Homogenization in Perforated Domains.

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Introduction

In this course we consider boundary–value problems in perforated domains. Such problems attracted much attention of mathematicians in connection with applications. The studying of behavior of perforated and sceleton structures is important to researchers working with cellular and composite materials (lightweight materials) such as honeycombs, foams, wood, cork. Other modern engineering applications are space antennas, solar panels, civil engineering technologies, permeability of membranes and porous materials. Such problems arise also in chemistry, oil exploitation and others.

The most interesting case is a singular case, when the limit (homogenized) problem has different structure in comparison with the initial problem.

We consider problems with "term etrange" in the limit equation.

1 Dirichlet Problem in Perforated Domain with Small Concentration of Holes

1.1 Notation and Setting of the Problems.

Consider a smooth domain Ω in $\mathbb{R}^3$ and the ball $G_0 = \{\xi \ : \ |\xi| < 1\}$ in $\mathbb{R}^3$. Suppose that $\Omega_\varepsilon = \Omega \setminus \bigcup\limits_{z\in\mathbb{Z}^3} (\varepsilon^3 G_0 + 2\varepsilon z)$, i.e. Ω_ε is a domain with small cavities situated periodically in Ω with period 2ε. The diameter of holes equals to $2\varepsilon^3$.

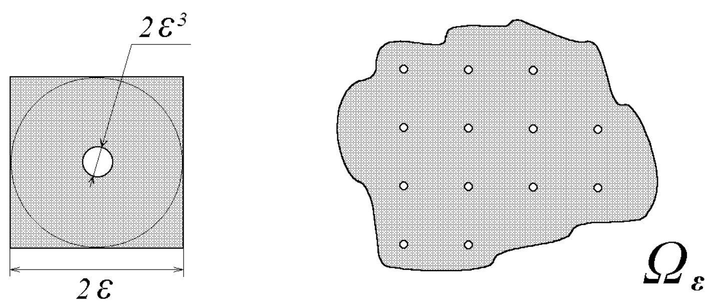

Figure 1:

Consider the following problems:

$$\begin{cases} \Delta u_\varepsilon = f(x) \quad \text{in} \quad \Omega_\varepsilon, \\ u_\varepsilon \in \overset{\circ}{H}{}^1(\Omega_\varepsilon), \end{cases} \tag{1}$$

$$\begin{cases} (\Delta + \mu) u = f(x) \quad \text{in} \quad \Omega, \\ u \in \overset{\circ}{H}{}^1(\Omega), \ \mu = -\frac{\pi}{2}, \end{cases} \tag{2}$$

The interesting effect of homogenization in this case is connected with the fact that solutions u_ε and u of problems (1) and (2) respectively, are close, i.e. the homogenized operator has the same high-order term but also it has a potential μI, where I is the identical operator. First this effect was discovered and studied in [1]. The proofs in [1] are very complicated but one can apply them to the general case with nonperiodically situated holes. They are based on the method of potentials for the solutions to boundary–value problems. Note that in [2] the authors introduced effective methods for studying the analogues problems in periodic media.

In this lecture we study one partial case of periodically perforated body, when holes are ball shaped. In this situation we can prove the estimate of deviation of solutions much easier.

1.2 Main Result.

Theorem 1 *For solutions u_ε and u of problems* (1) *and* (2) *the estimates*

$$\|u_\varepsilon - w_\varepsilon u\|_{H^1(\Omega_\varepsilon)} \le C\varepsilon \|f\|_{C^\alpha(\overline{\Omega})},$$

$$\|u_\varepsilon - u\|_{L_2(\Omega_\varepsilon)} \le C\varepsilon \|f\|_{C^\alpha(\overline{\Omega})},$$

are valid, where $w_\varepsilon(x)$ is 2ε-periodic function:

$$w_\varepsilon = \begin{cases} 0, & \text{if} \quad x \in \bigcup\limits_{z \in \mathbb{Z}^3} \left(\varepsilon^3 G_0 + 2\varepsilon z\right), \\ 1, & \text{if} \quad x \in \mathbb{R}^3 \setminus \bigcup\limits_{z \in \mathbb{Z}^3} \varepsilon \left(G_0 + 2z\right), \\ \dfrac{\left(\frac{1}{r} - \frac{1}{\varepsilon^3}\right)}{\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon^3}\right)}, & \text{if} \quad x \in \bigcup\limits_{z \in \mathbb{Z}^3} \left(\varepsilon G_0 \setminus \varepsilon^3 G_0 + 2\varepsilon z\right), \end{cases} \tag{3}$$

and $C, \alpha = \mathrm{const} > 0$, C does not depend on ε and $f(x)$.

1.3 Proof of Theorem 1.

Proof. The proof is based on the following statements:

Problem 1 *Prove that* $\|w_\varepsilon - 1\|_{L_2(\Omega_\varepsilon)} \to 0$ *and the estimate*

$$\|w_\varepsilon - 1\|_{L_2(\Omega_\varepsilon)} \leq C\varepsilon^2$$

takes place as $\varepsilon \to 0$*, where* C *does not depend on* ε*.*

Problem 2 * *Prove that for any smooth function* u *the following estimate*

$$\|(\Delta w_\varepsilon - \mu)u\|_{H^{-1}(\Omega_\varepsilon)} \leq C\varepsilon \|u\|_{\overset{\circ}{H}{}^1(\Omega)}$$

is valid, where the constant $C > 0$ *does not depend on* ε *and* $u \in \overset{\circ}{H}{}^1(\Omega)$*.*

Solving of this problems are based on the studying of properties of the function $w_\varepsilon(x)$.

Let us consider the identity

$$\Delta(u_\varepsilon - w_\varepsilon u) = f - \Delta u w_\varepsilon - 2(\nabla u, \nabla w_\varepsilon) - u\Delta w_\varepsilon =$$

$$= f(1 - w_\varepsilon) - (\Delta w_\varepsilon - \mu)u + \mu(w_\varepsilon - 1)u - 2(\nabla u, \nabla w_\varepsilon).$$

The right-hand side of the identity consists of four terms. If we will estimate the $H^{-1}(\Omega_\varepsilon)$-norm of each term using $\varepsilon C\|f\|_{C^\alpha(\overline{\Omega})}$, then we will prove Theorem. It is easy to obtain these estimates (prove it!) on the base of the a priori estimates of the solutions of (1) (2), and the following Schauder estimate for the solution of problem (2) (see, for instance, [3])

$$\|u\|_{C^{\alpha+2}(\overline{\Omega})} \leq K\|f\|_{C^\alpha(\overline{\Omega})},$$

where $\alpha > 0, K > 0$ are constants, and $K > 0$ does not depend on $f(x)$.

The second estimate of the theorem follows from the first and the definition of w_ε.

2 Fourier Problem in Perforated Domain.

This lecture is devoted to the results of the papers [4], [5], [6].

2.1 Basic Notation

Suppose that Ω is a smooth bounded domain in $\mathbb{R}^d$, $d \geq 2$.

$$J^\varepsilon = \{j \in \mathbb{Z}^d \ : \ \mathrm{dist}\,(\varepsilon j, \partial\Omega) \geq \varepsilon\sqrt{d}\},$$

$$\square \equiv \{\xi \mid -\frac{1}{2} < \xi_j < \frac{1}{2},\ j = 1, \dots, d\}.$$

Given $F(\xi)$, a $\square$-periodic smooth function, such that

$$F(\xi)\Big|_{\xi\in\partial\square} > 0,\ F(0) = -1,\ \nabla_\xi F\Big|_{\xi\in\square\setminus\{0\}} \neq 0,$$

we define

$$Q_j^\varepsilon = \{x \in \varepsilon\,(\square + j) \mid F(\frac{x}{\varepsilon}) \leq 0\},$$

consider

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{j\in J^\varepsilon} Q_j^\varepsilon.$$

Thus, $\partial\Omega^\varepsilon = \partial\Omega \cup S_\varepsilon$. Denote

$$Q = \{\xi \mid -\frac{1}{2} < \xi_j < \frac{1}{2},\ j = 1, \dots, d,\ F(\xi) \leq 0\},$$

$S = \{\xi \mid F(\xi) = 0\}$ and by ν the internal unit normal to S.

2.2 Setting of the Problem

Consider the following boundary value problem:

$$\begin{cases} -\mathcal{L}_\varepsilon u_\varepsilon := \dfrac{\partial}{\partial x_k}\left(a_{kj}(\dfrac{x}{\varepsilon})\dfrac{\partial u_\varepsilon}{\partial x_j}\right) = f(x) \,\text{in}\, \Omega^\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \gamma} + p(\dfrac{x}{\varepsilon})u_\varepsilon + \varepsilon q(\dfrac{x}{\varepsilon})u_\varepsilon = g(\dfrac{x}{\varepsilon}) \text{ on } S_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

where $\dfrac{\partial u_\varepsilon}{\partial \gamma} := a_{kj}\dfrac{\partial u_\varepsilon}{\partial x_j}\nu_k^\varepsilon$. We assume that all the functions $a_{kj}(\xi)$, $p(\xi)$, $q(\xi)$ and $g(\xi)$ are $\Box$-periodic, the matrix (a_{kj}) is symmetric, and

$$\kappa_1\eta^2 \le a_{kj}\eta_k\eta_j \le \kappa_2\eta^2 \quad \text{for any vector} \quad \eta.$$

We suppose that

$$<p(\xi)>_S = <g(\xi)>_S = 0, \tag{5}$$

here $<\cdot>_S := \int\limits_S \cdot \, d\sigma$.

We denote by $A^\varepsilon : \overset{\circ}{H}{}^1(\Omega^\varepsilon, \partial\Omega) \to H^{-1}(\Omega^\varepsilon)$ the operator associated with (4).

2.3 Formal Asymptotic Analysis

We search for the solution to (4) in the form

$$u_\varepsilon(x) \sim u_0(x) + \varepsilon u_1(x,\xi) + \varepsilon^2 u_2(x,\xi) + \dots, \tag{6}$$

where $u_i(x,\xi)$ are $\Box$-periodic with respect to ξ. Denote

$$-\mathcal{L}_{\alpha\beta}\varphi(x,\xi) := \frac{\partial}{\partial \alpha_k}\left(a_{kj}(\xi)\frac{\partial \varphi(x,\xi)}{\partial \beta_j}\right),$$

$$\frac{\partial \varphi(x,\xi)}{\partial \gamma_\alpha} := a_{kj}(\xi)\frac{\partial \varphi(x,\xi)}{\partial \alpha_j}\nu_k.$$

Substituting (6) in (4) and collecting the "highest order" terms, we arrive at

$$\begin{cases} \mathcal{L}_{\xi\xi}u_1 + \mathcal{L}_{\xi x}u_0 = 0 \quad \text{in} \quad \Box\backslash Q, \\ \frac{\partial u_1}{\partial \gamma_\xi} + \frac{\partial u_0}{\partial \gamma_x} + p(\xi)u_0 = g(\xi) \quad \text{on} \quad S. \end{cases} \tag{7}$$

The associated integral identity takes the form

$$\int\limits_{\Box\backslash Q} a_{kj}\frac{\partial u_1}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}d\xi + \int\limits_{\Box\backslash Q} a_{kj}\frac{\partial u_0}{\partial x_j}\frac{\partial v}{\partial \xi_k}d\xi +$$

$$+\int_S p(\xi) u_0 v \; d\sigma = \int_S g(\xi) v \; d\sigma, \tag{8}$$

where $v \in H^1_{\mathrm{per}}(\Box \backslash Q)$. Integral identity (8) suggests to look for the function $u_1(x,\xi)$ in the form

$$u_1(x,\xi) = L(\xi) + M(\xi) u_0(x) + N_i(\xi) \frac{\partial u_0(x)}{\partial x_i}. \tag{9}$$

Substituting (9) in (8) leads us to

$$\int_{\Box \backslash Q} a_{kj} \frac{\partial N_i}{\partial \xi_j} \frac{\partial v}{\partial \xi_k} d\xi + \int_{\Box \backslash Q} a_{ki} \frac{\partial v}{\partial \xi_k} d\xi = 0, \tag{10}$$

or, equivalently,

$$\begin{cases} \mathcal{L}_{\xi\xi} \left(N_i(\xi) + \xi_i \right) = 0 \quad \text{in} \quad \Box \backslash Q, \\ \frac{\partial N_i(\xi)}{\partial \gamma_\xi} = -a_{ki}(\xi) \nu_k \quad \text{on} \quad S, \end{cases}$$

where $i = 1, \ldots, d$;

$$\int_{\Box \backslash Q} a_{kj} \frac{\partial M}{\partial \xi_j} \frac{\partial v}{\partial \xi_k} d\xi + \int_S p(\xi) v \; d\sigma = 0, \tag{11}$$

or in the classical form

$$\begin{cases} \mathcal{L}_{\xi\xi} M(\xi) = 0 \quad \text{in} \quad \Box \backslash Q, \\ \frac{\partial M(\xi)}{\partial \gamma_\xi} = -p(\xi) \quad \text{on} \quad S \end{cases}$$

and

$$\int_{\Box \backslash Q} a_{kj} \frac{\partial L}{\partial \xi_j} \frac{\partial v}{\partial \xi_k} d\xi = \int_S g(\xi) v \; d\sigma, \tag{12}$$

or

$$\begin{cases} \mathcal{L}_{\xi\xi} L(\xi) = 0 \quad \text{in} \quad \Box \backslash Q, \\ \frac{\partial L(\xi)}{\partial \gamma_\xi} = g(\xi) \quad \text{on} \quad S. \end{cases}$$

The functions $L(\xi)$, $M(\xi)$, and $N_i(\xi)$ are defined up to an additive constant that can be fixed by

$$< L >_{\Box \backslash Q} = < M >_{\Box \backslash Q} = < N_i >_{\Box \backslash Q} = 0$$

$$\forall \ i = 1, \dots, d.$$

Similarly, collecting next terms leads to

$$\begin{cases} \mathcal{L}_{\xi\xi} u_2 + \mathcal{L}_{x\xi} u_1 + \mathcal{L}_{\xi x} u_1 + \mathcal{L}_{xx} u_0 = -f \quad \text{in} \quad \square \backslash Q, \\ \frac{\partial u_2}{\partial \gamma_\xi} + \frac{\partial u_1}{\partial \gamma_x} + p(\xi) u_1 + q(\xi) u_0 = 0 \quad \text{on} \quad S. \end{cases} \tag{13}$$

The solvability condition for problem (13) reads.

$$\widehat{a}_{kj} \frac{\partial^2 u_0(x)}{\partial x_k \partial x_j} - u_0(x) \Big(\int\limits_S p(\xi) M(\xi) \ d\sigma + \int\limits_S q(\xi) \ d\sigma \Big) =$$

$$= |\square \backslash Q| f(x) + \int\limits_S g(\xi) M(\xi) \ d\sigma, \tag{14}$$

where

$$\widehat{a}_{ik} := \int\limits_{\square \backslash Q} \Big(a_{ij}(\xi) \frac{\partial N_k(\xi)}{\partial \xi_j} + a_{ik}(\xi) \Big) \, d\xi.$$

Finally, the homogenized problem reads

$$\begin{cases} \widehat{a}_{kj} \dfrac{\partial^2 u_0(x)}{\partial x_k \partial x_j} (m - <q>_S) u_0(x) = |\square \backslash Q| f(x) - l \text{in } \Omega, \\ u_0(x) = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{15}$$

where $m := -<pM>_S$, $l := -<gM>_S$. Let $\widehat{A}$ be the operator of problem (15).

Remark 1 The coerciveness of the limit problem (15) is a delicate question since the constant m, as we will see later, is always positive. In particular, the coerciveness of (15) is provided by the inequality

$$m - <q>_S < \lambda_0,$$

where λ_0 is the first eigenvalue of the differential operator

$$-\widehat{a}_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

in the Sobolev space $\overset{\circ}{H}{}^1(\Omega, \partial\Omega)$.

2.4 Main Estimates and Results.

Here we obtain upper and lower bounds for the coefficient m.

We begin by considering an auxiliary spectral problem

$$\begin{cases} \dfrac{\partial}{\partial \xi_k}\left(a_{kj}(\xi)\dfrac{\partial \theta}{\partial \xi_j}\right) = 0 \quad \text{in} \quad \square \backslash Q, \\ \dfrac{\partial \theta}{\partial \gamma} = \Upsilon \theta \quad \text{on} \quad S, \\ \theta(\xi) \text{ is } \square\text{-periodic in } \xi, \ <\theta>_S = 0, \end{cases} \tag{16}$$

where Υ is a spectral parameter. The first eigenvalue Υ_1 of problem (16) can be found from the variational principle

$$\Upsilon_1 = \inf_{\substack{\psi \in H^1_{\rm per}(\square)\backslash\{0\}, \\ <\psi>_S = 0}} \frac{a(\psi,\psi)}{<\psi^2>_S},$$

where $\quad a(u,v) := \displaystyle\int_{\square\backslash Q} a_{kj}\frac{\partial u}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}\, d\xi.$

Lemma 1 *The constant m is positive. Moreover, it satisfies the estimates*

$$<p^2>_S \frac{<p^2>_S}{a(p,p)} \ \leq \ m \ \leq \ \frac{<p^2>_S}{\Upsilon_1}. \tag{17}$$

Remark 2 Note that the equalities in (17) are only attained if $p(\xi)$ happens to be the first eigenfunction in (16), i.e. the eigenfunction that corresponds to the eigenvalue Υ_1.

Proof. Substituting $M(\xi)$ as a test function in (11) we have

$$\int_{\square\backslash Q} a_{kj}\frac{\partial M}{\partial \xi_j}\frac{\partial M}{\partial \xi_k} d\xi + \int_S p(\xi) M \ d\sigma = 0.$$

Thus,

$$m = - <pM>_S = \left\langle a_{kj}\frac{\partial M}{\partial \xi_j}\frac{\partial M}{\partial \xi_k}\right\rangle_{\square\backslash Q} > 0,$$

if $M \not\equiv 0$. Note that $M \equiv 0$ iff $p \equiv 0$. Write down the variational problem for $M(\xi)$:

$$\inf_{\psi \in H^1_{\rm per}(\square)} H(\psi) \equiv \inf_{\psi \in H^1_{\rm per}(\square)} \Big\{ a(\psi,\psi) + 2 <p\psi>_S \Big\} = \tag{18}$$

$$= a(M,M) + 2 <pM>_S = <pM>_S = -m.$$

Substituting $\psi = -\frac{<p^2>_S}{a(p,p)} p$ in $H(\psi)$, we get

$$H(-\frac{<p^2>_S}{a(p,p)} p) = - <p^2>_S \frac{<p^2>_S}{a(p,p)} \geq -m.$$

Similarly, substituting $\psi = -\frac{<p^2>_S}{a(p,p)} \varphi$ with an arbitrary φ, we get

$$H(-\frac{<p^2>_S}{a(p,p)} \varphi) = -\frac{\Big(<p\varphi>_S \Big)^2}{a(\varphi,\varphi)}.$$

Since

$$m = \sup_{\varphi \in H^1_{\rm per}(\square)} \frac{\Big(<p\varphi>_S \Big)^2}{a(\varphi,\varphi)}, \tag{19}$$

then

$$\frac{1}{m} = \inf_{\varphi \in H^1_{\rm per}(\square) \setminus \{0\}} \frac{a(\varphi,\varphi)}{\Big(<p\varphi>_S \Big)^2} \geq$$

$$\inf_{\substack{\varphi \in H^1_{\rm per}(\square) \setminus \{0\}, \\ <\varphi>_S = 0}} \frac{a(\varphi,\varphi)}{<p^2>_S \ <\varphi^2>_S} = \frac{\Upsilon_1}{<p^2>_S}.$$

Finally, we have

$$m \leq \frac{<p^2>_S}{\Upsilon_1}.$$

Remark 3 To give a simple explanation of positiveness of the coefficient m arising in the homogenized problem (15), we consider the operator associated with the following modification of (4): $q \equiv 0$ and instead of the Dirichlet boundary condition on the exterior boundary $\partial\Omega$ the Neumann boundary condition is stated. In this case $-m$ is the first eigenvalue of the operator associated with the respective homogenized problem. In view of the convergence of spectrum results, the first eigenvalue of the operator introduced above is close to $-m$ for sufficiently small ε. Substitution of a constant function in the variational principle for the first eigenvalue of this operator gives zero. Thus, for any ε the said eigenvalue is negative and so is $-m$.

Theorem 2 *Let $f(x) \in C^1(\Omega)$, and suppose that $p(\xi)$, $q(\xi)$, and $g(\xi)$ are $\square$-periodic C^1 functions. Furthermore, assume that*

$$m < \lambda_0 + <q>_S, \tag{20}$$

where λ_0 is the first eigenvalue of the differential operator $-\widehat{a}_{ij}\dfrac{\partial^2}{\partial x_i \partial x_j}$. Then for all sufficiently small $\varepsilon > 0$ the problem (4) has a unique solution $u_\varepsilon(x)$ and the following estimate holds

$$\|u_0 + \varepsilon u_1 - u_\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq K_1 \sqrt{\varepsilon}. \tag{21}$$

Here K_1 does not depend on ε.

2.5 Auxiliary Lemmas

Lemma 2 *If $m < \lambda_0 + <q>_S$ then problem (15) is coercive.*

Proof. The variational principle for the first eigenvalue of the operator $\widehat{A}$ leads to the following relation

$$\inf_{\substack{v \in \overset{\circ}{H}{}^1(\Omega), \\ \|v\|_{L_2(\Omega)}=1}} \left(-\widehat{A}v, v\right)_{L_2(\Omega)} = \inf_{\substack{v \in \overset{\circ}{H}{}^1(\Omega), \\ \|v\|_{L_2(\Omega)}=1}} \int_\Omega \widehat{a}_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} +$$

$$+ \left(<q>_S - m\right) v^2 \, dx = \inf_{\substack{v \in \overset{\circ}{H}{}^1(\Omega), \\ \|v\|_{L_2(\Omega)}=1}} \int_\Omega \widehat{a}_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx +$$

$$+ \left(<q>_S - m\right) = \lambda_0 + <q>_S - m.$$

Thus

$$\left(-\widehat{A}v, v\right)_{L_2(\Omega)} \geq C \, \|v\|^2_{L_2(\Omega)}, \qquad C > 0,$$

that completes the proof.

Lemma 3 *The coerciveness of the homogenized problem implies the coerciveness of the original problem for all sufficiently small ε.*

Proof. First, it can be shown that

$$\int\limits_{S_\varepsilon} p(\frac{x}{\varepsilon})u^2(x)\ ds \le \alpha \int\limits_{\Omega^\varepsilon} |\nabla u|^2\ dx + \frac{1}{\alpha}\int\limits_{\Omega^\varepsilon} u^2\ dx$$

for any $\alpha > 0$. Then, there exists a sufficiently large Λ such that for any ε the operator associated with

$$\begin{cases} \mathcal{L}_\varepsilon u_\varepsilon + \Lambda u_\varepsilon = -f(x) \quad \text{in} \quad \Omega^\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \gamma} + p(\frac{x}{\varepsilon})u_\varepsilon + \varepsilon q(\frac{x}{\varepsilon})u_\varepsilon = g(\frac{x}{\varepsilon}) \text{ on } S_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{22}$$

is coercive. Consider the problems

$$(A^\varepsilon + \Lambda\cdot 1)^{-1}u_\varepsilon^k = \mu_k^\varepsilon u_\varepsilon^k, \quad (\widehat{A} + \Lambda\cdot 1)^{-1}u^k = \mu_k u^k, \tag{23}$$

Using the Oleinik–Shamaev–Yosifian Theorem (Theorem 1.4 from [7] (Section III.1)), we conclude that $\mu_0^\varepsilon \to \mu_0$ as $\varepsilon \to 0$. Then

$$\lambda_0^\varepsilon \equiv -\Lambda + \frac{1}{\mu_0^\varepsilon} \to \lambda_0 \equiv -\Lambda + \frac{1}{\mu_0} \qquad \text{as} \quad \varepsilon \to 0.$$

The lemma is proved.

2.6 Proof of Theorem 2.

To estimate $\|u_0 + \varepsilon u_1 - u_\varepsilon\|_{H^1(\Omega^\varepsilon)}$ let us substitute the expression

$$z_\varepsilon(x,\tfrac{x}{\varepsilon}) = u_0(x) + \varepsilon u_1(x,\tfrac{x}{\varepsilon}) - u_\varepsilon(x)$$

in (4). After proper rearrangments we obtain

$$\left| \int\limits_{\Omega^\varepsilon} \nabla z_\varepsilon\, \nabla v\ dx + \int\limits_{S_\varepsilon} (p+\varepsilon q)\, z_\varepsilon\, v\ ds \right| \le$$

$$\le \varepsilon \left| \varepsilon \int\limits_{S_\varepsilon} q\, u_1\, v(x)\ ds \right| + \left| \int\limits_{S_\varepsilon} \frac{\partial L}{\partial \gamma_\xi}\Big|_{\xi=\frac{x}{\varepsilon}} v\ ds - \int\limits_{S_\varepsilon} g\, v\ ds \right| +$$

$$
\begin{aligned}
&+\left|\int\limits_{S_\varepsilon} u_0 \frac{\partial M}{\partial \gamma_\xi}\Big|_{\xi=\frac{x}{\varepsilon}} v\ ds + \int\limits_{S_\varepsilon} p u_0 v\ ds\right| + \\
&+\left|\varepsilon \int\limits_{S_\varepsilon} q\, u_0\, v\ ds - \frac{1}{|\square \cap \omega|}\int\limits_{\Omega^\varepsilon} <q>_S u_0\, v(x)\ dx\right| + \\
&+\left|\varepsilon \int\limits_{\Omega^\varepsilon} \mathcal{L}_{xx} u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} v\ dx + O(\varepsilon)\|v\|_{H^1(\Omega^\varepsilon)}\right| + \\
&+\left|\int\limits_{S_\varepsilon}\left(\frac{\partial u_0}{\partial \gamma_x} + \frac{\partial u_0}{\partial x_i}\frac{\partial N_i(x,\xi)}{\partial \gamma_\xi}\Big|_{\xi=\frac{x}{\varepsilon}}\right) v\ ds\right| + \\
&+\left|\int\limits_{S_\varepsilon} \varepsilon M\, p\, u_0\, v\ ds + \frac{1}{|\square \cap \omega|}\int\limits_{\Omega^\varepsilon} m\, u_0\, v\, dx\right| + \\
&+\left|\frac{1}{|\square \cap \omega|}\int\limits_{\Omega^\varepsilon} l\, v\, d\hat{x} + \int\limits_{S_\varepsilon} \varepsilon\, p\, L\, v\ ds\right| + \\
&+\left|\int\limits_{S_\varepsilon} \varepsilon\, p \frac{\partial u_0}{\partial x_k} N_k\, v\ ds - \int\limits_{\Omega^\varepsilon} a_{ij}\frac{\partial u_0}{\partial x_i}\frac{\partial M(\xi)}{\partial \xi_j}\Big|_{\xi=\frac{x}{\varepsilon}} v\ dx\right| + \\
&+\Bigg|\int\limits_{\Omega^\varepsilon}\Big(\frac{\widehat{a}_{kj}}{|\square \cap \omega|}\frac{\partial^2 u_0}{\partial x_k \partial x_j}\, v - \\
&-a_{ij}(\xi)\frac{\partial^2 u_0}{\partial x_i\, \partial x_k}\frac{\partial N_k(\xi)}{\partial \xi_j}\Big|_{\xi=\frac{x}{\varepsilon}} v + \mathcal{L}_\varepsilon u_0\ v\Big)\ dx\Bigg| .
\end{aligned}
$$

Estimating all the terms in the right–hand side using Lemmas and definitions (10) – (12) leads us to inequality (21). Theorem is proved.

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