 : v_1 \geq 0, v_2 \geq 0, v_3 = 0, v \neq (0, 0, 0)\}$ into the cone $D_3' := \{v \in \mathbb{R}^3 : v_1 > 0, v_2 > 0, v_3 = 0\}$. Suppose now that $\lambda(\overline{E}, E_{(n)}) = \lambda(\overline{E}, E)$, and take $u_n \in \tilde{A}^{\pm}(\overline{E}, E_{(n)})$. Suppose that it attains its minimum at P_3. If it attains its minimum at another point, we can proceed similarly; moreover, a similar argument works if it attains its maximum at P_3. By Lemma 4.17 ii), we can and do assume $u_n(P_3) = 0$, so that $u_n \in D_3$, in particular u_n is nonconstant. We have $T_3^n(u_n) \in \tilde{A}^{\pm}$. In order to study the asymptotic behaviour of $w_n := T_3^n(u_n)$, note that it clearly tends to 0, but we can ask to what kind of configuration it tends, in other words, what is the limit of the normalized vector $v_n := \frac{w_n}{||w_n||}$ (if such a limit exists). Since T_3 in some sense has the effect of mixing the first two components in a symmetric way, we can expect that $v_n \underset{n\to\infty}{\longrightarrow} \frac{\sqrt{2}}{2}(1, 1, 0)$; however, since the point u_n depends on n, this cannot be simply proved on the base on the convergence of the iterated, but we need a sort of uniform convergence. We now come to give the precise statement of the convergence of T_j^n. Let

$$D_j := \{v \in \mathbb{R}^3 : v_l \geq 0 \text{ for } l \neq j, v_j = 0, v \neq (0, 0, 0)\},$$

$$D_j' := \{v \in \mathbb{R}^3 : v_l > 0 \text{ for } l \neq j, v_j = 0, \}\,.$$

Let $\overline{v}_1 = \big(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\big)$, $\overline{v}_2 = \big(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\big)$, $\overline{v}_3 = \big(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\big)$. Let $\widehat{T_j^n}(v) = \frac{T_j^n(v)}{||T_j^n(v)||}$ when $T_j^n(v) \neq 0$. Let $q(v) = \dfrac{\max v_i}{\min v_i}$, where the maximum and the minimum are taken over $i = 1, 2, 3$ with $v_i > 0$, when $v \in D_j'$. Then

Lemma 5.3. *We have* $\widehat{T_j^n}(D_j) \underset{n\to\infty}{\longrightarrow} \overline{v}_j$, *in the sense that*

$$\sup_{v \in D_j} \Big(q\big(T_j^n(v)\big)\Big) \underset{n\to\infty}{\longrightarrow} 1\,. \tag{5.2}$$

Proof. We suppose for example $j = 3$, the other cases being analogous. Define functions q_1, q_2 on D'_3 by $q_1(v) = \frac{v_2}{v_1}$, $q_2(v) = \frac{v_1}{v_2}$. Thus

$$q = \max\{q_1, q_2\}.$$

Note that q_1, q_2, and so q, are continuous with values in $]0, +\infty[$. By (5.1) we have

$$q_1(T_3(v)) = \frac{\frac{1}{5}v_1 + \frac{2}{5}v_2}{\frac{2}{5}v_1 + \frac{1}{5}v_2} = \frac{2q_1(v) + 1}{q_1(v) + 2}$$

and a similar formula holds for $q_2(T_3(v))$. In other words, for $i = 1, 2$, we have $q_i(T_3(v)) = f(q_i(v))$ where

$$f(t) = \frac{2t + 1}{t + 2}.$$

Note that f satisfies

a) $f(1) = 1$.
b) f is strictly increasing on $]0, +\infty[$.
c) $f(t) < t$ for every $t > 1$.

We deduce $q(T_3(v)) = f(q(v))$. Also, for every $t \geq 1$,

$$f^n(t) \underset{n\to\infty}{\longrightarrow} 1. \tag{5.3}$$

Moreover, the convergence in (5.3) is uniform on every interval of the form $[1, c]$ with $c > 1$, for, by b) we have $1 \leq f^n(t) \leq f^n(c)$ for $t \in [1, c]$. Now, q has a maximum $M \geq 1$ on $T_3(D_3)$ as q is continuous and 0-homogeneous, and the set of unit vectors in D_3, so its image via T_3, is compact. Since $q(T_3^n(v)) = f^n(q(v))$, we have

$$\sup_{v\in D_3} q\big(T_3^n(v)\big) \underset{n\to\infty}{\longrightarrow} 1.$$

∎

Corollary 5.4. *Given a strictly increasing sequence n_h of naturals, and $v_h \in \widehat{T_j^{n_h}}(D_j)$, we have $v_h \underset{h\to\infty}{\longrightarrow} \overline{v}_j$.*

Proof. Let for example $j = 3$. By Lemma 5.3 we have $q(v_h) \underset{h\to\infty}{\longrightarrow} 1$. If w is a limit point of v_h, then $w \in D'_3$, for, in the contrary case, a subsequence of $q(v_h) \underset{h\to\infty}{\longrightarrow}$ tends to $+\infty$. Hence, by continuity, $q(w) = 1$, which implies $w = \overline{v}_3$. Since $\overline{v}_3$ is the unique limit point of v_h, then $v_h \underset{h\to\infty}{\longrightarrow} \overline{v}_3$. ∎

Theorem 5.5. *Every $E \in \tilde{\mathcal{D}}$ goes to $\overline{E}$.*

Proof. By Lemma 5.2 it suffices to prove that given $E \in \tilde{\mathcal{D}}$ that does not approach $\overline{E}$, then E is a multiple of $\overline{E}$. Take $u_n \in \tilde{A}^+(\overline{E}, E_{(n)})$. Clearly, there exists a strictly increasing sequence of naturals n_h such that all u_{n_h} attain their minima at a unique point P_{j_1} and their maxima at a unique point P_{j_2}, with $j_1 \neq j_2$. By Prop. 4.17 ii we also have $w_{n_h} := u_{n_h} - u_{n_h}(P_{j_1}) \in \tilde{A}^+(\overline{E}, E_{(n_h)})$, $w'_{n_h} := u_{n_h}(P_{j_2}) - u_{n_h} \in \tilde{A}^+(\overline{E}, E_{(n_h)})$. Moreover, $w_{n_h} \in D_{j_1}$, $w'_{n_h} \in D_{j_2}$. Thus, $v_{n_h} := \widehat{T_{j_1}^{n_h}}(w_{n_h}) \in A^+$ by Prop. 5.1, and by Corollary 5.4, $v_{n_h} \underset{h\to\infty}{\longrightarrow} \overline{v}_{j_1}$, hence $\overline{v}_{j_1} \in A^+$. By proceeding in a similar way, $v'_{n_h} := \widehat{T_{j_2}^{n_h}}(w'_{n_h}) \in A^+$, and $v'_{n_h} \underset{h\to\infty}{\longrightarrow} \overline{v}_{j_2} \in A^+$. By the same argument there exist j'_1, j'_2, with $j'_1 \neq j'_2$ such that $\overline{v}_{j'_1}, \overline{v}_{j'_2} \in A^-$. Since we have only three points in $V^{(0)}$ there exists $\overline{v}_j \in \tilde{A}^+ \cap \tilde{A}^-$, and, by Prop. 4.17 iii, E is a multiple of $\overline{E}$. ∎

Remark 5.6. The argument in the proof of Theorem 5.5 heavily relies on the fact that $N = 3$. Aiming to possible extension of the proof to more general fractals, it could be useful to observe that a modification of that argument does not use $N = 3$, but uses the strong symmetry of the Gasket, which implies in particular that $\overline{v}_j$ is symmetric with respect to the components different from j. We saw in proof of Theorem 5.5 that there exists $j(= j(0)) = 1, 2, 3$ such that $\overline{v}_j \in A^+(\overline{E}, E)$. On the other hand, clearly, $E_{(n)}$ does not approach $\overline{E}$, so that we can apply the same argument to $E_{(n)}$ and there exists $j(n) = 1, 2, 3$ such that $\overline{v}_{j(n)} \in A^+(\overline{E}, E_{(n)})$. We can now repeat the previous argument. Let n_h be a strictly increasing sequence of naturals and let $\overline{j} = 1, 2, 3$ be such that $j(n_h) = \overline{j}$ for all h. Then, by Prop. 5.1, $\widehat{T_{\overline{j}}^{n_h}}(\overline{v}_{\overline{j}}) \in A^+$, and, since $\overline{v}_{\overline{j}} \in D_{\overline{j}}$, by Corollary 5.4

$$\widehat{T_{\overline{j}}^{n_h}}(\overline{v}_{\overline{j}}) \underset{h\to\infty}{\longrightarrow} \overline{v}_{\overline{j}} \in A^+ .$$

On the other hand, for $j \neq \overline{j}$, $\tilde{v}_{\overline{j}} := (1, 1, 1) - \sqrt{2}\,\overline{v}_{\overline{j}} \in D_j \cap A^+$, hence

$$\widehat{T_j^{n_h}}(\tilde{v}_{\overline{j}}) \underset{h\to\infty}{\longrightarrow} \overline{v}_j \in A^+ .$$

In conclusion, $\overline{v}_j \in \tilde{A}^+$ for every j, and by the same argument, $\overline{v}_j \in \tilde{A}^-$ for every j, so that $\tilde{A}^+ \cap \tilde{A}^- \neq \emptyset$. ∎

Corollary 5.7. *Every eigenform is a multiple of $\overline{E}$.*

Proof. It suffices to observe, that if $E \in \tilde{\mathcal{D}}$ is an eigenform, then $E = E_{(n)} \underset{n\to\infty}{\longrightarrow} a\overline{E}$, for some $a > 0$. ∎

We will consider Theorem 5.5 as a starting point to prove the Γ-convergence of $E^{\Sigma}_{(n)}$. However, Theorem 5.5 is interesting in itself. Let now $E \in \tilde{\mathcal{D}}$ and $\tilde{E} = \lim_{n\to+\infty} E_{(n)}$. We are going to prove that $\tilde{E}^{\Sigma}_{(\infty)} = \Gamma(X-) \lim_{n\to+\infty} E^{\Sigma}_{(n)}$, where $X = \mathbb{R}^K$ with the metric L^∞. Note that, since $\tilde{E}$ is an eigenform, $\tilde{E}^{\Sigma}_{(\infty)}$ is defined. The argument of the proof of Γ-convergence is due to S. Kozlov [7], who proved the Γ-convergence on the Gasket for forms E having two of the three coefficients equal, with respect to a topology which is different from L^∞, and induces a sort of Sobolev space on the Gasket. In any case the proof of [7] also works for general fractals once we know that the sequence $E_{(n)}$ is convergent. In these notes, we follow the approach of [15], while in [17] the problem is treated by a slightly more general point of view in the sense that the Γ-convergence with respect to different topologies is investigated there. Recall that, given a sequence of functionals F_n from a metric space X with values in $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, F_n are said to be $\Gamma(X-)$-convergent to a functional F if for every $x \in X$

i) there exist $x_n \underset{n\to+\infty}{\longrightarrow} x$ such that $F_n(x_n) \underset{n\to+\infty}{\longrightarrow} F(x)$.

ii) For every $x_n \underset{n\to+\infty}{\longrightarrow} x$ we have $\liminf_{n\to+\infty} F_n(x_n) \geq F(x)$.

In order to obtain the Γ-convergence result, for every $m, n \in \mathbb{N}$ with $n \geq m$ and for every $v : V^{(m)} \to \mathbb{R}$, we define $v_{(n,m)} : K \to \mathbb{R}$ to be the harmonic extension with respect to $\tilde{E}$ (or to $\overline{E}$, which is the same as $\tilde{E}$ is a multiple of $\overline{E}$) of $\tilde{v} : V^{(n)} \to \mathbb{R}$ defined by

$$\tilde{v}\big(\psi_{i_1,\dots,i_n}(P)\big) = H_{(n-m;E)}(v \circ \psi_{i_1,\dots,i_m}) \circ \psi_{i_{m+1},\dots,i_n}(P), \quad P \in V^{(0)}\,.$$

The definition of $\tilde{v}$ is correct by the same argument as in Section 2, via Lemma 2.5. Then, we define $v_{(n)} = v_{(n,[\frac{n}{2}])}$. We have

Lemma 5.8. *For every $v \in C(K)$, $v_{(n)} \underset{n\to\infty}{\longrightarrow} v$ with respect to L^∞.*

Proof. For every $Q \in K$ let $i_1, ..., i_n = 1, ..., k$, $P \in K$ be such that $Q = \psi_{i_1,\dots,i_n}(P)$. Then, by the definition of harmonic extension,

$$v_{(n)}(Q) = v_{(n)}(\psi_{i_1,\dots,i_n}(P)) = H_{(\infty;\tilde{E})}\Big(\tilde{v} \circ \psi_{i_1,\dots,i_n}\Big)(P)$$

and by Lemma 2.8 we have

$$v_{(n)}(Q) \in [\min_{V^{(0)}} \tilde{v} \circ \psi_{i_1,\dots,i_n}, \max_{V^{(0)}} \tilde{v} \circ \psi_{i_1,\dots,i_n}]\,.$$

Moreover, by Corollary 4.7 we get

$$v_{(n)}(Q) \in \Big[\min_{V^{(0)}} v \circ \psi_{i_1,\dots,i_{[\frac{n}{2}]}}, \max_{V^{(0)}} v \circ \psi_{i_1,\dots,i_{[\frac{n}{2}]}}, \Big]\,.$$

In conclusion,

$$v_{(n)}(Q), v(Q) \in \big[\inf_{K_{i_1,\dots,i_{[\frac{n}{2}]}}} v, \sup_{K_{i_1,\dots,i_{[\frac{n}{2}]}}} v\big]\,,$$

hence $|v_{(n)}(Q) - v(Q)| \leq \sup_{K_{i_1,\dots,i_{[\frac{n}{2}]}}} v - \inf_{K_{i_1,\dots,i_{[\frac{n}{2}]}}} v$, and by the uniform continuity of v on K and (2.12), $v_{(n;E)} \underset{n\to\infty}{\longrightarrow} v$ uniformly. ∎

Lemma 5.9. *For every $v \in C(K)$ we have*

$$E^{\Sigma}_{(n)}(v_{(n)}) = (E_{(n-[\frac{n}{2}])})^{\Sigma}_{([\frac{n}{2}])}(v) \leq E^{\Sigma}_{(n)}(v)\,.$$

Proof. We first observe that

$$S_n(E)(v_{(n)}) = \sum_{i_1,..,i_{[\frac{n}{2}]}=1}^{k} S_{n-[\frac{n}{2}]}(E)(v_{(n)} \circ \psi_{i_1,\ldots,i_{[\frac{n}{2}]}})\,,$$

$$S_{n-[\frac{n}{2}]}(E)(v_{(n)} \circ \psi_{i_1,\ldots,i_{[\frac{n}{2}]}}) = S_{n-[\frac{n}{2}]}(E)\big(H_{(n-[\frac{n}{2}];E)}(v \circ \psi_{i_1,\ldots,i_{[\frac{n}{2}]}})\big) =$$

$$M_{n-[\frac{n}{2}]}(E)(v \circ \psi_{i_1,\ldots,i_{[\frac{n}{2}]}}) \leq S_{n-[\frac{n}{2}]}(E)(v \circ \psi_{i_1,\ldots,i_{[\frac{n}{2}]}})\,,$$

$$S_n(E)(v) = \sum_{i_1,..,i_{[\frac{n}{2}]}=1}^{k} S_{n-[\frac{n}{2}]}(E)(v \circ \psi_{i_1,\ldots,i_{[\frac{n}{2}]}})\,.$$

It follows that

$$S_n(E)(v_{(n)}) = S_{[\frac{n}{2}]}\big(M_{n-[\frac{n}{2}]}(E)\big)(v) \leq S_n(E)(v)\,.$$

By dividing by ρ^n we get the Lemma. ∎

Theorem 5.10 *We have*

$$\Gamma(X-)\lim_{n\to+\infty} E^{\Sigma}_{(n)} = \tilde{E}^{\Sigma}_{(\infty)}$$

where $X = C(K)$ with the metric L^∞.

Proof. By Corollary 4.11, for every $\varepsilon > 0$ we have

$$(1-\varepsilon)\tilde{E} \leq E_{(n)} \leq (1+\varepsilon)\tilde{E} \tag{5.4}$$

for sufficiently large n. Now, given $v \in X$, by Lemma 5.8 $v_{(n)} \underset{n\to\infty}{\longrightarrow} v$ in X. Also, by Lemma 5.9 and (5.4),

$$E^{\Sigma}_{(n)}(v_{(n)}) = (E_{(n-[\frac{n}{2}])})^{\Sigma}_{([\frac{n}{2}])}(v) \in \big[(1-\varepsilon)\tilde{E}^{\Sigma}_{([\frac{n}{2}])}(v), (1+\varepsilon)\tilde{E}^{\Sigma}_{([\frac{n}{2}])}(v)\big]$$

for sufficiently large n, so that $E^{\Sigma}_{(n)}(v_{(n)}) \underset{n\to\infty}{\longrightarrow} \tilde{E}^{\Sigma}_{(\infty)}(v)$. It remains to prove that given $v_n \underset{n\to+\infty}{\longrightarrow} v$ in X, then

$$\liminf_{n\to+\infty} E^{\Sigma}_{(n)}(v_n) \geq \tilde{E}^{\Sigma}_{(\infty)}(v)\,,$$

and it is clearly sufficient to show that for every $m \in \mathbb{N}$

$$\liminf_{n\to+\infty} E^{\Sigma}_{(n)}(v_n) \geq \tilde{E}^{\Sigma}_{(m)}(v)\,. \tag{5.5}$$

By Lemma 5.9 and (5.4) and since $E^{\Sigma}_{(n)}$ is increasing, if $0 < \varepsilon < 1$, for sufficiently large n we have

$$E^{\Sigma}_{(n)}(v_n) \geq \left(E_{(n-[\frac{n}{2}])}\right)^{\Sigma}_{([\frac{n}{2}])}(v_n) \geq (1-\varepsilon)\tilde{E}^{\Sigma}_{([\frac{n}{2}])}(v_n) \geq (1-\varepsilon)\tilde{E}^{\Sigma}_{(m)}(v_n)\,.$$

Since $v_n \underset{n\to+\infty}{\longrightarrow} v$ and $\tilde{E}^{\Sigma}_{(m)}$ is continuous from X to $\mathbb{R}$, we get

$$\liminf_{n\to+\infty} E^{\Sigma}_{(n)}(v_n) \geq (1-\varepsilon)\liminf_{n\to+\infty} \tilde{E}^{\Sigma}_{(m)}(v_n) = (1-\varepsilon)\tilde{E}^{\Sigma}_{(m)}(v)\,.$$

Since this holds for any $\varepsilon \in]0,1[$, (5.5) follows. ∎

6. Homogenization on General Fractals

In this section we will extend the results of Section 5 to arbitrary finitely ramified fractals. The difficulty in imitating the proof in Section 5 consists in extending Theorem 5.5. Note however, that in general, we cannot expect that $E_{(n)}$ tends to a multiple of a fixed eigenform, for in such a case, we could prove as in Corollary 5.7, that the eigenform is unique up to a multiplicative constant, and this is no longer true in the general case. We remark that if $N = 2$, then for every $E \in \tilde{\mathcal{D}}$, we have

$$E = c\big(u(P_1) - u(P_2)\big)^2, \qquad M_1(E) = c'\big(u(P_1) - u(P_2)\big)^2$$

for some $c, c' > 0$. Hence, E is in any case an eigenform, and the convergence of $E_{(n)}$ takes trivially place. Thus, we can assume $N \geq 3$. What properties of the Gasket have we used in the proof of the convergence of $E_{(n)}$? We essentially used either $N = 3$ or the very strong symmetry of the Gasket. If the fractal is less symmetric, one could prove an analogous of Lemma 5.3, but the limit vector is not symmetric with respect to the components different from j. Hence such a vector does not attain its maximum at *all* $P \in V^{(0)}$, $P \neq P_j$, and the proof in Remark 5.6 does not work. Since in the present case, we want to prove the convergence of $E_{(n)}$ to an eigenform that, in case of nonuniqueness, may well depend on E, and we cannot expect to know in advance what eigenform is the limit, we will not try to prove that $\lambda(\tilde{E}, E_{(n)})$ tends to 0 for some specific eigenform $\tilde{E}$. We will instead try to prove that $\lambda_n := \lambda(E_{(n)}, E_{(n+1)})$ tends to 0. Note that $E_{(n+1)} = (E_{(1)})_{(n)}$, so that we can use Corollary 4.15 and Corollary 4.19 with $E_{(1)}$ in place of E'. In particular, λ_n is decreasing. We say that E is λ-*contracting* if we have $\lambda_n < \lambda_0$ for some, so for sufficiently large, n. We have the following analogous to Lemma 5.2.

Lemma 6.1. *If every $E \in \tilde{\mathcal{D}}$ which is not an eigenform is λ-contracting, then for every $E \in \tilde{\mathcal{D}}$ there exists an eigenform $\tilde{E}$ such that $E_{(n)} \underset{n\to\infty}{\longrightarrow} \tilde{E}$.*

Proof. Let $E \in \tilde{\mathcal{D}}$. By Corollary 4.14 there exists a strictly increasing sequence of naturals n_h, and $E' \in \tilde{\mathcal{D}}$ such that $E_{(n_h)} \underset{h\to\infty}{\longrightarrow} E'$. For every $m \in \mathbb{N}$ we have

$$\lambda(E'_{(m)}, E'_{(m+1)}) = \lambda\big(\lim_{h\to\infty}(E_{(n_h)})_{(m)}, \lim_{h\to\infty}(E_{(n_h)})_{(m+1)}\big) =$$

$$\lambda\big(\lim_{h\to\infty} E_{(m+n_h)}, \lim_{h\to\infty} E_{(m+1+n_h)}\big) = \lim_{h\to\infty} \lambda(E_{(m+n_h)}, E_{(m+1+n_h)}) = \lim_{n\to\infty} \lambda(E_{(n)}, E_{(n+1)}).$$

Thus E' is not λ-contracting, and by hypothesis it is an eigenform. By Corollary 4.16 $E_{(n)} \underset{n\to\infty}{\longrightarrow} E'$. ■

Actually, in the argument of the proof of convergence on the Gasket, when we stated that a and b in (5.1) are positive, we used also the strong maximum principle. The strong maximum principle, in fact, will play an important role also in the argument in this section. This will lead us to restrict the class of fractals. The convergence result can be proved for all finitely ramified fractals, using a variant of the strong maximum principle, but the proof is considerably more technical and will be omitted. I will hint the idea at the end of this section. We saw in Prop. 3.11 and Remark 3.12 that $H_{(1;E)}(u)$ satisfies the maximum principle, but in general not the strong maximum principle when $E \in \tilde{\mathcal{D}}$. We now see that, if

$$c_{j_1,j_2}(E) > 0 \quad \forall j_1, j_2 : j_1 \neq j_2 . \tag{6.1}$$

then $H_{(1;E)}(u)$ satisfies even the strong maximum principle. Note that, if $E \in \tilde{\mathcal{D}}$ satisfies (6.1), then E-close amounts to close and E-connected amounts to connected.

Proposition 6.2. *Suppose that $E \in \tilde{\mathcal{D}}$ and (6.1) holds. Suppose $u : V^{(0)} \to \mathbb{R}$, and $v := H_{(1;E)}(u)$ attains its maximum or its minimum at a point Q of $V^{(1)} \setminus V^{(0)}$. Then u is constant on $V^{(0)}$.*

Proof. We first prove that any point $Q \in V^{(1)} \setminus V^{(0)}$ is strongly E-connected to any point $Q' \in V^{(0)}$. The proof of Lemma 3.8 shows that there exists a path $(Q_1, ..., Q_m)$ E-connecting Q to Q'. We will prove that if such a path has minimum length among the paths E-connecting Q and Q', then $Q_2, ..., Q_{m-1} \notin V^{(0)}$. Suppose on the contrary $Q_{\bar{i}} = P_h \in V^{(0)}$ with $1 < \bar{i} < m$. As V_h is the unique 1-cell containing P_h, then $Q_{\bar{i}-1}, Q_{\bar{i}+1} \in V_h$. Hence the path $(Q_1, ..., Q_{\bar{i}-1}, Q_{\bar{i}+1}, ..., Q_m)$ E-connects Q and Q' and has length $m-1$, which contradicts our assumption. Now, from Lemma 3.10, if v attains its maximum or its minimum at Q, then $u = v = v(Q)$ on $V^{(0)}$. ■

We cannot apply directly Prop. 6.2 to prove the convergence of $E_{(n)}$, as E does not necessarily satisfy (6.1), but we now will see that a relatively mild condition on the fractal implies that every $M_1(E)$ satisfies (6.1). We first give a sufficient condition on E in order that $M_1(E)$ satisfy (6.1). It can be easily proved that it is also a necessary condition.

Lemma 6.3. *Suppose that $E \in \tilde{\mathcal{D}}$ and every two points in $V^{(0)}$ are strongly E-connected. Then $M_1(E)$ satisfies (6.1).*

Proof. The proof is a variant of that of b) in Theorem 3.15. Let j_1, j_2, u_{j_1,j_2}, v_{j_1,j_2}, w be as in Theorem 3.15. It suffices to prove that $M_1(E)(v_{j_1,j_2}) > M_1(E)(u_{j_1,j_2})$, hence that in (3.6) at least one of the two inequalities is strict. If the second one is not strict, we have $E(|w| \circ \psi_i) = E(w \circ \psi_i)$ for all $i = 1, ..., k$, and thus w cannot attain opposite signs at two points which are E-close, as the inequality in (3.7) is strict when a and b have opposite signs. Let now $(Q_1, ..., Q_m)$ be a path strongly E-connecting P_{j_1} to P_{j_2}. Here $m > 2$ as P_{j_1} and P_{j_2} cannot lie in the same 1-cell. Since $w(Q_1) = w(P_{j_1}) = v_{j_1,j_2}(P_{j_1}) = 1$ and $w(Q_m) = w(P_{j_2}) = v_{j_1,j_2}(P_{j_2}) = -1$, there exists $h = 2, ..., m-1$ such that $w(Q_h) = 0$, and, by taking the minimal h, we can assume $|w(Q_{h-1})| > 0$. Since Q_h and Q_{h-1} are E-close, $|w|$ does not satisfy (3.5) at Q_h. Hence, $|w| \neq H_{(1;E)}(u_{j_1,j_2})$, and the first inequality in (3.6) is strict. ∎

We now require that the fractal has a strong connectedness property. The argument in proof of Prop. 6.2 shows that any two points in $V^{(0)}$ are strongly connected. We require a slightly stronger condition. We say that the fractal is *strongly connected* if for every $Q, Q' \in V^{(0)}$ there exist $Q_1, ..., Q_m \in V^{(1)}$ such that $Q_1 = Q$, $Q_m = Q'$ and, for every $h = 1, ..., m-1$ there exists $i(h) = 1, ..., k$ such that $Q_h, Q_{h+1} \in V_{i(h)}$, and in addition, $i(h) > N$ for $h = 2, ..., m-2$. Note that the last condition means that all cells but the first and the last contain no points of $V^{(0)}$. It is easy to see that the Gasket, the Vicsek set and the Snowflake are strongly connected, while the tree-like Gasket is not so.

Proposition 6.4. *Suppose that the fractal is strongly connected and $E \in M_1(\tilde{\mathcal{D}})$. Then E satisfies (6.1). Thus $H_{(1;E)}(u)$ satisfies the strong maximum principle for every $u \in \mathbb{R}^{V^{(0)}}$.*

Proof. Let $E = M_1(E')$, $E' \in \tilde{\mathcal{D}}$. We will prove that every two points in $V^{(0)}$ are strongly E'-connected. The proof imitates that of Lemma 3.8. Fix $P_{j_1}, P_{j_2} \in V^{(0)}$. Let $(Q_1, ..., Q_m)$ be a path as in definition of strongly connected fractal. For any $h = 1, ..., m-1$, the points Q_h and Q_{h+1} are E'-connected by a path which remains in $V_{i(h)}$, in particular, for $h = 2, ..., m-2$, it remains in $V^{(1)} \setminus V^{(0)}$. Since $Q_2 \in V_{j_1}$ (the unique 1-cell containing P_{j_1}), by Remark 3.7 Q_2 is strongly E'-connected to P_{j_1}. For the same reason, Q_{m-1} is strongly E'-connected to P_{j_2}. In conclusion, we have found a path that strongly E'-connects P_{j_1} and P_{j_2}. ∎

From now on, unless specified otherwise, we will assume that the fractal is strongly connected.

We will write any $u \in \mathbb{R}^{V^{(0)}}$ as $(u(P_1), ..., u(P_N))$. In this way, u will be identified to a vector in $\mathbb{R}^N$. We will denote by $e_1, ..., e_N$ the vectors of the canonical basis in $\mathbb{R}^N$.

Corollary 6.5. *Suppose $E \in M_1(\tilde{\mathcal{D}})$. Then for every $i = 1, ..., k$, $j = 1, ..., N$ we have*

$$T_{i;E}(e_h)(P_j)\begin{cases} = 1 & \text{if } i = j = h \\ = 0 & \text{if } i = j \neq h \\ \in]0,1[& \text{if } i \neq j\,. \end{cases}$$

Proof. We have $T_{i;E}(e_h)(P_j) = H_{(1;E)}(e_h)(\psi_i(P_j))$. If $i \neq j$ we have $\psi_i(P_j) \in V^{(1)} \setminus V^{(0)}$, so the conclusion follows from the strong maximum principle. If $i = j$ the results is trivial. ∎

In previous section, in order to prove the convergence of $E_{(n)}$, we were lead, in view of Lemma 5.2, to investigate the implications of the equality $\lambda_\pm(\overline{E}, E_{(n)}) = \lambda_\pm(\overline{E}, E)$. In the present case, in order to prove the convergence of $E_{(n)}$, in view of Lemma 6.1, we have to investigate the implications of the equality $\lambda(E, E_{(1)}) = \lambda(E_{(n)}, E_{(n+1)})$. Consequently, instead of using a version of Corollary 4.19 in the case in which E is an eigenform, we will use a version of Corollary 4.19 in the case $E' = E_{(1)}$. However, also in the present case, it will be sufficient to consider the case $i_1 = ... = i_m$. We have

Lemma 6.6. *Suppose $E \in \tilde{\mathcal{D}}$. Let $\lambda_{\pm,n} = \lambda_\pm(E_{(n)}, E_{(n+1)})$ for all n. If we have $\lambda_{\pm,n} = \lambda_{\pm,0}$, then for all m with $0 \leq m \leq n$ we have $\lambda_{\pm,m} = \lambda_{\pm,0}$ and if $u \in A^{\pm,n}(E, E_{(1)})$*

$$T_{i,m} \circ \cdots \circ T_{i,n-1}(u) = T_{i,m+1} \circ \cdots \circ T_{i,n}(u) \in A^{\pm,m}(E, E_{(1)})\,.$$

Proof. It suffices to put $E' = E_{(1)}$, $i_1 = \cdots = i_m = i$ in Corollary 4.19, and to use Lemma 4.6, taking in account the definition of $H_{(n-m,m;E)}$. ∎

We need an analogous of Lemma 5.3 suitable for Lemma 6.6. Roughly speaking, we want to prove that the normalization of the composition of m linear operators, under suitable conditions, contracts the positive cone to a unique vector, which in this case may well be nonsymmetric. However, there is a more relevant difference with the case of Section 5, that is that, as suggested by Lemma 6.6 we need to consider the case where the operators are different. I will give a general theorem for operators in $\mathbb{R}^M$ and then I will apply this to our case. Let $M \geq 2$ be fixed for the following (in case $M = 1$, Theorem 6.8 is trivially satisfied with $w = 1$). We define

$$D = \{v \in \mathbb{R}^M : v_i \geq 0\ \forall i = 1, ..., M, v \neq 0\}\,,$$

$$D' = \{v \in \mathbb{R}^M : v_i > 0\ \forall i = 1, ..., M\}\,.$$

Let also $\tilde{D} = \{v \in D : ||v|| = 1\}$, $\tilde{D}' = \{v \in D' : ||v|| = 1\}$. Let $\mathcal{A}$ be the set of linear operators $T : \mathbb{R}^M \to \mathbb{R}^M$ that map D into D' or equivalently such that $T(e_i) \in D'$ for all

$i = 1, ..., M$. Let $\mathcal{N} : \mathbb{R}^M \setminus \{0\} \to \mathbb{R}^M$ be defined as $\mathcal{N}(v) = \frac{v}{||v||}$. We will now introduce on D' a semimetric that we implicitly used in the proof of Lemma 5.3 with $M = 2$. Given $u, v \in D'$ put

$$\lambda'_+(u,v) = \max_{i=1,\ldots,M} \frac{v_i}{u_i}, \quad \lambda'_-(u,v) = \min_{i=1,\ldots,M} \frac{v_i}{u_i}, \quad \lambda'(u,v) = \ln(\lambda'_+(u,v)) - \ln(\lambda'_-(u,v)) .$$

The definition of λ' resembles that of λ on $\tilde{\mathcal{D}}$, and in fact it is another case of Hilbert's projective metric. It satisfies the same properties as λ: we have $\lambda'(u, v) = 0$ if and only if v is a multiple of u, $\lambda'(au, bv) = \lambda'(u, v)$ for every $a, b > 0$. Thus, λ' induces a metric on the projective space $pr(D')$ generated by D', that is the space of equivalence classes on D', mod the relation where two elements are equivalent if they are multiple of each other. In particular, on $\tilde{D}'$, it induces a metric, which we denote as λ' as well. It is not difficult to see that such a metric is equivalent to the euclidean one, in the sense that they generate the same topology. The proof is rather simple. However, as we will not use that statement, I will not discuss it. We instead will use a weaker form in Theorem 6.8. We will denote the diameter of a subset A of $\tilde{D}$ with respect to λ' by diam$A$. The use of λ' is illustrated by the following lemma.

Lemma 6.7. *If $u, v \in D'$ and $T \in \mathcal{A}$ then we have $\lambda'(T(u), T(v)) \leq \lambda'(u, v)$ and the inequality is strict unless $\lambda'(u, v) = 0$.*

Proof. For all $i = 1, ..., M$, we have

$$\frac{T(v)_i}{T(u)_i} = \frac{\sum\limits_{j=1}^{M} a_{i,j} v_j}{\sum\limits_{j=1}^{M} a_{i,j} u_j}$$

where $a_{i,j} = \big(T(e_j)\big)_i > 0$. We have $a_{i,j} v_j \leq \lambda'_+(u, v) a_{i,j} u_j$ and, if $\lambda'(u, v) \neq 0$, i.e., v is not a multiple of u, the inequality is strict for at least one j. Suppose $\lambda'(u, v) \neq 0$, thus $\frac{T(v)_i}{T(u)_i} < \lambda'_+(u, v)$ for every $i = 1, ..., M$. Hence, $\lambda'_+(T(u), T(v)) < \lambda'_+(u, v)$ and similarly, $\lambda'_-(T(u), T(v)) > \lambda'_-(u, v)$, thus $\lambda'(T(u), T(v)) < \lambda'(u, v)$. If $\lambda'(u, v) = 0$, the result is trivial. ∎

The following theorem is specially interesting in the case in which the operators coincide. This is a form of the well-known Perron-Frobenius Theorem. For information on the Perron-Frobenius theory see for example [19].

Theorem 6.8. *Let $\mathcal{A}'$ be a compact subset of $\mathcal{A}$. Let $T_1, ..., T_n, ...$ be a sequence of operators in $\mathcal{A}'$. Then there exists $w \in \tilde{D}'$ such that*

$$\mathcal{N}\big(T_1 \circ ... \circ T_n(D)\big) \underset{n\to\infty}{\longrightarrow} w$$

in the sense that $\sup\left\{d(w,v) : v \in \mathcal{N}\big(T_1 \circ ... \circ T_n(D)\big)\right\} \underset{n\to\infty}{\longrightarrow} 0$.

Proof. For every $T \in \mathcal{A}$ let $\tilde{T} = \mathcal{N} \circ T$. Then, we have

$$\mathcal{N}\big(T_1 \circ ... \circ T_n(D)\big) = \tilde{T}_1 \circ ... \circ \tilde{T}_n(\tilde{D}) \quad \forall n \in \mathbb{N}.$$

Thus, putting $\tilde{T}_{m,n} = \tilde{T}_m \circ ... \circ \tilde{T}_n$ when $m \leq n$, we have to prove that there exists $w \in \tilde{D}'$ such that

$$\sup\left\{d(w,v) : v \in \tilde{T}_{1,n}(\tilde{D})\right\} \underset{n\to\infty}{\longrightarrow} 0\,. \tag{6.2}$$

Let $\alpha : \tilde{D} \times \mathcal{A}' \to \tilde{D}'$ be defined as $\alpha(v,T) = \tilde{T}(v)$. It is continuous, hence $B :=\mathrm{Im}\alpha$ is a compact subset of $\tilde{D}'$, thus, $M :=\mathrm{diam}(B) < +\infty$. We have $\tilde{T}_n(\tilde{D}) \subseteq B \subseteq \tilde{D}$. It follows

$$B \supseteq \tilde{T}_{1,1}(\tilde{D}) \supseteq \tilde{T}_{1,2}(\tilde{D}) \supseteq ... \supseteq \tilde{T}_{1,n}(\tilde{D}) \supseteq ... \tag{6.3}$$

We will prove that

$$\mathrm{diam}(\tilde{T}_{1,n}(\tilde{D})) \underset{n\to\infty}{\longrightarrow} 0\,. \tag{6.4}$$

For every $\eta > 0$ let $F_\eta = \{(u,v) \in B \times B : \lambda'(u,v) \geq \eta\}$ and let $\beta : \mathcal{A}' \times F_\eta \to \mathbb{R}$ be defined as

$$\beta(T,u,v) = \lambda'(u,v) - \lambda'\big(T(u),T(v)\big)\,.$$

Since $\mathcal{A}' \times F_\eta$ is compact, β has a minimum m_η on it which, by Lemma 6.7, is positive. We will now prove that, given $n = 1,2,3,...$ such that $M - nm_\eta < \eta$, we have

$$\mathrm{diam}\tilde{T}_{1,n+1}(\tilde{D}) \leq \eta\,. \tag{6.5}$$

As η is arbitrary, this implies (6.4). Let $u,v \in \tilde{D}$. As $\tilde{T}_{n+1}(u), \tilde{T}_{n+1}(v) \in B$, we have

$$\lambda'(\tilde{T}_{n+1}(u), \tilde{T}_{n+1}(v)) \leq M\,.$$

Now, if $\lambda'(\tilde{T}_{m,n+1}(u), \tilde{T}_{m,n+1}(v)) < \eta$ for some $m \leq n+1$, by Lemma 6.7 we have $\lambda'(\tilde{T}_{1,n+1}(u), \tilde{T}_{1,n+1}(v)) < \eta$. In the contrary case, we have $\big(\tilde{T}_{m,n+1}(u), \tilde{T}_{m,n+1}(v)\big) \in F_\eta$ for all $m \leq n+1$. Hence, in view of the definition of m_η, by a recursive argument we get $\lambda'(\tilde{T}_{1,n+1}(u), \tilde{T}_{1,n+1}(v)) \leq M - nm_\eta < \eta$, and $\lambda'(\tilde{T}_{1,n+1}(u), \tilde{T}_{1,n+1}(v)) < \eta$ again. Thus, we have proved (6.5), and hence also (6.4). Since by (6.3) $\tilde{T}_{1,n}(\tilde{D})$ is a decreasing sequence of nonempty compact subsets of B, there exists $w \in \bigcap_{n=1}^{\infty} \tilde{T}_{1,n}(\tilde{D})$, and, by (6.4) we have

$$\sup\left\{\lambda'(w,v) : v \in \tilde{T}_{1,n}(\tilde{D})\right\} \underset{n\to\infty}{\longrightarrow} 0\,.$$

On the other hand, since λ' generates on the euclidean compact set B a topology weaker than the euclidean one, λ' is equivalent to the euclidean metric. Hence, we get (6.2), and the theorem is proved. ∎

Corollary 6.9. *Under the hypotheses of Theorem 6.8, given a strictly increasing sequence n_h of naturals and $v_h \in D$ we have $\mathcal{N}\big(T_1 \circ ... \circ T_{n_h}(v_h)\big) \underset{h\to\infty}{\longrightarrow} w$.* ∎

Corollary 6.10. *Let $i = 1, ..., k$ be fixed. Let $D = \big\{v \in \mathbb{R}^{V^{(0)}} : v(P_i) = 0, v(P_{i'}) \geq 0 \quad \forall i' \neq i, v \neq 0\big\}$. Fix $E \in \tilde{\mathcal{D}}$. For every $m = 1, 2, ...$, there exists $w_m \in \mathbb{R}^{V^{(0)}}$ such that, for every strictly increasing sequence n_h of naturals and for every $v_h \in D$, we have*

$$\mathcal{N}\big(T_{i,m} \circ \cdots \circ T_{i,n_h}(v_h)\big) \underset{h\to\infty}{\longrightarrow} w_m .$$

Proof. As previously seen, the operators $T_{i,n}$ can be identified to linear operators from $\mathbb{R}^{N-1}$ into itself, as they map the $(N-1)$-dimensional linear space Π_i into itself. By this identification, D corresponds to the previously defined D, with $N-1$ in place of M. Given $a, b > 0$ such that $\frac{E}{\rho} \in U_{a,b}$, for all $m \geq 1$ we have $E_{(m)} = M_1\big((\frac{E}{\rho})_{(m-1)}\big) \in M_1(U_{a,b})$, hence $T_{i,m} = T_{i;E_{(m)}} \in T_{i;M_1(U_{a,b})}$. By Lemma 4.13 and Lemma 4.9, the set $T_{i;M_1(U_{a,b})}$ is compact, and by Corollary 6.5 it is contained in $\mathcal{A}$. We can now use Corollary 6.9. ∎

We are now ready to prove the main theorem in this section.

Theorem 6.11. *For every $E \in \tilde{\mathcal{D}}$ there exists $\tilde{E} \in \tilde{\mathcal{D}}$ such that $E_{(n)} \underset{n\to\infty}{\longrightarrow} \tilde{E}$.*

Proof. By Lemma 6.1 it suffices to prove that, given any non λ-contracting $E \in \tilde{\mathcal{D}}$, then E is an eigenform. For every natural n let $u_n \in \tilde{A}^{+,n}(E, E_{(1)})$. Let n_h be a strictly increasing sequence of naturals and let $i = 1, ..., N$ be such that $\min u_{n_h} = u_{n_h}(P_i)$ for all $h \in \mathbb{N}$. We can assume that $u_{n_h}(P_i) = 0$, so that $u_{n_h} \in D$ where D is defined as in Corollary 6.10. Since E is not λ-contracting, by Remark 4.20 we can apply Lemma 6.6, hence

$$u_{m,n_h} := \mathcal{N}\big(T_{i,m} \circ \cdots \circ T_{i,n_h-1}(u_{n_h})\big) = \mathcal{N}\big(T_{i,m+1} \circ \cdots \circ T_{i,n_h}(u_{n_h})\big) \in \tilde{A}^{+,m}(E, E_{(1)}) \quad (6.6)$$

for all $m \geq 1$. So, using Corollary 6.10, there exists $w_m \in \mathbb{R}^{V^{(0)}}$ such that

$$u_{m,n_h} \underset{h\to\infty}{\longrightarrow} w_m = w_{m+1} .$$

Hence, there exists a unit vector $w \in \mathbb{R}^{V^{(0)}}$ such that $w_m = w$ for all $m \geq 1$. It follows from (6.6) that $w \in \tilde{A}^{+,m}(E, E_{(1)})$ for all $m \geq 1$, so that $E_{(m+1)}(w) = \lambda_{+,0} E_{(m)}(w)$, hence

$$E_{(m)}(w) = \lambda_{+,0}^{m-1} E_{(1)}(w),$$

for all $m \geq 1$. By the same argument, there exists $w' \in \tilde{A}^{-,m}(E, E_{(1)})$ such that

$$E_{(m)}(w') = \lambda_{-,0}^{m-1} E_{(1)}(w')$$

for all $m \geq 1$. But we know that for some $a, b > 0$ we have $E_{(m)} \in U_{a,b}$ for all naturals m. Thus,

$$E_{(m)}(w) \leq b\hat{E}(w) = b\hat{E}(w')\frac{\hat{E}(w)}{\hat{E}(w')} \leq LE_{(m)}(w'),$$

where $L = \frac{b}{a}\frac{\hat{E}(w)}{\hat{E}(w')}$. This is possible only if $\lambda_{-,0} = \lambda_{+,0}$, i.e., only if E is an eigenform. ∎

We can now prove a Γ-convergence result in the exactly same way as in Section 5. Namely

Theorem 6.12. *Let $E \in \tilde{\mathcal{D}}$. We have*

$$\Gamma(X-) \lim_{n \to +\infty} E^{\Sigma}_{(n)} = \tilde{E}^{\Sigma}_{(\infty)}$$

where $\tilde{E} = \lim_{n\to\infty} E_{(n)}$ and $X = C(K)$ with the metric L^{∞}. ∎

As we previously seen, Theorem 6.11 holds, even if the fractal is not strongly connected. I now sketch the idea of the proof in this case. The problem is that, as previously hinted, the strong maximum principle does not hold. So, we cannot use Theorem 6.8 to deduce Corollary 6.10. However, we can prove the following variant of Theorem 6.8.

Theorem 6.13. *Suppose $T_1, .., T_n, ..., T_\infty$ are linear maps from $\mathbb{R}^N$ to $\mathbb{R}^N$, and there exists $B \subseteq \{1, ..., N\}$ such that*
i) $(T_n(e_j))_{j'} = 0$ if $j \in B, j' \notin B,\ n \in \mathbb{N} \cup \{\infty\}$.
ii) $B = B_{1,n} \cup B_{2,n}$ where $B_{1,n} = \{j \in B : (T_n(e_j))_{j'} > 0\ \forall j' \in B\}$, $B_{2,n} = \{j \in B : (T_n(e_j))_{j'} = 0\ \forall j' \in B\}$, and $B_{1,n} \neq \emptyset$, for every $n \in \mathbb{N} \cup \{\infty\}$.
iii) There exists $\sigma : \mathbb{N} \to \mathbb{N}$ strictly increasing with $T_{\sigma(n)} \underset{n\to\infty}{\longrightarrow} T_\infty$, and $B_{1,\sigma(n)} = B_{1,\infty}$ for every $n \in \mathbb{N}$.
Then, putting $D' = \{v \in \mathbb{R}^N : v_j > 0\ \forall j \in B, v_j = 0\ \forall j \notin B\}$, there exists $\overline{v} \in D'$ such that, for every strictly increasing sequence of naturals n_h, for every $v_h \in D'$ we have $\mathcal{N}(T_1 \circ ... \circ T_{n_h}(v_h)) \underset{h\to\infty}{\longrightarrow} \overline{v}$.

In other words, in Theorem 6.13, we have a linear subspace V of $\mathbb{R}^N$ that is mapped into itself by all T_n, i.e., $\{v \in \mathbb{R}^N : v_i = 0\ \ \forall i \notin B\}$. This is analogous to Π_i, in the case of operators $T_{i,n;E}$ on strongly connected fractals. Moreover, in place of the condition that all e_j are mapped into the interior of the positive cone, we have here the condition that they are mapped either into the interior of the positive cone or in 0, and the set of those e_j mapped into the interior of the positive cone may depend on n. Theorem 6.13 fits in the situation of fractals which are not strongly connected. In fact, the positivity of $T_{i,n;E}(e_j)(P_{j'})$ is related to the graph $\mathcal{G}(E_n)$. Usually, $\mathcal{G}(E_n)$ can change at any step, and it need not satisfy the hypotheses of Theorem 6.13 if we do not require some additional hypothesis. However, this is the case if E is not λ-contracting. More precisely, we have:

Lemma 6.14. *There exists $n_3 \geq 1$ such that, if $h \geq n_1+n_2+n_3$, $E \in \tilde{\mathcal{D}}$, $u \in \tilde{A}^{\pm,h}(E, E_{(1)})$ with $\lambda_{\pm,h}(E, E_{(1)}) = \lambda_{\pm}(E, E_{(1)})$, then there exist m with $0 \leq m \leq n_3$, $i_1, ..., i_m = 1, ..., k$, $j = 1, ..., N$, $B \subseteq V^{(0)} \setminus \{P_j\}$, $a = 1, -1$, such that*

i) $a\big(H_{m,h-m}(u) \circ \psi_{i_1,...,i_m} - c\big) \in D'$ where D' is as in Theorem 6.13, and c is a suitable constant. It turns out that $c = H_{m,h-m}(u) \circ \psi_{i_1,...,i_m}(P_j)$.

ii) Putting $T_n = T_{j,n+n_1}$, then for suitable $B_{1,n}, B_{2,n}$, i) and ii) of Theorem 6.13 are satisfied.

Of course, we here identify $\mathbb{R}^{V^{(0)}}$ with $\mathbb{R}^N$, and, by this identification, $v(P_j)$ corresponds to v_j. In other words, while in the case of strong maximum principle we can start with a function $u \in D$, which is mapped into D' by one operator $T_{j,n}$, in the present case i) tells that we have to use m operators before mapping u into D'. After doing this, on the base of ii), we are able to apply Theorem 6.13. We have, in fact, still to verify iii) of Theorem 6.13, which however will follow from a compactness argument. Lemma 6.14 plays in this more general case the role that Corollary 6.5 played in the case of strongly connected fractals . The proof of Theorem 6.13 is a not too complicated variant of the proof of Theorem 6.8. On the contrary, the proof of Lemma 6.14 is the most delicate step in the proof of convergence of $E_{(n)}$. At this moment, I do not know any simplification of that proof. Using Theorem 6.13 and Lemma 6.14 it is possible to prove again Theorem 6.11, and consequently, Theorem 6.12. In the proof of Theorem 6.11, however, there are some additional technical points with respect to the case of strongly connected fractals. Complete details can be found in [16], Section 4. There, Theorem 6.11 was proved in the more general setting of combinatorial fractal structures. The idea of investigating combinatorial fractal structures, firstly considered in [3], is motivated by the fact that $S_n(E)$, $M_n(E)$ and so on only depend on the graph $\mathcal{G}_1(E)$ and not on the geometry of the fractal. Theorem 6.11 was proved in [16] (or in [15]) in its full generality, i.e., fractal structures having an eigenform. For previous proofs in particular cases see for example [7] and [11], Example 8.8 and references therein for the case of Gasket, [12] for nested fractals with coefficient only depending on the distance; in [13] the result in strongly symmetric fractals (e.g., the Gasket) was announced without proof.

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