H–measures and semi–classical measures:
an introduction
by

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H–measures, also called microlocal defect measures —introduced independently by P. Gérard and by L. Tartar— and semi–classical measures —introduced by P. Gérard— are both designed to compute the weak limit of quadratic products of weakly oscillating quantities. When the oscillations do not occur at a specific scale, then H–measures are employed, whereas semi–classical measures are tuned to a specific oscillation scale. Although those measures only target quadratic non–linearities, their great asset is that, in contrast to Young–measures, they will satisfy a P.D.E. as soon as some combination of first order derivatives of the oscillating quantity is well-behaved, for example as soon as that quantity is solution to a first order hyperbolic system. That P.D.E. is usually a transport equation along the bicharacteristic rays (both in physical and phase space); as such it totally bypasses caustics and has a solution globally in time.

Such measures are of great interest in various settings where oscillations develop. In dynamic problems in mechanics, the weak limit of oscillating fields generated by, for example, a microstructure, or a vanishing dimension, usually possesses an energy density which is smaller than that of the oscillating fields. This is because part of the energy is locked into high frequencies which are lost (that is thermally disspated) in the limit process. The computation of this energy loss is quadratic in a mechanically linear framework and therefore can be performed through adequate use of such measures.

In homogenization, Landau & Lifschitz have developed universal formulae for the expansion to second order in the contrast of the overall conductivity of a two–phase mixture. These can be justified and generalized to other settings using such measures. Similarly, bounds on the overall properties of binary mixtures can be derived with the help of those measures.

Problems of controlability are also within the scope of such measures.

After a brief introduction and list of the properties of such measures I will focuss on applications spanning most of the above–mentioned topics. A non–exhaustive list of references is given below.

References

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