Gamma-convergence and its Applications to Some Problems in the Calculus of Variations

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CONTENTS

1. $\Gamma$-convergence: the general framework.
2. Limits of sequences of Riemannian metrics.
3. $\Gamma$-convergence for a class of singular perturbation problems.
4. A limit problem in phase transitions theory.
5. $\Gamma$-convergence in optimal control theory.
Lesson 1. Gamma-convergence: the general framework

We recall the definition of $\Gamma$-limits in metric spaces:

$\Gamma \lim \inf_{h \to +\infty} F_h(x) = \inf \{ \lim \inf_{h \to +\infty} F_h(x_h) : x_h \to x \}$

$\Gamma \lim \sup_{h \to +\infty} F_h(x) = \inf \{ \lim \sup_{h \to +\infty} F_h(x_h) : x_h \to x \}$

moreover the infima in formulas above are attained (more generally this holds in spaces with first countability axiom). Analogous definitions for families $(F_\varepsilon)$ with $\varepsilon \to 0$.

Coerciveness: $F : X \to \overline{\mathbb{R}}$ is said coercive if for every $t \in \mathbb{R}$ there exists a compact subset $K_t$ of $X$ such that

$\{ F \leq t \} \subset K_t$.

Equi-coerciveness: A sequence $(F_h)$ of functionals is said equi-coercive if for every $t \in \mathbb{R}$ there exists a compact subset $K_t$ of $X$ (independent of $h$) such that

$\{ F_h \leq t \} \subset K_t \quad \forall h \in \mathbb{N}$.

The main properties of $\Gamma$-convergence are (see the book of Dal Maso [Birkhäuser]):

- $(F_h)$ equicoercive, $F_h \Gamma \to F \Rightarrow \liminf_X F = \lim_{h \to +\infty} (\inf_X F_h)$;
- $F_h \Gamma \to F$, $x_h$ minimizer of $F_h$, $x_h \to x \Rightarrow x$ minimizer of $F$;
- $F_h \Gamma \to F$, $x_h$ minimizer of $F_h$, $(F_h)$ equicoercive, $F$ has a unique minimum point $x \Rightarrow x_h \to x$ (and $F_h(x_h) \to F(x)$);
- $F_h \Gamma \to F$, $G$ continuous $\Rightarrow F_h + G \Gamma \to F + G$;
- If $X$ is separable the $\Gamma$-convergence is a compact convergence, in the sense that from every sequence $(F_h)$ we may extract a subsequence $(F_{h_k})$ which $\Gamma$-converges.

Homogenization. Consider on the Sobolev space $W^{1,p}(\Omega)$ (with $1 < p < +\infty$) the family of functionals

$F_\varepsilon(u) = \int_\Omega f(x/\varepsilon, Du) \, dx \quad (\varepsilon \to 0)$

where $f(x, z)$ satisfies the assumptions:

- $f(x, \cdot)$ convex on $\mathbb{R}^n$;
- $f(\cdot, z)$ measurable and $Y$-periodic;
- $|z|^p \leq f(x, z) \leq C(1 + |z|^p)$.

Then $F_\varepsilon \Gamma \to F$ in the weak $W^{1,p}(\Omega)$ convergence, where

$F(u) = \int_\Omega f_0(Du) \, dx$

and $f_0$ is given by the formula

$f_0(z) = \inf \left\{ \frac{1}{|Y|} \int_Y f(x, z + Dw(x)) \, dx : w \in W^{1,p}_{per} \right\}$.

When $f(x, z)$ is a quadratic form

$f(x, z) = \sum_{i,j=1}^n a_{ij}(x) z_i z_j$
then \( f_0(z) \) is a quadratic form too
\[
f_0(z) = \sum_{i,j} \alpha_{ij} z_i z_j
\]
with constant coefficient \( \alpha_{ij} \) which can be computed by the formula above.

Other variations on the theme can be made: for instance the Attouch & Buttazzo [Ann. SNS] case of "periodic reinforcement"
\[
F_\varepsilon(u) = \int |D\varepsilon|^2 \, dx + k\varepsilon \int_{\Omega \cap S_\varepsilon} |Du|^2 \, d\sigma
\]
where \( S_\varepsilon \) is the \( \varepsilon \)-rescaling of a \( (n-1) \) dimensional manifold \( S \subset Y \). The \( \Gamma \)-limit \( F \) is then
\[
F(u) = \int \Omega f(Du) \, dx
\]
with
\[
f(z) = \inf \left\{ \frac{1}{Y} \int_Y |Dw|^2 \, dx + \frac{k}{|S|} \int_{S} |D\tau w|^2 \, d\sigma : w - \langle z, \cdot \rangle \in W_{per}^{1,2} \right\}.
\]

The homogenization has been widely treated in the other courses of this school. Therefore, the program we intend to follow in these lectures is to show some applications of \( \Gamma \)-convergence different from periodic homogenization. More precisely we shall treat the following topics:

- limits of periodic Riemannian metrics;
- limits of singular perturbation problems;
- a limit problem in phase transitions theory;
- \( \Gamma \)-convergence and optimal control problems.

Lesson 2. Limits of sequences of Riemannian metrics

We shall study the limit (as \( \varepsilon \to 0 \)) of the functionals
\[
F_\varepsilon(u) = \int_0^1 \sum_{i,j=1}^n a_{ij}(\varepsilon) u_i' u_j' \, dt
\]
where \( \{a_{ij}\} \) are the coefficients of a Riemannian metric, or more generally in the so called "Finsler case"
\[
F_\varepsilon(u) = \int_0^1 f(\varepsilon, u') \, dt
\]
where \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is a Borel function such that

- \( f(s, \cdot) \) is convex
- \( f(\cdot, z) \) is \( Y \)-periodic (\( Y = [0, 1]^n \))
- \( |z|^p \leq f(s, z) \leq C(1 + |z|^p) \) with \( p > 1 \).

**Theorem.** There exists a convex function \( \varphi \) with
\[
|z|^p \leq \varphi(z) \leq C(1 + |z|^p)
\]
such that \( F_\varepsilon \Gamma \to \Phi \) where
\[
\Phi(u) = \int_0^1 \varphi(u') \, dt.
\]
Moreover, $\varphi$ is given by
\[
\varphi(z) = \lim_{\varepsilon \to 0} \left[ \inf \left\{ F_{\varepsilon}(w) : w \in W^{1,p}(0,1), w(0) = 0, w(1) = z \right\} \right] = \\
= \lim_{T \to +\infty} \left[ \inf \left\{ \frac{1}{T} \int_{0}^{T} f(w, w') \, dt : w \in W^{1,p}(0,1), w(0) = 0, w(T) = Tz \right\} \right].
\]

We prove the theorem in several steps; for some technical details we refer to the original paper by Acerbi & Buttazzo [JAM]. It will be convenient to localize all functionals by setting for every open subset $A$ of $(0,1)$ (we denote by $\mathcal{A}$ such a class)
\[
F_{\varepsilon}(u, A) = \int_{A} f\left(\frac{u}{\varepsilon}, u'\right) \, dt \\
\Phi(u, A) = \int_{A} \varphi(u') \, dt.
\]

**Step 1.** There exists a sequence $\varepsilon_h \to 0$ such that for every open set $A$ belonging to a countable base $\mathcal{U}$ of open sets in $(0,1)$ the sequence $F_{\varepsilon_h}(\cdot, A)$ $\Gamma$-converges to some $\Gamma$-limit we denote by $G(\cdot, A)$.

It is enough to apply the compactness property of $\Gamma$-convergence and a diagonal procedure.

**Step 2.** The sequence $F_{\varepsilon_h}(\cdot, A)$ $\Gamma$-converges for all $A \in \mathcal{A}$ to
\[
F(u, A) = \sup \left\{ G(u, B) : B \in \mathcal{B}, B \subset\subset A \right\}.
\]

See Acerbi & Buttazzo [JAM].

**Step 3.** The set function $A \mapsto F(u, A)$ is a measure for all $u \in W^{1,p}(0,1)$.

We prove only the key fact that $F(u, \cdot)$ is a subadditive set function, that is for every $A, B, C \in \mathcal{U}$ with $C \subset\subset A \cup B$ and every $u \in W^{1,p}(A \cup B)$
\[
G(u, C) \leq G(u, A) + G(u, B).
\]

the remaining facts can be found in Acerbi & Buttazzo [JAM].

Let $K$ be a compact subset of $A$ containing $\overline{C} \setminus B$ in its interior, let $\delta = \text{dist}(K, \partial A)$, let $\nu \in \mathbb{N}$ be a fixed integer number, and let for $i = 1, \ldots, \nu$
\[
A_i = \left\{ t \in [0, 1] : \text{dist}(t, K) < i \frac{\delta}{\nu} \right\} \quad (A_0 = \int K)
\]
\[
\varphi_i \in C_{c}^{\infty}(A_i), \quad 0 \leq \varphi_i \leq 1, \quad \varphi_i = 1 \text{ on } A_{i-1}, \quad |\varphi_i'| \leq \frac{2\nu}{\delta}.
\]

Moreover let $u_h \to u$ in $L^{p}(A)$, $v_h \to v$ in $L^{p}(B)$ be such that
\[
G(u, A) = \lim_{h} F_{\varepsilon_h}(u_h, A) \\
G(u, B) = \lim_{h} F_{\varepsilon_h}(v_h, B).
\]

Setting $w_{i,h} = \varphi_i u_h + (1 - \varphi_i)v_h$ we have
\[
F_{\varepsilon_h}(w_{i,h}, C) \leq F_{\varepsilon_h}(u_h, A_{i-1}) + F_{\varepsilon_h}(v_h, C \setminus A_i) + \\
+ C \int_{C \cap (A_i \setminus A_{i-1})} (1 + |w_{i,h}'|^p) \, dt \leq \\
\leq F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B) + C\left(\frac{\nu}{\delta}\right)^p \int_{C} |u_h - v_h|^p \, dt + \\
+ C \int_{C \cap (A_i \setminus A_{i-1})} (1 + |u_h'|^p + |v_h'|^p) \, dt.
\]
For every $h \in \mathbb{N}$ choose $i_h \leq \nu$ such that

$$
\int_{C \cap (A_{i_h} \setminus A_{i_h-1})} (1 + |u_h'|^p + |v_h'|^p) dt \leq \frac{1}{p} \int_{C \cap A \cap B} (1 + |u_h'|^p + |v_h'|^p) dt \leq \frac{C}{\nu} [1 + F_{\varepsilon h}(u_h, A) + F_{\varepsilon h}(v_h, B)]
$$

so that

$$
F_{\varepsilon h}(w_{i_h,h}, C) \leq (1 + \frac{C}{\nu})[F_{\varepsilon h}(u_h, A) + F_{\varepsilon h}(v_h, B)] + C \frac{\nu}{\delta} \int_C |u_h - v_h|^p dt.
$$

It is easy to see that $w_{i_h,h} \to u$ in $L^p(C)$, so that as $h \to +\infty$

$$
G(u, C) \leq \limsup_h F_{\varepsilon h}(w_{i_h,h}, C) \leq (1 + \frac{C}{\nu})[G(u, A) + G(u, B)] + \frac{C}{\nu}
$$

and the proof follows by letting now $\nu \to +\infty$.

**Step 4.** For every $a \in \mathbb{R}^n$ we have

$$
F(u + a, A) = F(u, A).
$$

Take $a_h \to a$ in $\mathbb{R}^n$ such that $a_h/\varepsilon_h \in \mathbb{Z}^n$ and take $u_h \to u$ such that

$$
F(u, A) = \lim_h F_{\varepsilon h}(u_h, A).
$$

Then $u_h + a_h \to u + a$ and so

$$
F(u + a, A) \leq \liminf_h \int_A f \left( \frac{u_h}{\varepsilon_h} + \frac{a_h}{\varepsilon_h}, u_h' \right) dt = \liminf_h \int_A f \left( \frac{u_h}{\varepsilon_h}, u_h' \right) dt = F(u, A).
$$

The opposite inequality can be proved in a similar way.

**Step 5.** There exists a convex function $\varphi$ such that

$$
F(u, A) = \int_A \varphi(u') dt.
$$

This follows from the Buttazzo & Dal Maso [Nonl. An.] and [JMPA] integral representation theorem (valid also in the multiple integrals case):

Let $F : W^{1,p} \times A \to \mathbb{R}$ be a functional such that

(i) $F(\cdot, \cdot)$ is a measure (proved in Step 3);

(ii) $F(\cdot, A)$ is lower semicontinuous $L^p$ (because it is a $\Gamma$-limit);

(iii) $\int_A |u'|^p dt \leq F(u, A) \leq C \int_A (1 + |u'|^p) dt$ (by the assumptions on $f$);

(iv) $F$ is local, i.e. $u = v$ a.e. on $B \Rightarrow F(u, B) = F(v, B)$ (we refer to Acerbi & Buttazzo [JAM] for the proof);

(v) $F(u + a, A) = F(u, A)$ for every $a \in \mathbb{R}^n$ (proved in Step 4).

Then there exists a function $\varphi(t, z)$ convex in $z$ such that

$$
F(u, A) = \int_A \varphi(t, u') dt.
$$
The fact that in our case the function $\varphi$ does not depend on $t$ follows from the translations invariance of $F$ (easy to prove):

$$F(u, A) = F(u_\tau, A + \tau) \quad (u_\tau(t) = u(t - \tau)).$$

**Step 6.** Setting for every $z \in \mathbb{R}^n$ and $T > 0$

$$M_T(z) = \inf \left\{ \frac{1}{T} \int_0^T f(u, u') \, dt : u \in W^{1,p}(0, T), u(0) = 0, u(T) = Tz \right\}$$

there exists

$$\lim_{T \to +\infty} M_T(z) = M(z).$$

We refer to Acerbi & Buttazzo [JAM] for the proof.

**Step 7.** $M(z) = \varphi(z)$ for every $z \in \mathbb{R}^n$.

We refer to Acerbi & Buttazzo [JAM] for the proof.

We can conclude now the proof of the main $\Gamma$-convergence theorem because for every $\varepsilon_k \to 0$ we may extract $(\varepsilon_k)_k$ such that $F_{\varepsilon_k} \Gamma$-converges to some $\int_A \varphi(u') \, dt$ with $\varphi$ possibly depending on the subsequence choosen. By Step 7 the function $\varphi$ is identified in a way which is independent of the subsequence choosen; therefore the entire $(F_\varepsilon)$ $\Gamma$-converges to $\int_A \varphi(u') \, dt$.

**Example.** Let $n = 2$ and consider the chessboard structure with $f(s, z) = a(s)|z|^2$ ($a$ is considered extended periodically). Therefore $f_\varepsilon(s, z) = a(s/\varepsilon)|z|^2$ corresponds to the Riemannian metric with coefficients $a(s/\varepsilon)\delta_{ij}$.

The rescaled coefficient $a(s)$

We know that as $\varepsilon \to 0$ the limit functional is of the form

$$\int_0^1 \varphi(u') \, dt$$

with $\varphi$ convex. In the theorem above it is easy to prove that $f(s, z)$ is positively $p$-homogeneous in $z$ so is $\varphi(z)$; then in our case $\varphi(z)$ is positively 2-homogeneous with

$$a|z|^2 \leq \varphi(z) \leq \beta|z|^2.$$
The following fact hold.

• If \( \alpha \neq \beta \) then \( \varphi(z) \) is not a quadratic form (see Acerbi & Buttazzo [JAM]); therefore the variational limit of a sequence of Riemannian metrics may be not Riemannian but only a Finsler metric; the class of Finsler metrics on the contrary is closed under \( \Gamma \)-convergence.

• If \( \beta/\alpha \) is large enough then the function \( \varphi \) depends only on \( \alpha \) and has the form

\[
\varphi(z) = \alpha((\sqrt{2} - 1)|z_1| \wedge |z_2| + |z_1| \vee |z_2|)^2.
\]

Lesson 3. Gamma-convergence for a class of singular perturbation problems

We want to study the asymptotic behaviour (in terms of \( \Gamma \)-convergence) of problems of the form

\[
F_\varepsilon(u) = \int_\Omega f(x, u, \varepsilon Du, \varepsilon^2 D^2 u, \ldots, \varepsilon^m D^m u) \, dx.
\]

For instance the optimal control problem \((u \text{ is the state}, v \text{ is the control})\)

\[
\min \int_\Omega (k|v|^2 + |u - u_0(x)|^p) \, dx
\]

with state equation

\[
\begin{aligned}
\varepsilon^2 \Delta u + g(u) &= v \\
u &\in H^1_0(\Omega)
\end{aligned}
\]

reduces to the functional

\[
\int_\Omega k|\varepsilon^2 \Delta u + g(u)|^2 + |u - u_0(x)|^p \, dx.
\]

The first difficulty to overcome is the lack of equi-coerciveness in the Sobolev spaces; therefore we study the \( \Gamma \)-limit in the weak \( L^p \) topology.

We make the following assumptions on \( f(x, s, z) \) where \( s \) represents \( u \) and \( z \) represents \( (Du, D^2 u, \ldots, D^m u) \):

(i) there exist \( a \in L^1, \ c \geq 1, \ p > 1, \ 1 \leq r \leq p \) such that

\[
-a(x) + |s|^p \leq f(x, s, z) \leq a(x) + C[|s|^p + |z|^r];
\]

(ii) there exist continuity moduli \( w \) and \( \sigma \) such that

\[
|f(x, s, z) - f(y, t, w)| \leq w(x, |y - x|) + \sigma(|y - x| + |t - s| + |w - z|)(a(x) + f(x, s, z));
\]

(iii) \( f(x, s, z) + |s|^p + a(x) \geq \gamma(s, z) \) where \( \gamma \) is such that

\[
\int_A \sum_{|\alpha| \leq m} |D^\alpha u|^r \, dx \leq \lambda(A, A') \int_{A'} \gamma(u, Du, \ldots, D^m u) \, dx \quad \forall A \subset A'
\]

where \( \lambda(A, A') \) is such that

\[
\lim_{t \to +\infty} \lambda(tA, tA') < +\infty.
\]

For instance, if

\[
F_\varepsilon(u) = \int_\Omega k|\varepsilon^2 \Delta u + g(u)|^2 + |u - u_0(x)|^p \, dx
\]
the assumptions above are fulfilled with \((m = r = 2)\)

\[
f(x, s, z) = k\left|\sum_{i=1}^{n} z_{ii} + g(s)\right|^p + |s - u_0(x)|^p
\]

\[
\gamma(s, z) = C_1 \left[\sum_{i=1}^{n} z_{ii}^2 + |s|^2\right]
\]

\[
\lambda(A, A') = C_2 \max\{1, \text{dist}^{-4}(A, \partial A')\}
\]

provided \(g\) is such that

\[
|g(s)| \leq C(1 + |s|^{p/2})
\]

\[
|g(s) - g(t)| \leq \omega(t - s)(1 + |s|^{p/2}).
\]

**Theorem.** There exists a function \(\psi(x, s)\) convex in \(s\) such that

\[
F_\epsilon(u, A) \rightharpoonup \int_A \psi(x, u) \, dx \quad \text{(weakly in } L^p(A))
\]

for every \(A \in \mathcal{A}\). Moreover

\[
f^{**}_{s,z}(x, s, 0) \leq \psi(x, s) \leq f^{**}(x, s, 0)
\]

where \(f^{**}_{s,z}\) and \(f^{**}\) represent the convexification of \(f\) in \((s, z)\) and in \(z\) respectively. A representation formula for \(\psi\) is \((Y = [0,1]^n)\)

\[
\psi(x, s) = \lim_{\epsilon \to 0} \left[\inf \left\{ F_\epsilon(x, u) : \int_Y u \, dy = s \right\} \right] =
\]

\[
= \inf \left\{ F_\epsilon(x, u) : \epsilon > 0, \int_Y u \, dy = s \right\}
\]

where

\[
F_\epsilon(x, u) = \int_Y f(x, u(y), \epsilon Du(y), \ldots, \epsilon^m D^m u(y)) \, dy.
\]

We prove only the key fact that the \(\Gamma - \lim \sup\) is subadditive, by referring for all other details to Buttazzo and Dal Maso [CRAS], [Ann. SNS].

Setting

\[
F^+(u, A) = \inf_h \left\{ \limsup_{h \to 0} F_{\epsilon_h}(u_h, A) : u_h \to u \text{ in } L^p \right\}
\]

we have to prove for every \(u \in L^p(A \cup B)\) and \(C \subset\subset A \cup B\)

\[
F^+(u, C) \leq F^+(u, A) + F^+(u, B).
\]

Fix \(K = \overline{C} \setminus B\) and \(A_0, B_0\) with \(K \subset A_0 \subset B_0 \subset \subset A\). Fix an integer \(\nu\) and let \((A_i)_{1 \leq i \leq \nu}\) be such that \(A_i \subset \subset A_1 \subset \subset \ldots \subset \subset A_\nu \subset \subset B_0\). Denote by \(S = C \cap (B_0 \setminus A_0)\) and by \(S_i = C \cap (A_i \setminus A_{i-1})\) and let \(\varphi_i \in C_\infty^\infty(A_i)\) be such that \(0 \leq \varphi_i 1\) and \(\varphi_i = 1\) on \(A_i \setminus A_{i-1}\). We have

\[
F^+(u, A) = \limsup_h F_{\epsilon_h}(u_h, A)
\]

\[
F^+(u, B) = \limsup_h F_{\epsilon_h}(v_h, B)
\]
for suitable sequences \((u_h)\) and \((v_h)\) converging to \(u\). Setting
\[
 w_{i,h} = \varphi_i u_h + (1 - \varphi_i) v_h
\]
we have
\[
 F_{\varepsilon_h}(w_{i,h}, C) \leq F_{\varepsilon_h}(u_h, C \cap A_{i-1}) + F_{\varepsilon_h}(v_h, C \setminus A_i) +
 C \int_{S_i} \left[ a(x) + |w_{i,h}|^p + \sum_{k=1}^m \| F_{i,k} u_{i,h} \|^r \right] \, dx
 \leq F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B) +
 C \int_{S_i} \left[ a(x) + |u_h|^p + |v_h|^p + \sum_{k=1}^m \| F_{i,k} u_h \|^r + \| F_{i,k} v_h \|^r \right] \, dx
 + C_{\nu} \int_{S_i} \left[ \sum_{k=1}^m \varepsilon_i^k \sum_{j=0}^{k-1} \| D^j u_h \|^r + \| D^j v_h \|^r \right] \, dx
\]
where \(C_{\nu}\) depends on \(\| \varphi_i \|_{C^m}\) for \(i = 1, \ldots, \nu\). Let \(i_h\) be an index such that
\[
 \int_{S_{\epsilon_h}} \left[ a(x) + |u_h|^p + |v_h|^p + \sum_{k=1}^m \varepsilon_i^k \| D^k u_h \|^r + \| D^k v_h \|^r \right] \, dx \leq
t \int_{S} \left[ a(x) + |u_h|^p + |v_h|^p + \sum_{k=1}^m \varepsilon_i^k \| D^k u_h \|^r + \| D^k v_h \|^r \right] \, dx
\]
and set \(w_h = w_{i_h,h}\). Then \(w_h \to u\) and
\[
 F_{\varepsilon_h}(w_h, C) \leq F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B) +
 C_{\nu} \int_{S} \left[ \sum_{k=1}^m \varepsilon_i^k \sum_{j=0}^{k-1} \| D^j u_h \|^r + \| D^j v_h \|^r \right] \, dx.
\]
As \(h \to +\infty\)
\[
 F^+(u, C) \leq \limsup_h F_{\varepsilon_h}(w_h, C) \leq F^+(u, A) + F^+(u, B) +
 C_{\nu} \int_{S} \left[ \sum_{k=1}^m \varepsilon_i^k \sum_{j=0}^{k-1} \| D^j u_h \|^r + \| D^j v_h \|^r \right] \, dx.
\]
Set now \(U_h(x) = u_h(\varepsilon_h x)\) and \(V_h(x) = v_h(\varepsilon_h x)\); then, for every \(S' \subset A \cap B\)
\[
 \int_{S'} \left[ \sum_{k=1}^m \varepsilon_i^k \| D^k u_h \|^r + \| D^k v_h \|^r \right] \, dx = \varepsilon_h^m \int_{S'/\varepsilon_h} \left[ \sum_{k=1}^m \| D^k U_{h,h} \|^r + \| D^k V_{h,h} \|^r \right] \, dx \leq
\]
\[
 \leq \varepsilon_h^m \lambda(S'/\varepsilon_h, A \cap B/\varepsilon_h) \int_{A \cap B/\varepsilon_h} \left[ \gamma(U_{h,h}, \ldots, D^m U_{h,h}) + \gamma(V_{h,h}, \ldots, D^m V_{h,h}) \right] \, dx =
\]
\[
 = \lambda(S'/\varepsilon_h, A \cap B/\varepsilon_h) \int_{A \cap B/\varepsilon_h} \left[ \gamma(u_{h,h}, \ldots, D^m u_{h,h}) + \gamma(v_{h,h}, \ldots, D^m v_{h,h}) \right] \, dx \leq
\]
\[
 \leq \lambda(S'/\varepsilon_h, A \cap B/\varepsilon_h) \int_{A \cap B/\varepsilon_h} \left[ 2a(x) + |u_h|^p + |v_h|^p + f(x, u_h, \ldots, \varepsilon_h^m D^m u_h) + f(x, v_h, \ldots, \varepsilon_h^m D^m v_h) \right] \, dx \leq
\]
\[
 \leq \lambda(S'/\varepsilon_h, A \cap B/\varepsilon_h) \int_{A \cap B} \left[ |C + F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B) | \right] \, dx \leq C.
\]
By using inequalities as
\[ \int_S |D^k w|^r \, dx \leq \alpha \int_{S'} |D^m w|^r \, dx + C_\nu \int_{S'} |w|^r \, dx \]
for every \( 1 \leq k \leq m - 1 \) and every \( \alpha > 0 \), where \( S \subset \subset S' \), we have
\[
C_\nu \sum_{k=1}^{m} \epsilon_h^k \sum_{j=0}^{k-1} \int_S |D^j u_h|^r \, dx \leq \\
\leq C_\nu \sum_{k=1}^{m} \epsilon_h^k \int_{S'} |\sigma D^k u_h|^r + C_\sigma |u_h|^r \, dx \leq \\
\leq \sigma C_\nu \int_{S'} \sum_{k=1}^{m} \epsilon_h^k |D^k u_h|^r \, dx + \epsilon_h C_\nu C_\sigma
\]
and analogously for \( v_h \). Therefore, arguing as before we obtain
\[
C_\nu \limsup_h \int_S \sum_{k=1}^{m} \epsilon_h^k \sum_{j=0}^{k-1} |(D^j u_h)^r + |D^j v_h|^r \, dx \leq \sigma C_\nu
\]
so that
\[
F^+(u, C) \leq F^+(u, A) + F^+(u, B) + \frac{C}{\nu} + \sigma C_\nu
\]
and the subadditivity follows by taking first the limit as \( \sigma \to 0 \) and then the limit as \( \nu \to +\infty \).

Once the subadditivity is proved, standard methods prove that the \( \Gamma \)-limit \( F(u, \cdot) \) is a measure, and by the Buttazzo and Dal Maso [RM] integral representation theorem
\[
F(u, A) = \int_A \psi(x, u) \, dx
\]
for a suitable \( \psi(x, s) \) convex in \( s \). The inequality \( f_{s, z}^{**}(x, s, 0) \leq \psi(x, s) \) is trivial because if \( u_h \to u \) weakly in \( L^p \)
\[
\int_A f_{s, z}^{**}(x, u_h, 0) \, dx \leq \liminf_h \int_A f_{s, z}^{**}(x, u_h, \epsilon_h D u_h, \ldots, \epsilon_h^m D^m u_h) \, dx \leq \\
\leq \liminf_h \int_A f(x, u_h, \epsilon_h D u_h, \ldots, \epsilon_h^m D^m u_h) \, dx
\]
and so
\[
\int_A f_{s, z}^{**}(x, u, 0) \, dx \leq \int_A \psi(x, u) \, dx.
\]
For the inequality \( \psi(x, s) \leq f_{s}^{**}(x, s, 0) \) it is enough to show that
\[
\int_A \psi(x, u) \, dx \leq \int_A f(x, u, 0) \, dx
\]
for every \( u \). This can be proved by taking \( u_h = \rho_h * u \) where \( \rho_h(x) = \epsilon_h^{-n \theta} \rho(\epsilon_h^{-\theta} x) \); if \( \theta \) is small enough \( (\theta < 1/n + 1) \) we have \( u_h \to u \) strongly in \( L^p \) and \( \epsilon_h^k D^k u_h \to 0 \) strongly in \( L^p \), so that
\[
\int_A \psi(x, u) \, dx \leq \liminf_h \int_A f(x, u_h, \ldots, \epsilon_h^m D^m u_h) \, dx = \int_A f(x, u, 0) \, dx.
\]
In the case \( f(x, s, z) = k |\sum_{i=1}^{n} z_i + g(s)|^2 + |s - u_0(x)|^p \) it is possible to prove (see Buttazzo & Dal Maso [Ann. SNS])
• $g$ affine $\Rightarrow \psi(x,s) = f^*_s(x,s,0) = k|g(s)|^2 + |s - u_0(x)|^p$;
• $g$ decreasing $\Rightarrow \psi(x,s) = f^*_s(x,s,0) = k|g(s)|^2 + |s - u_0(x)|^p$;
• the equality $\psi(x,s) = f^*_s(x,s,0)$ is not true in general;
• $\geq 0$ convex $\Rightarrow \psi(x,s) = k|g(s)|^2 + |s - u_0(x)|^p$.

Note that in the case $F_\varepsilon(u) = \int_\Omega [\varepsilon^2 |Du|^2 + W(u)] dx$
we have $f(s,z) = |z|^2 + W(s)$ so that
\[ f^*_s(s,z) = f^*_s(s,z) = |z|^2 + W^*(s). \]

Hence $\psi(x,s) = W^*(s)$.

Lesson 4. A limit problem in phase transitions theory

Let $W : \mathbb{R} \to \mathbb{R}$ be a positive continuous function with only two zeros (say at $-1$ and at $1$); consider
the functionals
\[ F_\varepsilon(u) = \int_\Omega \left[ \varepsilon |Du|^2 + \frac{1}{\varepsilon} W(u) \right] dx \]
where $\Omega$ is a bounded open Lipschitz subset of $\mathbb{R}^n$. We shall prove that the $\Gamma$-limit (as $\varepsilon \to 0$) in the
topology $L^1(\Omega)$ is
\[ F(u) = \begin{cases} C_0 \int_\Omega |Du| & \text{if } |u(x)| = 1 \text{ for a.e. } x \in \Omega \\ +\infty & \text{otherwise} \end{cases} \]
defined for all $u \in BV(\Omega)$, where $C_0 = 2 \int_{-1}^1 \sqrt{W(s)} \, ds$.

It is convenient to introduce the function
\[ \phi(t) = \int_0^t \sqrt{W(s)} \, ds \]
and to write $F(u)$ for $|u| \equiv 1$ as
\[ F(u) = 2 \int_\Omega |D(\phi \circ u)|. \]
Then the inequality
\[ F(u) \leq \Gamma - \liminf_\varepsilon F_\varepsilon(u) \]
is rather easy to prove. Indeed, when $|u| \neq 1$ we have
\[ \liminf_\varepsilon F_\varepsilon(u_\varepsilon)) \geq \liminf_\varepsilon \frac{1}{\varepsilon} \int_\Omega W(u_\varepsilon) \, dx = +\infty \]
whereas if $|u| \equiv 1$
\[ \liminf_\varepsilon F_\varepsilon(u_\varepsilon) \geq \liminf_\varepsilon \int_\Omega 2|Du_\varepsilon| \sqrt{W(u_\varepsilon)} \, dx = \]
\[ = \liminf_\varepsilon \int_\Omega 2|D(\phi \circ u_\varepsilon)| \, dx \geq 2 \int_\Omega |D(\phi \circ u)| \]
where the first inequality follows from the standard $a^2 + b^2 \geq 2ab$ and the last one from the lower semicontinuity of the total variation functional.
The approximating sequence

We prove now the opposite inequality

\[ F(u) \geq \Gamma \limsup_{\varepsilon} F_{\varepsilon}(u) \]

only for functions \( u \) of the form \(-1_A + 1_{\Omega \setminus A}\) where \( A \) is an open set with a smooth boundary \( \Sigma \) transversal to \( \partial \Omega \). We refer to the original papers of Modica & Mortola \[\text{[BUMI], [BUMII]}\] for the proof that from this particular case we can deduce, by a density argument, the general case.

We want to construct an approximating sequence \( u_{\varepsilon} \) as in the picture, where the thickness of the transition layer and the transition itself have to be suitably choosen.

Set for every \( t \in \mathbb{R} \)

\[
\psi_{\varepsilon}(t) = \int_{-1}^{t} \frac{\varepsilon}{\sqrt{\varepsilon + W(s)}} ds
\]

\[
\varphi_{\varepsilon}(t) = \begin{cases} 
-1 & \text{if } t \leq 0 \\
\psi_{\varepsilon}^{-1}(t) & \text{if } 0 \leq t \leq \psi_{\varepsilon}(1) \\
1 & \text{if } t \geq \psi_{\varepsilon}(1)
\end{cases}
\]

and, if \( d(x) = dist(x,A) \)

\[ u_{\varepsilon}(x) = \varphi_{\varepsilon}(d(x)). \]

We have \( u_{\varepsilon} \to u \) in \( L^1(\Omega) \) and, if \( \Sigma_{\varepsilon} = \{x \in \Omega : 0 < d(x) < \psi_{\varepsilon}(1)\} \)

\[
F_{\varepsilon}(u_{\varepsilon}) = \int_{\Omega} \left[ \varepsilon |\varphi_{\varepsilon}^1(d(x))|^2 + \frac{1}{\varepsilon} W(\varphi_{\varepsilon}(d(x))) \right] dx = \\
= \int_{\Sigma_{\varepsilon}} \left[ \varepsilon |\varphi_{\varepsilon}^1(d)|^2 + \frac{1}{\varepsilon} W(\varphi_{\varepsilon}(d)) \right] dx = \\
= |\Sigma| \int_0^{\psi_{\varepsilon}(1)} \left[ \varepsilon |\varphi_{\varepsilon}^1(t)|^2 + \frac{1}{\varepsilon} W(\varphi_{\varepsilon}(t)) \right] dt.
\]

Since

\[ \varphi'_{\varepsilon} = \frac{1}{\psi_{\varepsilon}^2(\psi_{\varepsilon}^{-1})} = \frac{\sqrt{\varepsilon + W(\psi_{\varepsilon}^{-1})}}{\varepsilon} = \frac{1}{\varepsilon} \sqrt{\varepsilon + W(\varphi_{\varepsilon})} \]
we get
\[
F_\varepsilon(u_\varepsilon) = |\Sigma| \int_0^{\psi_\varepsilon(1)} \left[ \frac{\varepsilon + W(\varphi_\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon} W(\varphi_\varepsilon) \right] \, dt \leq
\]
\[
\leq \frac{2|\Sigma|}{\varepsilon} \int_0^{\psi_\varepsilon(1)} [\varepsilon + W(\varphi_\varepsilon)] \frac{dt}{d\varphi_\varepsilon} d\varphi_\varepsilon
\]
\[
= 2|\Sigma| \int_{-1}^{1} \sqrt{\varepsilon + W(s)} \, ds.
\]

Therefore
\[
\limsup_{\varepsilon} F_\varepsilon(u_\varepsilon) \leq C_0 |\Sigma|.
\]

Other cases have been considered in the Modica and Mortola paper; for instance if \( W \) is periodic and \( t_\varepsilon \to +\infty \)
\[
F_\varepsilon(u) = \int_{\Omega} \left[ \varepsilon |Du|^2 + \frac{1}{\varepsilon} W(t_\varepsilon u) \right] \, dx
\]
\( \Gamma \)-converge to
\[
F(u) = C_0 \int_{\Omega} |Du| \quad (\forall u \in BV(\Omega))
\]
where \( T \) is the period of \( W \)
\[
C_0 = \frac{2}{T} \int_0^T \sqrt{W(s)} \, ds.
\]

**Lesson 5. Gamma-convergence in optimal control theory**

The abstract framework is the following:
- \( Y \) space of states;
- \( U \) space of controls;
- \( J(u, y) \) cost functional;
- \( E \subset U \times Y \) admissible set given by the state equation.

The optimal control problem is then
\[
\min \{ J(u, y) : (u, y) \in E \}
\]
or equivalently
\[
\min \{ F(u, y) : (u, y) \in U \times Y \} \quad \text{where} \quad F = J + \chi_E.
\]

When we deal with sequences of problems
\[
\min \{ F_\varepsilon(u, y) : (u, y) \in U \times Y \} \quad \text{where} \quad F_\varepsilon = J_\varepsilon + \chi_{E_\varepsilon}
\]
we have to study the \( \Gamma \)-convergence of \( F_\varepsilon \) in the product space \( U \times Y \).

The typical case is:
- \( U = L^p(0, T; \mathbb{R}^m) \) topology \( w = L^p \);
- \( Y = W^{1,1}(0, T; \mathbb{R}^m) \) topology strong \( L^\infty \);
- \( J_\varepsilon(u, y) = \int_0^T f_\varepsilon(t, y, u) \, dt \);
- \( E_\varepsilon = \{ y' = a_\varepsilon(t, y) + b_\varepsilon(t, y)u, \ y(0) = \xi_\varepsilon \} \).

We would like to study the \( \Gamma \)-limits of \( J_\varepsilon \) and of \( \chi_{E_\varepsilon} \) separately, but the equality
\[
\Gamma \lim F_\varepsilon = \Gamma \lim J_\varepsilon + \Gamma \lim \chi_{E_\varepsilon}
\]
is false in general. To bypass this difficulty we introduce the multiple \( \Gamma \)-limits for functions on a product space.

\[
\Gamma(U^-, Y^-) \liminf_{\varepsilon} F_\varepsilon(u, y) = \inf_{u \in U} \inf_{y \in Y} \liminf_{\varepsilon} F_\varepsilon(u, y)
\]

\[
\Gamma(U^-, Y^+) \liminf_{\varepsilon} F_\varepsilon(u, y) = \inf_{u \in U} \sup_{y \in Y} \liminf_{\varepsilon} F_\varepsilon(u, y)
\]

\[
\Gamma(U^+, Y^-) \liminf_{\varepsilon} F_\varepsilon(u, y) = \sup_{u \in U} \inf_{y \in Y} \liminf_{\varepsilon} F_\varepsilon(u, y)
\]

\[
\Gamma(U^+, Y^+) \liminf_{\varepsilon} F_\varepsilon(u, y) = \sup_{u \in U} \sup_{y \in Y} \liminf_{\varepsilon} F_\varepsilon(u, y)
\]

and analogously for the \( \Gamma \)-limits with \( \limsup \). When two of them coincide we use notations as

\[
\Gamma(U, Y^-) \liminf_{\varepsilon} F_\varepsilon, \quad \Gamma(U, Y) \limsup_{\varepsilon} F_\varepsilon, \quad \Gamma(U^-, Y) \liminf_{\varepsilon} F_\varepsilon.
\]

In this way it is possible to sum with the \( \Gamma \)-limits. More precisely we have:

\[
\Gamma(U^-, Y^-) \lim_{\varepsilon} (F_\varepsilon + G_\varepsilon) = \Gamma(U^-, Y) \lim_{\varepsilon} F_\varepsilon + \Gamma(U, Y^-) \lim_{\varepsilon} G_\varepsilon
\]

(see Buttazzo and Dal Maso [JOTA]). Since the \( \Gamma \)-limits which we want to study is the

\[
\Gamma(U^-, Y^-) \lim_{\varepsilon}(J_\varepsilon + \chi_{E_\varepsilon})
\]

we have to identify the limits

\[
\Gamma(U^-, Y) \lim_{\varepsilon} J_\varepsilon, \quad \Gamma(U, Y^-) \lim_{\varepsilon} \chi_{E_\varepsilon}.
\]

We restrict our analysis to the case (for other cases see Buttazzo and Dal Maso [JOTA])

\[
J_\varepsilon(u, y) = \int_0^T f_\varepsilon(t, y, u) dt
\]

\[
E_\varepsilon = \{ y' = a_\varepsilon(t, y) + b_\varepsilon(t, y)u, \ y(0) = \xi_\varepsilon \}.
\]

**Case when \( b_\varepsilon \) is strongly convergent.**

Assumptions on \( f_\varepsilon : [0, T] \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R} \) Borel functions:

(i) \( f_\varepsilon(t, s, \cdot) \) is convex and l.s.c. on \( \mathbb{R}^m \);

(ii) \( f_\varepsilon(t, s, z) \geq |z|^p \) \( (p > 1) \);

(iii) for every \( R > 0 \) there exists a continuity modulus \( \omega_R \) such that

\[
|f_\varepsilon(t, s_1, z) - f_\varepsilon(t, s_2, z)| \leq \omega_R(|s_1 - s_2|)(1 + f_\varepsilon(t, s, z))
\]

for every \( t \in [0, T], s_1, s_2 \in \mathbb{R}^m \) with \( |s_1|, |s_2| \leq R \);

(iv) there exists \( u_\varepsilon \in L^p \) such that \( f_\varepsilon(t, 0, u_\varepsilon(t)) \) is weakly compact in \( L^1 \).

Then the \( \Gamma(U^-, Y) \lim_{\varepsilon} J_\varepsilon \) can be computed in the following way (see Marcellini and Sbordone [Ric. Mat. 1977]): for every \( s \in \mathbb{R}^m \) and \( z^* \in \mathbb{R}^m \)

\[
\varphi(\cdot, s, z^*) = w - L^1 \liminf_{\varepsilon} f^*_\varepsilon(\cdot, s, z^*)
\]

\[
f(t, s, z) = \varphi(t, s, z)
\]

\[
\Gamma(U^-, Y) \lim_{\varepsilon} J_\varepsilon(u, y) = \int_0^T f(t, y, u) dt.
\]
For instance if $f_{\varepsilon}(t, s, z) = a_{\varepsilon}(t)|z|^2 + |s - y_0(t)|^2$ we have $f(t, s, z) = a(t)|z|^2 + |s - y_0(t)|^2$ where

$$\frac{1}{a_{\varepsilon}} \rightarrow \frac{1}{a} \text{ weakly in } L^1(0, T).$$

Concerning the differential state equations we assume:

(i) $|a_{\varepsilon}(t, s_1) - a_{\varepsilon}(t, s_2)| \leq \alpha_{\varepsilon}(t)|s_1 - s_2| \quad \text{with } \sup_{t} \int_{0}^{T} \alpha_{\varepsilon} \, dt < +\infty$;

(ii) $|b_{\varepsilon}(t, s_1) - b_{\varepsilon}(t, s_2)| \leq \beta_{\varepsilon}(t)|s_1 - s_2| \quad \text{with } \sup_{t} \int_{0}^{T} \beta_{\varepsilon} \, dt < +\infty$;

(iii) $\sup_{t} \int_{0}^{T} |a_{\varepsilon}(t, 0)| \, dt < +\infty$;

(iv) $\sup_{t} \int_{0}^{T} |b_{\varepsilon}(t, 0)| \, dt < +\infty$;

(v) $a_{\varepsilon}(-, s) \Rightarrow a(-, s)$ weakly in $L^1$ $\forall s \in \mathbb{R}^n$;

(vi) $b_{\varepsilon}(-, s) \Rightarrow b(-, s)$ strongly in $L^p$ $\forall s \in \mathbb{R}^n$;

(vii) $\xi_{\varepsilon} \Rightarrow \xi$ in $\mathbb{R}^n$.

Then $\Gamma(U, Y^-) \lim_{\varepsilon} \chi E_{\varepsilon} = \chi_R$ where

$$E = \{ y' = a(t, y) + b(t, y)u, \; y(0) = \xi \}.$$

Therefore the limit control problems is

$$\min \left\{ \int_{0}^{T} f(t, y, u) \, dt : \; y' = a(t, y) + b(t, y)u, \; y(0) = \xi \right\}.$$

**Case when $b_{\varepsilon}$ is only weakly convergent.**

Assume for the sake of simplicity that $b_{\varepsilon} = b_{\varepsilon}(t)$ and that (vi) is substituted by

(vi) $b_{\varepsilon} \rightharpoonup b$ weakly in $L^p$.

The simplest situation is when $|b_{\varepsilon}|^p$ is equi-uniformly integrable (we shall remove later this assumption). In this case it is convenient to introduce an auxiliary variable $v \in V = L^1(0, T)$ and rewrite the control problems in the form

$$\min \left\{ \int_{0}^{T} [f_{\varepsilon}(t, y, u) + \chi_{v=b_{\varepsilon}(t)}u] \, dt : \; y' = \frac{a}{\varepsilon}(t, y) + v, \; y(0) = \xi_{\varepsilon} \right\}.$$

We can now apply the previous analysis with

$$\tilde{Y} = Y, \quad \tilde{U} = U \times V, \quad \tilde{f}_{\varepsilon}(t, s, z, w) = f_{\varepsilon}(t, s, z) + \chi_{w=b_{\varepsilon}(t)}z, \quad \tilde{a}_{\varepsilon}(t, s) = a_{\varepsilon}(t, s), \quad \tilde{b}_{\varepsilon}(t, s) \cdot (z, w) = w,$$

obtaining as a limit problem

$$\min \left\{ \int_{0}^{T} \tilde{f}(t, y, u, v) \, dt : \; y' = a(t, y) + v, \; y(0) = \xi \right\}$$

being

$$\tilde{f}(t, s, z, w) = (w - L^1 \lim_{\varepsilon} (f_{\varepsilon}(t, s, z) + \chi_{w=b_{\varepsilon}(t)}z)^*)^*$$
where the * operator is now made with respect to the variables \((z, w)\). Finally we eliminate the variable \(v\) by solving \(v = y' - a(t, y)\) and plugging into the cost functional

\[
\min \left\{ \int_0^T \tilde{f}(t, y, u, y' - a(t, y)) \, dt : y(0) = \xi \right\}.
\]

Note that

\[
(f_\varepsilon(t, s, z) + \chi_{w=b_\varepsilon(t)z})^* (t, s, z^*, w^*) = f_\varepsilon^*(t, s, z^* + b_\varepsilon(t)w^*)
\]

and in some cases the function \(\tilde{f}\) is finite everywhere, that is the state equation may disappear in the limit problem. Consider for instance the case

\[
f_\varepsilon(t, s, z) = |z|^2 + |s - y_0(t)|^2 \quad \text{(for every } \varepsilon\)
\]

and

\[
\begin{cases}
y' = a_\varepsilon(t, y) + b_\varepsilon(t)u \\
y(0) = \xi_\varepsilon
\end{cases}
\]

with \(a_\varepsilon(\cdot, s)\) weakly \(L^1\) convergent to \(a(\cdot, s)\) and \(b_\varepsilon \to b\) weakly \(L^2\) with \(b_\varepsilon^2 \to \beta^2\) weakly \(L^1\). Then some easy computations give

\[
\tilde{f}(t, s, z, w) = |z|^2 + \frac{(w - b(t)z)^2}{\beta^2(t) - b^2(t)}
\]

so that the limit problem is

\[
\min \left\{ \int_0^T \left[ |u|^2 + |y - y_0(t)|^2 + \frac{|y' - a(t, y) - b(t)u|^2}{\beta^2(t) - b^2(t)} \right] \, dt : y(0) = \xi \right\}
\]

and the relaxed form of the limit state equation is now in a penalization term.

For instance \(b_\varepsilon(t) = \sin(t/\varepsilon)\) gives \(b \equiv 0\), \(\beta^2 \equiv 1/2\) so that the limit problem becomes

\[
\min \left\{ \int_0^T [|u|^2 + |y - y_0|^2 + 2|y' - a(t, y)|^2] \, dt : y(0) = \xi \right\}.
\]

We want now to drop the assumption that \(|b_\varepsilon|^p\) is equi-uniformly integrable. In this case we may only obtain (up to extracting subsequences) that \(|b_\varepsilon|^p\) converges to a suitable measure \(\mu\) in the weak* convergence of measures. Assume for simplicity that the cost integrands are of the form

\[
f_\varepsilon(t, s, z) = \varphi_\varepsilon(t, z) + \psi(t, s).
\]

In this case the limit problem is expressed by means of the measure \(\mu\) in the following way (see Buttazzo and Freddi [AMSA]). As before consider the auxiliary variable \(v = b_\varepsilon(t)u\) and the polar integrand (with respect to \((z, w)\))

\[
(\varphi_\varepsilon(t, z) + \chi_{w=b_\varepsilon(t)z})^* (t, z^*, w^*).
\]

It is possible to show that (up to subsequences) this integrand converges weakly* in \(\mathcal{M}(\overline{\Omega})\) to a measure of the form

\[
g(t, z^*, w^*) \cdot \nu \quad \text{(with } \nu = dt + \mu_x)\]

where \(g(t, \cdot, \cdot)\) is convex. Then the limit problem is with cost

\[
\int_{\overline{\Omega}} q^*(t, w, \frac{dv}{d\nu}) \, d\nu + \int_\Omega \psi(t, y) \, dt + \chi_{\{v<<\nu\}}.
\]
and state equation
\[
\begin{align*}
    y' &= a(t,y) + v & \text{(in the sense of } \mathcal{M}(\Omega)) \\
    y(0) &= \xi.
\end{align*}
\]
Eliminating the variable \( v \) (which varies in the space of measures) we obtain again that the limit differential state equation may disappear becoming a penalization:
\[
\int_\Omega g^*(t,u,y'_{r} - a(t,y)) \, dt + \int_\Omega g^*(0,0, \frac{dy'_{s}}{d\mu_s}) \, d\mu_s + \int_\Omega \psi(t,y) \, dt + \chi(y'_{s} \ll d\mu_s)
\]
where \( y' = y'_{r} \cdot dt + y'_{s} \) is the decomposition of the measure \( y' \) into absolutely continuous and singular parts with respect to the Lebesgue measure \( dt \), and the last term is the constraint that \( y'_{s} \) must be absolutely continuous with respect to \( \mu_s \).

In the previous example
\[
f_{\varepsilon}(t,s,z) = |z|^2 + |s - y_0(t)|^2
\]
with \( a_{\varepsilon}(\cdot,s) \to a(\cdot,s) \) weakly \( L^1 \), \( b_{\varepsilon} \to b \) weakly \( L^2 \) but now \( b_{\varepsilon}^2 \to \mu \) weakly* in the sense of measures, we get at the limit
\[
\int_0^T \left[ |u|^2 + |y - y_0(t)|^2 + \frac{|y'_{r} - a(t,y) - b(t)u|^2}{\mu_r(t) - b^2(t)} \right] \, dt +
\int_{[0,T]} \left[ \frac{|dy'_{s}}{d\mu_s}^2 + \frac{|y(0^+) - \xi|^2}{\mu_s(\{0\})} \right] + \chi(y'_{s} \ll d\mu_s).
\]
For instance, if \( b_{\varepsilon}(t) = \sin(t/\varepsilon) + \frac{1}{\sqrt{\varepsilon}} 1_{[0,\varepsilon]}(t) \) we have \( b \equiv 0 \) and \( \mu = \frac{1}{2} dt + \delta_0 \) so that the limit problem is
\[
\min_{u \in L^2, y \in W^{1,1}} \left\{ \int_0^T \left[ |u|^2 + |y - y_0(t)|^2 + 2|y' - a(t,y)|^2 \right] dt + |y(0^+) - \xi|^2 \right\}.
\]

\section*{Bibliography}


