Andrea Braides
(Università di Roma
`Tor Vergata', Italy)

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Spa2015@oxford-man.ox.ac.uk

Variational Percolation Problems

Andrea Braides (Roma *Tor Vergata*)

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Model Setting: Discrete Variational Problems

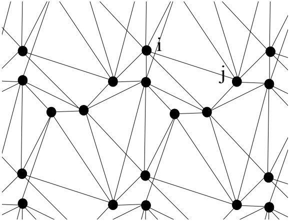
(bond formulation)

$\mathcal{L}$ = lattice in $\mathbb{R}^d$

u_i = parameter describing the system, with $i \in \mathcal{L}$

$$E(\{u_i\}) = \sum_{(i,j)} \Phi_{ij}(u_i, u_j) \qquad \text{energy}$$

(Φ_{ij} = bond interaction energy - pairwise for simplicity)
The sum runs over a given set of bonds (e.g., nearest neighbours)



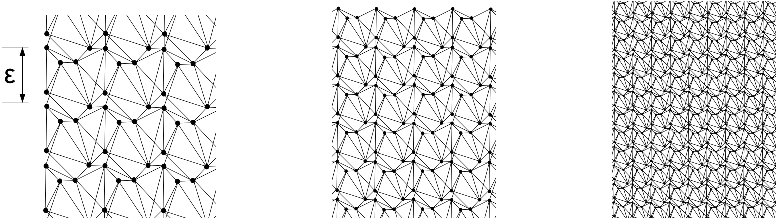
(more in general we may have three-point interactions, etc.)

Model: $u_i \in \mathbb{R}^d$ atomistic displacement

$\phi_{ij}(u_i, u_j) = \phi(|u_i - u_j|)$ interatomic pair potential

Discrete-to-Continuum Analysis

cf. Chambolle, B-Gelli, Blanc-Le Bris-Lions, Friesecke-Theil, Caffarelli-de la Llave, etc.



$$\varepsilon \longrightarrow 0$$

- Scale the lattice: $\varepsilon\mathcal{L}$, with $\varepsilon > 0$ a small parameter
- (Possibly) localize the analysis on a bounded domain $D \cap \varepsilon\mathcal{L}$ (equivalent to considering the original energy on a large domain $\frac{1}{\varepsilon}D$)

- Scale the energy

$$E_\varepsilon(\{u_i\}) = \sum_{(i,j)} \phi^\varepsilon_{ij}(u_i, u_j)$$

(u_i value at εi; the definition of ϕ^ε_{ij} from ϕ_{ij} depends on the relevant energy scale for the analysis)

- define a discrete-to-continuum convergence $u_\varepsilon \to u$ of functions defined on $D \cap \varepsilon\mathcal{L}$ to functions defined on D (e.g., convergence of piecewise-constant interpolations on Voronoi cells, or convergence of the empirical measures $\mu_\varepsilon = \sum \varepsilon^d u_i \delta_{\varepsilon i}$)

- Compute a **limit continuum energy**.

Γ-limit (zero-temperature limit)

In this context the limit continuum energy F is defined as the Γ-limit of E_ε with respect to the L^1 convergence of interpolations.

The formal definition is that for all u

- $F(u) \leq \liminf_\varepsilon E_\varepsilon(u_\varepsilon)$ whenever $u_\varepsilon \to u$ (lower bound)
- there exists $\overline{u}_\varepsilon \to u$ such that $F(u) = \lim_\varepsilon E_\varepsilon(u_\varepsilon)$ (optimality)

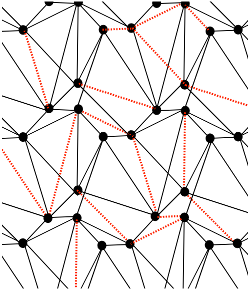
Fundamental properties:

- if E_ε is equicoercive (i.e., energy-bounded sequences are precompact) and $E_\varepsilon \to F$ then minimum problems for E_ε converge to minimum problems for F (convergence of minimum values and minimizers)
- (stability) for a class of perturbations G the Γ-convergence of E_ε to F implies the Γ-convergence of $E_\varepsilon + G$ to $F + G$ (this allows e.g. to treat minimum problems with given boundary conditions, integral constraints, etc.)

(...some extensions to positive temperature possible)

Discrete Variational Problems w/ Random Defects

Introduce i.i.d. random variable that model the presence of random defects with probability p



Correspondingly, consider two types of energies
ϕ_{s} = strong bond interaction, ϕ_{w} = weak bond interaction
and define ϕ_{ij} by

$$\phi_{ij} = \phi_{ij}^{\omega} = \begin{cases} \phi_{\mathrm{w}} & \text{on defected bonds (w/ prob. } p) \\ \phi_{\mathrm{s}} & \text{on non-defected bonds (w/ prob. } 1-p) \end{cases}$$

(ω = realization of the random variable)

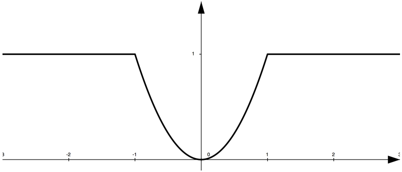
1st Model Case: Blake-Zisserman weak membrane

(computer vision/fracture mechanics)

Consider: $\mathcal{L} = \mathbb{Z}^2$, only nearest-neighbour interactions, parameter $u_i \in \mathbb{R}$

$$\phi_{\mathrm{s}}(u_i, u_j) = (u_i - u_j)^2, \quad \phi_{\mathrm{w}}(u_i, u_j) = \min\{(u_i - u_j)^2, 1\}$$

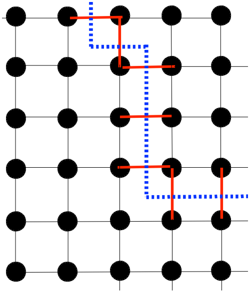
(*truncated quadratic potential*)



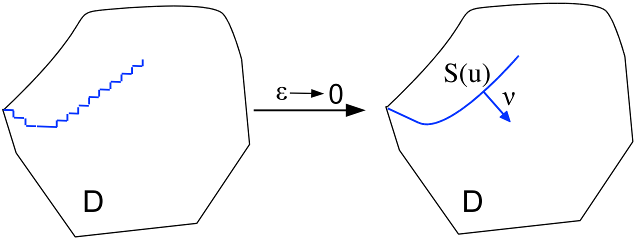
(this model can be derived from Lennard-Jones interactions)

Strong energy (if $\phi_{ij} = \phi_{\mathrm{s}}$ for all ij): the discretization of the Dirichlet energy $\int_D |\nabla u|^2 \, dx$ with domain $H^1(D)$

Weak energy ($\phi_{ij} = \phi_{\rm s}$ for all i, j): for all $\{u_i\}$ we can consider the set of bonds such that $|u_i - u_j| \geq 1$ and associate to it a path in the dual lattice representing a discrete fracture site



We scale $\phi_{\rm w}$ so that the energy on discrete fractures scales as a surface energy: $\phi^{\varepsilon}_{ij} = \min\{(u_i - u_j)^2, \varepsilon\}$



The Γ-limit of E_ε is an anisotropic continuum fracture energy

$$F(u) = \int_{D \setminus S(u)} |\nabla u|^2 \, dx + \int_{S(u)} (|\nu_1| + |\nu_2|) d\mathcal{H}^1$$

defined on functions $u \in SBV(D)$; i.e., functions with a (sufficiently smooth) **discontinuity set** $S(u)$ and (weakly) differentiable outside that set.

Notation: ν = normal to $S(u)$, $\mathcal{H}^k$= k-dimensional surface (Hausdorff) measure

Random weak-membrane percolation theorem

(B-Piatnitski ARMA 2008)

With fixed a realization ω *let scaled energy densities with*

$$\phi^{\varepsilon}_{ij} = \begin{cases} \min\{(u_i - u_j)^2, \varepsilon\} & \text{on defected bonds (w/ prob. } p) \\ (u_i - u_j)^2 & \text{on non-defected bonds (w/ prob. } 1-p) \end{cases}$$

and E^{ω}_{ε} *the corresponding discrete energies. Then the* Γ*-limit is deterministic, depends almost surely only on* p *and*

- *if* $p \le \frac{1}{2}$ *then* $F(u) = \int_D |\nabla u|^2\, dx$ *(negligible defects)*
- *if* $p > 1/2$ *then* $F(u) = \int_{D\setminus S(u)} |\nabla u|^2\, dx + \int_{S(u)} \varphi_p(\nu) d\mathcal{H}^1$,

where $\varphi_p(\nu)$ *is the asymptotic chemical distance on the weak cluster in direction* ν (e.g. Garet-Marchand '04-'07)

$$\varphi_p(\nu) = \lim_{T\to+\infty} \frac{1}{T} \text{ length of the minimal path of weak bonds joining } 0 \text{ and } T\nu$$

Q.: Asymptotic behaviour of φ_p as $p \to 1/2^+$?

Variational percolation issues

Subcritical regime: Poincaré inequality on the strong cluster. Let $p < 1/2$. Then the “channel property” of the strong cluster gives a.s. a uniform Poincaré inequality on the strong cluster

$\Rightarrow$ compactness in Sobolev spaces

$\Rightarrow H^1(S(u)) = 0$

Supercritical regime: a Quantitative Percolation Lemma (Kesten) Let $p > 1/2$. For a.e. realization ω and for T sufficiently large “paths of bonds joining 0 and $T\nu$ with length less than $T(\varphi_p(\nu) - \eta)$ must contain a percentage $c_\eta > 0$ of strong bonds”

$\Rightarrow$ may suppose that bonds with $|u_i - u_j| > 1$ lie in the weak cluster

$\Rightarrow$ length of discontinities are estimated by chemical distance

2nd Model Case: Dilute Spin Systems

(statistical mechanics/ continuum mechanics for perforated domains)

Consider: $\mathcal{L} = \mathbb{Z}^d$ $(d = 2, 3)$, nearest-neighbour interaction, $u_i \in \{-1, +1\}$ (spin variable)

$$\phi_{\rm s}(u_i, u_j) = -u_i u_j \qquad \text{(ferromagnetic interaction)}$$

$$\phi_{\rm w}(u_i, u_j) = 0 \qquad \text{(noninteracting spins)}$$

Ferromagnetic Γ-limit: If $\phi_{ij} = \phi_{\rm s}$ for all ij and $\phi^\varepsilon_{ij} = \varepsilon^{d-1}\phi_{ij}$ (up to additive constants) we have coerciveness with respect to the strong convergence $u_\varepsilon \to u \in \{\pm 1\}$, and u can be identified with the set of finite perimeter $A = \{u = 1\}$. Then the Γ-limit is simply

$$F(A) = \int_{\partial A} (|\nu_1| + |\nu_2|) d\mathcal{H}^1$$

$(F(A) = \int_{\partial A} (|\nu_1| + |\nu_2| + |\nu_3|) d\mathcal{H}^2$ if $d = 3$) that is, a crystalline perimeter energy with the Wulff shape a square (resp., a cube)

Dilute-spin variational percolation theorem

(B-Piatnitski J. Stat. Phys. 2012)

As above, introduce iid random variable such that

$$\phi_{ij} = \phi_{ij}^{\omega} = \begin{cases} 0 & \text{on defected bonds w/ prob. } p \\ -u_i u_j & \text{on non-defected bonds w/ prob. } 1-p \end{cases}$$

(ω realization of the random variable) and consider the energies

$$E_\varepsilon(u) = \sum_{(i,j)} \varepsilon^{d-1} \phi_{ij}^{\omega}(u_i, u_j).$$

2D Result The Γ-limit is deterministic, depends almost surely only on p and

- If $p < 1/2$ then $F(A) = \int_{\partial A} \psi_p(\nu) d\mathcal{H}^1$ and we have compactness in L^1 on the strong cluster (ψ_p = first-passage percolation formula; cf. eg Grimmet-Kesten)

- if $p \geq 1/2$ then $F(A) = 0$.

3D Result

The Γ-limit is deterministic and depends a.s. only on p.
Let p_* (resp., p^*) be the percolation threshold below which (resp., above which) the weak (resp., strong) cluster is not connected.

- If $p < p_*$ then $F(A) = \displaystyle\int_{\partial A} \psi_p(\nu) d\mathcal{H}^2$ and we have coerciveness in L^1 on the strong cluster;
(*surface tension* ψ_p as in Wouts '09, Cerf-Theret '11)
- if $p \geq p^*$ then $F(A) = 0$
- (partially) open problem if $p_* \leq p < p^*$ then $F(A) = \displaystyle\int_{\partial A} \psi_p(\nu) d\mathcal{H}^2$ and ψ_p is positive, but coerciveness is not known (even though $\psi_p > 0$)

Percolation coerciveness lemma.

(3d case) Let $p < p_*$. Then a.s. "if we have a connected set composed of N bonds and containg 0 then it contains a fixed percentage of strong bonds for N large "

$\Rightarrow$ BV estimates of sets $\{u = 1\}$ on the strong cluster

Conclusions

Variational percolation problems involve interaction between variational techniques and probabilistic issues. We have examined two model cases.
Many more interesting variational model involve random quantities:
- random lattices (e.g., Poisson clouds)
- long-range interactions
- anti-ferromagnetic inclusions
- etc.

On one hand Percolation Theory provides relevant objects and techniques for the a.s. description of limit variational problems
On the other hand variational questions introduce new types of issues in the percolation context.

Thank you for your attention!